The Myth of Long Horizon Predictability:

An Asset Allocation Perspective.

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Abstract:

We analyse the effects of asset return predictability on an investor's portfolio strategy and welfare by comparing the portfolio's certainty equivalent rates of return obtained by exploiting predictability at various horizons. Using overlapping observations, we show that the impact of predictability on the investor's welfare is stronger for shorter return periods than for longer ones when the dividend yield and the 3-month Treasury bill rate are used as predictors. However, when we correct, as in Valkanov (2003), for the persistence in the predictive regression residuals brought about by overlapping observations, the conclusion is overturned. Predictability is strong across predictive horizons, and tend to be stronger at longer horizons, although the relationship is not monotonous but U-shaped. Overall, we find that the welfare losses suffered by investors following sub-optimal, myopic strategies enlarge across the board as both the investment and the prediction horizons increase.

1 INTRODUCTION

The issue of stock return predictability, initially investigated by Fama and Schwert (1987) and Campbell (1987), is still hotly debated in the literature. The empirical evidence as to short term predictability is at best mixed and drastically depends on the frequency of observations and the overall period under scrutiny. In addition, out-of-sample results may conflict with in-sample evidence, as shown for instance by Goyal and Welch (2005) and Ang and Bekaert (2007). Recent papers by Menzly, Santos and Veronesi (2004), Santos and Veronesi (2006), and Lettau and Van Nieuwerburgh (2008) have explained variously this time instability, in particular that of the aggregate dividend yield. Menzly, Santos and Veronesi (2004) offer that the latter exerts two contradictory influences on future stock returns, which may eliminate predictability if they happen to cancel out. Lettau and Van Nieuwerburgh (2008) claim that predictability reappears once regime switching on the part of the dividend yield is taken into account.

In view of this evidence, attention swiftly shifted to predictability over longer horizons, following the lead of Fama and French (1988) and Campbell and Shiller (1988). Long term predictability appears much more robust, as the R^2s of the predictive regressions increase considerably with the length of the return period. Many authors believe and use in theoretical and empirical analyses what now constitutes the "conventional wisdom" of weak predictability at short horizons and strong one at long horizons.[1] Useful predictors range from some measure of dividend yield, the short term interest rate, the yield spread, the default spread, and market volatility. However, support for this view is not universal. Other researchers have recently claimed that the improvement in R^2s is largely spurious, the main culprit being the persistence in variables brought about by the use of overlapping observations. When the thus created persistence is cleansed away, long term predictability seems to become even more problematic than short term predictability. For instance, Ang and Bekaert (2007) argued that it disappears when Hodrick's (1992) correction procedure is applied. Boudoukh, Richardson and Whitelaw (2008), using both an indirect approach and econometric tests, even claim that long run stock return predictability must be discarded as a myth.

So far, the debate in that strand of literature has centered on estimation and inference issues, not on applications to actual portfolio management, Kandel and Stambaugh (1996), Barberis (2000), and Brennan and Xia (2010) being notable exceptions. Also, scant attention has been

[1] See for instance the standard textbooks by Campbell, Lo, and MacKinlay (1997) and Cochrane (2001), or the surveys by Fama (1998), Campbell (2003) and Barberis and Thaler (2003) among many others.

devoted to the bond market, in spite of the early lead of Fama and Bliss (1987). In this paper, not only do we revisit the predictability issue by investigating both the U.S. stock and bond markets over a long time span at a monthly frequency, but we focus our analysis on the effect of predictability on an investor's portfolio strategy and welfare. In particular, we compare the portfolio's certainty equivalent returns obtained by exploiting predictability at various horizons. In a first-round analysis, we show that the impact of predictability on the investor's welfare is stronger for shorter prediction horizons than for longer ones when the dividend yield and the 3-month Treasury bill rate are used as predictors. The relationship, however is not monotonous and exhibits an inverse U-shaped pattern. We then make a drastic correction, as in Valkanov (2003), for the persistence in the predictive regression residuals induced by the use of overlapping observations, which invalidates our previous results. Long term predictability is seemingly not a myth and has a stronger impact than short term predictability. The relationship, however, is not monotonous and presents a U-shaped pattern, except for very short term investors for whom it is more linear. Overall, predictability at long horizons is valuable to investors, and seems rather robust. We thus tend, cautiously, to side with the conventional wisdom provided the parameter estimates are corrected for persistence and the prediction horizon increases with the investment horizon.

The remainder of the paper is structured as follows. Section 2 provides a description of the economy in which the individual's investment decisions take place and derives her optimal (and sub-optimal) dynamic portfolio strategies. Section 3 explains the procedure followed to take the continuous time model to the data and obtain estimates for the parameters of the properly discretized versions of the relevant continuous processes. Section 4 describes the data. Section 5 reports the empirical results regarding our bivariate predictive regressions and the estimated parameters for the predictor and excess return processes. Section 6 presents our empirical evidence as to the impact of stock and bond return predictability on (i) the investor's optimal asset allocation strategy and (ii) two different sub-optimal portfolio strategies. [Section 7 provides additional results regarding the level of risk aversion and the stability of the portfolio composition over time]. Section 8 proposes a way, inspired by Valkanov (2003), to correct for the persistence in the residuals of the regressions of asset excess returns on the predictors created by using overlapping data. Section 9 provides concluding remarks and discusses possible extensions that can be pursued. All proofs and technical derivations are left to the Appendix for readability.

2 DYNAMIC ASSET ALLOCATION

We describe first the economy in which the investor operates. We then derive her optimal portfolio strategy and compute the normalized certainty equivalent of her optimal terminal wealth. Finally, we assess the (percentage) loss of welfare incurred should she follow a sub-optimal strategy and ignore intertemporal hedging.

2.1 The economy

We consider a frictionless and arbitrage-free financial market in which trading takes place continuously. There are three assets available for trade, a riskless asset (the money market account), a stock (the market index or portfolio, or more simply the market) and a long lived, constant maturity bond. The value C_t of the money market account (initially set at \$1) evolves through time according to

$$\frac{dC_t}{C_t} = r_t dt, \tag{1}$$

where r_t is the instantaneous interest rate assumed to follow the Ornstein-Ulhenbeck process

$$dr_t = \theta_r \left[\overline{r} - r_t\right] dt + \sigma_r dZ_{r,t}. \tag{2}$$

The parameters θ_r and σ_r are constant and $\overline{r}$ is the long term value of the riskless rate. $Z_{r,t}$ is a one-dimensional Brownian motion under the actual (historical) probability measure. Note that all our subsequent Brownian motions $Z_{x,t}$ are also one-dimensional and defined under the historical measure, and that all are generically correlated.

The value M_t of the market portfolio is assumed to evolve according to the stochastic differential equation (SDE):

$$\frac{dM_t}{M_t} - r_t dt = \mu_{M,t} dt + \sigma_M dZ_{M,t}, \tag{3}$$

where $\mu_{M,t}$ denotes the expected risk premium on the market, which is assumed to be time-varying, and σ_M is the market volatility assumed constant for simplicity. $Z_{M,t}$ is a Brownian motion whose correlation with $Z_{r,t}$ is denoted by $\rho_{r,M}$.

The value B_t of the constant maturity bond is assumed to obey the SDE:

$$\frac{dB_t}{B_t} - r_t dt = \mu_{B,t} dt + \sigma_B dZ_{B,t}, \tag{4}$$

where $\mu_{B,t}$ denotes the time-varying, expected excess return on the bond and σ_B its volatility which is assumed constant,[2] and $Z_{B,t}$ is a Brownian motion whose correlations with $Z_{M,t}$ and $Z_{r,t}$ are denoted by $\rho_{B,M}$ and $\rho_{r,B}$, respectively. Note that, unlike in Vasicek's (1977) model, returns on fixed income instruments of different maturities are not perfectly correlated because predictors exert their influence variously, which justifies the use of distinct Z_r and Z_B.

We impose some structure on the stock and bond risk premia by assuming they vary linearly in some predictors. It is generally accepted in the extant literature that a univariate predictive regression generally suffers from a missing variable problem. To keep the model tractable, we then follow Ang and Bekaert (2007), among many others, and consider bivariate predictive regressions, although the theoretical framework could accommodate any number of predictors.[3] Using $z_{1,t}$ and $z_{2,t}$ as predictors, we assume the following affine structure:

$$\mu_{M,t} = \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t}, \tag{5}$$

$$\mu_{B,t} = \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t}. \tag{6}$$

where $\mu_{i,0}, \mu_{i,1}$ and $\mu_{i,2}$ are constants for $i = M,\ B$.

Usually a single predictor is assumed to obey a Gaussian AR(1) process and a set of predictors a Gaussian VAR(1) process. As Amihud and Hurvich (2004) have shown, however, adopting a VAR process leads to far more involved expressions and more parameters to be estimated, which deteriorates the model's performance if the matrix of the regressors' vector autoregressive coefficients is in fact diagonal or nearly so. Therefore, we keep the simpler AR(1) approach and assume the Ornstein-Ulhenbeck processes:

$$dz_{i,t} = \theta_i \left[\overline{z_i} - z_{i,t}\right] dt + \sigma_{z_i} dZ_{z_i,t},\ \forall i = 1, 2, \tag{7}$$

where the parameters θ_i, σ_{z_i}, $\rho_{i,M}$, $\rho_{i,B}$, $\rho_{i,r}$ $(i = 1, 2)$ and $\rho_{1,2}$ (the correlation between the

[2] This assumption does not make much violence to the data since the bond has a constant maturity.

[3] The literature on asset return predictability has become very large. Following the lead of Fama and French (1988, 1989), Kandel and Stambaugh (1988) and Campbell and Shiller (1988), Campbell (1991), Hodrick (1992), Kandel and Stambaugh (1996), Kothari and Shanken (1997), Pontiff and Schall (1998), Stambaugh (1999), Keim and Madhavan (2000), Lettau and Ludvingson (2001), Lewellen (2004) and Amihud and Hurvich (2004, 2008), among others, have investigated the predictive power of return related macro variables such as the dividend/price ratio, dividend growth rate, book/market ratio, earnings/price ratio, default risk premium, term yield premium and short interest rate. Another strand of literature considers as predictors volatility and/or liquidity variables.

Brownian increments dZ_{z_1} and dZ_{z_2}) are constant, and $\overline{z_i}$ is the long term value towards which $z_{i,t}$ converges.

Given the above specification, the two predictors are correlated through their own respective correlation with the stock, the bond and the interest rate, as well as through their direct correlation (if $\rho_{1,2} \neq 0$). Note that the use of Ornstein-Uhlenbeck processes is standard in portfolio theory[4] and that the presence of either one of the two sources of risk $Z_{z_i,t}$ makes the financial market incomplete.

2.2 The investor's optimal portfolio strategy

Our individual has an investment horizon denoted by T and maximizes the expected utility of her terminal wealth V_T, subject to adopting an admissible, self-financing strategy. Her utility function, $U(V_T)$, is assumed to be iso-elastic:

$$U(V_T) = \frac{V_T^{1-\gamma}}{1-\gamma}, \tag{8}$$

where the constant γ (> 0 and $\neq 1$) is her relative risk aversion coefficient.[5]

We denote by $\omega_{M,t}$ and $\omega_{B,t}$ the proportion of her wealth invested in the stock and in the bond, respectively. She solves the program:

$$\begin{aligned} &\underset{\{\omega_t\}}{Max} E_t \left[\frac{V_T^{1-\gamma}}{1-\gamma} \right] \\ &s.t. \\ &\frac{dV_t}{V_t} = r_t dt + \omega_{M,t} \left[\frac{dM_t}{M_t} - r_t dt \right] + \omega_{B,t} \left[\frac{dB_t}{B_t} - r_t dt \right], \end{aligned} \tag{9}$$

and subject also to Eqs. (1) to (7).

A decisive advantage of ignoring intermediate consumption in the expected utility framework is that we can obtain an exact and quasi-explicit solution (see for instance Kim and Omberg (1996)) even though the financial market is incomplete.[6]

As shown in Appendix A [Eq. (27)], the investor's optimal portfolio strategy is characterized by:

[4] See for instance Brennan and Xia (2002) for the interest rate process and Kim and Omberg (1996) or Wachter (2002) for the market price of risk process.

[5] Log utility ($\gamma = 1$) is a special case that changes the structure and the nature of the solutions, e.g. implies a myopic behavior of no interest here.

[6] In contrast, under recursive utility for instance an approximation is needed at some point of the derivation. See for instance Hansen et al. (2008).

$$
\begin{aligned}
\begin{bmatrix} \omega_{M,t} \\ \omega_{B,t} \end{bmatrix} &= \frac{1}{\gamma}\Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\
&+ \frac{\left(A_1(T-t) + A_{11}(T-t)\, r_t + A_5(T-t)\, z_{1,t} + A_6(T-t)\, z_{2,t}\right)}{\gamma}\Sigma^{-1}\Sigma_r \\
&+ \frac{\left(A_2(T-t) + A_{22}(T-t)\, z_{1,t} + A_4(T-t)\, z_{2,t} + A_5(T-t)\, r_t\right)}{\gamma}\Sigma^{-1}\Sigma_1 \\
&+ \frac{\left(A_3(T-t) + A_{33}(T-t)\, z_{2,t} + A_4(T-t)\, z_{1,t} + A_6(T-t)\, r_t\right)}{\gamma}\Sigma^{-1}\Sigma_2.
\end{aligned} \tag{10}
$$

where the (2x2) variance-covariance matrix Σ of asset returns and the (2x1) vectors of covariances Σ_r, $\Sigma_1$and Σ_2 are defined in Appendix A, and the complicated $A_i(T-t)$ and $A_{ij}(T-t)$ functions (along with the function $A_0(T-t)$) are obtained from the system of ten ordinary differential equations derived in Appendix A. These equations are solved using Matlab.[7] The structure of this strategy is standard in an affine model. The first component is the traditional mean-variance term and the last three terms are Merton-Breeden intertemporal hedges against unfavorable shifts in the predictors and the interest rate that affect the stock and bond risk premia. The portfolio weight affected to the money market account is equal to $(1 - \omega_{M,t} - \omega_{B,t})$.

To compute the value at time t of the indirect utility function induced by the optimal strategy, we need in particular to know $\omega_{M,t}$ and $\omega_{B,t}$. The latter weights depend on the variables $z_{1,t}$ and $z_{2,t}$ and, directly or through the $A_i(T-t)$ and $A_{ij}(T-t)$ functions, on the following set of 27 parameters: $\mu_{M,0}$, $\mu_{M,1}$, $\mu_{M,2}$, $\mu_{B,0}$, $\mu_{B,1}$, $\mu_{B,2}$, σ_M, σ_B, σ_r, σ_{z_1}, σ_{z_2}, $\rho_{B,M}$, $\rho_{r,M}$, $\rho_{r,B}$, $\rho_{1,M}$, $\rho_{2,M}$, $\rho_{1,B}$, $\rho_{2,B}$, $\rho_{1,r}$, $\rho_{2,r}$, $\rho_{1,2}$, θ_r, θ_1, θ_2, $\overline{r}$, $\overline{z_1}$ and $\overline{z_2}$. These will be obtained by means of the estimation procedure described below. It is important to note that the set of parameters is derived at a given date t (we will always reports results for $t = 0$), for given *investment horizon* T and risk aversion coefficient γ, and for a given *prediction horizon*, denoted below by h. There is indeed no a priori reasons why the investment and predictive horizons should be identical, in spite of what most of the literature in fact does.

Once this step is completed, we can compute the optimal strategy and the certainty equivalent of the random (annualized) rate of return achieved by the strategy over the investment horizon. More precisely, denoting by $J(t,.)$ the investor's value function at date t $(0 \leq t \leq T)$

[7] These Ricatti Ordinary Differential Equations possess a highly non-linear structure as they are all interdependant. The terminal conditions (for $t = T$) are that all functions have zero value such that $e^{G(T-T,.)} = 1$. Details are available upon request to the authors.

and by $CEOW_t$ the certainty equivalent of the investor's optimal wealth, we have:

$$\frac{CEOW_t^{1-\gamma}}{1-\gamma} = J(t,.) = \frac{V_t^{1-\gamma}}{1-\gamma} e^{G(T-t,r_t,z_{1,t},z_{2,t})}, \tag{11}$$

where

$$\begin{aligned} G\left(T-t, r_t, z_{1,t}, z_{2,t}\right) \equiv\ & A_0\left(T-t\right) + A_1\left(T-t\right) r_t + \frac{1}{2} A_{11}\left(T-t\right) r_t^2 + A_2\left(T-t\right) z_{1,t} \\ & + \frac{1}{2} A_{22}\left(T-t\right) z_{1,t}^2 + A_3\left(T-t\right) z_{2,t} + \frac{1}{2} A_{33}\left(T-t\right) z_{2,t}^2 \\ & + A_4\left(T-t\right) z_{1,t} z_{2,t} + A_5\left(T-t\right) z_{1,t} r_t + A_6\left(T-t\right) r_t z_{2,t}. \end{aligned} \tag{12}$$

Evaluating at initial date $t = 0$, and normalizing initial wealth by setting $V_0 = \$1$, we can compute the certainty equivalent of the investor's optimal terminal wealth in dollar terms. $CEOW_0$ is the amount of wealth to be obtained at the investment horizon T with certainty that would yield the same expected utility as the one derived from investing \$1 optimally at date $t = 0$. Then calculating $(\ln CEOW_0)/T \equiv CE_{opt}$ yields the annualized "certainty equivalent" rate of return. The latter can for instance be compared to the average value of the riskfree rate of interest over the investment horizon T or, better still, the current market yield on the (default-free) zero-coupon bond maturing at date T. The main advantage of translating the result of a strategy in a certainty equivalent rate of return is that comparing results across different risk aversion parameters and predictability horizons (see Section 6 below for the latter point) is immediate and leads to obvious interpretation.[8] It also eases greatly the comparison with two sub-optimal strategies analyzed below. Its slight drawback is that, as an internal rate of return, it does not in general lead to meaningful comparison across T: one cannot say, for instance, that 10% per year over 8 years is better or worst than 9% per year over 10 years. Only if the annualized certainty equivalent rate of return is monotonously increasing with T can we conclude that the investor is better off with a larger rate for a longer horizon. Since the main objective of the paper is to compare results across h (the prediction horizon) for a *given* T, this potential drawback is not a concern.

[8] Brennan and Xia (2010) among others also use a certainty equivalent rate of return to assess the economic significance of asset return predictability in simulated portfolios.

2.3 The investor's sub-optimal strategies

A worth exploring aspect of portfolio allocation is the magnitude of the cost associated with a sub-optimal strategy. We investigate two such strategies. The first, which we call "myopic", ignores the hedging terms associated with the time variation in the riskless rate and the predictors. After all, if this cost is negligible or of second order only, it may not be economical in real world situations to indulge in the complex computation of intertemporal hedging. More precisely, we assess the magnitude of the welfare loss that results from the investor taking into account the mean-variance term only when she builds her portfolio. This loss will conveniently be measured by the *ratio* of the certainty equivalent rate of return of her optimal strategy over that achieved with the myopic one.

We therefore assume here that she ignores the three intertemporal hedging terms present in Eq.(10). The proportion of her wealth invested in risky assets according to her sub-optimal strategy thus reduces to:

$$\begin{bmatrix} \omega_{M,t} \\ \omega_{B,t} \end{bmatrix} = \frac{1}{\gamma}\Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix}. \tag{13}$$

As the market premia continue to be predicted by the variables z_i, the investor's HJB equation, while simpler than in the optimal case, also contains the derivatives of the value function with respect to these predictors (see Appendix B).

Note that the (sub-optimal) investor still has power utility ($\gamma \neq 1$).The myopic (in the usual sense) case of log utility ($\gamma = 1$) of course would also lead to ignoring the intertemporal hedging terms, but the rest of the derivation would be completely different. This is because the terms in the HJB equation involving the cross derivatives of the value function with respect to r and the z_i variables would vanish, and the log investor's strategy would be optimal.

We then compute the certainty equivalent of the investor's sub-optimal wealth ($CEMW$) in a way analogous to that followed for the optimal strategy. Setting $(\ln CEMW)/T \equiv CE_{myo}$ yields the annualized certainty equivalent rate of return of the myopic strategy.

Our second sub-optimal strategy consists in ignoring asset return predictability. We investigate the magnitude of the welfare loss the investor may suffer if she thus ignores time variation in expected risk premia. This loss will also be measured by a *ratio*, as above. We thus fix the market risk premia $\overline{\mu}_M$ and $\overline{\mu}_B$ to their long term values and ignore all terms connected to the

variability of the two predictors. We call this the "no predictability" strategy.

While the latter rests on the investor's assumption of constant risk premia, the dynamics of her wealth still depends on the stochastic evolution of the interest rate. Consequently, this sub-optimal strategy differs from the myopic one discussed above. Were the interest rate deterministic, the two strategies would be identical.

We then compute the certainty equivalent of the investor's sub-optimal wealth ($CENPW$). Then computing $(\ln CENPW)/T \equiv CE_{np}$ provides the certainty equivalent rate of the strategy.

3 THE ESTIMATION PROCEDURE

To compute the certainty equivalent rates of return on optimal or sub-optimal strategies, we must beforehand estimate the model parameters using actual data. We first have to identify the parameters for the dynamics of the predictors, the interest rate, the stock returns and the bond returns. As observed data are discrete, we must use a discretized version of the continuous process for each variate. We show in Appendix C how to make the discrete time parameters of the actual regressions consistent with the parameters of the continuous time processes.

We start by identifying the parameters of the predicting processes. For so doing, we integrate the continuous time dynamics (7) over the discrete time interval $[t, t+h]$, where h is the prediction horizon deemed relevant to the investor's problem. We will set h equal to $\Delta t,\ 2\Delta t,\ 3\Delta t, ...,$ where Δt is the length of the discrete interval adopted for the data (a month therein), so that we can later assess the relative merits of various horizons and compare the short and long run powers of the predictors. We thus obtain:

$$z_{i,t+h} = \overline{z_i}\left(1 - e^{-\theta_i h}\right) + e^{-\theta_i h} z_{i,t} + \sigma_{z_i} e^{-\theta_i (t+h)} \int_t^{t+h} e^{\theta_i s} dZ_{z_i,s} \qquad \forall i = 1, 2. \tag{14}$$

We then identify the integrated process parameters with those of the discrete time regressions:

$$z_{i,t+h} = a_{z_i} + A_{z_i} z_{i,t} + \upsilon_{z_i,t+h} \qquad \forall i = 1, 2. \tag{15}$$

This is a key point of the paper. We know that the optimal strategy involves intertemporal hedging terms that protect the investor from unfavorable moves in the state variables.

Therefore we need to estimate the correlations between the equity and bond returns with the *contemporaneous* innovations in the predictors. For instance, to obtain an estimate of the 2 years ahead stock excess return, one should use the estimates for the values of the predictors 2 years ahead also. Therefore we posit Eq. (15) and will check that the estimated A_Z coefficient for a given h and a given predictor i remains significant. In practice, we will have h ranging from 1 (Δt) to 120 (Δt), i.e. one month to 10 years.

As shown in Appendix C, our procedure yields, for a given h, the following seven parameters: θ_1, θ_2, $\overline{z}_1$, $\overline{z}_2$, $\sigma^2_{z_1}$, $\sigma^2_{z_2}$, and $\rho_{1,2}$. We apply the same procedure to the interest rate, integrate the process $r_t e^{\theta_r t}$ over the time interval $[t, t+h]$ and obtain:

$$r_{t+h} = \overline{r}\left(1 - e^{-\theta_r h}\right) + e^{-\theta_r h} r_t + \sigma_r e^{-\theta_r (t+h)} \int_t^{t+h} e^{\theta_r s} dZ_{r,s}. \tag{16}$$

Identifying the integrated process parameters with those of the discrete time regression

$$r_{t+h} = a_r + A_r r_t + \upsilon_{r,t+h} \tag{17}$$

yields, for a given h, the following five parameters: θ_r, $\overline{r}$, σ^2_r, $\rho_{1,r}$ and $\rho_{2,r}$ (see Appendix C).

Identifying the parameters for the stock returns relies on the same approach as we integrate the continuous time processes (3), using (5), over the discrete interval $[t, t+h]$ for all relevant prediction horizons h. We obtain (see Appendix C):

$$\begin{aligned}
\ln M_{t+h} - \ln M_t - \int_t^{t+h} r_s ds = & \left(\mu_{M,0} - \frac{\sigma^2_M}{2}\right) h \\
& + \mu_{M,1}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right)\overline{z_1} + \mu_{M,2}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right)\overline{z_2} \\
& - \mu_{M,1}\frac{1}{\theta_1}\left(e^{-\theta_1 h} - 1\right) z_{1,t} - \mu_{M,2}\frac{1}{\theta_2}\left(e^{-\theta_2 h} - 1\right) z_{2,t} \\
& + \int_t^{t+h} \sigma_M dZ_{M,s} \\
& + \mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\int_t^{t+h}\left(1 - e^{-\theta_1 (t+h)} e^{\theta_1 s}\right) dZ_{z_1,s} \\
& + \mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\int_t^{t+h}\left(1 - e^{-\theta_2 (t+h)} e^{\theta_2 s}\right) dZ_{z_2,s}.
\end{aligned} \tag{18}$$

As evidenced by this equation, the stock index risk premium is affected by three sources

of risk: its own idiosyncratic component and the two components due to the variability in the predictors. Therefore, when the return period length h increases, the decline in the (annualized) volatility of the stock market, which is the cornerstone of the long run predictability argument, is not guaranteed.

We apply the same technique to the constant maturity bond returns to obtain a similar expression for $\ln B_{t+h} - \ln B_t$.

We then identify the integrated process parameters with those of the discrete time regressions:

$$r_{M,t+h} - \int_t^{t+h} r_s ds = a_{M,h} + \beta_{M,h,1} z_{1,t} + \beta_{M,h,2} z_{2,t} + \upsilon_{M,t+h}, \tag{19}$$

$$r_{B,t+h} - \int_t^{t+h} r_s ds = a_{B,h} + \beta_{B,h,1} z_{1,t} + \beta_{B,h,2} z_{2,t} + \upsilon_{B,t+h}, \tag{20}$$

where $r_{M,t+h}$ (resp., $r_{B,t+h}$) is the continuously compounded h-period return on the stock market (resp., bond) between dates t and $t+h$, the return on the stock possibly including a dividend.

Note that the innovations $\upsilon_{M,t+h}$ and $\upsilon_{B,t+h}$ both obey a MA$(h-1)$ process under the null hypothesis of no predictability ($\beta_{M,h,1} = \beta_{M,h,2} = \beta_{B,h,1} = \beta_{B,h,2} = 0$) when observations are overlapping.

This procedure yields the fifteen needed parameters: $\mu_{M,0}$, $\mu_{M,1}$, $\mu_{M,2}$, σ_M^2, $\rho_{r,M}\sigma_M\sigma_r$, $\rho_{1,M}\sigma_M\sigma_{z_1}$, $\rho_{2,M}\sigma_M\sigma_{z_2}$ for the market portfolio, $\mu_{B,0}$, $\mu_{B,1}$, $\mu_{B,2}$, σ_B^2, $\rho_{r,B}\sigma_B\sigma_r$, $\rho_{1,B}\sigma_B\sigma_{z_1}$, $\rho_{2,B}\sigma_B\sigma_{z_2}$ for the bond and $\rho_{B,M}\sigma_M\sigma_B$ for their direct correlation [see Appendix C].

It thus remains to obtain through time-series regressions empirical estimates for the parameters appearing in Eqs. (15), (17), (19) and (20). The problems associated with these estimates are discussed in Section 4 below.

4 DATA AND PREDICTORS

This study uses U.S. monthly data for the period 1963:M7 to 2009:M12 (558 observations). [9]Data relative to the returns on the market portfolio have been downloaded from Kenneth French's website. The stock market return is computed as the value-weighted return ("*mkt*") on all NYSE, AMEX, and NASDAQ stocks (obtained from the CRSP data files). In addition,

[9]We selected June 1963 as the starting month because the value premium has been shown not to co-move with innovations in investment opportunities in the pre-1963 period as much as in the post-1963 period. See for instance Campbell and Vuolteenaho (2004), Petkova and Zhang (2005), Fama and French (2006) and Ang and Chen (2007).

we have selected the bond having a (constant) maturity of 20 years.[10] Its monthly (annualized) returns have been downloaded from the CRSP files and designated by "*bond*". The riskless rate is the 3-month Treasury bill rate ("*tb*"), downloaded from the FRED®database of the Federal Reserve Bank of St. Louis.

As to the predictors, we adopted variables that are variously but commonly used in the literature.[11] These variables lend themselves to a clear financial interpretation and have become common since the early start of Fama and French (1989). Adopting them will ease the comparison with previous research. The level of the yield curve determining investment opportunities, we retained as our first predictor the three-month Treasury bill rate ("*tb*"). As there is a convincing evidence that asset returns are predicted by the aggregate dividend yield, we adopted the dividend yield ("*dy*") measured as the total dividends paid off during the last 12 months divided by the actual value of the value-weighted market portfolio. We selected these measures as monthly or quarterly dividend yields cannot be used because seasonality is predominant. The dividend yield series was downloaded from the CRSP database.[12]

Insert Table I about here

Panel A of Table I reports various statistics for the 3-month Treasury bill rate ("*tb*") and the returns and excess returns on the risky assets ("*mkt*" and "*bond*"). All data are monthly, expressed in percent and annualized. For instance, the average risk premium is 3.9% for the equity market and 1.6% for the long term bond. All distributions have fat tails. They are negatively skewed for the equity index and positively skewed for the constant maturity bond.

Panel B of Table I reports the same statistics for the set of predictors. A salient feature of particular relevance and concern emerges. The large first-order auto-regression coefficients for the predictive variables, about 0.99, reflects a high level of persistence that will be discussed below. These figures are comparable to what is found in the literature. In contrast, the coefficient for the stock market was found in Panel A to be very small (0.10 for "*mkt*" and 0.03

[10] The 20-year rate seems to be the usual long term reference. Results are not significantly different with bonds of constant maturities 30, 10 and 7 years.

[11] See for instance Dumas and Solnik (1995), Jagannathan and Wang (1996), Ferson and Harvey (1999), Petkova (2006) or Ang and Bekaert (2007), in addition to the references quoted in footnote 1.

[12] We also performed tests with "*term*", the level of a term spread measured as the difference between the 10-year constant maturity Treasury bond yield and the 1-year constant maturity T-bond yield, and "*def*", a default spread measured as the difference between the yield of a 10-year Baa-rated bond and that of a 10-year Aaa-rated bond. For long prediction horizons, however, the coefficients of the postulated AR(1) processes turn out to be negative, which is not compatible with our set up.We are also investigating a measure of the market volatility. Preliminary results are encouraging.

for "*bond*"). Also, the correlation between the dividend yield and the short term rate is rather large (0.68).

We start the empirical analysis by the estimation of Eqs. (15) to observe the pattern followed by the coefficient of the lagged value of the predictor and assess whether it remains positive for any prediction horizon h, where $h = 1$, 3, 6 months and 1 to 10 years. We call $\hat{H}$ the set of values taken by h. Table II displays the results for the two predictors. As expected, the coefficients and the regression R^2s decay rapidly with h, but for both predictors, even for $h = 10$ years, the coefficients are significant and (sufficiently) positive. This implies that we can later test our portfolio strategies with any prediction horizon $h \epsilon \hat{H}$.

Insert Table II about here

As we selected "*dy*" and "*tb*" as our predictors, it should be noted that in this case the short term interest rate plays a dual role as an asset which the household will invest in, and as a predictor. We estimated the time-series regressions (19) and (20) where the h-month period excess returns for the stock and the bond were computed for $h \epsilon \hat{H}$.

4.1 PARAMETER ESTIMATES

Predictive Regressions

Results for the OLS bivariate regressions are reported in Table III. The first t-stats (t_{NW}) are corrected for both autocorrelation and heteroskedasticity using the Newey-West estimator with a number of lags equal to the prediction horizon minus one. This is the standard procedure to assess the statistical significance of the independent variable. However, predictive regressions potentially suffer from two features that violate the standard assumptions underlying OLS: biased coefficient estimates in small samples on the one hand, and serial correlation created in the regression residuals by using overlapping observations, on the other. Although we stress that our objective is not to dwell on statistical inference issues, we nonetheless need to make further adjustments.

When a regressor z obeys a Gaussian AR(1) process with innovations that are strongly persistent and correlated with the innovations in the regression, the estimate of the slope of the predictive regression is biased in finite samples. Stambaugh (1999) first suggested a correction to this bias which, however, was shown by Lewellen (2004) to be too large. Moreover, Stambaugh's

method is mute as to the proper way to adjust the t-stats of the corrected estimators. Amihud and Hurvich (2004) and Amihud, Hurvich and Wang (2008) proposed another bias correction method that improves on Stambaugh's in two ways. First, it provides an adjustment to the t-stats, thereby mitigating the seriousness of inference problems. This is all the more valuable here because the predictors exhibit a higher degree of persistence. Second, it can be applied to multiple regressor models, which is useful for our bivariate cases.

However, their procedure is ineffective when observations are overlapping, which is always the case except for one-month returns ($h = \Delta t$), and we will use it only when we deal with non-overlapping data. To illustrate, when returns are computed over a 5-year period ($h = 60\,\Delta t$), two consecutive observations have 59 months in common. This violates the assumption of absence of serial correlation in the residuals of the OLS regression and makes both the t-stats of the estimated coefficient(s) and the regression R^2 increase mechanically with the width of the return period. One way around this problem has been proposed by Hodrick (1992) who developed standard errors (and thus a t-statistics) that are corrected for serial correlation. Exploiting covariance stationarity, and using the innovations in regressions that involve one-period returns only, his method eliminates the overlapping nature of the innovations in the estimation of the standard error, thereby avoiding the serial correlation induced by longer horizon returns. The recent study by Ang and Bekaert (2007) on stock return predictability vindicates the superiority of Hodrick's standard errors over the Newey-West and the robust GMM generalization of Hansen and Hodrick (1980) standard errors. Therefore, we use Hodrick's procedure, which produces the $t_{Hodrick}$ statistics reported in Table III.[13]

Insert Table III about here

We discuss the predictability of the stock market excess return first (Panel A). Our results are close, in particular, to those reported in Campbell, Lo and MacKinley's (1997) and Cochrane's (2001) textbooks and confirm that when observations are overlapping, the adjusted R^2s increase mechanically (with a small slump, however, for $h = 3$ and 4 years) with the prediction horizon.[14]

[13] Boudoukh, Richardson and Whitelaw (2008) follow a completely different approach that does not correct for serial correlation but allows for assessing whether the problem is serious. Assuming no predictability, they estimate in the univariate case the β and R^2 of the OLS regression using yearly overlapping observations. Then they re-run the regression without the no predictability assumption. If the β and R^2 are significantly different from the previous ones, serial correlation is not deemed to be a significant issue. We do not pursue this route for two reasons. The first is that it applies to univariate regressions only. The second is that our basic return period being one month, instead of one year, the R^2s we found for return periods beyond 24 (months) increase to a value much beyond one.

[14] See Boudoukh et al. (2008) for a lucid explanation and details.

Recall that this would be true even in absence of return predictability and therefore is not *per se* evidence in favor of or against it. As to the first predictor, the aggregate dividend yield "dy", its coefficients are positive as expected from the literature, and their associated t_{NW}-stats virtually always significant. When we use Hodrick's (1992) procedure, the $t_{Hodrick}$ -stats exhibit a completely different pattern. They are significant and stable from the one-month return period (2.5) up to the one-year return period (2.3), then become insignificant (1.7 for $h = 2$ years) and steadily decrease as the prediction horizon increases (up to 10 years). Hodrick's correction then appears rather drastic although we cannot at this stage reject the assumption of predictability over horizons up to one year. This finding thus seems at odds with the "conventional wisdom" but consistent with the claim of both Ang and Bekaert (2007), who used the Morgan Stanley Capital International (MSCI) total return index for the market portfolio, and Boudoukh, Richardson and Whitelaw (2008) that the aggregate dividend yield has no predictive power at long horizons. The second predictor, "tb", has the expected negative impact when significant. Interestingly, it exhibits essentially the same pattern as the dividend yield: according to both t-stats, it is significant up to the one-year prediction horizon only. Its significance gradually decays as the return period lengthens and it even reverses its sign for $h \geq 7$ years. This result is roughly in accordance with former findings by Fama and Schwert (1977), Campbell (1987), Breen, Glosten, and Jagannathan (1989), and Lee (1992), and with the recent one by Ang and Bekaert (2007). Any discrepancies may be attributed in part to the adopted return frequency and in part to the fact that the Federal Reserve pegged interest rates from the Great Depression up to 1951, a sub-period not covered by our sample. Finally, these findings are broadly consistent with the claim by most authors that bivariate predictive regressions generally improve on univariate ones, data mining and snooping notwithstanding. In particular, the predictability of the dividend yield is enhanced at short horizons by the presence of the interest rate.

Results for the predictability of excess returns on the 20-year constant maturity bond are displayed in Panel B of Table III. The adjusted R^2s exhibit an inverted U-shape. They increase sharply from zero to 0.35 up to a prediction horizon of 6 years, then decrease steadily (down to 0.24) while remaining higher than that of the 3-year horizon. The aggregate dividend yield is understandably less significant than for the stock market. Its influence on the bond return is essentially negative, and its significance, measured by either t-stats, offers the same pattern as the adjusted R^2: "dy" is significant only for prediction horizons ranging from 2 to 6 years.

The second predictor, "tb", has the expected positive sign, except when it is insignificant. It becomes significant only at long horizons ($h \geq 2$ years) and then becomes gradually more so. This result is roughly in accordance with former findings on bond portfolios by Fama and Bliss (1987).

4.2 Predictor and excess return processes

Panel A of Table IV reports the estimated parameters of the discretized continuous processes followed by the two predictors for our various prediction horizons. The estimation procedure is the one described in section 3 above and detailed in Appendix C. Panel A essentially checks whether the coefficients θ, $\overline{z}$, and σ characterizing the Ornstein-Ulhenbeck processes (7) are stable across return periods (from 1 month to 10 years). This stability is required to ensure realism of the model and consistency of the estimates. Inspection of the results shows that, for both the dividend yield and the short term rate, it is indeed the case for $\overline{z}$ and σ. Their estimated long run ($\overline{z}$) values and volatilities σ are reassuringly stable across the board. The mean reversion coefficient θ for the interest rate varies somewhat with the predictive horizon but with no discernable pattern. That of the dividend yield, however, decreases rather markedly from $h = 6$ months to $h = 5$ years and re-increases for longer prediction horizons. This is noteworthy as a smaller θ implies (see Eq. (34) in Appendix C) a higher persistence, hence a stronger predictability power. This could hint at a potentially better performance, for any investment horizon T, attained with short run predictability.

Insert Table IV about here

Panel B of Table IV displays the estimated parameters of the process followed by the stock excess return predicted by "dy" and "tb". We check here for any sizeable discrepancy between the estimated parameters across return periods and what can be directly observed from the data.

We have computed the annualized long run equity risk premium (EP) implied by our estimates to check for robustness and stability across return periods. EP is computed as the weighted sum $(\mu_{M,0} + \mu_{M,1}\overline{z}_1 + \mu_{M,2}\overline{z}_2)$. We obtain estimates for EP around 5% across the board. This result is comforting as being both rather stable and in line with the historical market excess return found in the data (4%). The correlation of the stock market return with

the dividend yield is stable around -0.9, a result in line with previous research.[15] Its correlation with the interest rate, however, is more hectic and tends to sharply increase in absolute value with the prediction horizon. Its volatility σ_M tends to be stable around an annualized rate of 16%, then increases (up to 19%) for long run horizons ($h = 9$ and 10 years). This finding, however, is a concern as it is at odds with what is found in the data (where σ_M slightly decreases monotonously with the return period) and what was previously noted by Pastor and Stambaugh (2010).

For the bond excess return, results are qualitatively the same, as evidenced in Panel C of Table IV. The patterns followed by σ_B and the long run bond premium BP ($= \mu_{B,0} + \mu_{B,1}\overline{z}_1 + \mu_{B,2}\overline{z}_2$) are close to their historical values. However, the correlation of the bond return with both the dividend yield and the short term rate is essentially increasing (in absolute value) with the return period length, a pattern that remains to be explained.

4.3 ASSET ALLOCATION STRATEGIES

Endowed with our estimated parameters, we can implement the investor's effective portfolio strategies. We compute by way of Eq. (10) the optimal portfolio composition (that includes the mean-variance term MV and the intertemporal hedging components IH), and, using Eq. (11), the annualized certainty equivalent rate of return of the optimal strategy, CE_{opt}, which is to be compared to 5.53%, the average riskfree rate. Similarly, we compute CE_{np}, the certainty equivalent return of the "no predictability" strategy, and CE_{myo}, that of the myopic strategy ignoring the hedge terms. We set the individual's risk aversion coefficient γ to 2, the lower bound, so as to magnify the differences in certainty equivalents .[16] We consider many different cases by varying her investment horizon T and her prediction horizon h. Usually in the literature devoted to portfolio allocation, both horizons coincide. This of course need not be the case. On the contrary, we investigate the relationship between the two horizons, for a given risk aversion. For instance, there is no reason why a long term investor should not use short horizon predictability if the latter turns out to be more valuable than longer run ones. By symmetry, why should a short term investor not exploit long term predictability if it is stronger? Consequently, we use for h the same range of values as in the predictive regressions, namely 1, 3 and 6 months and 1 to 10 years (i.e. $h \epsilon \hat{H}$), and for T a wider range that includes two more horizons, namely

[15] See for instance Campbell (1996) or Petkova (2006).

[16] For huge risk aversion coefficients (above 10), the weight of the riskless asset in the portfolio tends to one, making irrelevant the predictability issue.

20 and 30 years. This yields a (13×15) matrix of certainty equivalent return rates.

Results are exhibited in Table V (Panels A to C). Panel A displays the matrix of optimal certainty equivalents. Two general patterns strongly emerge. The first one was expected: for a *given* prediction horizon, the annualized CE_{opt} return rate dramatically and monotonously increases with the investment period.[17] For the 30-year horizon, CE_{opt} is approximately thrice as large as it is for the one-month horizon, depending on the prediction horizon. According to the conventional professional wisdom in asset allocation, the weight of risky assets in the optimal portfolio should depend positively on the investment's period length. As we will show below, this is the case for all prediction horizons. The sharp increase in the proportion of risky assets entails that of the certainty equivalent return rates, because the investor can benefit more from the positive market risk premia. Note that CE_{opt} is across the board well above the average riskless rate (5.53%) because the weight of the global risky position in the portfolio never falls below 70%.

Reading now Panel A of Table V columnwise, the second, and more crucial to our investigation, striking pattern that emerges is the rather strong tendency for CE_{opt} to *decrease* with the prediction horizon, for all investment horizons. For instance, for $T = 1$, 7 and 30 years, respectively, it is equal to 11.1%, 21.2% and 25.7% when h is one month, and to 9.6%, 17.2% and 22.7% only when h is ten years. This result seems contrary to the conventional academic wisdom regarding stock return long term predictability, although the relationship is far from being monotonous: actually, a maximum is reached for a prediction horizon of about 4-5 years. Nonetheless, even for short run investors, it seems better to use predictors one month ahead rather than seven years ahead or more, implying that (very) long term predictability is poorer.

Insert Table V about here

Panel B of Table V reports the ratio CE_{opt}/CE_{np} which gauges the welfare loss, if any, suffered by an investor who assumes that market risk premia are constant (and set to their long run average values). As expected, the ratio essentially depends on the investment horizon. It ranges from 1.02 (very short term investors) to a staggering 3.48 (very long run ones) and does not vary much with the prediction horizon. Even though we should not take these ratios too literally, taking into account the time variation in risk premia is undoubtedly rewarding.

[17] We keep in mind that comparing (annualized certainty equivalent) rates of return across investment horizons is not in general meaningful. However, it is legitimate here to state that the longer the investment horizon, the better off is the investor since CE_{opt} starts at a level higher than the riskless rate and increases monotonously with T.

The ratio CE_{opt}/CE_{myo}, which measures the welfare loss induced by ignoring the intertemporal hedging terms in portfolio allocation, is displayed in Panel C. Although the weights of the IH components can be substantial, as will be seen above, they tend to cancel the MV terms, so that the global weights invested in the risky assets do not change dramatically when the investor's strategy is sub-optimal. Therefore, the loss is negligible (less than 2%) at all predictive horizons for short term investors ($T \leq 3$ years), very small or small (less than 10%) for medium term investors (up to 10 years) and substantial only for long term ones (20 and 30 years). Only in the latter case is there a material difference between short run and long run predictability, with a pattern that reflects that of Panel A (a tendency for the ratio to increase with the prediction horizon, but not monotonously, with a peak for h around 5 years). Even so, it reaches a maximum value of 1.17 only, which may cast some doubt as to the economic significance of intertemporal hedging, at least for weak risk aversion levels, except for very long term investors.

Turning now to the portfolio composition, Panel A of Table VI reports the percentages of wealth invested in the risky assets that help explain, and reflect, the certainty equivalents of Table V. The percentage increases with the investment horizon, for a given prediction horizon, in accordance with the professional conventional wisdom. It tends to decrease with the prediction horizon, although not monotonously. Panel B provides the optimal stock/bond mix. The latter follows no discernable patterns and is extremely sensitive to the prediction horizon and to the short or long duration of the investment period. Panel C displays the ratio of risky assets devoted to intertemporal hedging (IH) over risky assets held for speculative motives (mean-variance term MV). The ratio depends positively on both the investment and the prediction horizons. For short term investors (up to 12 months), the IH components have understandably a negligible weight, for all prediction horizons but the longest ones. The weights of IH terms relative to MV increase monotonously with the investment horizon, as the investor is more exposed to unfavorable shifts in her opportunity set and presumably wants to hedge against such occurrences. If we except the row relative to one-month predictive returns, for a given horizon these weights also increase with the length of the prediction period, and for the same reason.

Insert Table VI about here

Overall, although some findings are difficult to interpret, our results tend to side against the

academic conventional wisdom and along with the skeptics who discard long run predictability as a myth, provided "long run" means 7 years or more. When the prediction horizon enlarges, the investor misses the expected path that predictors will follow up to the horizon. Therefore, she does not fully exploit the predictability and her performance declines. It remains to check whether these results are robust to the correction for the persistence of the regressors created by the use of overlapping observations. that remains to be explained.

4.4 ADDITIONAL RESULTS

To be added.

4.5 CORRECTING FOR PERSISTENT RESIDUALS

One potential issue regarding the above procedure is that, for return periods h longer than one month, observations overlap. This creates undue persistence in the residuals of the regression of the stock and bond returns on the predictors. To overcome this problem, we adapt the method proposed by Valkanov (2003). According to his Theorem 3 (on p. 208), one may obtain consistent estimates of the slope(s) of the predictive regression if the stock excess return is regressed on the predictors' "long period" values, the latter being computed as the *sum* of the predictors' monthly values over the h-long period. We thus have to re-identify the parameters of the stock (and bond) excess return process. The procedure is described in detail in Appendix D. The results are reported in Tables VII and VIII, which are analogous to Tables IV and V, respectively.

Using the same parameters for the dynamics of the predictors as above since they do not change, we re-estimate those for the stock and bond returns for prediction periods $h \epsilon \hat{H}$. As shown in Panel A of Table VII, the correction has little impact on the long run equity premium (EP), as evidenced by comparing with Table IV, except for a prediction horizon of 10 years. This part of the estimation thus seems robust to the persistence of residuals issue. In contrast, the pattern followed by the correlation between the market and both predictors is affected: correlations are much lower in absolute terms for predictive horizons larger than 1 year, as intuition suggests. The main result, however, is that the stock market volatility now decreases (from $h = 6$ months) monotonously with the predictive horizon, which is completely in line with what we found in the data and what is argued by Pastor and Stambaugh (2010). This

improvement in the stock market's Sharpe ratio at long term predictions is a major source of revision of the previous results.

As to the bond parameters, reported in Panel B, the conclusions are similar: analogous long term premium (BP), smaller correlations in absolute value with the predictors and an almost monotonous decline in the return volatility, hence an improvement of the Sharpe ratio.

Insert Table VII about here

The result relative to the certainty equivalent rates of return of the optimal portfolio strategy is the second major finding of this paper.

Unlike what was observed in Table V, the CE_{opt} rates exhibited in Table VIII (Panel A) have a strong tendency to *increase* with the prediction horizon, although not monotonously (except when the investment horizon is one month). This is all the more true that investment horizons are shorter. Actually, for any given investment horizon (but T =1 month), the pattern is U-shaped with a trough at h equal to 2 or 3 years and a sharp rise as h increases. For instance, for $T = 1$, 7 and 30 years, respectively, CE_{opt} is equal to 11.1%, 21.2% and 25.7% when h is one month (as in Table V), and to 19.7%, 26.9% and 28.0% when the return period h is ten years. Why then does Valkanov's correction rehabilitates (at least partially) long run predictability in accordance with most of the literature? The answer is that by taking the whole expected path for the predictors into account for predicting the market returns, the investor in effect recovers the information she looses when she does not. As corollaries, Panels B and C of Table VIII vindicate that the welfare losses suffered by investors following sub-optimal strategies enlarge across the board as both the investment and the prediction horizons increase.

Insert Table VIII about here

Overall then, in spite of the recent results by Ang and Bekaert (2007) and some of those reported by Boudoukh et al. (2008),[18] we find that at both short and long horizons asset returns are in fact predictable by the dividend yield and the 3-month T-bill rate, and that the beneficial impact on optimal portfolio performance is sizeable, decreasing and then sharply increasing with the prediction horizon.

[18] The title of the Boudoukh et al. (2008) paper may be somewhat misleading. They claim that, with *most* the predictors they use, long-horizon predictability is "a myth". However, they report in the text (on p. 1599) that two variables, the net payout yield and the equity share of new issuances, have strong (individual) predictive power across horizons. Ang and Bekaert (2007) report predictability of dividend yields and interest rates at short, but not at long, horizons.

5 Concluding remarks

In accordance with conventional wisdom of weak predictability at short horizons and a stronger one at long horizons, and in spite of some recent results to the contrary, we find that the U.S. stock index and 20-year constant maturity bond returns have been predictable for the period 1963:06 to 2009:12. The predictors we used are the aggregate dividend yield and the 3-month Treasury bill rate. Predictability is strong across predictive horizons, and tend to be stronger at longer horizons, although the relationship is not monotonous but U-shaped. The influence predictability exerts on the composition and outcome of optimal (or even sub-optimal) portfolio strategies is also generally more pronounced for longer return periods, although the extent of this result depends on the individual's risk aversion and investment horizon. It also crucially depends on whether estimates of the parameters for the market risk premia are corrected or not for the persistence in residuals brought about by overlaps in observations. One serious challenge we now face is to assess to what extent what seems true in-sample will still hold out-of-sample.

6 Appendix

6.1 Appendix A

We derive first the investor's optimal portfolio strategy and the associated value function used to compute the certainty equivalent discussed in the text when the predictability of asset returns is fully taken into account.

The investor's wealth dynamics writes:

$$\frac{dV_t}{V_t} = r_t dt + \left[\omega_{M,t}\mu_{M,t} + \omega_{B,t}\mu_{B,t}\right] dt + \omega_{M,t}\sigma_M dZ_{M,t} + \omega_{B,t}\sigma_B dZ_{B,t}, \tag{21}$$

or else:

$$\begin{aligned}\frac{dV_t}{V_t} = r_t dt + & \left[\begin{array}{c} \omega_{M,t}\mu_{M,0} + \omega_{B,t}\mu_{B,0} + \left(\omega_{M,t}\mu_{M,1} + \omega_{B,t}\mu_{B,1}\right) z_{1,t} \\ + \left(\omega_{M,t}\mu_{M,2} + \omega_{B,t}\mu_{B,2}\right) z_{2,t} \end{array}\right] dt \\ & + \omega_{M,t}\sigma_M dZ_{M,t} + \omega_{B,t}\sigma_B dZ_{B,t}.\end{aligned} \tag{22}$$

The Hamilton-Jacobi-Bellman (HJB) equation for the investor's program (9) writes:

$$\max_{\left\{\omega_{M,t},\omega_{B,t}\right\}_{t=0}^{t=T}} \left\{\begin{array}{c} \frac{\partial J}{\partial t} + V_t J_V \left[r_t + \left[\begin{array}{c} \omega_{M,t}\mu_{M,0} + \omega_{B,t}\mu_{B,0} + \left(\omega_{M,t}\mu_{M,1} + \omega_{B,t}\mu_{B,1}\right) z_{1,t} \\ + \left(\omega_{M,t}\mu_{M,2} + \omega_{B,t}\mu_{B,2}\right) z_{2,t} \end{array}\right]\right] \\ + \frac{1}{2} V_t^2 J_{VV} \left[\omega_{M,t}^2\sigma_M^2 + \omega_{B,t}^2\sigma_B^2 + 2\omega_{B,t}\omega_{M,t}\sigma_M\sigma_B\rho_{B,M}\right] \\ + J_r\theta_r\left[\overline{r} - r_t\right] + \frac{1}{2}J_{rr}\sigma_r^2 + J_{z_1}\theta_1\left[\overline{z}_1 - z_{1,t}\right] + \frac{1}{2}J_{z_1z_1}\sigma_{z_1}^2 \\ + J_{z_2}\theta_2\left[\overline{z}_2 - z_{2,t}\right] + \frac{1}{2}J_{z_2z_2}\sigma_{z_2}^2 \\ + V_t J_{Vr}\left[\omega_{M,t}\sigma_M\sigma_r\rho_{r,M} + \omega_{B,t}\sigma_B\sigma_r\rho_{r,B}\right] \\ + V_t J_{Vz_1}\left[\omega_{M,t}\sigma_M\sigma_{z_1}\rho_{1,M} + \omega_{B,t}\sigma_B\sigma_{z_1}\rho_{1,B}\right] \\ + V_t J_{Vz_2}\left[\omega_{M,t}\sigma_M\sigma_{z_2}\rho_{2,M} + \omega_{B,t}\sigma_B\sigma_{z_2}\rho_{2,B}\right] \\ + J_{rz_1}\sigma_{z_1}\sigma_r\rho_{1,r} + J_{rz_2}\sigma_r\sigma_{z_2}\rho_{2,r} + J_{z_1z_2}\sigma_{z_1}\sigma_{z_2}\rho_{1,2} \end{array}\right\} = 0. \tag{23}$$

where $J = J(t, V_t, r_t, z_{1,t}, z_{2,t})$ denotes the investor's indirect utility (value function), J_x and J_{xx} denote first and second partial derivatives of J with respect to its last three arguments (with their time dependency removed, e.g. $J_{Vz_1} = \partial^2 J(.)/\partial V_t.\partial z_{1,t}$).

The following terminal condition is imposed on the HJB equation:

$$J(T,.) = \frac{V_T^{1-\gamma}}{1-\gamma}. \tag{24}$$

The first order conditions for optimality write:

$$\begin{aligned} 0 &= V_t J_V \left[\mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t}\right] + V_t^2 J_{VV} \left[\omega_{M,t}\sigma_M^2 + \omega_{B,t}\sigma_M\sigma_B\rho_{B,M}\right] \\ &+ V_t J_{Vr}\sigma_M\sigma_r\rho_{r,M} + V_t J_{Vz_1}\sigma_M\sigma_{z_1}\rho_{1,M} + V_t J_{Vz_2}\sigma_M\sigma_{z_2}\rho_{2,M}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} 0 &= V_t J_V \left[\mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t}\right] + V_t^2 J_{VV} \left[\omega_{M,t}\sigma_M\sigma_B\rho_{B,M} + \omega_{B,t}\sigma_B^2\right] \\ &+ V_t J_{Vr}\sigma_B\sigma_r\rho_{r,B} + V_t J_{Vz_1}\sigma_B\sigma_{z_1}\rho_{1,B} + V_t J_{Vz_2}\sigma_B\sigma_{z_2}\rho_{2,B}, \end{aligned} \tag{26}$$

so that the optimal portfolio strategy is given by

$$\begin{aligned} \begin{bmatrix} \omega_{M,t} \\ \omega_{B,t} \end{bmatrix} &= \frac{J_V}{-V_t J_{VV}} \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\ &+ \frac{J_{Vr}}{-V_t J_{VV}} \Sigma^{-1}\Sigma_r + \frac{J_{Vz_1}}{-V_t J_{VV}} \Sigma^{-1}\Sigma_1 + \frac{J_{Vz_2}}{-V_t J_{VV}} \Sigma^{-1}\Sigma_2, \end{aligned} \tag{27}$$

where

$$\Sigma = \begin{bmatrix} \sigma_M^2 & \sigma_M\sigma_B\rho_{B,M} \\ \sigma_M\sigma_B\rho_{B,M} & \sigma_B^2 \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} \sigma_M\sigma_r\rho_{r,M} \\ \sigma_B\sigma_r\rho_{r,B} \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} \sigma_M\sigma_{z_1}\rho_{1,M} \\ \sigma_B\sigma_{z_1}\rho_{1,B} \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} \sigma_M\sigma_{z_2}\rho_{2,M} \\ \sigma_B\sigma_{z_2}\rho_{2,B} \end{bmatrix}.$$

Plugging back these optimal values for $\omega_{M,t}$ and $\omega_{B,t}$ into the HJB equation yields:

$$
\begin{aligned}
0 = & \frac{\partial J}{\partial t} + V_t J_V r_t \\
& - \frac{1}{2}\frac{J_V J_V}{J_{VV}} \left(\begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \right)' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\
& + \left(\frac{J_V J_{Vr}}{-J_{VV}} \Sigma_r + \frac{J_V J_{Vz_1}}{-J_{VV}} \Sigma_1 + \frac{J_V J_{Vz_2}}{-J_{VV}} \Sigma_2 \right)' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\
& - \frac{1}{2}\frac{J_{Vr} J_{Vr}}{J_{VV}} \Sigma_r' \Sigma^{-1} \Sigma_r - \frac{1}{2}\frac{J_{Vz_1} J_{Vz_1}}{J_{VV}} \Sigma_1' \Sigma^{-1} \Sigma_1 - \frac{1}{2}\frac{J_{Vz_2} J_{Vz_2}}{J_{VV}} \Sigma_2' \Sigma^{-1} \Sigma_2 \\
& - \frac{J_{Vz_1} J_{Vr}}{J_{VV}} \Sigma_r' \Sigma^{-1} \Sigma_1 - \frac{J_{Vz_2} J_{Vr}}{J_{VV}} \Sigma_r' \Sigma^{-1} \Sigma_2 - \frac{J_{Vz_2} J_{Vz_1}}{J_{VV}} \Sigma_1' \Sigma^{-1} \Sigma_2 \\
& + J_r \theta_r \left[\overline{r} - r_t \right] + \frac{1}{2} J_{rr} \sigma_r^2 + J_{z_1} \theta_1 \left[\overline{z}_1 - z_{1,t} \right] + \frac{1}{2} J_{z_1 z_1} \sigma_{z_1}^2 \\
& + J_{z_2} \theta_2 \left[\overline{z}_2 - z_{2,t} \right] + \frac{1}{2} J_{z_2 z_2} \sigma_{z_2}^2 + J_{r z_1} \sigma_r \sigma_{z_1} \rho_{1,r} + J_{r z_2} \sigma_r \sigma_{z_2} \rho_{2,r} + J_{z_1 z_2} \sigma_{z_1} \sigma_{z_2} \rho_{1,2}.
\end{aligned}
\tag{28}
$$

Given the affine structure of the model, we guess the following form for the value function:

$$
J(t, .) = \frac{V_t^{1-\gamma}}{1-\gamma} e^{G(T-t, r_t, z_{1,t}, z_{2,t})}, \tag{29}
$$

$$
\begin{aligned}
\text{with } G\left(T-t, r_t, z_{1,t}, z_{2,t}\right) = & A_0\left(T-t\right) + A_1\left(T-t\right) r_t + \frac{1}{2} A_{11}\left(T-t\right) r_t^2 + A_2\left(T-t\right) z_{1,t} \\
& + \frac{1}{2} A_{22}\left(T-t\right) z_{1,t}^2 + A_3\left(T-t\right) z_{2,t} + \frac{1}{2} A_{33}\left(T-t\right) z_{2,t}^2 \\
& + A_4\left(T-t\right) z_{1,t} z_{2,t} + A_5\left(T-t\right) z_{1,t} r_t + A_6\left(T-t\right) r_t z_{2,t}.
\end{aligned}
\tag{30}
$$

We have to find the $A_i(T-t)$ and $A_{ij}(T-t)$ functions so as to solve for the optimal strategy $(\omega_{M,t}\,; \omega_{B,t})$ and the investor's expected utility. We first plug the various first and second order derivatives of J_t computed from Eq.(29) into the HJB equation (23). We then regroup terms involving the same power of r, z_1 or z_2 (r^0 (or z_1^0 or z_2^0), r, z_1, z_2, r^2, z_1^2, z_2^2, rz_1, rz_2, $z_1 z_2$), and set their sums to zero, which yields the following ten equations:

$$
\begin{aligned}
-\frac{\partial A_0}{\partial t} &= \theta_r \overline{r} A_1 + A_2 \theta_1 \overline{z}_1 + \theta_2 \overline{z}_2 A_3 + \frac{1}{2}\sigma_r^2 A_{11} + \frac{1}{2}\sigma_r^2 A_1^2 \\
&+ \frac{1}{2}\sigma_{z_1}^2 A_{22} + \frac{1}{2}\sigma_{z_1}^2 A_2^2 + \frac{1}{2}\sigma_{z_2}^2 A_{33} + \frac{1}{2}\sigma_{z_2}^2 A_3^2 \\
&+ A_2 A_1 \sigma_{z_1}\sigma_r \rho_{1,r} + A_1 A_3 \sigma_r \sigma_{z_2} \rho_{2,r} + A_2 A_3 \sigma_{z_1}\sigma_{z_2}\rho_{1,2} \\
&+ \frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + \frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_r A_1 \\
&+ \frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_1 A_2 + \frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_2 A_3 \\
&+ \frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_1 + A_1 A_2 \frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_1 \\
&+ A_1 \frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_r A_1 + A_1 A_3 \frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_2 \\
&+ \frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_2 + A_2 \frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_2 A_3 \\
&+ A_2 A_1 \frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_r + A_2 A_2 \frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_1 \\
&+ \frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_3 + A_1 \frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_r A_3 \\
&- \frac{1}{2}\frac{1-\gamma}{\gamma}\left(\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' + A_1\Sigma_r' + A_2\Sigma_1' + A_3\Sigma_2'\right) \\
&\times \left(\Sigma^{-1}\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_1\Sigma^{-1}\Sigma_r + A_2\Sigma^{-1}\Sigma_1 + \Sigma^{-1}\Sigma_2 A_3\right) \\
&+ A_2 \frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_1 A_3 + A_3 \frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_2 A_3.
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_1}{\partial t} &= (1-\gamma) - \theta_r A_1 + \theta_r \overline{r} A_{11} + A_5 \theta_1 \overline{z}_1 + \sigma_{z_1}^2 A_2 A_5 \\
&+ \theta_2 \overline{z}_2 A_6 + \sigma_r^2 A_1 A_{11} + \sigma_{z_2}^2 A_3 A_6 \\
&+ A_5 A_1 \sigma_{z_1} \sigma_r \rho_{1,r} + A_2 A_{11} \sigma_{z_1} \sigma_r \rho_{1,r} + A_3 A_{11} \sigma_r \sigma_{z_2} \rho_{2,r} \\
&+ A_1 \sigma_r \sigma_{z_2} \rho_{2,r} A_6 + A_2 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_6 + A_3 A_5 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_r A_{11} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_1 A_5 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_2 A_6 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_1 A_{11} + A_1 A_6 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 + A_1 A_5 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_{11} + A_1 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_{11} \\
&+ A_2 A_{11} \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r + A_2 A_5 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 + A_2 A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 \\
&+ A_2 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_{11} + A_3 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_{11} \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' + A_1 \Sigma_r' + A_2 G \Sigma_1' + A_3 \Sigma_2' \right) \\
&\times \left(\Sigma^{-1} \Sigma_1 A_5 + \Sigma^{-1} \Sigma_r A_{11} + G \Sigma^{-1} \Sigma_2 A_6 \right) \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_1 G \Sigma^{-1} \Sigma_r + A_2 G \Sigma^{-1} \Sigma_1 + G \Sigma^{-1} \Sigma_2 A_3 \right) \\
&\times \left(A_5 G \Sigma_1' + A_{11} G \Sigma_r' + A_6 G \Sigma_2' \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_6 + A_1 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_6 + A_2 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_6 \\
&+ A_3 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_6 + A_{11} G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_3 \\
&+ A_5 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_3 + A_6 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_3 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_5 + A_1 G \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_5 \\
&+ A_2 G \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_5 + G \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_3 A_5.
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_{11}}{\partial t} &= -2\theta_r A_{11} + \sigma_r^2 A_{11}^2 + \sigma_{z_1}^2 A_5^2 + \sigma_{z_2}^2 A_6^2 \\
&+ 2A_5 A_{11}\sigma_{z_1}\sigma_r\rho_{1,r} + 2A_{11}\sigma_r\sigma_{z_2}\rho_{2,r}A_6 + 2A_5\sigma_{z_1}\sigma_{z_2}\rho_{1,2}A_6 \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_r A_{11}A_{11} + 2A_5\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_1 A_{11} \\
&+ 2A_6\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_2 A_{11} \\
&+ 2A_{11}\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_r A_5 + 2A_5\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_1 A_5 \\
&+ 2A_6\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_2 A_5 \\
&+ 2A_{11}\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_r A_6 + 2A_5\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_1 A_6 + 2A_6\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_2 A_6 \\
&- \frac{1-\gamma}{\gamma}\left(\Sigma^{-1}\Sigma_1 A_5 + \Sigma^{-1}\Sigma_r A_{11} + \Sigma^{-1}\Sigma_2 A_6\right)\left(A_5\Sigma_1' + A_{11}\Sigma_r' + A_6\Sigma_2'\right).
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_2}{\partial t} &= \theta_r \overline{r} A_5 - \theta_1 A_2 + A_{22}\theta_1 \overline{z}_1 + \theta_2 \overline{z}_2 A_4 + \sigma_r^2 A_1 A_5 + \sigma_{z_1}^2 A_2 A_{22} + \sigma_{z_2}^2 A_3 A_4 \\
&+ A_2 A_5 \sigma_{z_1}\sigma_r \rho_{1,r} + A_{22} A_1 \sigma_{z_1}\sigma_r \rho_{1,r} + A_1 \sigma_r \sigma_{z_2} \rho_{2,r} A_4 \\
&+ A_3 A_5 \sigma_r \sigma_{z_2} \rho_{2,r} + A_3 A_{22} \sigma_{z_1}\sigma_{z_2}\rho_{1,2} + A_2 \sigma_{z_1}\sigma_{z_2}\rho_{1,2} A_4 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_r A_5 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_1 A_{22} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1}\Sigma_2 A_4 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_1 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1}\Sigma_r \\
&+ A_3 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1}\Sigma_2 + A_2 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1}\Sigma_1 \\
&- \frac{1}{2}\frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_1 \Sigma^{-1}\Sigma_r + A_2 \Sigma^{-1}\Sigma_1 + \Sigma^{-1}\Sigma_2 A_3 \right) \\
&\times \left(\begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' + A_5 \Sigma_r' + A_4 \Sigma_2' + A_{22}\Sigma_1' \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_1 + A_1 A_{22} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_1 + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_r A_1 A_5 \\
&+ A_1 A_4 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_2 + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_5 + A_2 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_1 A_5 \\
&+ A_1 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_r A_5 + A_3 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1}\Sigma_2 A_5 \\
&- \frac{1}{2}\frac{1-\gamma}{\gamma} \left(\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' + A_1 \Sigma_r' + A_2 \Sigma_1' + A_3 \Sigma_2' \right) \\
&\times \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} + \Sigma^{-1}\Sigma_r A_5 + \Sigma^{-1}\Sigma_1 A_{22} + \Sigma^{-1}\Sigma_2 A_4 \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_2 + A_2 A_5 G \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1}\Sigma_r \\
&+ \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_4 + A_1 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1}\Sigma_r A_4 + A_2 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1}\Sigma_1 A_4 \\
&+ A_3 G \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1}\Sigma_2 A_4
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_{22}}{\partial t} &= -2\theta_1 A_{22} + \sigma_r^2 A_5^2 + \sigma_{z_1}^2 A_{22}^2 \\
&+ \sigma_{z_2}^2 A_4^2 + 2A_{22}A_5\sigma_{z_1}\sigma_r\rho_{1,r} + 2A_5\sigma_r\sigma_{z_2}\rho_{2,r}A_4 \\
&+ 2A_{22}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}A_4 \\
&+ 2\frac{1-\gamma}{\gamma}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}'\Sigma^{-1}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix} + 2A_{22}\frac{1-\gamma}{\gamma}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}'\Sigma^{-1}\Sigma_1 \\
&+ 2A_5\frac{1-\gamma}{\gamma}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}'\Sigma^{-1}\Sigma_r + 2A_4\frac{1-\gamma}{\gamma}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}'\Sigma^{-1}\Sigma_2 \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}A_5 + 2\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_r A_5 A_5 \\
&+ 2A_{22}\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_1 A_5 + 2A_4\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_2 A_5 \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}A_{22} + 2A_5\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_r A_{22} \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}A_4 + 2A_5\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_r A_4 \\
&+ 2A_{22}\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_1 A_4 \\
&+ 2A_4\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_2 A_4 \\
&+ 2A_{22}\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_1 A_{22} + 2\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_2 A_4 A_{22} \\
&- \frac{1-\gamma}{\gamma}\left(\Sigma^{-1}\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix} + \Sigma^{-1}\Sigma_r A_5 + \Sigma^{-1}\Sigma_1 A_{22} + \Sigma^{-1}\Sigma_2 A_4\right) \\
&\times\left(\begin{bmatrix}\mu_{M,1}\\ \mu_{B,1}\end{bmatrix}' + A_5\Sigma_r' + A_4\Sigma_2' + A_{22}\Sigma_1'\right).
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_3}{\partial t} &= \theta_r \overline{r} A_6 + A_4 \theta_1 \overline{z}_1 - \theta_2 A_3 + \theta_2 \overline{z}_2 A_{33} + \sigma_r^2 A_1 A_6 + \sigma_{z_1}^2 A_2 A_4 \\
&+ \sigma_{z_2}^2 A_3 A_{33} + A_2 A_6 \sigma_{z_1} \sigma_r \rho_{1,r} + A_4 A_1 \sigma_{z_1} \sigma_r \rho_{1,r} + A_3 A_6 \sigma_r \sigma_{z_2} \rho_{2,r} \\
&+ A_3 A_4 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} + A_2 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_{33} + A_1 \sigma_r \sigma_{z_2} \rho_{2,r} A_{33} \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_r A_6 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_1 A_4 + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' \Sigma^{-1} \Sigma_2 A_{33} \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_3 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_2 \\
&+ A_2 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_1 + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_r A_1 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_1 + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_1 A_6 \\
&+ A_1 A_4 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 + A_1 A_{33} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_2 + A_2 A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} + A_1 \Sigma^{-1} \Sigma_r + A_2 \Sigma^{-1} \Sigma_1 + \Sigma^{-1} \Sigma_2 A_3 \right) \\
&\times \left(\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' + A_6 \Sigma_r' + A_4 \Sigma_1' + A_{33} \Sigma_2' \right) \\
&+ A_2 A_4 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 + \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_2 A_{33} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_4 + A_1 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_4 + A_2 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_4 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_3 A_4 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix} A_6 + A_1 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_6 \\
&+ A_2 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_6 + A_3 \frac{1-\gamma}{\gamma} \Sigma_{33}^{r\prime} \Sigma^{-1} \Sigma_2 A_6 \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\begin{bmatrix} \mu_{M,0} \\ \mu_{B,0} \end{bmatrix}' + A_1 \Sigma_r' + A_2 \Sigma_1' + A_3 \Sigma_2' \right)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_{33}}{\partial t} &= -2\theta_2 A_{33} + \sigma_r^2 A_6^2 + \sigma_{z_1}^2 A_4^2 \\
&+ \sigma_{z_2}^2 A_{33}^2 + 2A_4 A_6 \sigma_{z_1}\sigma_r \rho_{1,r} + 2A_6 \sigma_r \sigma_{z_2} \rho_{2,r} A_{33} \\
&+ 2A_4 \sigma_{z_1}\sigma_{z_2}\rho_{1,2} A_{33} \\
&+ 2\frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} + 2\frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1}\Sigma_2 A_{33} \\
&+ 2\frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1}\Sigma_1 A_4 + 2\frac{1-\gamma}{\gamma}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1}\Sigma_r A_6 \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_6 + 2\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_r A_6 A_6 \\
&+ 2A_4\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_1 A_6 + 2A_{33}\frac{1-\gamma}{\gamma}\Sigma_r'\Sigma^{-1}\Sigma_2 A_6 \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_4 + 2A_6\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_r A_4 \\
&+ 2A_4\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_1 A_4 + 2\frac{1-\gamma}{\gamma}\Sigma_1'\Sigma^{-1}\Sigma_2 A_{33} A_4 \\
&- \frac{1-\gamma}{\gamma}\left(\Sigma^{-1}\Sigma_1 A_4 + \Sigma^{-1}\Sigma_2 A_{33} + \Sigma^{-1}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} + \Sigma^{-1}\Sigma_r A_6\right) \\
&\times \left(\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' + A_6\Sigma_r' + A_4\Sigma_1' + A_{33}\Sigma_2'\right) \\
&+ 2\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_{33} + 2A_6\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_r A_{33} \\
&+ 2A_4\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_1 A_{33} + 2A_{33}\frac{1-\gamma}{\gamma}\Sigma_2'\Sigma^{-1}\Sigma_2 A_{33}.
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_4}{\partial t} &= -\theta_1 A_4 - \theta_2 A_4 + \sigma_r^2 A_5 A_6 + \sigma_{z_1}^2 A_{22} A_4 + \sigma_{z_2}^2 A_{33} A_4 \\
&+ A_4 A_5 \sigma_{z_1} \sigma_r \rho_{1,r} + A_{22} A_6 \sigma_{z_1} \sigma_r \rho_{1,r} + A_5 \sigma_r \sigma_{z_2} \rho_{2,r} A_{33} \\
&+ A_6 \sigma_r \sigma_{z_2} \rho_{2,r} A_4 + A_{22} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_{33} + A_4 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_4 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} + A_6 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_r \\
&+ A_4 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_1 + A_{33} \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_2 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_2 A_4 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_1 A_{22} + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_r A_5 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_4 + A_5 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_4 \\
&+ A_{22} \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_4 + \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_4 A_4 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_6 + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_5 A_6 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_{33} + A_5 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_{33} \\
&+ A_{22} \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_{33} + A_4 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_{33} \\
&+ A_{22} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_6 + A_4 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_6 \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} + \Sigma^{-1} \Sigma_r A_5 + \Sigma^{-1} \Sigma_1 A_{22} + \Sigma^{-1} \Sigma_2 A_4 \right) \\
&\times \left(\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' + A_6 \Sigma_r' + A_4 \Sigma_1' + A_{33} \Sigma_2' \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_5 + A_{33} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_5 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_{22} + A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_{22} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} \qquad \frac{1-\gamma}{\gamma} \Sigma_1'
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_5}{\partial t} &= -\theta_r A_5 - \theta_1 A_5 + \sigma_r^2 A_{11} A_5 + \sigma_{z_1}^2 A_{22} A_5 + \sigma_{z_2}^2 A_4 A_6 + A_{22} A_{11} \sigma_{z_1} \sigma_r \rho_{1,r} \\
&+ A_5 A_5 \sigma_{z_1} \sigma_r \rho_{1,r} + A_{11} \sigma_r \sigma_{z_2} \rho_{2,r} A_4 + A_5 \sigma_r \sigma_{z_2} \rho_{2,r} A_6 \\
&+ A_{22} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_6 + A_5 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_4 \\
&+ A_{11} \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_r + A_5 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_1 \\
&+ A_6 \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' \Sigma^{-1} \Sigma_2 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_{11} + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_5 A_{11} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_{11} A_5 + A_5 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_5 + A_6 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_5 \\
&+ A_{22} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_{11} + A_4 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_{11} \\
&+ A_{11} \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_4 + A_5 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_4 + A_6 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_4 \\
&+ A_{11} \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_{22} + A_5 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_{22} + A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_{22} \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \Sigma_1 A_5 + \Sigma^{-1} \Sigma_r A_{11} + \Sigma^{-1} \Sigma_2 A_6 \right) \\
&\times \left(\begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix}' + A_5 \Sigma_r' + A_4 \Sigma_2' + A_{22} \Sigma_1' \right) \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} + \Sigma^{-1} \Sigma_r A_5 + \Sigma^{-1} \Sigma_1 A_{22} + \Sigma^{-1} \Sigma_2 A_4 \right) \\
&\times \left(A_5 \Sigma_1' + A_{11} \Sigma_r' + A_6 \Sigma_2' \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_6 + A_5 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_6 + A_{22} \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_6 \\
&+ A_4 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_6 \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,1} \\ \mu_{B,1} \end{bmatrix} A_5 + A_5 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_5 + \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_4 A_5 \\
&+ A_{22} \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_5.
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial A_6}{\partial t} &= -\theta_r A_6 - \theta_2 A_6 + \sigma_r^2 A_{11} A_6 + \sigma_{z_1}^2 A_4 A_5 + \sigma_{z_2}^2 A_{33} A_6 + A_4 A_{11} \sigma_{z_1} \sigma_r \rho_{1,r} \\
&+ A_5 A_6 \sigma_{z_1} \sigma_r \rho_{1,r} + A_6 \sigma_r \sigma_{z_2} \rho_{2,r} A_6 + A_{11} \sigma_r \sigma_{z_2} \rho_{2,r} A_{33} \\
&+ A_5 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_{33} + A_4 \sigma_{z_1} \sigma_{z_2} \rho_{1,2} A_6 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_1 A_5 + \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_2 A_6 \\
&+ \frac{1-\gamma}{\gamma} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' \Sigma^{-1} \Sigma_r A_{11} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_{11} + \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_6 A_{11} \\
&+ A_{11} \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_4 + A_5 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_4 + A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_4 \\
&+ A_4 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_{11} + A_{33} \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_{11} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_r A_{11} A_6 + A_5 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_1 A_6 \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \Sigma_1 A_5 + \Sigma^{-1} \Sigma_r A_{11} + \Sigma^{-1} \Sigma_2 A_6 \right) \\
&\times \left(\begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix}' + A_6 \Sigma_r' + A_4 \Sigma_1' + A_{33} \Sigma_2' \right) \\
&+ A_6 \frac{1-\gamma}{\gamma} \Sigma_r' \Sigma^{-1} \Sigma_2 A_6 + A_4 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_1 A_5 \\
&- \frac{1}{2} \frac{1-\gamma}{\gamma} \left(\Sigma^{-1} \Sigma_1 A_4 + \Sigma^{-1} \Sigma_2 A_{33} + \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} + \Sigma^{-1} \Sigma_r A_6 \right) \\
&\times \left(A_5 \Sigma_1' + A_{11} \Sigma_r' + A_6 \Sigma_2' \right) \\
&+ \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_6 + A_6 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_6 + A_4 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_6 \\
&+ A_{33} \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_6 \\
&+ A_{11} \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_r A_{33} + A_5 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_1 A_{33} + A_6 \frac{1-\gamma}{\gamma} \Sigma_2' \Sigma^{-1} \Sigma_2 A_{33} \\
&+ \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \begin{bmatrix} \mu_{M,2} \\ \mu_{B,2} \end{bmatrix} A_5 + \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_2 A_{33} A_5 + A_6 \frac{1-\gamma}{\gamma} \Sigma_1' \Sigma^{-1} \Sigma_r A_5.
\end{aligned}
$$

6.2 Appendix B

We redo now the exercise (i) when the investor follows a sub-optimal strategy that ignores intertemporal hedging, and (ii) when she assumes constant the market risk premia and thus ignores predictability.

(i) To solve for the sub-optimal strategy that ignores hedging terms, we remark that:

$$\begin{bmatrix} \omega_{M,t} \\ \omega_{B,t} \end{bmatrix} = \frac{J_V}{-V_t J_{VV}} \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix}. \tag{31}$$

Plugging in the HJB equation yields:

$$\begin{aligned} 0 = & \frac{\partial J}{\partial t} + V_t J_V r_t \\ & - \frac{1}{2} \frac{J_V J_V}{J_{VV}} \left(\begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \right)' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\ & + \left(\frac{J_V J_{Vr}}{-J_{VV}} \Sigma_r + \frac{J_V J_{Vz_1}}{-J_{VV}} \Sigma_1 + \frac{J_V J_{Vz_2}}{-J_{VV}} \Sigma_2 \right)' \Sigma^{-1} \begin{bmatrix} \mu_{M,0} + \mu_{M,1} z_{1,t} + \mu_{M,2} z_{2,t} \\ \mu_{B,0} + \mu_{B,1} z_{1,t} + \mu_{B,2} z_{2,t} \end{bmatrix} \\ & + J_r \theta_r \left[\overline{r} - r_t \right] + \frac{1}{2} J_{rr} \sigma_r^2 + J_{z_1} \theta_1 \left[\overline{z}_1 - z_{1,t} \right] + \frac{1}{2} J_{z_1 z_1} \sigma_{z_1}^2 \\ & + J_{z_2} \theta_2 \left[\overline{z}_2 - z_{2,t} \right] + \frac{1}{2} J_{z_2 z_2} \sigma_{z_2}^2 + J_{r z_1} \sigma_r \sigma_{z_1} \rho_{1,r} + J_{r z_2} \sigma_r \sigma_{z_2} \rho_{2,r} + J_{z_1 z_2} \sigma_{z_1} \sigma_{z_2} \rho_{1,2}. \end{aligned} \tag{32}$$

We proceed as for the optimal strategy in Appendix A to obtain ten equations, compute the certainty equivalent of the sub-optimal, myopic portfolio allocation, and then the annualized "certainty equivalent" rate of return. The $A_0(T-t)$, $A_i(T-t)$ and $A_{ij}(T-t)$ functions above are replaced by the functions $B_0(T-t)$, $B_i(T-t)$ and $B_{ij}(T-t)$, not shown here but available upon request.

(ii) To solve for the sub-optimal strategy that ignores predictability, we force in the HJB equation the parameters $\mu_{M,1}$, $\mu_{M,2}$, $\mu_{B,1}$, $\mu_{B,2}$ and the terms involving $J_{Vz_1}$and J_{Vz_2} to zero. The market risk premia $\overline{\mu}_M$ and $\overline{\mu}_B$ are fixed to their long term values, but the inteest rate still is stochastic. The $A_0(T-t)$, $A_i(T-t)$ and $A_{ij}(T-t)$ functions of Appendix A are replaced by $C_0(T-t)$, $C_i(T-t)$ and $C_{ij}(T-t)$ functions.

6.3 Appendix C

To estimate the model, we have to identify the parameters for the dynamics of the predictors, the interest rate (money market account), the stock return and the bond return. To do so, we must use a discretized version of the continuous process for each variate, as described below.

6.3.1 Identifying the predictor parameters

We identify the parameters governing the predictor dynamics (7) using a discrete time representation. Recall that the Ornstein-Uhlenbeck (O-U hereafter) processes for the predictors write:

$$dz_{i,t} = \theta_i \left[\overline{z_i} - z_{i,t}\right] dt + \sigma_{z_i} dZ_{z_i,t}, \tag{33}$$

for $i = 1, 2$. Applying Ito's lemma to $z_{i,t}e^{\theta_i t}$ and then setting $u - t = h$ (with $u > t$) yields:

$$z_{i,t+h} = \overline{z_i}\left(1 - e^{-\theta_i h}\right) + e^{-\theta_i h} z_{i,t} + \sigma_{z_i} e^{-\theta_i (t+h)} \int_t^{t+h} e^{\theta_i s} dZ_{z_i,s} \qquad \forall i = 1, 2. \tag{34}$$

To identify the continuous time parameters, we run the following regressions (for $i = 1, 2$):

$$z_{i,t+h} = a_{z_i} + A_{z_i} z_{i,t} + \upsilon_{z_i,t+h}. \tag{35}$$

Identifying (34) with (35) (for $i = 1, 2$) yields:

$$\begin{aligned} \theta_i &= -\frac{\ln A_{z_i}}{h}, \\ \overline{z}_i &= \frac{a_{z_i}}{1 - A_{z_i}}, \\ \sigma_{z_i}^2 &= -\frac{2}{1 - A_{z_i}^2} \frac{\ln A_{z_i}}{h} \sigma_{\upsilon_{z_i}}^2. \end{aligned} \tag{36}$$

Furthermore, we have:

$$\begin{aligned} Cov_t\left(z_{1,t+h}, z_{2,t+h}\right) &= \sigma_{z_1}\sigma_{z_2}\rho_{1,2}\frac{1}{\theta_1 + \theta_2}\left(1 - e^{-(\theta_1+\theta_2)h}\right) \\ &= Cov_t\left(\upsilon_{z_1,t+h}, \upsilon_{z_2,t+h}\right). \end{aligned} \tag{37}$$

Solving for $\rho_{1,2}$ then yields:

$$\rho_{1,2} = Cov_t\left(\upsilon_{z_1,t+h}, \upsilon_{z_2,t+h}\right) \frac{\theta_1 + \theta_2}{\sigma_{z_1}\sigma_{z_2}} \left(1 - e^{-(\theta_1+\theta_2)h}\right)^{-1}. \tag{38}$$

The correlations of the predictors with the interest rate and the traded assets are provided below.

6.3.2 Identifying the interest rate parameters

Applying Ito's lemma to $r_t e^{\theta_r t}$ yields:

$$r_{t+h} = \overline{r}\left(1 - e^{-\theta_r h}\right) + e^{-\theta_r h} r_t + \sigma_r e^{-\theta_r (t+h)} \int_t^{t+h} e^{\theta_r s} dZ_{r,s}. \tag{39}$$

We run the following regression for r using market data:

$$r_{t+h} = a_r + A_r r_t + \upsilon_{r,t+h}. \tag{40}$$

Identifying (39) and (40) yields:

$$\begin{aligned} \theta_r &= -\frac{\ln A_r}{\Delta t}, \\ \overline{r} &= \frac{a_r}{1 - A_r}, \\ \sigma_r^2 &= -\frac{2}{1 - A_r^2} \frac{\ln A_r}{\Delta t} \sigma_{\upsilon_r}^2. \end{aligned} \tag{41}$$

Furthermore, we have:

$$\begin{aligned} Cov_t\left(z_{1,t+h}, r_{t+h}\right) &= \sigma_{z_1} \sigma_r \rho_{1,r} \frac{1}{\theta_1 + \theta_r}\left(1 - e^{-(\theta_1 + \theta_r)h}\right) \\ &= Cov_t\left(\upsilon_{z_1,t+h}, \upsilon_{r,t+h}\right), \end{aligned} \tag{42}$$

$$\begin{aligned} Cov_t\left(z_{2,t+h}, r_{t+h}\right) &= \sigma_{z_2} \sigma_r \rho_{2,r} \frac{1}{\theta_2 + \theta_r}\left(1 - e^{-(\theta_2 + \theta_r)h}\right) \\ &= Cov_t\left(\upsilon_{z_2,t+h}, \upsilon_{r,t+h}\right). \end{aligned} \tag{43}$$

Solving this system yields:

$$\begin{aligned} \rho_{1,r} &= Cov_t\left(\upsilon_{z_1,t+h}, \upsilon_{r,t+h}\right) \frac{\theta_1 + \theta_r}{\sigma_{z_1} \sigma_r}\left(1 - e^{-(\theta_1 + \theta_r)h}\right)^{-1}, \\ \rho_{2,r} &= Cov_t\left(\upsilon_{z_2,t+h}, \upsilon_{r,t+h}\right) \frac{\theta_2 + \theta_r}{\sigma_{z_2} \sigma_r}\left(1 - e^{-(\theta_2 + \theta_r)h}\right)^{-1}. \end{aligned} \tag{44}$$

6.3.3 Identifying the stock return parameters

We apply the same procedure to the stock market index and integrate the continuous time process (3), using (5), over the discrete interval $[t, t+h]$ for all prediction horizons $h = \Delta t,\ 2\Delta t,\ 3\Delta t, ...$

Applying Ito's lemma to $\ln M_t$ from Eq.(3) and using (5) yields, for any positive h :

$$\ln M_{t+h} - \ln M_t - \int_t^{t+h} r_s ds = \left(\mu_{M,0} - \frac{\sigma_M^2}{2}\right) h + \int_t^{t+h} \left(\mu_{M,1} z_{1,s} + \mu_{M,2} z_{2,s}\right) ds + \int_t^{t+h} \sigma_M dZ_{M,s}. \tag{45}$$

Since

$$\int_t^{t+h} z_{1,s} ds = \left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right) \overline{z_1} - \frac{1}{\theta_1}\left(e^{-\theta_1 h} - 1\right) z_{1,t} + \frac{1}{\theta_1}\sigma_{z_1} \int_t^{t+h} \left(1 - e^{-\theta_1 (t+h)} e^{\theta_1 s}\right) dZ_{z_1,s}, \tag{46}$$

$$\int_t^{t+h} z_{2,s} ds = \left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right) \overline{z_2} - \frac{1}{\theta_2}\left(e^{-\theta_2 h} - 1\right) z_{2,t} + \frac{1}{\theta_2}\sigma_{z_2} \int_t^{t+h} \left(1 - e^{-\theta_2 (t+h)} e^{\theta_2 s}\right) dZ_{z_2,s}, \tag{47}$$

then we get:

$$\begin{aligned} \ln M_{t+h} - \ln M_t - \int_t^{t+h} r_s ds = & \left(\mu_{M,0} - \frac{\sigma_M^2}{2}\right) h \\ & + \mu_{M,1}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right) \overline{z_1} + \mu_{M,2}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right) \overline{z_2} \\ & - \mu_{M,1}\frac{1}{\theta_1}\left(e^{-\theta_1 h} - 1\right) z_{1,t} - \mu_{M,2}\frac{1}{\theta_2}\left(e^{-\theta_2 h} - 1\right) z_{2,t} \\ & + \int_t^{t+h} \sigma_M dZ_{M,s} \\ & + \mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1} \int_t^{t+h} \left(1 - e^{-\theta_1 (t+h)} e^{\theta_1 s}\right) dZ_{z_1,s} \\ & + \mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2} \int_t^{t+h} \left(1 - e^{-\theta_2 (t+h)} e^{\theta_2 s}\right) dZ_{z_2,s}. \end{aligned} \tag{48}$$

A natural regression for identifying these parameters is:

$$r_{M,t+h} - \int_t^{t+h} r_s ds = a_{M,h} + \beta_{M,h,1} z_{1,t} + \beta_{M,h,2} z_{2,t} + \upsilon_{M,t+h}, \tag{49}$$

where $r_{M,t+h}$ is the continuously compounded h-period return on the stock index between dates t and $t+h$, the stock market return possibly including a dividend. Identifying (48) and (49) yields:

$$\mu_{M,0} = \frac{1}{h}a_{M,h} + \frac{\sigma_M^2}{2} - \frac{1}{h}\mu_{M,1}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right)\overline{z_1} - \frac{1}{h}\mu_{M,2}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right)\overline{z_2},$$
$$\mu_{M,1} = \frac{\theta_1}{1 - e^{-\theta_1 h}}\beta_{M,h,1},$$
$$\mu_{M,2} = \frac{\theta_2}{1 - e^{-\theta_2 h}}\beta_{M,h,2}.$$

We also have:

$$\begin{aligned}\sigma^2_{\upsilon_{M,t+h}} &= \int_t^{t+h} \sigma_M^2 ds \\ &+ \left(\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\right)^2 \int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right)^2 ds \\ &+ \left(\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\right)^2 \int_t^{t+h}\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right)^2 ds \\ &+ 2\frac{1}{\theta_1}\sigma_M\sigma_{z_1}\rho_{1,M}\mu_{M,1}\int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right) ds \\ &+ 2\frac{1}{\theta_2}\sigma_M\sigma_{z_2}\rho_{2,M}\mu_{M,2}\int_t^{t+h}\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right) ds \\ &+ 2\frac{1}{\theta_1}\frac{1}{\theta_2}\mu_{M,1}\mu_{M,2}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right)\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right) ds,\end{aligned} \tag{50}$$

$$\begin{aligned}Cov_t\left(\upsilon_{M,t+h}, \upsilon_{r,t+h}\right) &= \sigma_M\sigma_r\rho_{r,M}e^{-\theta_r(t+h)}\int_t^{t+h} e^{\theta_r s}ds \\ &+ \mu_{M,1}\frac{1}{\theta_1}\sigma_r\sigma_{z_1}\rho_{1,r}e^{-\theta_r(t+h)}\int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right)e^{\theta_r s}ds \\ &+ \mu_{M,2}\frac{1}{\theta_2}\sigma_r\sigma_{z_2}\rho_{2,r}e^{-\theta_r(t+h)}\int_t^{t+h}\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right)e^{\theta_r s}ds,\end{aligned} \tag{51}$$

$$\begin{aligned}Cov_t\left(\upsilon_{M,t+h}, \upsilon_{z_1,t+h}\right) &= \sigma_M\sigma_{z_1}\rho_{1,M}e^{-\theta_1(t+h)}\int_t^{t+h} e^{\theta_1 s}ds \\ &+ \mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}^2 e^{-\theta_1(t+h)}\int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right)e^{\theta_1 s}ds \\ &+ \mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\sigma_{z_1}\rho_{1,2}e^{-\theta_1(t+h)}\int_t^{t+h}\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right)e^{\theta_1 s}ds,\end{aligned} \tag{52}$$

$$
\begin{aligned}
Cov_t\left(\upsilon_{M,t+h},\upsilon_{z_2,t+h}\right) &= \sigma_M\sigma_{z_2}\rho_{2,M}e^{-\theta_2(t+h)}\int_t^{t+h}e^{\theta_2 s}ds \\
&+ \mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\int_t^{t+h}\left(1-e^{-\theta_1(t+h)}e^{\theta_1 s}\right)e^{-\theta_2(t+h)}e^{\theta_2 s}ds \\
&+ \mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}^2e^{-\theta_2(t+h)}\int_t^{t+h}\left(1-e^{-\theta_2(t+h)}e^{\theta_2 s}\right)e^{\theta_2 s}ds.
\end{aligned} \tag{53}
$$

Solving this system yields:

$$
\begin{aligned}
\sigma_M^2 &= \frac{1}{h}\sigma_{\upsilon_{M,t+h}}^2 \\
&- \frac{1}{h}\left(\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\right)^2\left(h-2\frac{1}{\theta_1}\left(1-e^{-\theta_1 h}\right)+\frac{1}{2\theta_1}\left(1-e^{-2\theta_1 h}\right)\right) \\
&- \frac{1}{h}\left(\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\right)^2\left(h-2\frac{1}{\theta_2}\left(1-e^{-\theta_2 h}\right)+\frac{1}{2\theta_2}\left(1-e^{-2\theta_2 h}\right)\right) \\
&- 2\frac{1}{h}\frac{1}{\theta_1}\sigma_M\sigma_{z_1}\rho_{1,M}\mu_{M,1}\left(h-\frac{1}{\theta_1}\left(1-e^{-\theta_1 h}\right)\right) \\
&- 2\frac{1}{h}\frac{1}{\theta_2}\sigma_M\sigma_{z_2}\rho_{2,M}\mu_{M,2}\left(h-\frac{1}{\theta_2}\left(1-e^{-\theta_2 h}\right)\right) \\
&- 2\frac{1}{h}\frac{1}{\theta_1}\frac{1}{\theta_2}\mu_{M,1}\mu_{M,2}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\left(h-\frac{1}{\theta_2}\left(1-e^{-\theta_2 h}\right)-\frac{1}{\theta_1}\left(1-e^{-\theta_1 h}\right)+\frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)h}\right)\right),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\sigma_M\sigma_r\rho_{r,M} &= \frac{\theta_r}{1-e^{-\theta_r h}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{r,t+h}\right) \\
&- \mu_{M,1}\frac{1}{\theta_1}\frac{\theta_r}{1-e^{-\theta_r h}}\sigma_r\sigma_{z_1}\rho_{1,r}\left(\frac{1}{\theta_r}\left(1-e^{-\theta_r h}\right)-\frac{1}{\theta_r+\theta_1}\left(1-e^{-(\theta_r+\theta_1)h}\right)\right) \\
&- \mu_{M,2}\frac{1}{\theta_2}\frac{\theta_r}{1-e^{-\theta_r h}}\sigma_r\sigma_{z_2}\rho_{2,r}\left(\frac{1}{\theta_r}\left(1-e^{-\theta_r h}\right)-\frac{1}{\theta_r+\theta_2}\left(1-e^{-(\theta_r+\theta_2)h}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_M\sigma_{z_1}\rho_{1,M} &= \frac{\theta_1}{1-e^{-\theta_1 h}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{z_1,t+h}\right) \\
&- \mu_{M,1}\frac{1}{\theta_1}\frac{\theta_1}{1-e^{-\theta_1 h}}\sigma_{z_1}^2\left(\frac{1}{\theta_1}\left(1-e^{-\theta_1 h}\right)-\frac{1}{2\theta_1}\left(1-e^{-2\theta_1 h}\right)\right) \\
&- \mu_{M,2}\frac{1}{\theta_2}\frac{\theta_1}{1-e^{-\theta_1 h}}\sigma_{z_2}\sigma_{z_1}\rho_{1,2}\left(\frac{1}{\theta_1}\left(1-e^{-\theta_1 h}\right)-\frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)h}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_M\sigma_{z_2}\rho_{2,M} &= \frac{\theta_2}{1-e^{-\theta_2 h}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{z_2,t+h}\right) \\
&- \mu_{M,1}\frac{1}{\theta_1}\frac{\theta_2}{1-e^{-\theta_2 h}}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\left(\frac{1}{\theta_2}\left(1-e^{-\theta_2 h}\right)-\frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)h}\right)\right) \\
&- \mu_{M,2}\frac{1}{\theta_2}\frac{\theta_2}{1-e^{-\theta_2 h}}\sigma_{z_2}^2\left(\frac{1}{\theta_2}\left(1-e^{-\theta_2 h}\right)-\frac{1}{2\theta_2}\left(1-e^{-2\theta_2 h}\right)\right).
\end{aligned}
$$

6.3.4 Identifying the bond return parameters

The approach is similar to that followed for the stock return. We note that:

$$\begin{aligned}\ln B_{t+h} - \ln B_t - \int_t^{t+h} r_s ds = & \left(\mu_{B,0} - \frac{\sigma_B^2}{2}\right) h \\ & + \mu_{B,1}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right)\overline{z_1} + \mu_{B,2}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right)\overline{z_2} \\ & - \mu_{B,1}\frac{1}{\theta_1}\left(e^{-\theta_1 h} - 1\right) z_{1,t} - \mu_{B,2}\frac{1}{\theta_2}\left(e^{-\theta_2 h} - 1\right) z_{2,t} \\ & + \int_t^{t+h} \sigma_B dZ_{B,s} + \mu_{B,1}\frac{1}{\theta_1}\sigma_{z_1}\int_t^{t+h}\left(1 - e^{-\theta_1(t+h)}e^{\theta_1 s}\right) dZ_{z_1,s} \\ & + \mu_{B,2}\frac{1}{\theta_2}\sigma_{z_2}\int_t^{t+h}\left(1 - e^{-\theta_2(t+h)}e^{\theta_2 s}\right) dZ_{z_2,s}.\end{aligned} \tag{55}$$

A natural regression for identifying these parameters is:

$$r_{B,t+h} - \int_t^{t+h} r_s ds = a_{B,h} + \beta_{B,h,1} z_{1,t} + \beta_{B,h,2} z_{2,t} + \upsilon_{B,t+h}. \tag{56}$$

The parameters are $\mu_{B,0}$, $\mu_{B,1}$, $\mu_{B,2}$, σ_B^2, $\rho_{r,B}\sigma_B\sigma_r, \rho_{1,B}\sigma_B\sigma_{z_1}$ and $\rho_{2,B}\sigma_B\sigma_{z_2}$. They have a structure identical to that for the stock index. Finally, to obtain the correlation between bond and stock returns, we note that:

$$\begin{aligned}\sigma_B\sigma_M\rho_{B,M} = & \, Cov_t\left(\upsilon_{M,t+h}, \upsilon_{B,t+h}\right) \\ & - \frac{1}{\theta_1}\mu_{M,1}\sigma_{z_1}\sigma_B\rho_{B,1}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right) \\ & - \frac{1}{\theta_2}\mu_{M,2}\sigma_{z_2}\sigma_B\rho_{B,2}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right) \\ & - \frac{1}{\theta_1}\mu_{B,1}\sigma_M\sigma_{z_1}\rho_{1,M}\left(h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right)\right) \\ & - \frac{1}{\theta_2}\mu_{B,2}\sigma_M\sigma_{z_2}\rho_{2,M}\left(h - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right)\right) \\ & - \frac{1}{\theta_1^2}\mu_{M,1}\mu_{B,1}\sigma_{z_1}^2\left(h - 2\frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right) + \frac{1}{2\theta_1}\left(1 - e^{-2\theta_1 h}\right)\right) \\ & - \frac{1}{\theta_2^2}\mu_{M,2}\mu_{B,2}\sigma_{z_2}^2\left(h - 2\frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right) + \frac{1}{2\theta_2}\left(1 - e^{-2\theta_2 h}\right)\right) \\ & - \frac{1}{\theta_2}\frac{1}{\theta_1}\left(\mu_{M,2}\mu_{B,1} + \mu_{M,1}\mu_{B,2}\right)\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\begin{pmatrix} h - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 h}\right) - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 h}\right) \\ + \frac{1}{\theta_1+\theta_2}\left(1 - e^{-(\theta_1+\theta_2)h}\right)\end{pmatrix}.\end{aligned}$$

This ends our quest for the needed parameters.

6.4 Appendix D: correcting for persistent residuals

Since for return periods h longer than one month ($h \geq 2\Delta t$) data are in general overlapping, we need to correct for the thus created persistence in residuals. We describe here the procedure used for the stock index only, the case of the bond being similar.

The correction we use is that of Valkanov (2003). It consists in regressing the h-period stock return on the *sum* of the h predictor values, for each predictor. We thus have to re-identify the parameters of the stock excess return process (as in Appendix C but) using Valkanov's correction.

Writing Eq. (48) for a unit time interval Δt and then summing over $h/\Delta t$ such intervals yields:

$$\begin{aligned}
&\sum_{i=0}^{h/\Delta t-1}\left(\ln M_{t+(i+1)\Delta t}-\ln M_{t+i\Delta t}\right)-\sum_{i=0}^{h/\Delta t-1}\int_{t+i\Delta t}^{t+(i+1)\Delta t} r_s ds \qquad (57)\\
&=\sum_{i=0}^{h/\Delta t-1}\left(\mu_{M,0}-\frac{\sigma_M^2}{2}\right)\Delta t+\sum_{i=0}^{h/\Delta t-1}\mu_{M,1}\left(\Delta t-\frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right)\right)\overline{z_1}\\
&+\sum_{i=0}^{h/\Delta t-1}\mu_{M,2}\left(\Delta t-\frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right)\overline{z_2}\\
&-\sum_{i=0}^{h/\Delta t-1}\mu_{M,1}\frac{1}{\theta_1}\left(e^{-\theta_1\Delta t}-1\right)z_{1,t+i\Delta t}-\sum_{i=0}^{h/\Delta t-1}\mu_{M,2}\frac{1}{\theta_2}\left(e^{-\theta_2\Delta t}-1\right)z_{2,t+i\Delta t}\\
&+\sum_{i=0}^{h/\Delta t-1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\sigma_M dZ_{M,s}\\
&+\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\sum_{i=0}^{h/\Delta t-1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1-e^{-\theta_1(t+i\Delta t)}e^{\theta_1 s}\right)dZ_{z_1,s}\\
&+\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\sum_{i=0}^{h/\Delta t-1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1-e^{-\theta_2(t+i\Delta t)}e^{\theta_2 s}\right)dZ_{z_2,s},
\end{aligned}$$

or else

$$
\begin{aligned}
&\ln M_{t+h} - \ln M_t - \int_t^{t+h} r_s ds \\
&= \frac{h}{\Delta t}\left\{\left(\mu_{M,0} - \frac{\sigma_M^2}{2}\right)\Delta t + \mu_{M,1}\left(\Delta t - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 \Delta t}\right)\right)\overline{z_1} + \mu_{M,2}\left(\Delta t - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right)\overline{z_2}\right\} \qquad (58) \\
&- \mu_{M,1}\frac{1}{\theta_1}\left(e^{-\theta_1 \Delta t} - 1\right)\sum_{i=0}^{h/\Delta t - 1} z_{1,t+i\Delta t} - \mu_{M,2}\frac{1}{\theta_2}\left(e^{-\theta_2 \Delta t} - 1\right)\sum_{i=0}^{h/\Delta t - 1} z_{2,t+i\Delta t} + \sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t} \sigma_M dZ_{M,s} \\
&+ \mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1 - e^{-\theta_1(t+i\Delta t)}e^{\theta_1 s}\right) dZ_{z_1,s} \\
&+ \mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1 - e^{-\theta_2(t+i\Delta t)}e^{\theta_2 s}\right) dZ_{z_2,s}.
\end{aligned}
$$

A natural regression to run is:

$$
r_{M,t+h} - \int_t^{t+h} r_s ds = a_{M,h} + \beta_{M,h,1}\sum_{i=0}^{h/\Delta t - 1} z_{1,t+i\Delta t} + \beta_{M,h,2}\sum_{i=0}^{h/\Delta t - 1} z_{2,t+i\Delta t} + \upsilon_{M,t+h}.
$$

Identifying the parameters, we then obtain:

$$
\mu_{M,0} = \frac{1}{h}a_{M,h} + \frac{\sigma_M^2}{2} - \mu_{M,1}\frac{1}{\Delta t}\left(\Delta t - \frac{1}{\theta_1}\left(1 - e^{-\theta_1 \Delta t}\right)\right)\overline{z_1} - \mu_{M,2}\frac{1}{\Delta t}\left(\Delta t - \frac{1}{\theta_2}\left(1 - e^{-\theta_2 \Delta t}\right)\right)\overline{z_2}, \qquad (59)
$$

$$
\mu_{M,1} = \frac{\theta_1}{1 - e^{-\theta_1 \Delta t}}\beta_{M,h,1}, \qquad (60)
$$

$$
\mu_{M,2} = \frac{\theta_2}{1 - e^{-\theta_2 \Delta t}}\beta_{M,h,2}, \qquad (61)
$$

$$
\begin{aligned}
\sigma^2_{\upsilon_{M,t+h}} = &\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\sigma_M^2 ds + \left(\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\right)^2\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1 - e^{-\theta_1(t+i\Delta t)}e^{\theta_1 s}\right)^2 ds \\
&+ \left(\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\right)^2\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1 - e^{-\theta_2(t+i\Delta t)}e^{\theta_2 s}\right)^2 ds \\
&+ 2\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\sigma_M\left(1 - e^{-\theta_1(t+i\Delta t)}e^{\theta_1 s}\right) ds \\
&+ 2\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\sigma_M\left(1 - e^{-\theta_2(t+i\Delta t)}e^{\theta_2 s}\right) ds \\
&+ 2\mu_{M,1}\frac{1}{\theta_1}\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\sigma_{z_1}\sum_{i=0}^{h/\Delta t - 1}\int_{t+i\Delta t}^{t+(i+1)\Delta t}\left(1 - e^{-\theta_1(t+i\Delta t)}e^{\theta_1 s}\right)\left(1 - e^{-\theta_2(t+i\Delta t)}e^{\theta_2 s}\right) ds.
\end{aligned}
$$

Also using Eq.(39) and Eq.(34) for $i = 1, 2$ yields:

$$
\begin{aligned}
Cov_t\left(\upsilon_{M,t+h}, \upsilon_{r,t+h}\right) = {} & \sigma_M \sigma_r \rho_{r,M} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} e^{-\theta_r(t+i\Delta t)} e^{\theta_r s} ds \\
& + \mu_{M,1} \frac{1}{\theta_1} \sigma_{z_1} \rho_{1,r} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s}\right) \sigma_r e^{-\theta_r(t+i\Delta t)} e^{\theta_r s} ds \\
& + \mu_{M,2} \frac{1}{\theta_2} \sigma_{z_2} \rho_{2,r} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s}\right) \sigma_r e^{-\theta_r(t+i\Delta t)} e^{\theta_r s} ds,
\end{aligned}
$$

$$
\begin{aligned}
Cov_t\left(\upsilon_{M,t+h}, \upsilon_{z_1,t+h}\right) = {} & \sigma_M \sigma_{z_1} \rho_{1,M} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s} ds \\
& + \mu_{M,1} \frac{1}{\theta_1} \sigma_{z_1}^2 \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s}\right) e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s} ds \\
& + \mu_{M,2} \frac{1}{\theta_2} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s}\right) e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s} ds,
\end{aligned}
$$

$$
\begin{aligned}
Cov_t\left(\upsilon_{M,t+h}, \upsilon_{z_2,t+h}\right) = {} & \sigma_M \sigma_{z_2} \rho_{2,M} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s} ds \\
& + \mu_{M,1} \frac{1}{\theta_1} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_1(t+i\Delta t)} e^{\theta_1 s}\right) e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s} ds \\
& + \mu_{M,2} \frac{1}{\theta_2} \sigma_{z_2}^2 \sum_{i=0}^{h/\Delta t-1} \int_{t+i\Delta t}^{t+(i+1)\Delta t} \left(1 - e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s}\right) e^{-\theta_2(t+i\Delta t)} e^{\theta_2 s} ds.
\end{aligned}
$$

Explicit computation gives:

$$
\begin{aligned}
\sigma_M^2 = & \frac{1}{h}\sigma^2_{\upsilon_{M,t+h}} \\
& - \frac{1}{\Delta t}\left(\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\right)^2\left(\Delta t - 2\frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right) + \frac{1}{2\theta_1}\left(1-e^{-2\theta_1\Delta t}\right)\right) \\
& - \frac{1}{\Delta t}\left(\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_2}\right)^2\left(\Delta t - 2\frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right) + \frac{1}{2\theta_2}\left(1-e^{-2\theta_2\Delta t}\right)\right) \\
& - 2\frac{1}{\Delta t}\mu_{M,1}\frac{1}{\theta_1}\sigma_M\sigma_{z_1}\rho_{1,M}\left(\Delta t - \frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right)\right) \\
& - 2\frac{1}{\Delta t}\mu_{M,2}\frac{1}{\theta_2}\sigma_M\sigma_{z_2}\rho_{2,M}\left(\Delta t - \frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right)\right) \\
& - 2\frac{1}{\Delta t}\mu_{M,1}\mu_{M,2}\frac{1}{\theta_1}\frac{1}{\theta_2}\sigma_{z_2}\sigma_{z_1}\rho_{1,2} \\
& \times\left(\Delta t - \frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right) - \frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right) + \frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)\Delta t}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_M\sigma_r\rho_{r,M} = & \frac{\Delta t}{h}\frac{\theta_r}{1-e^{-\theta_r\Delta t}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{r,t+h}\right) \\
& - \frac{\theta_r}{1-e^{-\theta_r\Delta t}}\mu_{M,1}\frac{1}{\theta_1}\sigma_r\sigma_{z_1}\rho_{1,r}\left(\frac{1}{\theta_r}\left(1-e^{-\theta_r\Delta t}\right) - \frac{1}{\theta_1+\theta_r}\left(1-e^{-(\theta_1+\theta_r)\Delta t}\right)\right) \\
& - \frac{\theta_r}{1-e^{-\theta_r\Delta t}}\mu_{M,2}\frac{1}{\theta_2}\sigma_r\sigma_{z_2}\rho\left(\frac{1}{\theta_r}\left(1-e^{-\theta_r\Delta t}\right) - \frac{1}{\theta_r+\theta_2}\left(1-e^{-(\theta_r+\theta_2)\Delta t}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_M\sigma_{z_1}\rho_{1,M} = & \frac{\Delta t}{h}\frac{\theta_1}{1-e^{-\theta_1\Delta t}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{z_1,t+h}\right) \\
& - \frac{\theta_1}{1-e^{-\theta_1\Delta t}}\mu_{M,1}\frac{1}{\theta_1}\sigma^2_{z_1}\left(\frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right) - \frac{1}{2\theta_1}\left(1-e^{-2\theta_1\Delta t}\right)\right) \\
& - \frac{\theta_1}{1-e^{-\theta_1\Delta t}}\mu_{M,2}\frac{1}{\theta_2}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\left(\frac{1}{\theta_1}\left(1-e^{-\theta_1\Delta t}\right) - \frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)\Delta t}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_M\sigma_{z_2}\rho_{2,M} = & \frac{\Delta t}{h}\frac{\theta_2}{1-e^{-\theta_2\Delta t}}Cov_t\left(\upsilon_{M,t+h},\upsilon_{z_2,t+h}\right) \\
& - \frac{\theta_2}{1-e^{-\theta_2\Delta t}}\mu_{M,1}\frac{1}{\theta_1}\sigma_{z_1}\sigma_{z_2}\rho_{1,2}\left(\frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right) - \frac{1}{\theta_1+\theta_2}\left(1-e^{-(\theta_1+\theta_2)\Delta t}\right)\right) \\
& - \frac{\theta_2}{1-e^{-\theta_2\Delta t}}\mu_{M,2}\frac{1}{\theta_2}\sigma^2_{z_2}\left(\frac{1}{\theta_2}\left(1-e^{-\theta_2\Delta t}\right) - \frac{1}{2\theta_2}\left(1-e^{-2\theta_2\Delta t}\right)\right).
\end{aligned}
$$

Finally we obtain the correlation between the stock and bond returns:

$$
\begin{aligned}
\sigma_B \sigma_M \rho_{M,B} = & \frac{1}{h} Cov_t \left(\upsilon_{M,t+h}, \upsilon_{B,t+h} \right) \qquad (62) \\
& - \frac{1}{\Delta t} \mu_{B,1} \frac{1}{\theta_1} \sigma_{z_1} \sigma_M \rho_{1,M} \left(\Delta t - \frac{1}{\theta_1} \left(1 - e^{-\theta_1 \Delta t} \right) \right) \\
& - \frac{1}{\Delta t} \mu_{B,2} \frac{1}{\theta_2} \sigma_{z_2} \sigma_M \rho_{2,M} \left(\Delta t - \frac{1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) \right) \\
& - \frac{1}{\Delta t} \mu_{M,1} \frac{1}{\theta_1} \sigma_B \sigma_{z_1} \rho_{1,B} \left(\Delta t - \frac{1}{\theta_1} \left(1 - e^{-\theta_1 \Delta t} \right) \right) \\
& - \frac{1}{\Delta t} \mu_{M,1} \frac{1}{\theta_1} \sigma_{z_1}^2 \mu_{B,1} \frac{1}{\theta_1} \left(\Delta t - 2 \frac{1}{\theta_1} \left(1 - e^{-\theta_1 \Delta t} \right) + \frac{1}{2\theta_1} \left(1 - e^{-2\theta_1 \Delta t} \right) \right) \\
& - \frac{1}{\Delta t} \mu_{M,1} \frac{1}{\theta_1} \mu_{B,2} \frac{1}{\theta_2} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} \\
& \times \left(\Delta t - \frac{1}{\theta_1} \left(1 - e^{-\theta_1 \Delta t} \right) - \frac{1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) + \frac{1}{\theta_1 + \theta_2} \left(1 - e^{-(\theta_1 + \theta_2) \Delta t} \right) \right) \qquad (63) \\
& - \frac{1}{\Delta t} \mu_{M,2} \frac{1}{\theta_2} \sigma_B \sigma_{z_2} \rho_{2,B} \left(\Delta t - \frac{1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) \right) \\
& - \frac{1}{\Delta t} \mu_{M,2} \frac{1}{\theta_2} \mu_{B,1} \frac{1}{\theta_1} \sigma_{z_1} \sigma_{z_2} \rho_{1,2} \\
& \times \left(\Delta t - \frac{1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) - \frac{1}{\theta_1} \left(1 - e^{-\theta_1 \Delta t} \right) + \frac{1}{\theta_1 + \theta_2} \left(1 - e^{-(\theta_1 + \theta_2) \Delta t} \right) \right) \qquad (64) \\
& - \frac{1}{\Delta t} \mu_{M,2} \frac{1}{\theta_2} \mu_{B,2} \frac{1}{\theta_2} \sigma_{z_2}^2 \left(\Delta t - 2 \frac{1}{\theta_2} \left(1 - e^{-\theta_2 \Delta t} \right) + \frac{1}{2\theta_2} \left(1 - e^{-2\theta_2 \Delta t} \right) \right).
\end{aligned}
$$

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To be completed.

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Table I: Summary Statistics

Panel A reports various statistics for the returns on the value weighted stock market portfolio ("mkt") and on a synthetic bond with constant 20-year maturity. For each asset, we report the annualized continuously compounded return and excess return when the latter is computed with the continuously compounded 3-month Treasury bill rate. In Panel B, we report the summary statistics for two predictors: the dividend yield measured as the total dividends paid off during the last 12 months divided by the actual price of the value weighted stock market portfolio ("dy") and the 3-month Treasury bill rate ("tb"). "Auto" stands for the first-order auto-regression coefficient. Panel C shows the correlation matrix for these two predictors. The data cover the period 1963:07 to 2009:12 (558 monthly observations).

Panel A: Summary statistics for the market and the bond returns and excess returns

	return mkt	excess return mkt	return bond	excess return bond
Mean	9.22%	3.87%	6.95%	1.61%
Std	15.72%	15.74%	10.21%	10.19%
skewness	-0.844	-0.852	0.240	0.178
kurtosis	6.089	6.034	5.279	5.119
Auto	0.102	0.104	0.029	0.030

Panel B: Summary statistics for the predictors

	dyvw	tb
mean	2.97%	5.53%
std	1.07%	2.83%
skewness	0.192	0.899
kurtosis	2.305	4.659
min	1.06%	0.03%
max	5.82%	16.30%
auto	0.990	0.988

Panel C: Correlation matrix of the predictors

	dyvw	tb
dyvw	1	
tb	0.680	1

Table II: Long Horizon Auto Regressive Processes for the Predictors

This Table reports the results for the auto-regressive processes followed by the three predictors defined in Table I. The data being monthly, a prediction horizon *h* = 2 years, for instance, means that the lag is 24. The data cover the period 1963:07 to 2009:12 (558 observations).

		dyvw			tb		
h		**constant**	**lagged**	**Adj. R^2**	**constant**	**lagged**	**Adj. R^2**
1m	**Coefficient**	0.00	0.99	0.98	0.00	0.99	0.97
	t(OLS)	1.45	159.85		1.32	138.49	
3m	**Coefficient**	0.00	0.97	0.93	0.00	0.95	0.88
	t(OLS)	2.62	88.80		2.99	65.05	
6m	**Coefficient**	0.00	0.93	0.86	0.01	0.90	0.78
	t(OLS)	4.00	58.45		4.22	44.10	
1y	**Coefficient**	0.00	0.87	0.76	0.01	0.81	0.60
	t(OLS)	5.55	41.59		5.86	28.85	
2y	**Coefficient**	0.01	0.82	0.65	0.02	0.55	0.27
	t(OLS)	6.53	31.68		9.95	13.96	
3y	**Coefficient**	0.01	0.80	0.61	0.03	0.40	0.14
	t(OLS)	6.26	28.38		12.18	9.36	
4y	**Coefficient**	0.01	0.76	0.54	0.04	0.36	0.12
	t(OLS)	6.30	24.51		12.62	8.23	
5y	**Coefficient**	0.01	0.66	0.40	0.03	0.38	0.12
	t(OLS)	7.71	18.10		11.92	8.47	
6y	**Coefficient**	0.01	0.58	0.29	0.04	0.35	0.10
	t(OLS)	8.62	13.99		11.46	7.38	
7y	**Coefficient**	0.01	0.52	0.22	0.04	0.29	0.06
	t(OLS)	8.97	11.60		11.32	5.72	

8y	**Coefficient**	0.02	0.44	0.15	0.04	0.21	0.03
	t(OLS)	9.43	9.17		11.85	3.77	
9y	**Coefficient**	0.02	0.31	0.07	0.05	0.12	0.01
	t(OLS)	10.64	5.77		12.90	2.02	
10y	**Coefficient**	0.02	0.21	0.03	0.05	0.11	0.01
	t(OLS)	11.09	3.55		12.78	1.94	

Table III: Predictive Regressions

This Table reports the results of bivariate predictive regressions of the stock and bond excess returns over different periods (from 1 to 120 months). OLS regressions produced the estimated "Coefficients". To correct for the serial correlation that stems from using overlapping returns, we used the procedure suggested by Hodrick (1992) which led to the t(Hodrick) stats. The t(NW)-stats have been corrected for autocorrelation and heteroskedasticity using the Newey-West estimator with a number of lags equal to the horizon of the predictive regression minus 1. All data cover the period 1963:07 to 2009:12.

		Mkt				Bond			
		constant	**dy**	**tb**	**Adj. R^2**	**constant**	**dy**	**tb**	**Adj. R^2**
1m	**Coefficient**	0.00	0.75	-0.27	0.02	0.00	0.07	-0.03	0.00
	t(NW)	-0.67	2.60	-2.54		0.25	0.46	-0.38	
	t(Hodrick)	-0.66	2.53	-2.45		0.25	0.46	-0.38	
3m	**Coefficient**	-0.02	2.15	-0.70	0.04	0.01	-0.09	-0.02	0.00
	t(NW)	-0.99	3.20	-2.62		0.84	-0.24	-0.14	
	t(Hodrick)	-0.84	2.55	-2.25		0.74	-0.20	-0.10	
6m	**Coefficient**	-0.03	4.17	-1.27	0.07	0.02	-0.42	0.01	0.00
	t(NW)	-1.09	3.48	-2.54		1.14	-0.67	0.05	
	t(Hodrick)	-0.96	2.63	-2.16		0.93	-0.53	0.03	
1y	**Coefficient**	-0.06	6.85	-1.90	0.09	0.05	-1.60	0.35	0.01
	t(NW)	-0.97	2.83	-2.05		1.31	-1.44	0.69	
	t(Hodrick)	-0.89	2.32	-1.78		1.03	-1.08	0.45	
2y	**Coefficient**	-0.07	9.50	-2.43	0.10	0.09	-6.41	2.35	0.12
	t(NW)	-0.64	1.98	-1.71		1.44	-2.62	2.78	
	t(Hodrick)	-0.53	1.73	-1.40		1.02	-2.29	1.92	
3y	**Coefficient**	-0.04	9.57	-2.26	0.08	0.12	-10.22	4.07	0.22
	t(NW)	-0.30	1.71	-1.26		1.49	-2.61	3.19	
	t(Hodrick)	-0.22	1.20	-1.01		0.93	-2.31	2.67	
4y	**Coefficient**	-0.02	9.57	-1.94	0.07	0.15	-13.47	5.67	0.29

	t(NW)	-0.14	1.85	-1.14		1.50	-2.85	3.91	
	t(Hodrick)	-0.08	0.94	-0.73		0.85	-2.26	3.18	
5y	**Coefficient**	-0.05	11.30	-1.60	0.10	0.17	-16.04	6.98	0.34
	t(NW)	-0.36	3.17	-0.95		1.68	-3.69	4.48	
	t(Hodrick)	-0.17	0.95	-0.54		0.80	-2.16	3.35	
6y	**Coefficient**	-0.08	10.83	-0.39	0.11	0.18	-17.12	7.74	0.35
	t(NW)	-0.45	4.56	-0.20		1.77	-3.89	4.54	
	t(Hodrick)	-0.20	0.81	-0.12		0.70	-1.94	3.27	
7y	**Coefficient**	-0.12	11.18	0.71	0.15	0.15	-16.27	8.23	0.32
	t(NW)	-0.64	3.80	0.29		1.28	-3.98	5.28	
	t(Hodrick)	-0.27	0.78	0.19		0.50	-1.66	3.27	
8y	**Coefficient**	-0.17	13.87	0.78	0.19	0.08	-14.19	8.62	0.29
	t(NW)	-0.81	4.43	0.32		0.51	-3.97	5.26	
	t(Hodrick)	-0.35	0.89	0.20		0.24	-1.30	3.44	
9y	**Coefficient**	-0.26	17.36	1.06	0.23	0.04	-13.22	9.01	0.28
	t(NW)	-1.02	4.73	0.42		0.20	-3.07	4.10	
	t(Hodrick)	-0.49	1.02	0.28		0.12	-1.09	3.70	
10y	**Coefficient**	-0.27	15.23	3.04	0.24	0.05	-12.71	9.03	0.24
	t(NW)	-0.92	3.25	0.91		0.19	-2.22	3.82	
	t(Hodrick)	-0.47	0.83	0.81		0.13	-0.96	3.86	

Table IV: Parameters for the Dynamics of the Predictors and the Stock and Bond Returns

Panel A reports the estimated parameters of the processes followed by the two predictors "dy" and "tb" defined in Table I, for various return periods. These processes are given by equations (2) to (7). Panels B and C display the estimated parameters of the process followed by the stock market return and the 20-year bond return, respectively. The relevant equations for these parameters are given in Appendix B. Also on display are the long run equity market premium (EP) and 20-year bond premium (BP). The period is 1963:07 to 2009:12.

Panel A: Parameters for the predictors

		1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
dy	θ	0.123	0.130	0.146	0.134	0.102	0.075	0.067	0.082	0.091	0.094	0.101	0.130	0.155
	z	0.028	0.028	0.029	0.029	0.029	0.028	0.027	0.027	0.028	0.028	0.028	0.028	0.029
	σ	0.005	0.006	0.006	0.006	0.005	0.004	0.004	0.005	0.005	0.005	0.005	0.006	0.007
tb	θ	0.139	0.213	0.215	0.216	0.301	0.304	0.254	0.193	0.174	0.175	0.196	0.240	0.218
	z	0.051	0.052	0.053	0.053	0.055	0.056	0.056	0.056	0.055	0.055	0.055	0.056	0.057
	σ	0.017	0.020	0.020	0.020	0.023	0.023	0.021	0.019	0.018	0.018	0.019	0.021	0.021
	corr	0.219	0.208	0.195	0.189	0.263	0.304	0.351	0.448	0.517	0.557	0.618	0.656	0.681

Panel B: Parameters for the stock market

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
μ_0	-0.037	-0.047	-0.056	-0.050	-0.014	0.009	0.013	-0.003	-0.015	-0.026	-0.036	-0.060	-0.076
μ_1	9.007	8.723	8.655	7.319	5.251	3.562	2.727	2.754	2.345	2.179	2.531	3.269	3.001
μ_2	-3.242	-2.893	-2.683	-2.112	-1.620	-1.147	-0.771	-0.499	-0.106	0.176	0.193	0.288	0.748
EP	0.050	0.048	0.052	0.054	0.049	0.045	0.043	0.044	0.044	0.044	0.044	0.048	0.052
σ	0.157	0.167	0.175	0.177	0.170	0.156	0.148	0.150	0.145	0.146	0.155	0.179	0.190
Corr with tb	-0.134	-0.081	-0.040	0.035	0.075	0.110	0.095	-0.061	-0.239	-0.378	-0.465	-0.531	-0.639
Corr with dy	-0.901	-0.904	-0.896	-0.870	-0.841	-0.790	-0.771	-0.822	-0.869	-0.918	-0.953	-0.978	-0.996

Panel C: Parameters for the 20-year bond

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
μ_0	0.016	0.038	0.046	0.052	0.040	0.019	0.009	0.016	0.016	0.004	-0.016	-0.035	-0.017
μ_1	0.877	-0.357	-0.880	-1.705	-3.544	-3.806	-3.838	-3.908	-3.706	-3.172	-2.373	-2.490	-2.345
μ_2	-0.382	-0.100	0.025	0.386	1.567	2.070	2.256	2.179	2.078	2.037	1.808	2.441	1.904
BP	0.021	0.023	0.023	0.022	0.023	0.028	0.033	0.031	0.029	0.029	0.019	0.032	0.023
σ	0.102	0.101	0.098	0.101	0.108	0.110	0.111	0.110	0.111	0.120	0.120	0.157	0.142
Corr with tb	-0.307	-0.472	-0.532	-0.612	-0.721	-0.746	-0.786	-0.839	-0.868	-0.908	-0.951	-0.981	-0.975
Corr with dy	-0.239	-0.196	-0.185	-0.305	-0.327	-0.341	-0.363	-0.407	-0.458	-0.540	-0.658	-0.735	-0.685
Corr with mkt	0.175	0.096	0.081	0.162	0.136	0.124	0.119	0.194	0.314	0.435	0.506	0.605	0.640

Table V: Optimal and Sub-Optimal Strategies

This Table reports the results obtained for the optimal and the two sub-optimal investment strategies when the market return (computed over 1 up to 60 months) is predicted as in Table IV. The investor's risk aversion is 2. Her investment horizon *T* ranges (horizontally) from 1 month to 30 years. The prediction horizon ranges (vertically) from 1 month to 10 years. Certainty equivalent rates are expressed on a per annum basis in Panel A. The average riskless rate is 0.053. In Panels B and C are displayed the ratios of certainty equivalent rates. The period is 1963:07 to 2009:12.

Panel A: Optimal certainty equivalent (annualized)

	1m	**3m**	**6m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**	**10y**	**20y**	**30y**
1m	0.085	0.090	0.097	0.111	0.137	0.159	0.177	0.191	0.203	0.212	0.219	0.225	0.230	0.251	0.257
3m	0.085	0.089	0.096	0.109	0.132	0.151	0.168	0.181	0.192	0.201	0.208	0.214	0.220	0.244	0.252
6m	0.087	0.091	0.097	0.109	0.131	0.149	0.165	0.179	0.189	0.199	0.206	0.212	0.218	0.243	0.251
1y	0.085	0.088	0.093	0.102	0.120	0.135	0.149	0.161	0.171	0.180	0.187	0.194	0.199	0.227	0.236
2y	0.086	0.091	0.098	0.112	0.137	0.158	0.176	0.191	0.203	0.214	0.222	0.230	0.236	0.270	0.283
3y	0.091	0.097	0.106	0.121	0.149	0.172	0.190	0.205	0.217	0.227	0.236	0.243	0.249	0.282	0.295
4y	0.097	0.102	0.110	0.125	0.151	0.173	0.191	0.206	0.218	0.228	0.237	0.244	0.250	0.283	0.296
5y	0.093	0.098	0.104	0.117	0.141	0.162	0.181	0.197	0.211	0.223	0.234	0.243	0.251	0.292	0.308
6y	0.088	0.093	0.098	0.110	0.132	0.152	0.170	0.186	0.200	0.212	0.223	0.232	0.240	0.284	0.300
7y	0.083	0.086	0.091	0.101	0.120	0.137	0.152	0.166	0.178	0.188	0.198	0.206	0.213	0.253	0.267
8y	0.077	0.081	0.085	0.094	0.111	0.126	0.140	0.152	0.162	0.171	0.179	0.186	0.192	0.227	0.239
9y	0.078	0.082	0.089	0.102	0.126	0.146	0.164	0.180	0.193	0.204	0.214	0.222	0.229	0.267	0.281
10y	0.077	0.081	0.086	0.096	0.114	0.130	0.143	0.154	0.164	0.172	0.179	0.186	0.191	0.218	0.227

Panel B: Optimal certainty equivalent (annualized) over certainty equivalent from the no-predictability strategy

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	1.029	1.088	1.175	1.343	1.651	1.913	2.128	2.300	2.438	2.547	2.635	2.707	2.766	3.041	3.135
3m	1.027	1.080	1.158	1.308	1.578	1.808	1.998	2.155	2.283	2.387	2.474	2.546	2.606	2.896	2.993
6m	1.024	1.071	1.140	1.274	1.519	1.731	1.911	2.060	2.184	2.286	2.371	2.442	2.502	2.788	2.885
1y	1.019	1.057	1.113	1.222	1.426	1.610	1.771	1.911	2.031	2.133	2.221	2.296	2.360	2.685	2.800
2y	1.030	1.089	1.176	1.340	1.632	1.879	2.086	2.259	2.403	2.524	2.627	2.715	2.790	3.185	3.332
3y	1.033	1.097	1.190	1.364	1.667	1.912	2.110	2.270	2.401	2.509	2.599	2.675	2.741	3.093	3.230
4y	1.027	1.081	1.160	1.310	1.576	1.797	1.977	2.125	2.245	2.345	2.428	2.498	2.558	2.874	2.998
5y	1.023	1.070	1.139	1.272	1.519	1.739	1.930	2.095	2.236	2.357	2.460	2.549	2.626	3.026	3.173
6y	1.022	1.067	1.132	1.260	1.499	1.713	1.904	2.071	2.217	2.344	2.454	2.549	2.632	3.067	3.221
7y	1.021	1.062	1.122	1.239	1.456	1.650	1.823	1.974	2.107	2.224	2.325	2.414	2.491	2.908	3.058
8y	1.021	1.061	1.121	1.236	1.448	1.635	1.801	1.946	2.073	2.184	2.281	2.366	2.441	2.852	3.003
9y	1.030	1.089	1.176	1.340	1.636	1.890	2.107	2.292	2.449	2.583	2.698	2.797	2.882	3.322	3.478
10y	1.023	1.069	1.136	1.261	1.485	1.676	1.838	1.976	2.094	2.194	2.279	2.352	2.415	2.730	2.835

Panel C: Optimal certainty equivalent (annualized) over certainty equivalent from the myopic strategy

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	1.000	1.000	1.000	1.002	1.007	1.015	1.024	1.032	1.040	1.047	1.052	1.057	1.061	1.080	1.086
3m	1.000	1.000	1.000	1.002	1.006	1.014	1.023	1.032	1.041	1.049	1.056	1.063	1.068	1.097	1.106
6m	1.000	1.000	1.000	1.001	1.007	1.015	1.025	1.035	1.045	1.055	1.063	1.070	1.077	1.109	1.119
1y	1.000	1.000	1.000	1.001	1.005	1.011	1.020	1.030	1.039	1.049	1.058	1.066	1.073	1.112	1.126
2y	1.000	1.000	1.000	1.001	1.007	1.015	1.026	1.037	1.047	1.057	1.066	1.074	1.081	1.121	1.137
3y	1.000	1.000	1.000	1.002	1.010	1.020	1.032	1.043	1.053	1.062	1.070	1.077	1.083	1.118	1.133
4y	1.000	1.000	1.001	1.003	1.011	1.023	1.035	1.048	1.059	1.069	1.078	1.086	1.092	1.129	1.144
5y	1.000	1.000	1.000	1.002	1.009	1.019	1.031	1.044	1.057	1.069	1.080	1.090	1.099	1.152	1.174
6y	1.000	1.000	1.000	1.002	1.007	1.015	1.026	1.037	1.049	1.060	1.071	1.081	1.090	1.148	1.172
7y	1.000	1.000	1.000	1.002	1.006	1.013	1.022	1.031	1.041	1.051	1.061	1.070	1.079	1.136	1.162
8y	1.000	1.000	1.000	1.001	1.005	1.010	1.017	1.025	1.033	1.041	1.049	1.056	1.064	1.115	1.140
9y	1.000	1.000	1.001	1.002	1.008	1.017	1.027	1.038	1.048	1.058	1.068	1.077	1.085	1.136	1.157
10y	1.000	1.000	1.000	1.002	1.006	1.013	1.020	1.029	1.037	1.045	1.053	1.061	1.068	1.113	1.132

Table VI: Relative Weights of Risky Assets for the Optimal Strategy

This Table provides details as to the composition of the portfolio when the optimal strategy of Table V (Panel A) is followed. Panel A reports the overall weight of the risky assets (stocks and bonds) in the portfolio. Weights larger than one imply that the riskless asset is held negatively. Panel B displays the stock/bond mix, i.e. the proportion of stocks relative to bonds in the risky part of the portfolio. Panel C reports the ratio of the intertemporal hedging terms over the speculative, mean-variance ones. The period is 1963:07 to 2009:12.

Panel A: Optimal total risky assets position

	1m	**3m**	**6m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**	**10y**	**20y**	**30y**
1m	1.713	1.729	1.753	1.800	1.893	1.976	2.048	2.108	2.158	2.199	2.233	2.261	2.285	2.401	2.431
3m	1.792	1.801	1.814	1.840	1.893	1.946	1.995	2.040	2.081	2.116	2.147	2.174	2.197	2.311	2.338
6m	1.865	1.874	1.886	1.911	1.960	2.008	2.054	2.095	2.132	2.164	2.191	2.215	2.236	2.331	2.351
1y	1.632	1.642	1.657	1.686	1.743	1.797	1.849	1.897	1.941	1.980	2.015	2.046	2.073	2.208	2.240
2y	1.597	1.611	1.631	1.668	1.730	1.782	1.827	1.869	1.909	1.947	1.983	2.017	2.049	2.237	2.291
3y	1.855	1.875	1.904	1.957	2.040	2.101	2.146	2.182	2.213	2.242	2.269	2.295	2.320	2.502	2.573
4y	2.076	2.101	2.137	2.204	2.316	2.399	2.460	2.505	2.540	2.569	2.594	2.617	2.639	2.807	2.884
5y	1.904	1.930	1.967	2.039	2.167	2.272	2.358	2.428	2.485	2.534	2.576	2.613	2.647	2.872	2.962
6y	1.702	1.728	1.767	1.841	1.978	2.096	2.197	2.283	2.357	2.420	2.476	2.526	2.571	2.849	2.950
7y	1.398	1.423	1.461	1.534	1.669	1.790	1.896	1.989	2.071	2.142	2.206	2.262	2.313	2.619	2.725
8y	0.983	1.006	1.040	1.107	1.231	1.345	1.447	1.539	1.622	1.698	1.767	1.830	1.887	2.251	2.379
9y	0.863	0.886	0.920	0.986	1.107	1.216	1.310	1.394	1.466	1.530	1.586	1.636	1.679	1.910	1.967
10y	0.697	0.721	0.757	0.826	0.955	1.072	1.176	1.269	1.351	1.423	1.487	1.543	1.592	1.834	1.882

Panel B: Optimal stock/bond mix

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	1.195	1.232	1.290	1.412	1.671	1.928	2.162	2.362	2.526	2.657	2.762	2.846	2.913	3.192	3.253
3m	0.812	0.835	0.870	0.944	1.098	1.253	1.397	1.525	1.635	1.728	1.806	1.870	1.924	2.160	2.210
6m	0.758	0.778	0.808	0.871	1.006	1.145	1.277	1.398	1.503	1.593	1.669	1.732	1.784	2.001	2.041
1y	0.942	0.955	0.975	1.020	1.122	1.233	1.346	1.454	1.554	1.642	1.718	1.783	1.838	2.071	2.110
2y	0.946	0.931	0.911	0.879	0.842	0.832	0.838	0.854	0.873	0.894	0.914	0.933	0.949	1.020	1.032
3y	0.798	0.771	0.733	0.668	0.577	0.521	0.489	0.473	0.466	0.465	0.467	0.471	0.476	0.516	0.529
4y	0.728	0.698	0.657	0.584	0.476	0.405	0.358	0.328	0.310	0.298	0.292	0.289	0.288	0.305	0.315
5y	0.776	0.747	0.705	0.633	0.521	0.442	0.388	0.351	0.326	0.310	0.299	0.293	0.289	0.293	0.296
6y	1.017	0.977	0.923	0.829	0.688	0.590	0.522	0.474	0.441	0.419	0.403	0.393	0.385	0.370	0.365
7y	1.445	1.380	1.294	1.150	0.943	0.806	0.713	0.648	0.601	0.568	0.544	0.527	0.514	0.472	0.459
8y	9.580	8.020	6.475	4.735	3.193	2.503	2.122	1.888	1.732	1.623	1.543	1.482	1.435	1.215	1.139
9y	2.976	2.722	2.417	1.985	1.491	1.225	1.066	0.963	0.894	0.846	0.811	0.785	0.766	0.684	0.661
10y	-13.161	-18.993	-52.680	22.851	6.567	4.146	3.184	2.672	2.356	2.142	1.987	1.869	1.777	1.382	1.285

Panel C: Optimal intertemporal hedging over the mean-variance term

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	0.005	0.014	0.028	0.056	0.110	0.159	0.201	0.236	0.265	0.289	0.309	0.326	0.340	0.408	0.425
3m	0.002	0.007	0.014	0.029	0.059	0.088	0.116	0.141	0.164	0.184	0.201	0.216	0.229	0.292	0.308
6m	0.002	0.007	0.013	0.027	0.053	0.079	0.103	0.125	0.145	0.162	0.177	0.190	0.201	0.253	0.263
1y	0.003	0.009	0.018	0.036	0.071	0.104	0.136	0.166	0.193	0.217	0.238	0.257	0.274	0.357	0.376
2y	0.005	0.013	0.026	0.049	0.088	0.121	0.149	0.176	0.201	0.225	0.248	0.269	0.289	0.407	0.441
3y	0.006	0.017	0.032	0.061	0.106	0.139	0.163	0.183	0.200	0.215	0.230	0.244	0.257	0.356	0.395
4y	0.006	0.018	0.036	0.068	0.122	0.163	0.192	0.214	0.231	0.245	0.257	0.269	0.279	0.360	0.398
5y	0.007	0.021	0.040	0.078	0.146	0.202	0.247	0.284	0.314	0.340	0.362	0.382	0.400	0.519	0.567
6y	0.008	0.024	0.047	0.091	0.171	0.241	0.301	0.352	0.396	0.434	0.467	0.496	0.523	0.687	0.747
7y	0.009	0.028	0.055	0.108	0.206	0.293	0.370	0.437	0.495	0.547	0.593	0.634	0.671	0.891	0.968
8y	0.012	0.036	0.071	0.140	0.268	0.385	0.490	0.585	0.671	0.749	0.820	0.884	0.944	1.318	1.450
9y	0.014	0.041	0.080	0.157	0.300	0.427	0.539	0.636	0.722	0.797	0.863	0.921	0.972	1.243	1.310
10y	0.018	0.053	0.105	0.206	0.395	0.565	0.718	0.853	0.973	1.079	1.172	1.253	1.325	1.679	1.749

Table VII: Parameters for the Dynamics of the Predictors and the Stock and Bond Returns After Valkanov's Correction

This Table is analogous to Table IV except that the correction suggested by Valkanov (2003) has been applied to obtain the estimates of the relevant parameters for taking into account the persistence created by overlapping observations (see Appendix D), which then excludes the case h = 1 month. Parameters for the predictors are not reported as they are identical to those displayed in Panel A of Table IV. Panels A and B display the estimated parameters of the process followed by the stock market return and the 20-year bond return, respectively. The period is 1963:07 to 2009:12.

Panel A: Parameters for the stock market

	1m	**3m**	**6m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**	**10y**
μ_0	-0.037	0.024	0.038	0.039	0.045	0.048	0.050	0.051	0.049	0.046	0.043	0.043	0.061
μ_1	9.007	5.068	3.436	1.445	0.230	-0.571	-1.554	-2.683	-3.779	-4.795	-5.656	-6.475	-6.373
μ_2	-3.242	-2.211	-1.584	-0.607	-0.056	0.299	0.775	1.351	1.905	2.459	2.934	3.350	3.367
EP	0.050	0.051	0.053	0.050	0.048	0.049	0.052	0.053	0.050	0.048	0.049	0.050	0.069
σ	0.157	0.166	0.171	0.169	0.159	0.144	0.134	0.132	0.120	0.112	0.108	0.112	0.115
Corr with tb	-0.134	-0.080	-0.034	0.048	0.098	0.117	0.125	0.074	0.018	-0.024	-0.064	-0.054	-0.086
Corr with dy	-0.901	-0.895	-0.866	-0.799	-0.711	-0.643	-0.587	-0.546	-0.497	-0.471	-0.450	-0.404	-0.373

Panel B: Parameters for the 20-year bond

	1m	**3m**	**6m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**	**10y**
μ_0	0.016	0.042	0.053	0.066	0.078	0.081	0.081	0.081	0.081	0.080	0.062	0.075	0.068
μ_1	0.877	0.512	0.530	0.221	-0.646	-1.454	-2.212	-2.746	-3.652	-4.161	-4.505	-5.168	-3.905
μ_2	-0.382	-0.633	-0.851	-0.894	-0.647	-0.280	0.126	0.417	0.897	1.172	1.494	1.779	1.262
BP	0.021	0.023	0.024	0.025	0.024	0.025	0.028	0.029	0.029	0.030	0.020	0.030	0.028
σ	0.102	0.100	0.097	0.099	0.099	0.097	0.098	0.097	0.093	0.089	0.075	0.087	0.084
Corr with tb	-0.307	-0.456	-0.487	-0.510	-0.462	-0.356	-0.315	-0.325	-0.300	-0.275	-0.253	-0.206	-0.234
Corr with dy	-0.239	-0.188	-0.167	-0.286	-0.375	-0.413	-0.386	-0.337	-0.290	-0.271	-0.248	-0.244	-0.255
Corr with mkt	0.175	0.101	0.094	0.197	0.242	0.276	0.297	0.336	0.391	0.459	0.415	0.542	0.605

Table VIII: Optimal and Sub-Optimal Strategies Using Valkanov's Correction

This Table is analogous to Table V except that the correction suggested by Valkanov (2003) has been applied to obtain the estimates of the relevant parameters (see Appendix D) displayed in Table VII. The investor's risk aversion is 2. Her investment horizon *T* ranges (horizontally) from 1 month to 30 years. The prediction horizon ranges (vertically) from 1 month to 10 years. Certainty equivalent rates are expressed on a per annum basis in Panel A. The average riskless rate is 0.053. In Panels B and C are displayed the ratios of certainty equivalent rates. The period is 1963:07 to 2009:12.

Panel A: Optimal certainty equivalent (annualized)

	1m	**3m**	**6m**	**1y**	**2y**	**3y**	**4y**	**5y**	**6y**	**7y**	**8y**	**9y**	**10y**	**20y**	**30y**
1m	0.085	0.090	0.097	0.111	0.137	0.159	0.177	0.191	0.203	0.212	0.219	0.225	0.230	0.251	0.257
3m	0.087	0.089	0.093	0.099	0.110	0.118	0.125	0.131	0.135	0.139	0.142	0.145	0.147	0.158	0.161
6m	0.089	0.091	0.093	0.097	0.104	0.109	0.113	0.116	0.119	0.121	0.122	0.124	0.125	0.131	0.133
1y	0.085	0.086	0.087	0.089	0.092	0.094	0.096	0.097	0.098	0.099	0.100	0.101	0.101	0.104	0.106
2y	0.086	0.086	0.087	0.088	0.089	0.090	0.091	0.092	0.092	0.092	0.092	0.093	0.093	0.093	0.093
3y	0.092	0.092	0.092	0.093	0.093	0.094	0.094	0.094	0.094	0.094	0.093	0.093	0.093	0.092	0.091
4y	0.102	0.102	0.103	0.103	0.104	0.104	0.104	0.104	0.104	0.104	0.103	0.103	0.102	0.099	0.097
5y	0.104	0.105	0.107	0.109	0.112	0.114	0.115	0.116	0.116	0.116	0.116	0.116	0.115	0.113	0.111
6y	0.106	0.108	0.112	0.117	0.126	0.132	0.136	0.139	0.141	0.142	0.143	0.143	0.144	0.144	0.144
7y	0.109	0.114	0.120	0.130	0.146	0.157	0.164	0.170	0.173	0.176	0.178	0.179	0.180	0.185	0.186
8y	0.114	0.122	0.134	0.154	0.183	0.203	0.216	0.225	0.231	0.236	0.239	0.242	0.244	0.254	0.258
9y	0.115	0.126	0.141	0.167	0.203	0.225	0.240	0.250	0.257	0.262	0.265	0.268	0.271	0.281	0.285
10y	0.153	0.163	0.176	0.197	0.227	0.244	0.255	0.262	0.266	0.269	0.271	0.272	0.274	0.278	0.280

Panel B: Optimal certainty equivalent (annualized) over certainty equivalent from the no-predictability strategy

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	1.029	1.088	1.175	1.343	1.651	1.913	2.128	2.300	2.438	2.547	2.635	2.707	2.766	3.041	3.135
3m	1.052	1.080	1.119	1.192	1.316	1.415	1.494	1.558	1.610	1.653	1.689	1.719	1.744	1.871	1.914
6m	1.047	1.064	1.089	1.133	1.205	1.261	1.305	1.339	1.367	1.389	1.408	1.424	1.437	1.507	1.532
1y	1.019	1.027	1.039	1.059	1.093	1.118	1.137	1.153	1.165	1.176	1.185	1.192	1.199	1.235	1.251
2y	1.031	1.035	1.041	1.051	1.066	1.075	1.082	1.086	1.089	1.091	1.092	1.094	1.094	1.097	1.096
3y	1.039	1.040	1.041	1.042	1.043	1.042	1.040	1.038	1.035	1.033	1.030	1.028	1.025	1.005	0.991
4y	1.080	1.082	1.084	1.087	1.087	1.085	1.080	1.075	1.069	1.063	1.057	1.051	1.046	1.006	0.982
5y	1.147	1.155	1.166	1.184	1.208	1.221	1.227	1.229	1.227	1.224	1.220	1.215	1.210	1.169	1.147
6y	1.225	1.250	1.284	1.343	1.430	1.486	1.523	1.546	1.560	1.568	1.572	1.575	1.575	1.560	1.549
7y	1.352	1.403	1.474	1.596	1.776	1.893	1.969	2.019	2.053	2.075	2.090	2.101	2.108	2.125	2.126
8y	1.505	1.612	1.759	2.013	2.383	2.623	2.780	2.886	2.959	3.011	3.050	3.079	3.102	3.200	3.231
9y	1.525	1.668	1.863	2.188	2.637	2.909	3.079	3.188	3.262	3.313	3.351	3.380	3.402	3.494	3.522
10y	2.032	2.154	2.319	2.589	2.952	3.160	3.281	3.353	3.396	3.423	3.440	3.451	3.459	3.481	3.485

Panel C: Optimal certainty equivalent (annualized) over certainty equivalent from the myopic strategy

	1m	3m	6m	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	20y	30y
1m	1.000	1.000	1.000	1.002	1.007	1.015	1.024	1.032	1.040	1.047	1.052	1.057	1.061	1.080	1.086
3m	1.000	1.000	1.000	1.001	1.002	1.005	1.008	1.011	1.015	1.018	1.021	1.024	1.026	1.043	1.051
6m	1.000	1.000	1.000	1.000	1.001	1.003	1.004	1.006	1.008	1.010	1.011	1.013	1.014	1.025	1.030
1y	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.001	1.002	1.002	1.002	1.002	1.004	1.006
2y	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.001	1.001	1.001
3y	1.000	1.000	1.000	1.000	1.000	1.001	1.001	1.001	1.002	1.002	1.002	1.003	1.003	1.005	1.007
4y	1.000	1.000	1.000	1.000	1.001	1.002	1.003	1.004	1.005	1.006	1.006	1.007	1.008	1.013	1.016
5y	1.000	1.000	1.000	1.000	1.002	1.003	1.005	1.006	1.008	1.009	1.010	1.011	1.012	1.018	1.020
6y	1.000	1.000	1.000	1.001	1.002	1.004	1.006	1.008	1.010	1.012	1.013	1.014	1.015	1.020	1.021
7y	1.000	1.000	1.000	1.001	1.003	1.005	1.007	1.009	1.011	1.013	1.014	1.015	1.016	1.019	1.020
8y	1.000	1.000	1.000	1.001	1.003	1.005	1.007	1.009	1.010	1.011	1.012	1.013	1.014	1.017	1.018
9y	1.000	1.000	1.000	1.001	1.003	1.005	1.007	1.008	1.010	1.011	1.011	1.012	1.012	1.014	1.015
10y	1.000	1.000	1.000	1.001	1.003	1.005	1.006	1.007	1.008	1.009	1.010	1.010	1.010	1.012	1.013