Mad crowds and corrupt governments

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" Mere opinions, in fact, were as likely to govern people's actions as hard evidence, and were subject to sudden reversals as hard evidence could never be."

(K. Vonnegut, Galapagos)

1 Madness of crowds or wisdom of crowds?

Rousseau's theory of general will, formalized by Condorcet through the law of large numbers, asserts, roughly, that a large assembly would give better decisions than a single individual. However, in a famous passage, Rousseau already pointed the potential failures in expressing the general will, too:

If when the people, being furnished with adequate information, held its deliberations, ***the citizens had no communication one with another****, the grand total of the small differences would always give the general will, and the decision would always be good. ... It is therefore essential, if the general will is to be able to express itself, that there should be* ***no partial society*** *within the State, and that* ***each citizen should think only his own thoughts****: which was indeed the sublime and unique system established by the great Lycurgus.*

The reference to Lycurgus is a little oscure. In any case it was a commonplace in classic literature to compare the political stability of "silent" Spartans to the frequent revolutions and corrupt rulers generated from noisy athenian conventions (Sweden vs. Italy ?)

Failure of Condorcet theorem for large aggregates was studied in the XIX century psychological and sociological literature by Le Bon, ("Psychologie des foules",1905), Tarde ("L'Opinion et la foule", 1901), Freud ("Massenpsychologie und ich analyse",1921) and others.

It was remarked that often not only crowds were ineffective in taking the right decision but that they were impulsive and voluble exhibiting sudden and apparently unmotivated reversals of behaviour. Explanations were given in terms of "collective" psycology.

A rich popular literature originated from them (e.g. Ch. Mackay who used the term "madness of crowds", Surowiecki "wisdom of crowds").

I will investigate these two

QUESTIONS

1. **What is a precise definition of "Madness of crowds"?**
2. **Under what conditions is a crowd "mad" ?**

Informally

- "Wiseness of crowds" means effectiveness in aggregating information, e.g. almost sure convergence to the truth when the crowd size goes to infinity.
- "Madness" will mean ineffectiveness in aggregating information *and* sudden changes. We shall see that these two features are strongly related.

Results in the existing literature suggest that:

1. Full rationality of agents is not sufficient for almost sure convergence to the truth (e.g. Banerjee, Rational herding), nor necessary (Ben Golub, Jackson, Wiseness of crowds)
2. The form of the communication mechanism, instead, is very relevant (see again Ben Golub and Jackson).

In the model discussed in this paper voters form an opinion about the government putting together noisy signals on its behavior and the opinions expressed by their friends. In this way each voter gets a new opinion that is expressed, weighted against a new noisy signal and so on. After several iterations the process will be seen to converge with high probability.

This seems to be a reasonable way to aggregate information, however the fact that they do not "use their own thoughts", as Rousseau says, may lead to "madness of crowds".

Analysis of counterexamples shows that this happens because of two facts:

- The message set is discrete
- The communication pattern exhibit many redundancies. (Technically: graph of interactions has many loops)

The intuition is that discreteness of messages prevents agents from fully transferring all the information available to them, loops (redundancies) in the interaction graph results in overweighting other agents' opinions. (see blackboard)

2 Notation and model

Consider a community $\mathcal{N}$ consisting of N voters, indexed by $i = 1, 2..N$ and a ruler.

Time is discrete and divided into periods called *legislatures*, they are indexed by $t = 1, 2, ...$

Each legislature is divided into D subperiods, the *days*, indexed by $d = 1, 2, ...D$.

A subperiod will be denoted by (t, d).

To each voter i is associated a set $F(i) \subset \mathcal{N}$, the set of his *friends*.

At the beginning of each legislature t, the ruler has access to a budget X_t and splits it between welfare for the voters W_t and "bribes" that are kept by her, b_t.

The voters do not observe X_t. Every day they make noisy observations of W_t. During the legislature the voters discuss the ruler's behaviour among they friends. Their opinions change according to a stochastic process called the opinion formation process. On day (t, D) they vote for or against the ruler.

2.1 The opinion formation process

During the legislature the voters' opinions about the government evolve according the following mechanism.

At the beginning of each day, (t, d), each voter, i, has an opinion about the government that can be favourable or contrary, it will be denoted by $o^i_{t,d}$.

$o^i_{t,d} = +1$ means that the voter is in favour, $o^i_{t,d} = -1$ against.

During the day she hears the opinions of her friends $o^j_{t,d}$, $j \in F(i)$, communicates her opinion to them and makes a noisy observation of W_t getting a signal $W_t + \omega^i_{t,d}$, where $\omega^i_{t,d}$ are i.i.d. random variables.

At the end of the day with probability α she revises her opinion. The revised opinion will be favourable with a probability that is increasing in the amount of perceived welfare and in the number of favourable opinions heard from friends.

Formally we shall have:

$$o^i_{t,d+1} = \begin{cases} o^i_{t,d} & \text{with probability } 1-\alpha \\ r^i_{t,d+1} & \text{with probability } \alpha \end{cases} \tag{1}$$

Where the random variable $r^i_{t,d}$ has values in $\{+1, -1\}$ and is distributed accordingly to:

$$P(r^i_{t,d+1} = +1) = \frac{1}{2} + \frac{1}{2} f(X_t - b_t + \omega^i_{t,d} - W_o + \sum_{j \in F(i)} h_{i,j} o^j_{t,d})$$

Where $f : \mathbf{R} \to R$ is an *S-shaped function* with $f' > 0$, $f'' \gtreqless 0$ for $x \lesseqgtr 0$, $\lim_{x\to\infty} f(x) = 1, \lim_{x\to-\infty} f(x) = -1$.

The parameters $W_{o,i}$ measures how demanding a citizen is about the government. The other parameter, $h_{i,j}$ measures how much voter's i opinion is influenced by voter's j one.

For simplicity, we shall assume that all the voters have the same value for W_o, that they give equal weight to the opinions of all their friends and that "everybody knows everybody" i.e. that $F(i) = \mathcal{N}$ for all i.

The last assumption is a strong one and is certainly not realistic in a large country.

It can be considerably relaxed, without changing the results: for instance it may be assumed that each voter connects in each period to individuals randomly chosen with uniform probability among the population (internet forums) or that listens to a poll giving a noisy observation of the average opinion.

With these assumptions, if we denote by $O_{t,d}$ the average opinion $\frac{\sum_{i \in \mathcal{N}} o^i_{t,d}}{N}$ we have,

$$P(r^i_{t,d+1} = +1) = \frac{1}{2} + \frac{1}{2} f(X_t - b_t + \omega^i_{t,d} - W_o + \sigma O_{t,d})$$

where σ measures the influence of the average opinion on a voter's one.

2.2 Change of ruler

At the last day of a legislature, D, an election is held, with two candidates one of which is the old ruler.

All voters turn out and they vote according to their opinion on that day: if they are in favour of the ruler they vote for her, otherwise they choose the opponent.

Majority voting is applied: the ruler is re-elected if at least 50% of the population is in favour, namely if

$$O_{t,D} \geq 0$$

If reconfirmed for legislature $t+1$, the ruler's initial support is $O_{t+1,1} = O_{t,D}$.

If the new ruler is the opponent , her initial support is $O_{t+1,1} = -O_{t,D}$.

The initial consensus at the first legislature will be denoted by $O = O_{1,1}$.

3 A definition of "madness of crowds"

The stochastic process of the $O_{t,d}$ will be called the *opinion formation process*, it will be denoted by $\mathcal{P}_\sigma$ to emphasize its dependence on σ.

We shall concentrate on a fixed legislature, so the subscript t will be dropped.

To emphasize the dependence of the random variable O_d on N, W and the initial value O we shall write $O_d^N(W,O)$, this will be the value of the average opinion of a population of N individuals in day d, given that the average opinion on day 0 was O and that the amount of public good has been W.

Definition 1. *We say that, for a given value of* W*,* $\mathcal{P}_\sigma$ reaches consensus *above* O_f *starting from* O_i*, if for any* ϵ *there exists an* N_0 *such that*

$$\forall O > O_i, \forall N > N_0,\ P(O_D^N(W,O) \geq O_f) > 1 - \epsilon.$$

If $O_i = O_f$ we shall say that $\mathcal{P}_\sigma$ *keeps consensus* O_i.

Definition 2. *We say that, for a given value of* W*,* $\mathcal{P}_\sigma$ drops consensus below O_d *if for any* ϵ *there exists an* N_0 *such that*

$$\forall O, \forall N > N_0, P(O_D^N(W,O) < O_d) > 1 - \epsilon.$$

Definition 3. *We say that* $\mathcal{P}_\sigma$ *has* satisfaction value $\bar{W}(\sigma)$ *if there exist* $O_c(\sigma) \geq 0$ *and* $O_r(\sigma) \leq 0$ *such that:*

- *For all* $W > \bar{W}(\sigma)$*,* $\mathcal{P}_\sigma$ *keeps consensus* O_c*.*
- *For all* $W < W(\sigma) - \delta(D)$*,* $\mathcal{P}_\sigma$ *drops consensus below* $O_r(\sigma)$*.*

The maximal Oc and minimal Or are called the satisfaction value and the rejection value.

Note that in the definition $O_r(\sigma)$ must be the same for all δ.

One proves that the conditions in definition 3 are always satisfied, i.e.

Proposition 1. *A satisfaction value $W(\sigma)$ exists for every σ.*

Definition 4. *We say that $\mathcal{P}_\sigma$ exhibits* sudden reversals *if the consensus level is strictly larger than* 0 *and the rejection level is strictly less than* 0*.*

Definition 5. *We say that $\mathcal{P}_\sigma$ is* ineffective *if the satisfaction level $\bar{W}$ is strictly less than W_0.*

Definition 6. *We say that $\mathcal{P}_\sigma$ exhibits* madness *if it is ineffective and exhibits sudden reversals.*

4 Intense communication induces ”madness” in crowds

We then have:

Proposition 2. *There exists a σ_0 such that for $\sigma > \sigma_0$, $\mathcal{P}_\sigma$ exhibits crowd behaviour, more precisely:*

- *The satisfaction value $W(\sigma)$ is constant and equal to W_0 for $\sigma < \sigma_0$, it is a strictly decreasing function of σ for $\sigma > \sigma_0$.*
- *The consensus level $O_c(\sigma)$ is zero for $\sigma < \sigma_0$ and is strictly increasing in σ for $\sigma > \sigma_0$.*
- *The rejection level $O_r(\sigma)$ is zero for $\sigma < \sigma_0$ and is strictly decreasing in σ for $\sigma > \sigma_0$.*

SEE PICTURE

In words: when the crowd is large enough, the ruler can almost be sure of reelection if the citizens begin by being favorable to her and she gives to the citizens just a little above the satisfaction value. This value decreases when the influence of friends' opinions on an agent increases.

On the other side, in presence of crowd behaviour, if the ruler gives just a little less than the satisfaction value consensus will dramatically drop.

The proof is a simple exercise in the theory of random perturbation of dynamical systems and large deviations. (See blackboard)

WARNING: The invariant measure of the Markov chain is NOT the object of interest here. We are interested in the behavior in the long run. It can be shown (see blackboard) that convergent to the invariant distribution happens in the very long run only and that, with realistic parameters, the very long run is indeed very very long. (see simulation)

5 The ruler's optimal policy

Suppose the ruler has been elected in period 0, has a utility function $u(b_t)$ and discount factor ρ, so her problem is to maximize the sum

$$\sum_{t=0}^{T_f} \rho^t u(b_t)$$

where the random variable T_f is the last legislature in which the ruler is in power:$T_f = min\{t|O_{t,D} \geq 0, O_{t+1,D} < 0\}$.

The non trivial case is $W_0 < \rho E(X_t)$. (otherwise the ruler steals away everything in the first period and runs away).

Proposition 3. *Given N and σ, there is an amount $W(N,\sigma)$ such that if $X_t > W(N,\sigma)$,the ruler leaves to the voters $W(N,\sigma)$ and steals the rest. If $X_t < W(N,\sigma)$ steals everything. We have:*

$$\lim_{N\to\infty} W(N,\sigma) = W(\sigma)$$

6 Comments

Objection (actually made):

"Your result is trivial: it is obvious that, if agents rely strongly enough on their friends opinion and begin by being favourable to the government, they will never change their mind"'

It is not obvious, because it is not even true: in several other cases "wisdom of crowds" emerges.

In fact the result depends strongly on several features of the model:

1. expressed opinions are discrete, voters do not discuss their precise observations but just express a rough evaluation. If a continuum of opinion is possible voters converge to the truth. (DeMarzo)

2. the interactions are long range, if agents interact only with close neighbours, crowd behaviour does not arise. (Ben Golub, Matthew Jackson). See also the example on the blackboard, with infinite confidence in the neighbours.

3. voters are "'opinion taker"' they do not realize that favourable opinions may come from other voters' influence without being related to an actual observation.