A General Measure of Risk

with a Behavioral Foundation

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Abstract

We present a general measure of risk that is rooted in behavioral considerations about the way individuals value uncertain outcomes, and that fulfills fundamental requirements for prescriptive use. The psychological postulate is that, in contemplating a risky situation, individuals care about how they will come out relative to all prospective outcomes of the situation, not just a specific benchmark as is commonly assumed in measuring risk. Our model includes, or relates to, some classic measures of risk, such as the variance, Gini mean difference, or Fishburn's (1977) α-t, but it is distinct from the traditional families of risk measures. We provide necessary and sufficient conditions for the risk measure to satisfy first and second order stochastic dominance, two essential criteria for ordering prospects, thereby generalizing important results on the Gini risk measure. We also provide conditions for our risk measure to be a coherent or a convex risk measure, thus providing an alternative to other measures used for the management and regulation of risk. We show that the risk measure can produce both corner or interior solutions to the asset allocation problem, thus allowing more flexibility than traditional models. Finally, we obtain some evidence on the empirical form assumed by the risk measure by fitting it to experimental preference data.

Key words: Risk-Value Models; Stochastic Dominance; Gini Mean Difference; Coherent Risk Measures; Convex Risk Measures; Decision under Risk.

1. Introduction

Measuring the value of gambles as a function of their rewards and their risks is an appealing approach to decision under risk. This is because these two criteria get right to the core of decision makers' concerns, in a direct, transparent manner. There seems to be general agreement that the potential reward of a gamble can be captured adequately by its expected value, i.e., its mean. There is less accord about what constitutes an acceptable measure of risk. The challenge is to balance desirable normative considerations against intuitively or behaviorally appealing properties. This tension is ultimately at the heart of any prescriptive theory of choice under risk.

We present a generic Risk-Value model derived from our previous work on Disappointment. The model is built from scratch from psychological considerations regarding how an individual might value the receipt of a risky outcome in the context of other possible outcomes, which could have been more or less favorable. The model defines a new class of risk measures, distinct from the classic families of risk measures widely considered throughout the literature. It generalizes some prominent measures of risk, and extends previous results concerning stochastic dominance. Under appropriate conditions, the model satisfies first order and, more importantly, second order stochastic dominance. It can also satisfy the axioms of coherent risk measures and convex risk measures.

This introductory section highlights selected background on risk measures relevant to the present work. In Section 2, we present our general Risk-Value model and show how it relates to other classic families of risk measures. In Section 3, we analyze monotonicity with respect to first and second degree stochastic dominance. In Section 4, we determine the necessary and sufficient conditions for the model to provide a coherent or a convex risk measure. The model's prescriptions for optimal asset allocation are examined in Section 5. Finally, Section 6 provides a non-parametric calibration of the risk measure on experimental data, and Section 7 concludes.

The literature on risk measures is voluminous and thoroughly detailed. It emanates from several intellectual traditions, notably Statistics (measures of dispersion and the moments approach), Economics (the Expected Utility approach, but also the inequality measurement approach), Finance (the portfolio efficient set approach), and Psychology (the behavioral/cognitive approach). Sarin and

Weber (1993) present an overview of the Risk-Value models literature at the time of their writing; Pederson and Satchell (1998) provide a fairly detailed review of risk measures.

Expected Utility (EU) stands as *the* ultimately normative approach to rational choice under risk, however, there is no explicit construction of a risk index in EU. Although the risk of a gamble X can be measured as its risk premium, defined as $\pi(X) = E[X] - u^{-1}(Eu[X])$ (Pratt 1964), the valuation of a gamble cannot, in general, be calculated directly from its expected value and its risk premium in a Risk-Value spirit, because the estimation of $\pi(X)$ requires to calculate the certainty equivalent, leading to a circularity. Under particular conditions on the u function and/or the distribution of X, EU can take a Risk-Value form. For example, if the utility function is exponential and gambles have a normal distribution, or if the utility function is quadratic, then EU is equivalent to a Mean-Variance model. Bell (1995) and Jia and Dyer (1996) have explored in some depth alternative ways to cast EU as a function of risk and return: the possibilities seem limited to a well defined set. Because the notion of risk in EU is entirely tied to the concavity of the utility function, it is too intertwined with the concept of diminishing marginal utility of money. Requiring the valuation of each and every risk to be only and entirely determined by the pattern of utility for wealth may be too rigid for some decision makers. That is, EU may leave out some aspects of risk that legitimately matter to the decision maker.

The so-called Risk-Value framework may offer more flexibility in dealing with risk by allowing to define a risk measure "from scratch," that is, unconstrained by whether it is consistent with the maximization of an EU function or not. Because risk is associated with the presence of uncertainty in the payoffs, that is, the extent to which a payoff distribution departs from a sure outcome, risk measures are germane with measures of dispersion. Risk is traditionally measured as the propensity of an uncertain outcome to deviate from some reference level. Stone (1973) proposes that three basic ingredients are relevant to devising a risk measure: (i) a reference level, relative to which deviations are measured; (ii) the range of deviations taken into account; and (iii) how deviations are weighed. For example, Fishburn (1977) considers a family of risk measures in which risk is measured as a probability weighted function of the deviations below a specified target return, defined as follows:

$$\rho_t(F) = \int_{-\infty}^{t} \varphi(t-x)dF(x), \qquad (1)$$

where F is the cumulative distribution of the random payoff, t the target level, and φ measures how deviations below the target are weighed. Fishburn (1977) examines the special case

$\varphi(t - x) = (t - x)^{\alpha}$, for $x \leq t$, so-called the 'α-t' model. The α-t model belongs to a general family considered by Stone (1973), which also includes the standard risk measures used in Finance: variance, semi-variance, mean absolute deviation, and the probability of a loss worse than some specified level.

Variance, perhaps the most written about measure of risk, has a long history of use in portfolio selection (Markowitz 1952) and economic analysis (Tobin 1958). The variance has great computational advantages, but it presents theoretical shortcomings: it is not consistent with a number of compelling definitions of risk, including second order stochastic dominance (Rothschild and Stiglitz 1970). Attempts to avoid this problem have led to the use of fixed-target semivariance, as it is consistent with second order stochastic dominance (Porter 1974, Fishburn 1977).

2. The Proposed Risk Measure

In previous work (Delquié and Cillo 2006), we developed a Disappointment model of choice under risk based on the postulate that individuals are liable to experience a mixture of disappointment and contentment from comparing the outcome received from a gamble to the *other* possible outcomes, rather than a single prior expectation. Our purpose was to provide an alternative, appealing psychological rationale for disappointment. The basic idea is that the decision maker (DM) cares about how well off he/she comes out, not just in absolute terms, but also relative to each of the achievable outcomes. Thus, each and every outcome in the gamble plays the role of a reference point. This idea is also independently developed by Kőszegi and Rabin (2007) in their model of reference-dependent risk attitudes.

We further showed that our model of Disappointment without prior expectation could be mathematically reformulated as a Risk-Value model, with the following form:

$$V(X) = \sum_{i=1}^{n} p_i v(x_i) - \sum_{i=1}^{n} \sum_{j>i} p_i p_j H\big(v(x_i) - v(x_j)\big), \qquad (2)$$

where X is a gamble that yields payoff x_i with probability p_i, $i = 1, \ldots, n$, $\sum p_i = 1$ and $x_1 \geq x_2 \geq \ldots \geq x_n$; $v(\cdot)$ is an increasing function that describes the subjective value of outcomes; and H captures the disappointment from receiving x_j instead of x_i.

Here, we focus on a special case of (2): for the sake of having a Risk-Value representation comparable to those that have appeared before, we will assume v linear throughout this paper. Besides, this assumption does not play a role in the essential results and claims developed in the paper. Thus, the model we are interested in here is:

$$V(X) = \mathrm{E}[X] - R(X)$$
$$\text{with} \quad R(X) = \sum_{i=1}^{n} \sum_{j>i} p_i p_j H(x_i - x_j), \tag{3}$$

where E[X] is the mean of X, a measure of its potential reward, and $R(X)$ is the amount by which this reward should be discounted to account for the risk of X. If F denotes the cumulative distribution of X, the continuous form of $R(X)$ is:

$$R(X) = \int_{-\infty}^{+\infty} \int_{-\infty}^{x} H(x-y)dF(y)dF(x) = \mathrm{E}_X \left[\int_{-\infty}^{x} H(x-y)dF(y) \right]. \tag{4}$$

From here on, we will refer to the model expressed in (3) as the M-R model.

The function H describes the pattern of how the DM values discrepancies between achieved and missed outcomes, that is, the losses associated with less than desired outcomes. Specifically, $H(x_i - x_j)$ captures how the individual subjectively weighs the "opportunity loss" from receiving the outcome x_j instead of the superior outcome x_i. The immutable properties of H, that stem from its very definition, are: (i) $H(0) = 0$ and (ii) H is defined on the domain $y \geq 0$, that is, by construction H takes only positive deviations as arguments, that is, *differences* between ordered outcomes. It weighs the relative impact of large and small deviations. If H is linear, the deviations are weighed proportionally to their magnitudes. If H is convex, large deviations are penalized more strongly relative to small ones. If H is concave, there is diminishing sensitivity to deviation. For further details on the origins of H and its psychological interpretation, see Delquié and Cillo (2006).

A sensible property of H is that it be increasing. This would imply $H(y) \geq 0$ for all $y \geq 0$ (since $H(0) = 0$), leading to a positive risk premium $R(X)$ for any gamble. Under this assumption, a sure payoff equal to the expected value of the gamble will always be preferred to the gamble itself. Monotonicity is not required, however: H could be flat or decreasing over some initial range, then become increasing, meaning that deviations below a certain threshold are acceptable or even welcome, but large deviations are averted. Thus, the shape of H may allow the decision maker to be risk averse for some gambles, while being risk neutral or even seeking for some others.

$R(X)$ measures the riskiness of a gamble with no regard to the location, that is, irrespective of how good or bad the outcomes are in the absolute. We regard this as a desirable feature for a risk measure, because a risk judgment itself should be distinct from the overall desirability of a gamble. Risk is one ingredient in the evaluation of a gamble, which should be kept separate of, indeed untainted by, other considerations. Ranking gambles by riskiness or by reward is not enough: to have a complete preference ordering of gambles, we need a decision rule to integrate risk and reward. The risk measure $R(X)$ comes along with a decision rule: Equation (3) specifies how risk should be traded off against the reward. This choice rule is naturally inherited from the behavioral hypothesis on which the model is built.

$V(X)$ in (3) has constant risk aversion for any form of H. That is, if a constant is added to a gamble, the valuation of the gamble increases by the same constant. This means that we do not need to worry about whether gambles describe total wealth outcomes or incremental gains and losses. Thus, decreasing risk aversion, often deemed a plausible behavioral property, can only and entirely be due to a non linear v in (2). This makes sense because the behavioral rationale behind decreasing risk aversion is that changes in wealth will matter less when you have a lot, so the fact that risk aversion may decrease with wealth should be solely linked to the valuation of wealth, which is captured by v. The essence of a Risk-Value representation is to accept risk as a primitive construct, independent of how wealth under certainty is valued: in this mind-set, the property of constant risk aversion is not shocking. If the random variables considered are returns, i.e. relative payoffs, the M-R model would then imply constant *relative* risk aversion, that is, *diminishing* risk aversion in the absolute payoffs.

Finally, the M-R model in (3) can account for commonly observed risk preference patterns that are not consistent with EU, such as the Allais Paradox and the common ratio effect. This means that the M-R model possesses flexibility to make it behaviorally realistic. As we will see in Section 3, for prescriptive ends, this flexibility needs to be limited to avoid violating certain monotonicity axioms.

Relationship to Other Risk Measures

Notice from (3) that every pairwise difference between two outcomes enters exactly once in the make up of $R(X)$. For H non decreasing, $R(X)$ constitutes a general measure of dispersion, which includes two important cases: the Variance and the Gini measure. Variance is obtained for $H(y) = y^2$, as shown in Delquié and Cillo (2006, Appendix E).

When H is linear, $R(X)$ yields the Gini Mean Difference (up to a positive multiplicative constant), also known in Statistics as the 'Absolute Mean Difference' measure of dispersion. The Gini measure of dispersion is most prominently used as a measure of inequality of income or wealth distribution among a population, but it has also been used as a risk measure (Yitzhaki 1982). Gini's Mean Difference is defined as the average of the absolute differences between all possible pairs of observations of a random variable.[1] For $H(y) = y$, $R(X) = ½G(X)$, where $G(X)$ is the Gini mean difference. Thus we would have to take $H(y) = 2y$ to get exactly Gini. However, as we will see in the next section, it is wiser to keep $H'(y) \leq 1$.

$R(X)$ is neither a particular case of the general measure considered by Fishburn (1977), Equation (1), nor of the family proposed by Stone (1973). Indeed, one essential difference is that these traditional risk measures are sprung from the outcomes' deviations from a fixed reference level, whereas $R(X)$ is built on the deviations among outcomes from one another. Nevertheless, Equation (4) makes a relationship to Equation (1) apparent: the expression $\int_{-\infty}^{x} H(x-y)dF(y)$ within the expectation in (4) is nothing but the Fishburn (1977) measure of risk, $\rho_x(X)$, which represents the risk of failing to achieve the particular outcome x in gamble X. Thus, $R(X)$ can be thought of as the expectation of the collection of Fishburn's risk measures generated by taking as target level each and every value of X in turn. In $R(X)$, each outcome of X can be viewed as playing the role of a target and contributing its own 'à la Fishburn' risk to the gamble: the total risk $R(X)$ of the gamble is just the average of all the individual outcome risks. Thus (3) can be written as:

$$V(X) = E[X] - E[\rho_x(X)] = E[X - \rho_x(X)] = E[u_X(X)]$$

with $u_X(x) = x - \rho_x(X)$,

where $u_X(x)$ can be interpreted as the risk-adjusted utility of outcome x in gamble X. In other words, $\rho_x(X)$ is the risk premium associated with just outcome x in gamble X, and $E[\rho_x(X)] = R(X)$ is the risk premium of the whole gamble.

More Risk Averse Than…

[1] The Gini *Index*, or Gini coefficient, is a normalized, unit-free measure obtained by dividing the Gini Mean Difference by twice the mean of the distribution.

In studying risk taking (investment) or avoidance (insurance), it is often of interest to compare individuals on their risk attitudes, that is, make statements of the kind "individual *a* is more risk averse than individual *b*." Under Expected Utility, Pratt (1964) shows that individual *a* is more risk averse than individual *b*, in the sense that *b* would accept all the gambles that *a* accepts, if and only if there exists a concave function v such that: $u_a = v(u_b)$, where u_a and u_b are the utility functions of *a* and *b* respectively. This means that u_a is everywhere more concave than u_b. In the M-R model it is easy to show that individual *a* is more risk averse than individual *b*, in the same sense (that is, *b* has a lower risk premium than *a* for any gamble), if and only if there exists a function h taking non-negative values everywhere such that: $H_a = H_b + h$. This means that H_a lies everywhere above H_b.

3. Efficiency of the M-R Model

To avoid the difficulties connected with knowing decision makers' utility functions, several authors have examined the merits of ordering prospects in terms of dominance rules (Hadar and Russell 1969). Screening gambles (or portfolios) according to dominance criteria is sound, but the issue is that dominance relations are only a partial order. Therefore, other decision rules need to be invoked to choose among alternatives in the efficient set that meet the dominance criteria. Thus, it is a computationally difficult approach to building or identifying optimal portfolios.

Stochastic dominance establishes a partial ordering of probability distributions for which the distribution F dominates distribution G in the sense of n^{th}-order stochastic dominance if and only if every individual with a utility function u such that $\operatorname{sign} u^{(j)} = (-1)^{j+1}$ for $j = 1, \ldots, n$ prefers F to G (Ingersoll 1987). Such a utility function is said to satisfy stochastic-dominance preference of order n.

In what follows we will focus on the two essential dominance criteria: first order stochastic dominance (FSD), and second order stochastic dominance (SSD). FSD implies SSD, that is, if $X >_{FSD} Y$, then $X >_{SSD} Y$. SSD is more important than FSD because it can rank more prospects, and because it lies at the heart of fundamental notions of risk and risk aversion.

In developing a risk measure ad hoc, we risk —and presumably tolerate— violating some normative principles of Expected Utility, but we would like to keep others. For example, we do not want to give up monotonicity with respect to larger payoffs and decreasing risk, that is, respectively, first and second order stochastic dominance. Thus, a key property of any Risk-Value model is: does it rank prospects consistently with FSD and SSD?

PROPOSITION 1. *Assume that H is differentiable and all expectations exist. The M-R model (3) satisfies first order stochastic dominance (FSD) if and only if* $0 \le H'(y) \le 1$ *for all* $y \ge 0$.

Proof.

Sufficiency. Consider X and Y such that $X >_{FSD} Y$. Let F (f) and G (g) denote the cumulative distribution (probability density) functions of X and Y, respectively. We want to show that: $V(X) \ge V(Y)$, that is:

$$V(X) = \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dF(t) \right) dF(x) \ge \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dG(t) \right) dG(x) = V(Y).$$

Define $u_X(x) = x - \int_{-\infty}^{x} H(x-t)dF(t)$. The first derivative of u_X is:

$$u'_X(x) = 1 - \int_{-\infty}^{x} H'(x-t)dF(t) - H(0)f(x) = 1 - \int_{-\infty}^{x} H'(x-t)f(t)dt. \tag{5}$$

Because $H'(y) \le 1$ for all y, $\int_{-\infty}^{x} H'(x-t)f(t)dt \le \int_{-\infty}^{x} f(t)dt \le 1$, and therefore $u'_X(x) \ge 0$, that is, u_X is increasing. Because $X >_{FSD} Y$ and u_X increasing, we have:

E[$u_X(X)$] = $\int_{-\infty}^{+\infty} u_X(x)dF(x) \ge \int_{-\infty}^{+\infty} u_X(x)dG(x)$ = E[$u_X(Y)$] (Hanoch and Levy 1969). Let us now show that

$\int_{-\infty}^{+\infty} u_X(x)dG(x) \ge V(Y)$. For this, we define u_Y similarly as u_X, that is: $u_Y(x) = x - \int_{-\infty}^{x} H(x-t)dG(t)$,

and we show that $u_X(x) \ge u_Y(x)$ for all x.

For x given, consider the function $v_x(t)$ defined as:

$$v_x(t) = \begin{cases} -H(x-t) & \text{for } t \le x \\ 0 & \text{for } t > x \end{cases} \tag{6}$$

For any x, $v_x(t)$ is (weakly) increasing since H is increasing. Indeed for $t \le x$, $v'_x(t) = H'(x-t) \ge 0$, and for $t > x$, $v'_x(t) = 0$.

Because $X >_{FSD} Y$ and v_x is increasing, we have for all x: $E[v_x(X)] = \int_{-\infty}^{+\infty} v_x(t)dF(t) \geq \int_{-\infty}^{+\infty} v_x(t)dG(t) =$ $E[v_x(Y)]$ that is, $\int_{-\infty}^{x} -H(x-t)dF(t) \geq \int_{-\infty}^{x} -H(x-t)dG(t)$. Adding x on both sides of the preceding inequality, we get: $u_X(x) = x - \int_{-\infty}^{x} H(x-t)dF(t) \geq x - \int_{-\infty}^{x} H(x-t)dG(t) = u_Y(x)$, for all x, and hence $\int_{-\infty}^{+\infty} u_X(x)dG(x) \geq \int_{-\infty}^{+\infty} u_Y(x)dG(x) = V(Y)$.

In sum, we have shown that: $V(X) = \int_{-\infty}^{+\infty} u_X(x)dF(x) \geq \int_{-\infty}^{+\infty} u_X(x)dG(x) \geq \int_{-\infty}^{+\infty} u_Y(x)dG(x) = V(Y)$.

Necessity. Suppose there exists x such that $H'(x) > 1$, that is, $H'(x) = 1+h$ with $h > 0$. We construct a pair of gambles X, Y such that $Y >_{FSD} X$ and $V(Y) < V(X)$. Going back to the definition of the derivative of a function, we have:

$$H'(x) = \lim_{\varepsilon \to 0} \left(\frac{H(x+\varepsilon) - H(x)}{\varepsilon} \right) = 1 + h .$$

Now going back to the definition of a limit, we know that there exists $\delta > 0$ such that: for $\varepsilon < \delta$

$\left| \frac{H(x+\varepsilon) - H(x)}{\varepsilon} - (1+h) \right| < h/2$. That is, there exists ε such that:

$$H(x+\varepsilon) > H(x) + (1 + h/2)\varepsilon . \tag{7}$$

Let us take such an ε and consider the following binary gambles: $X = \{x, p; 0, 1-p\}$, $Y = \{x+\varepsilon, p; 0, 1-p\}$, with $p = h/(h+2)$; so $Y >_{FSD} X$. For these gambles we have:

$V(Y) = p(x+\varepsilon) - p(1-p)H(x+\varepsilon)$

$< p(x+\varepsilon) - p(1-p)(H(x)+(1+h/2)\varepsilon)$ by (7)

$= V(X) + p\varepsilon(1 - (1-p)(1+h/2))$

$= V(X)$ because $(1-p)(1+h/2) = 1$.

This completes the proof.[2]

[2] In Delquié and Cillo (2006, Appendix B), we provided a different proof of Proposition 1, using Machina's (1982) concept of "local utility function." The previous proof involved an alternative formulation of model (2), and non-linear v. There, we showed that FSD is satisfied if and only if $-1 \leq H'(y) \leq 1$. Because we are restricting ourselves to the case H increasing here, this condition becomes $0 \leq H'(y) \leq 1$. The present proof is parallel to, and lays the ground for, the approach used to prove Proposition 2 below.

The intuitive interpretation of the condition is that the sensitivity to differences between outcomes should not exceed the sensitivity to the outcomes themselves. Indeed, if the weight placed on deviations should ever exceed the weight put on the absolute payoffs, it would be possible to have a situation in which a strict increase in a payoff (making the gamble strictly better) would increase the risk $R(X)$ of the gamble more than its return E[X].

PROPOSITION 2. *Assume that H is twice differentiable and all expectations exist. The M-R model (3) satisfies second order stochastic dominance (SSD) if and only if H is such that, for all y ≥ 0:* $0 \le H'(y) \le 1$ *and* $H''(y) \ge 0$, *that is, H is increasing, convex, and never grows faster than the identity function.*

Proof.

Sufficiency. Consider X and Y such that $X >_{SSD} Y$. Denote F (f) and G (g) the cumulative distribution (probability density) functions of X and Y, respectively. We want to show that: $V(X) \ge V(Y)$, that is:

$$V(X) = \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dF(t) \right) dF(x) \ge \int_{-\infty}^{+\infty} \left(x - \int_{-\infty}^{x} H(x-t)dG(t) \right) dG(x) = V(Y).$$

Define $u_X(x) = x - \int_{-\infty}^{x} H(x-t)dF(t)$. The first derivative of $u_X(.)$ is given in (5) and we saw that, if $H'(y) \le 1$ for all y, then $u'_X(x) \ge 0$. Thus u_X is increasing.

Taking the derivative of (5), we obtain the second derivative of u_X:

$$u''_X(x) = -\int_{-\infty}^{x} H''(x-t)dF(t) - H'(0)f(x).$$

Because $H''(y) \ge 0$ and $H'(0) \ge 0$, $u''_X(x) \le 0$. Thus u_X is concave.

Because $X >_{SSD} Y$, we have $\int_{-\infty}^{+\infty} u_X(x)dF(x) \ge \int_{-\infty}^{+\infty} u_X(x)dG(x)$ for u_X increasing, concave (Hanoch and Levy 1969). Let us now show that $\int_{-\infty}^{+\infty} u_X(x)dG(x) \ge V(Y)$. For this, we show that $u_X(x) \ge u_Y(x) = x - \int_{-\infty}^{x} H(x-t)dG(t)$ for all x.

For any given x, consider again $v_x(t)$ defined in (6). Because H is increasing, convex, v_x is increasing, concave. As $X >_{SSD} Y$, we have: $E[v_x(X)] = \int_{-\infty}^{+\infty} v_x(t)dF(t) \geq \int_{-\infty}^{+\infty} v_x(t)dG(t) = E[v_x(Y)]$ for all x.

This implies: $\int_{-\infty}^{x} -H(x-t)dF(t) \geq \int_{-\infty}^{x} -H(x-t)dG(t)$ for all x. Adding x on both sides of the preceding inequality, we get:

$u_X(x) = x - \int_{-\infty}^{x} H(x-t)dF(t) \geq x - \int_{-\infty}^{x} H(x-t)dG(t) = u_Y(x)$, for all x, and hence

$\int_{-\infty}^{+\infty} u_X(x)dG(x) \geq \int_{-\infty}^{+\infty} u_Y(x)dG(x) = V(Y)$.

In sum, we have shown that: $V(X) = \int_{-\infty}^{+\infty} u_X(x)dF(x) \geq \int_{-\infty}^{+\infty} u_X(x)dG(x) \geq \int_{-\infty}^{+\infty} u_Y(x)dG(x) = V(Y)$.

Necessity. Suppose there exists x_0 such that $H''(x_0) < 0$. Then, there exists $\varepsilon > 0$ such that: $H(x_0 + \varepsilon) - H(x_0) < H(x_0) - H(x_0 - \varepsilon)$. The idea is to construct a mean preserving spread around x_0 (where H is concave) that would cause a *decrease* in risk $R(X)$.
Define: $H'_+(x_0) = (H(x_0 + \varepsilon) - H(x_0))/\varepsilon$ and $H'_-(x_0) = (H(x_0) - H(x_0 - \varepsilon))/\varepsilon$. Thus $H'_+(x_0) < H'_-(x_0)$. Take the binary gamble $X = \{x_0, p; 0, 1-p\}$, with $p/(1-p) < H'_-(x_0) - H'_+(x_0)$. Now consider the three-outcome gamble $Y = \{x_0+\varepsilon, p/2; x_0-\varepsilon, p/2; 0, 1-p\}$. Y is a mean-preserving spread of X, that is, $X >_{SSD} Y$, yet $R(Y) < R(X)$, hence, $V(Y) > V(X)$.
This completes the proof.

The convexity condition on H essentially guarantees that adding an independent, zero-mean risk to X will not cause $R(X)$ to decrease. Behaviorally, it reflects increasing sensitivity to deviations.

Yitzhaki (1982) shows that the Mean-½Gini model satisfies FSD and SSD. Propositions 1 and 2 extend this result to the more general M-R model, which includes Mean-½Gini. Also, the present analysis is quite different from Yitzhaki's (1982) approach for the Gini measure of risk.

Remark: it seems possible to extend the above results to stochastic dominance of order $n > 2$ by requiring that successive derivatives of H alternate in sign and $H^{(n-1)}(0) = 0$. The latter part of this condition does not seem to have a clear behavioral interpretation, therefore we will not explore it further here.

4. Conditions for a Coherent and a Convex Risk Measure

Several axiomatic approaches to constructing risk measures have been proposed. Some are more psychological in nature, that is, they state intuitively plausible principles for the subjective judgment of risk (Pollatsek and Tversky 1970). Other approaches try to account for empirically observed features of risk perception (Jia et al. 1999). Still others have a prescriptive orientation, that is, they attempt to outline general properties deemed desirable or necessary for adequate management of risk. For example, Kijima and Ohnishi (1993) proposed the following four properties for a measure of risk $r(.)$: for all X, Y

P1. Translation invariance: $r(X+\delta) = r(X)$ for all δ

P2. Subadditivity: $r(X+Y) \leq r(X) + r(Y)$

P3. Positive homogeneity: $r(\lambda X) = \lambda r(X)$ for all $\lambda \geq 0$

P4. Nonnegativity: $r(X) \geq 0$

Coherent Risk Measures

Artzner et al. (1999) proposed and justified a similar set of axioms, albeit stated in terms of the overall preference function, $V(X)$, rather than the risk measure itself, $r(X)$. They measure risk as the amount of cash that should be added to a risk position (a gamble) to make it acceptable (their formulation also incorporates the interest rate earned on the cash position). Instead of Nonnegativity (P4), Artzner et al. (1999) required (recasting their axioms in terms of the risk measure proper, instead of the preference function):

P4'. Monotonicity: If $X \leq Y$, $r(X) \geq r(Y)$

Artzner et al. (1999) argued P1-P3 and P4' to be necessary for the proper management and regulation of risk, and they called measures satisfying them *Coherent risk measures*.

Coherent risk measures are generally not consistent with second order stochastic dominance (de Giorgi 2005), but some are. For example, Ogryczak and Ruszczyński (1999) show that certain risk measures based on semi-deviations (standard or absolute) preserve second order stochastic dominance (see also Leitner 2005).

Let us briefly review what is required of the M-R model to comply with the axioms of coherent risk measures.

Translation invariance: holds for any H. Note that this, of course, implies constant risk aversion, which we discussed previously.

Subadditivity: holds if H is subadditive. That is the case if H is everywhere concave, for example. (For the proof: X and Y have to be defined on the same state space. If not, take the Cartesian product space, and redefine X and Y on the product space. Then rank order the outcomes of $Z = X + Y$, use the triangular inequality, and the subadditivity of H.)

Positive homogeneity: holds if and only if H is linear.

Monotonicity: holds if and only if $H'(y) \leq 1$, as seen in Proposition 1.

Thus, to satisfy the four axioms of coherent risk measures simultaneously, H has to be a seminorm, that is, we have to take H linear: $H(y) = \beta y$, with $0 \leq \beta \leq 1$. In that case, we have the Gini measure, $R(X) = \beta/2 \cdot \mathrm{G}(X)$, where the single parameter β reflects the decision maker's trade-off between risk and reward. In Delquié and Cillo (2006), we showed that for H linear, the Risk-Value model in (2) is equivalent to Rank Dependent Utility (RDU) with a quadratic cumulative probability weighting function (pwf), where β (the slope of H) controls the degree of probability transformation, specifically: $w(p) = \beta p^2 + (1-\beta)p$. Thus, the Mean-½Gini model considered by Yitzhaki (1982) is equivalent to RDU with linear utility and pwf $w(p) = p^2$. It is well known that RDU complies with FSD, indeed, this is what it was designed for (Quiggin 1982). What the above shows is that RDU with a quadratic pwf (note that $0 \leq \beta \leq 1$ insures that the pwf is increasing on the probability interval) also complies with SSD. This agrees with Yaari (1987, Theorem 2), who showed that his "Dual Theory" model (an axiomatic RDU model with linear utility) is consistent with Rothschild and Stiglitz (1970) definition of risk whenever the pwf is convex.

Convex Risk Measures

The property of positive homogeneity (P3) can be considered as rather restrictive. One could argue that doubling the position in a gamble will at least double the risk incurred, that is:

P3'. $r(\lambda X) \geq \lambda r(X)$ for all $\lambda \geq 1$ (this implies that $r(\lambda X) \leq \lambda r(X)$ for $\lambda \leq 1$)

The property P3' may be deemed more compelling, and more flexible, than P3. In this spirit, Föllmer and Schied (2002) propose to replace P2 and P3 by the weaker property:

P2'. Convexity: $r(\lambda X + (1-\lambda)Y) \leq \lambda r(X) + (1-\lambda)r(Y)$ for $0 \leq \lambda \leq 1$

Property P2' just provides that diversification should not increase risk, a cornerstone principle of risk management. Föllmer and Schied (2002) define risk measures satisfying Translation Invariance (P1), Convexity (P2') and Monotonicity (P4') as *Convex risk measures*. They show a representation theorem for convex risk measures parallel to that obtained by Artzner et al (1999) for coherent risk measures. See de Giorgi (2005) and Brown and Sim (2008) for further characterization of convex risk measures. In the M-R model, the convexity axiom (P2') will be fulfilled by requiring H convex. Thus, it turns out that the conditions for SSD (Proposition 2) ensure that M-R is, in addition, a convex risk measure.

In sum, the M-R model possesses the built-in flexibility to accommodate the general axioms reviewed above, either individually, or collectively for coherence or convexity.

5. The Asset Allocation Problem

A question of interest for any model of choice under risk is the kind of solutions it provides to the optimal asset allocation problem. Expected utility provides that a risk-averse individual should always invest a strictly positive amount of money in a risky asset that has a positive expected value, no matter how risky the asset, or how risk-averse the individual. Other models, such as Yaari's (1987) Dual Theory model, predict "plunging," that is, for any risky asset, invest either nothing or the full capital available in the risky asset. Yaari (1987) argues that these two classes of solutions (always an interior solution, or always a corner solution) are extreme, and that an intermediate between these two extremes would be both normatively and empirically more satisfactory. The M-R model is able to provide a more nuanced solution between these two extremes: it allows plunging/abstention for some risky assets, and interior solutions for others.

To illustrate this, it suffices to consider a simple asset allocation problem, involving a safe asset with 0 rate of return and a risky asset with a random rate of return that has a two-outcome distribution: $\theta = \{a, p; -b, 1-p\}$ with $a, b > 0$. Assume that the risky asset has a positive expected value $E[\theta] > 0$. Let K be the total amount available to invest, and x the amount to be invested in the risky asset, $0 \leq x \leq K$.

Thus, the net payoff is described by the random variable $X = K + \theta x$. According to the M-R model, the investor's utility for the portfolio is given by:

$$V(K + \theta x) = K + \mathrm{E}[\theta]x - p(1-p)H((a+b)x) = \Psi(x)\,. \tag{8}$$

To examine how $V(K + \theta x)$ varies with x, let us take its derivative with respect to x:

$$\Psi'(x) = \mathrm{E}[\theta] - p(1-p)(a+b)H'((a+b)x)$$

$$\Psi'(x) \gtreqless 0 \quad \Leftrightarrow \quad H'((a+b)x) \lesseqgtr \frac{\mathrm{E}[\theta]}{p(1-p)(a+b)}\,. \tag{9}$$

Note that $p(1-p)(a+b) = \tfrac{1}{2}\mathrm{G}(\theta)$, where G is the Gini measure of risk of the rate of return. Thus, the right-hand side of the second inequality in (9) can be viewed as a measure of performance of the risky asset: its return per "unit of risk," akin to the Sharpe ratio. It depends solely on the characteristics of the risky asset, not on the investor's risk preferences, which are captured by H. In what follows we assume that H fulfills the conditions of Proposition 2, that is, SSD.

Denote: $\mathrm{S}(\theta) = \dfrac{\mathrm{E}[\theta]}{\tfrac{1}{2}\mathrm{G}(\theta)} > 0$. $\mathrm{S}(\theta)$ so defined also happens to be the inverse of the Gini coefficient.

The following cases may be considered —remember that since H is convex, H' is increasing, therefore $H'(0) \le H'((a+b)x) \le H'((a+b)K)$ for $0 \le x \le K$:

- $H'(0) < \mathrm{S}(\theta) < H'((a+b)K)$, then there exists a unique, interior solution $0 < x^* < K$. It is easy to verify that the convexity of H ensures the second order condition for a maximum.
- $\mathrm{S}(\theta) \le H'(0)$, then we have the corner solution $x^* = 0$.
- $\mathrm{S}(\theta) \ge H'((a+b)K)$, then we have the corner solution $x^* = K$.

Note that if $H'(0) = 0$ (H is flat at the origin), then we have $\mathrm{S}(\theta) > H'(0)$. Therefore, the solution necessarily involves a strictly positive investment, $x^* > 0$, for a risky asset with $\mathrm{E}[\theta] > 0$, as under the EU model.

If $\mathrm{S}(\theta) > 1$, then $\Psi'(x) > 0$ for all x, since $0 \le H'(y) \le 1$ for all y. Thus for any security with $\mathrm{S}(\theta) > 1$, the maximum amount should be invested in the risky asset.[3]

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[3] For H linear: $H'(y) = c$, with $0 \le c \le 1$, we have plunging, i.e., corner solutions, for *any* risky asset. If the risky asset is such that $\mathrm{S}(\theta) = \mathrm{E}[\theta]/(\tfrac{1}{2}\mathrm{G}(\theta)) > c$, the optimal solution is $x^* = K$; if the risky asset has $\mathrm{S}(\theta) < c$, the optimal solution is $x^* = 0$; if the risky asset is such that $\mathrm{S}(\theta) = c$, the investor is indifferent toward any level of investment between 0 and K. This, of course, concords with Yaari (1987), since the case H linear yields a model equivalent to Yaari's Dual Theory with quadratic (convex) probability weighting.

In sum, the value of the portfolio (Eq. 8) can be monotone decreasing, monotone increasing, or non-monotone single peaked over the range of possible investment levels. Thus, the M-R model is able to produce all-or-nothing solutions as well as (unique) interior solutions to the optimal asset allocation problem, depending entirely on the features of the risky asset relative to the investor's pattern of risk aversion over the range of the portfolio's outcome.

6. Empirical Evidence on the Shape of the *H* Function

In this section, we fit the M-R model (3) to a sample of individuals' preferences for gambles. Our purpose is to obtain evidence on the empirical shape of the *H* function if the M-R model is assumed, not to test the descriptive merits of this model as compared to alternative models of choice under risk. The data set is from de Neufville & Delquié (1988). It consists of certainty equivalent responses to 24 binary gambles $X_{ij} = \{x_i, p_j ; 0, 1-p_j\}$ in a factorial design with 4 outcome levels: x_i = \$5000, \$7000, \$8500, \$10000, and 6 probability levels p_j = 0.50, 0.67, 0.75, 0.80, 0.85, 0.90. This data set allows for a rich non-parametric estimation of the *H* function: for each subject, we get six independent estimates for each of four values for *H*.

From each certainty equivalent response CE_{ij} to a gamble X_{ij}, we can calculate a point value of *H* as follows:

$$V(CE_{ij}) = V(X_{ij})$$

$$CE_{ij} = p_j x_i + (1-p_j)0 \; - \; p_j(1-p_j)H(x_i - 0) = p_j x_i - p_j(1-p_j)H(x_i)$$

Hence: $H(x_i) = (p_j x_i - CE_{ij})/(p_j(1-p_j))$.

For each x_i = \$5000, \$7000, \$8500, \$10000, we calculate six independent values of $H(x_i)$ corresponding to the six p_j levels. The average of these six values is taken as an estimate of $H(x_i)$.[4]

Results

The summary measures of $H(x_i)$ obtained are given in Table 1.

[4] We used two other methods of calibrating *H*: solving for values of $H(x_i)$ to minimize the sum of either squared, or absolute, differences between predicted and actual CE_{ij}. These alternative methods produce results similar to the approach of averaging point estimates presented here.

Table 1. Nonparametric estimate of *H* values of 24 individuals

N=24	*H*(5000)	*H*(7000)	*H*(8500)	*H*(10000)
Mean	8,198	12,717	17,010	20,761
Median	7,852	12,025	20,011	22,003
Std Dev	5,870	9,215	11,451	13,413

Recalling that *H*(0) = 0 by definition, the non-parametric estimate of the function *H* can be plotted in Figure 1. Visually, the mean *H* function seems close to linear, with a mild convexity. This pattern holds for 16 out of 24 subjects; 7 subjects have a non-monotonic *H* function, typically presenting a (sometimes pronounced) dip at 7000 or 8500; one subject has a mildly concave *H* function.

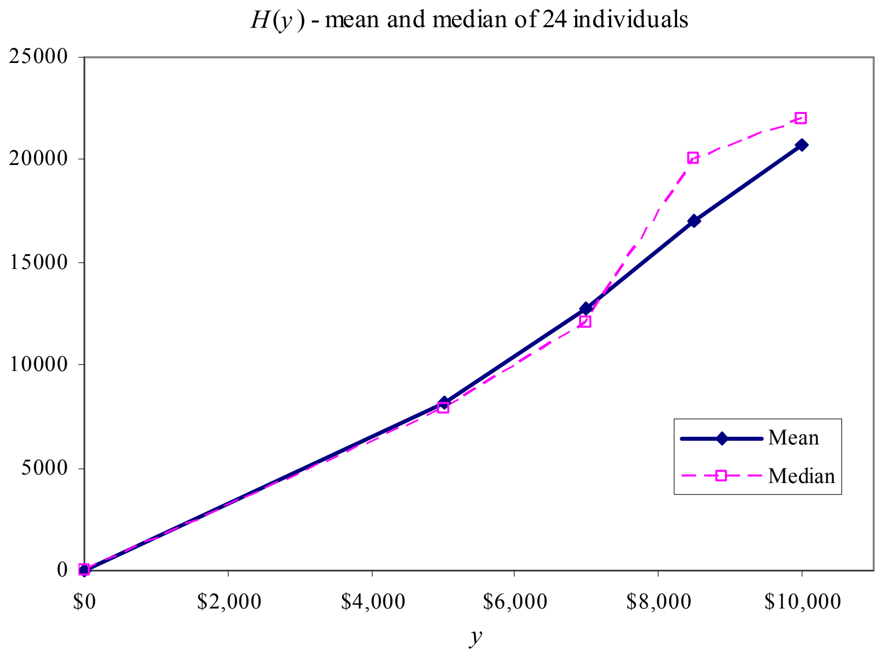

Figure 1. Nonparametric estimate of *H* function of 24 individuals

Although a mildly convex or linear shape would be consistent with the requirements derived in Section 3 for stochastic dominance, it is clear that the average *H* function obtained has a slope greater than 1, which is not consistent with the SD conditions. Indeed, these conditions imply $H(y) \le y$ for all $y \ge 0$, which is clearly not the case in Figure 1 because the average *H* lies above, not below, the diagonal. 7 subjects have a *H* function that satisfies $H(y) \le y$ throughout the range [$0, $10000].

This empirical finding elicits a couple of remarks. First, it is conceivable that the subjects (graduate students) did not have a linear valuation of monetary outcomes in the 0-\$10,000 range. In the presence of non-linear v, the M-R model (3) is of course not entirely adequate and it forces H to carry all of the non-linearity in preferences. In particular, the concavity of v would transpire through H by making it steeper. Second, the observation that $H' > 1$ for some subjects suggests that these individuals *could* violate FSD, not that they systematically will, but rather they may be prone to doing so in some cases. Although it is unlikely that individuals would violate transparent FSD in direct choice between simple gambles, it is quite possible that their responses as a set may violate FSD, due just to response error. As a matter of fact, we do observe such violations: for example, some subjects have a lower CE for the gamble {\$8500, 0.85} than for {\$8500, 0.80}, or a higher CE for {\$7000, 0.80} than for {\$8500, 0.80}.[5] Violations of FSD embedded in the data are liable to be reflected in the resulting H function.

The above empirical observations raise the question whether the M-R model will allow classic non-EU preference patterns, all within the limits imposed by stochastic dominance. The answer is yes: for example, the M-R model can explain the Allais paradox and common ratio effects—which are among the most robust findings in behavioral decision research—with $H(y) = \beta y$ with $0 \leq \beta \leq 1$, which satisfies the conditions of Propositions 1 and 2.

7. Conclusion

Riskiness of a gamble, like intelligence of a person, is a complex, multifaceted concept. Reducing it to a single index will necessarily miss some aspects of it. The risk measure we proposed here is founded on the single behavioral hypothesis that individuals may care about the differences between what they got and what they could have had from a gamble. This creates a new class of risk measure capable of allying realistic behavioral features with some indispensible normative properties. A summary diagram of the relationships among the different models discussed in this paper is proposed in Figure 2. Among other things, the M-R model provides an appealing generalization of Mean-Gini, which was shown by Yitzhaki (1982) to produce advantages over the widely used Mean-Variance model. In

[5] These inconsistencies would not be obvious to the subjects because the 24 gambles were presented in arbitrary orders with the constraint that consecutive gambles differ in *both* probability and payoff.

addition, the M-R model provides flexible —perhaps empirically more plausible— solutions to the asset allocation problem.

For those who would like to use a Risk-Value framework for prescriptive ends, the M-R model with H increasing convex (but less accelerated than 1) may offer an appealing approach, because it complies with FSD and SSD, two fundamental rules for ordering gambles, and it is a convex risk measure, which is normatively desirable for the practice of risk management. Examples of satisfactory one-parameter H functions are: $H(y) = y^2/(y+\alpha)$; or $H(y) = y + \alpha(e^{-y/\alpha} - 1)$ with $\alpha \geq 0$; or simply a piecewise linear function $H(y) = 0$ for $0 \leq y \leq \alpha$, $H(y) = y - \alpha$ for $y > \alpha$, which indicates that deviations within a certain tolerance are ignored.

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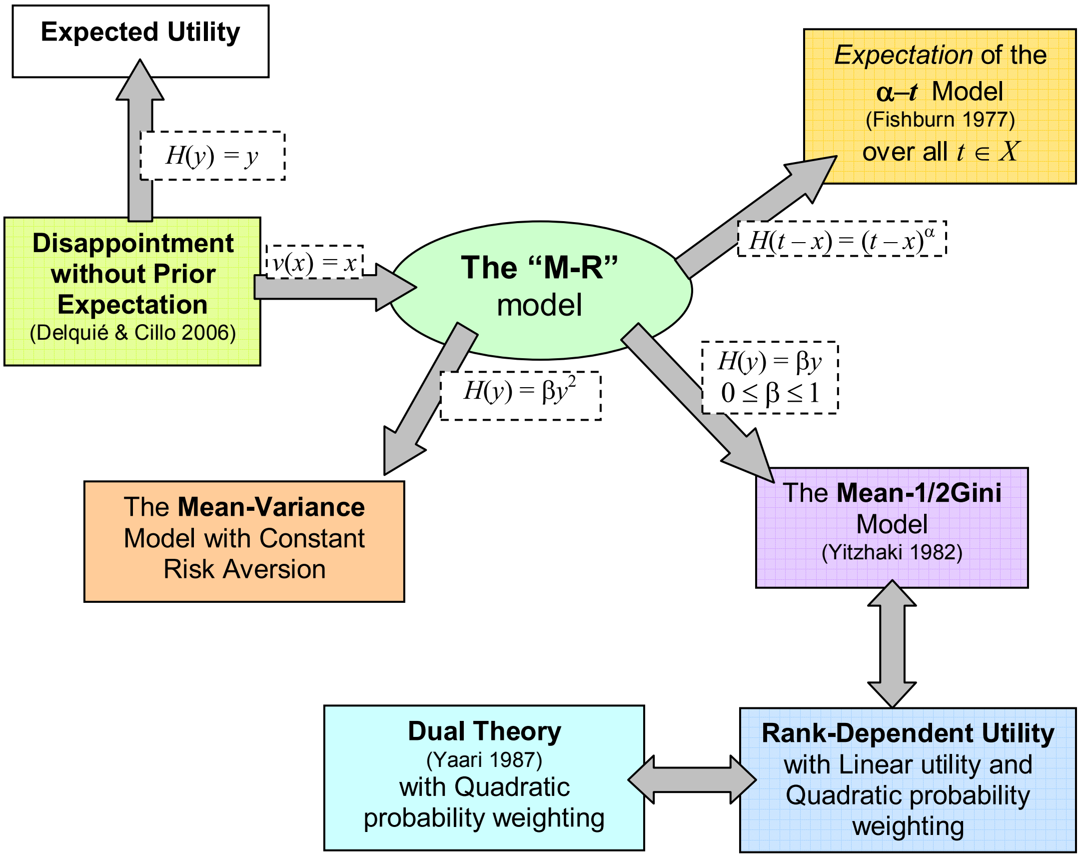

Figure 2. The relationship between the M-R model and other risk models. Dashed boxes indicate the assumptions, if any, associated with each link.