Revealed Reversals of Preferences

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Abstract

We weaken the implicit assumption of rational choice theory that imposes that preferences do not depend on the choice set. We concentrate on the cases where the preferences change monotonically when the choice set expands. We show that rationalizability is then equivalent to state the usual strong axiom of revealed preferences to the binary relation of "revealed reversals of preferences".

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1 Introduction

Classical rational choice theory is based on the axiom of revealed preferences. In short, we say that an individual reveals that he prefers an alternative a to an alternative b if there exists a choice set S such that b is in S and a is chosen from S. One of the most important results of classical choice theory is that a choice function is rationalized by a linear order[1] if and only if the revealed preferences derived from it are asymmetric (weak axiom of revealed preferences, introduced by [Samuelson, 1938]). Moreover, in this framework, the asymmetry of the revealed preferences is equivalent to their acyclicity (strong axiom of revealed preferences, introduced by [Houthakker, 1950]).

However, many experiments tend to show that the traditional rational choice theory is violated by most individuals (see [Loomes *et al.*, 1991], [Roelofsma and Read, 2000] and [Tversky, 1969] for instance). Then, in order to study the choice patterns of the observed individuals, one need to weaken the axioms of rationality or equivalently the representations of the theory. In particular, one recent article has studied rational choices in this line of analysis. [Manzini and Mariotti, 2007] studied sequentially rationalizable choice functions.[2] In order to characterize those choice functions, a weak version of revealed preferences is used. This axiom still requires the revealed preferences to be asymmetric but it defines a revealed preference as follows: a is said to be revealed preferred to b if there exist two sets S and S' such that $a, b \in S \subseteq S'$ and such that b is chosen in S whereas a is chosen in S'.[3] Hence, in order to state that a is revealed preferred to b it is not enough

[1]For the sake of simplicity, we will deal with single-valued choices.

[2]For more studies in this line, see [Ballester and Apesteguia, 2008], [Houy, 2007], [Houy and Tadenuma, 2007] and [Tadenuma, 2002].

[3]The original axiom of [Manzini and Mariotti, 2007] is not stated in those terms but is equivalent to the version we give here.

to state that a is chosen from a set S' of which b is an element. Indeed, if such a set S' exists, there are still three possibilities:

- if there exists $S \subseteq S'$ such that b is chosen from S, then a is revealed preferred to b,
- if there exists $S'' \supseteq S'$ such that b is chosen from S, then b is revealed preferred to a,
- if for any S''' such that $a \in S'''$, b is not chosen from S''', then b is not revealed preferred to a nor is a revealed preferred to b.

Hence, imposing an axiom that states that revealed preferences should be asymmetric allows for choice reversals but imposes that once a reversal has taken place, there is no reversal in the other direction for larger sets.

Another way to see the axiom stated above is as follows: by choosing b in $S \subseteq S'$ and a in S', the agent reveals a preference reversal between a and b, let us say in favor of a. Then, [Manzini and Mariotti, 2007]'s axiom states that the binary relation of "revealed reversals of preferences" should be asymmetric. Hence, [Manzini and Mariotti, 2007]'s axiom is analogous to [Samuelson, 1938]'s WARP when one is dealing with revealed reversals of preferences.

In this paper, we deal with the acyclicity of the binary relation of "revealed reversals of preferences". We show that imposing that the revealed reversals of preferences being acyclic is equivalent to stating that choices should be rationalized by some monotonous linear order.

This article is closely related to a few recent articles that all try to weaken the classical rational choice theory in the same direction. The basic observation is that the classical choice theory is too strong because it considers rational choices as the maximal elements of one preference binary relation that is choice set independent. In addition to [Manzini and Mariotti, 2007] are a few studies that try to investigate the problem of choice set dependent preferences. Obviously, if no structure is imposed on this dependence, all choice functions are rationalizable. [Tyson, 2008] imposes that the preferences be nested. [Houy, 2008] looks at the anti-nested component of the choice set dependent binary relations and of course, at the intersection between [Tyson, 2008] and [Houy, 2008] is classical rational choice theory. As we will show, the characterization we give in this article correponds to the choice set dependent binary relations being monotonous.

2 Notation and Result

Let X be a finite set of alternatives. Let $\mathcal{X}$ be the set of non-empty subsets of X, $\mathcal{X} = 2^X \setminus \emptyset$. A choice function is a function $C : \mathcal{X} \to X$ such that $\forall S \in \mathcal{X}, C(S) \in S$.

Let $S \in \mathcal{X}$. A binary relation on S is a subset of $S \times S$. A binary relation $P \subseteq S \times S$ is asymmetric if $\forall a, b \in S$ with $a \neq b$, $(a, b) \in P \Rightarrow (b, a) \notin P$. It is complete if $\forall a, b \in S$ with $a \neq b$, $(a, b) \in P$ or $(b, a) \in P$. It is transitive if $\forall a, b, c \in S$, $(a, b), (b, c) \in P \Rightarrow (a, c) \in P$. It is acyclic if $\forall n \in \mathbb{N} \setminus \{1\}$, $\forall a_1, ..., a_n \in S$, $(a_i, a_{i+1}) \in P$ for all $i \in \{1, ..., n-1\}$ implies $a_1 \neq a_n$. A linear order is an asymmetric, transitive and complete binary relation.

$\pi : \mathcal{X} \to 2^{X \times X}$ is a binary relation (resp. linear order) function if for all $S \in \mathcal{X}$, $\pi(S)$ is a binary relation (resp. linear order) on S. We say that π is monotonous if $\forall S, S', S'' \in \mathcal{X}$ such that $S \subseteq S' \subseteq S''$, $\forall a, b \in S$, $[(a, b) \in \pi(S) \cap \pi(S'') \Rightarrow (a, b) \in \pi(S')]$. We say that π is choice set independent if $\forall S, S' \in \mathcal{X}$, $\forall a, b \in S \cap S'$, $[(a, b) \in \pi(S) \Leftrightarrow (a, b) \in \pi(S')]$.

Let π be a binary relation function. A choice function C is rationalized by π if $\forall S \in \mathcal{X}, C(S) = a$ if and only if $\forall b \in S, (b, a) \notin \pi(S)$.

Our axiom is similar to what is known as the Strong Axiom of Revealed Preferences (SARP). For this axiom, we usually define the revealed preferences binary relation. Namely, we say that an alternative a is revealed prefered to b if there exists a choice set from which a is chosen and of which b is an alternative. Then, SARP imposes that this revealed preferences binary relation be acylic on X.

As in SARP, we impose that some revealed binary relation be acylic on X. However, we will define this revealed binary relation as the revealed reversals of preferences binary relation.

Formally, let C be a choice function. We define the revealed reversals of preferences binary relation on X, P_C^r, by: $\forall a, b \in X, (a, b) \in P_C^r$ if and only if $\exists S, S' \in \mathcal{X}$ such that $a, b \in S \subseteq S'$, $b = C(S)$ and $a = C(S')$.

Then, Reversal SARP is a formulation of SARP for the revealed reversals of preferences binary relation.

Axiom 1 (RSARP, Reversal SARP)
A choice function C satisfies Reversal SARP if P_C^r is acyclic.

Obviously, RSARP is weaker than SARP since P_C^r is included in the usual revealed preferences binary relation.

Our Theorem shows that a choice function satisfies RSARP if and only if it is rationalized by a monotonic linear order function.

Theorem 1
Let C be a choice function. C satisfies RSARP if and only if C is rationalized by a monotonic linear order function.

Notice that in a study addressing a quite different question, [Manzini and Mariotti, 2007] have introduced an axiom, weak WARP, that imposes the asymmetry of P_C^r. The following example shows that it is not enough to state that a choice function satisfies weak WARP to be rationalized by a monotonic linear order function.

Example 1

Let $X = \{a, b, c, d\}$*. Let* C *be defined as* $C(\{a\}) = a$*,* $C(\{b\}) = b$*,* $C(\{c\}) = c$*,* $C(\{d\}) = d$*.* $C(\{a, b\}) = b$*,* $C(\{a, c\}) = a$*,* $C(\{a, d\}) = a$*,* $C(\{b, c\}) = c$*,* $C(\{b, d\}) = b$*,* $C(\{c, d\}) = c$*.* $C(\{a, b, c\}) = a$*,* $C(\{a, b, d\}) = a$*,* $C(\{a, c, d\}) = c$*,* $C(\{b, c, d\}) = b$*.* $C(\{a, b, c, d\}) = d$*.*

Then, we have $P^r_C = \{(a, b), (b, c), (c, a), (d, a), (d, b), (d, c)\}$*.* P^r_C *is cyclic which by Theorem 1 implies that* C *cannot be rationalized by a monotonic linear order function even though in this case,* P^r_C *is asymmetric.*

Finally, the next result is a restatement of the traditional rational choice theory in our setting.

Theorem 2

Let C *be a choice function.* $P^r_C = \emptyset$ *if and only if* C *is rationalized by a choice set independent linear order function.*

A Proof of Theorem 1

A.1 Only If:

Let C be a choice function satisfying RSARP.

Let $S \in \mathcal{X}$. Let us define the irreflexive binary relations $P_1(S)$ and $P_2(S)$ as follows: $\forall a, b \in S$,

- $(a, b) \in P_1(S)$ if and only if $a \neq b$ and $\exists S', S'' \in \mathcal{X}$ such that $a, b \in S' \subseteq S'' \subseteq S$, $b = C(S')$ and $a = C(S'')$.
- $(a, b) \in P_2(S)$ if and only if $a \neq b$ and $[(a, b), (b, a) \notin P_1(S)$ and $\exists S', S'' \in \mathcal{X}$ such that $a, b \in S' \subseteq S \subseteq S''$, $a = C(S')$ and $a = C(S'')]$.

I. Let us show that $P_1(X) \cup P_2(X)$ is acyclic.

1) $P_1(X)$ is acyclic. Indeed, by definition, $P_1(X) = P_C^r$. Hence, $P_1(X)$ is acyclic by RSARP.

2) $P_2(X)$ is acyclic. Assume on the contrary that $P_2(X)$ is cyclic. Then, there exists $n \in \mathbb{N} \setminus \{1, 2\}$, there exist $a_1, ..., a_n \in X$ such that $\forall i \in \{1, ..., n-1\}$, $(a_i, a_{i+1}) \in P_2(X)$ and $a_n = a_1$. Then, by definition, $\forall i \in \{1, ..., n-1\}$, $[(a_i, a_{i+1}), (a_{i+1}, a_i) \notin P_1(X)$ and $\exists S', S'' \in \mathcal{X}$ such that $a_i, a_{i+1} \in S' \subseteq X \subseteq S''$, $a_i = C(S')$ and $a_i = C(S'') = C(X)]$. Hence, $\forall i \in \{1, ..., n-1\}$, $a_i = C(X)$ which, by definition of a choice function, is impossible. Hence, $P_2(X)$ is acyclic.

3) $P_1(X) \cup P_2(X)$ is acyclic. By 1) and 2), if $P_1(X) \cup P_2(X)$ is cyclic, then, there exists $a, b, c \in X$ such that $(a, b) \in P_1(X)$ and $(b, c) \in P_2(X)$. By definition, a, b, c are mutually different. Moreover, by definition, $(b, c) \in P_2(X)$ implies $b = C(X)$. Now, since $(a, b) \in P_1(X)$, $\exists S', S'' \in \mathcal{X}$ such that $a, b \in S' \subseteq S'' \subseteq X$, $b = C(S')$ and $a = C(S'')$. But then, by definition, $a = C(S'')$ and $b = C(X)$ with $S'' \subseteq X$ implys $(b, a) \in P_1(X)$. This contradicts RSARP since $(b, a), (a, b) \in P_1(X)$.

Then, $P_1(X) \cup P_2(X)$ is acyclic. Then, the transitive closure of $P_1(X) \cup P_2(X)$ is asymmetric and transitive. By Szpilrajn Theorem, see [Szpilrajn, 1930], since the transitive closure of $P_1(X) \cup P_2(X)$ is asymmetric and transitive, there exists a linear order $\overline{P}$ such that $P_1(X) \cup P_2(X) \subseteq \overline{P}$. Let us define the irreflexive binary relations $P_3(S)$ as follows: $\forall a, b \in S$,

- $(a, b) \in P_3(S)$ if and only if $[(a, b), (b, a) \notin P_1(S) \cup P_2(S)$ and $(a, b) \in \overline{P}]$.

II. Let us show that for all $S \in \mathcal{X}$, $P_1(S) \cup P_2(S) \cup P_3(S)$ is a linear order.

1) $P_1(S) \cup P_2(S) \cup P_3(S)$ is complete by definition since $\overline{P}$ is.

2) $P_1(S) \cup P_2(S) \cup P_3(S)$ is irreflexive by definition.

3) $P_1(S) \cup P_2(S) \cup P_3(S)$ is asymmetric. i) By definition, $P_1(S) \subseteq P_C^r$. Hence, $P_1(S)$ is asymmetric by RSARP. ii) Assume that $P_2(S)$ is not asymmetric. Then, there exist $a, b \in X$ (with $a \neq b$ by 2)) such that $[\exists S_1', S_1'' \in \mathcal{X}$ such that $a, b \in S_1' \subseteq S \subseteq S_1''$, $a = C(S_1')$ and $a = C(S_1'')]$ and $[\exists S_2', S_2'' \in \mathcal{X}$ such that $a, b \in S_2' \subseteq S \subseteq S_2''$, $b = C(S_2')$ and $b = C(S_2'')]$. Then, by definition, $a = C(S_1')$ and $b = C(S_2'')$ imply that $(b, a) \in P_1(X)$. Moreover, by definition, $b = C(S_2')$ and $a = C(S_1'')$ imply that $(a, b) \in P_1(X)$. Hence, a contradiction with the fact, that we showed above, that $P_1(X)$ is acyclic. iii) By definition, $P_3(S) \subseteq \overline{P}$. Since $\overline{P}$ is asymmetric, so is $P_3(S)$. Then, by definition, we have $P_1(S) \cup P_2(S) \cup P_3(S)$ asymmetric (since by definition, we cannot have $(a, b) \in P_i(S)$ and $(b, a) \in P_j(S)$ with $i \neq j$).

4) $P_1(S) \cup P_2(S) \cup P_3(S)$ is transitive. Since $P_1(S) \cup P_2(S) \cup P_3(S)$ is complete and asymmetric, $P_1(S) \cup P_2(S) \cup P_3(S)$ is transitive if and only if there exist no $a, b, c \in S$ mutually different such that $(a, b), (b, c), (c, a) \in P_1(S) \cup P_2(S) \cup P_3(S)$. Let us assume on the contrary that there exist $a, b, c \in S$ mutually different such that $(a, b), (b, c), (c, a) \in P_1(S) \cup P_2(S) \cup P_3(S)$. Let us have $(a, b) \in P_i(S), (a, b) \in P_j(S), (a, b) \in P_k(S)$ with $i, j, k \in \{1, 2, 3\}$. By definition, i, j, k are uniquely defined. i) $\neg(i, j, k \in \{1, 3\})$. By definition,

$P_1(S) \subseteq P_1(X) \subseteq \overline{P}$. Moreover, $P_3(S) \subseteq \overline{P}$. Hence, if $i, j, k \in \{1, 3\}$, $\overline{P}$ is cyclic which is a contradiction. ii) $\neg(i = j = k = 2)$. On the contrary, assume that $i = j = k = 2$. By definition, $\exists S_1', S_1'' \in \mathcal{X}$ such that $a, b \in S_1' \subseteq S \subseteq S_1''$, $a = C(S_1')$ and $a = C(S_1'')$, $\exists S_2', S_2'' \in \mathcal{X}$ such that $b, c \in S_2' \subseteq S \subseteq S_2''$, $b = C(S_2')$ and $b = C(S_2'')$, $\exists S_3', S_3'' \in \mathcal{X}$ such that $a, c \in S_3' \subseteq S \subseteq S_3''$, $c = C(S_3')$ and $c = C(S_3'')$. Assume that $b = C(\{a, b\})$. Then, $b = C(\{a, b\})$ and $a = C(S_1')$ with $a, b \in S_1' \subseteq S$ imply that $(a, b) \in P_1(S)$ which contradicts the fact that $(a, b) \in P_2(S)$. Hence, $a = C(\{a, b\})$. The same reasoning shows that $b = C(\{b, c\})$ and $c = C(\{a, c\})$. Now assume that $a = C(\{a, b, c\})$ (the proof is identical if $b = C(\{a, b, c\})$ or $c = C(\{a, b, c\})$). Then, by definition, $a = C(\{a, b, c\})$ and $c = C(\{a, c\})$ with $\{a, b, c\} \subseteq S$ implies $(a, c) \in P_1(S)$. This contradicts $(c, a) \in P_2(S)$. iii) $\neg((i, j) = (1, 2))$. On the contrary, assume that $(a, b) \in P_1(S)$ and $(b, c) \in P_2(S)$. Then, by definition, $\exists S_1', S_1'' \in \mathcal{X}$ such that $a, b \in S_1' \subseteq S_1'' \subseteq S$, $b = C(S_1')$ and $a = C(S_1'')$ and $\exists S_2', S_2'' \in \mathcal{X}$ such that $b, c \in S_2' \subseteq S \subseteq S_2''$, $b = C(S_2')$ and $b = C(S_2'')$. By definition, $(a, b) \in P_1(S)$ implies $(a, b) \in P_1(X)$. Moreover, $a = C(S_1'')$ and $b = C(S_2'')$ with $a, b \in S_1'' \subseteq S_2'' \subseteq X$ implies $(b, a) \in P_1(X)$. This contradicts the asymmetry of $P_1(X)$ that we showed above. For the same reasons, $\neg((j, k) = (1, 2))$, $\neg((k, i) = (1, 2))$. iv) $\neg((i, j) = (3, 2))$. On the contrary, assume that $(a, b) \in P_3(S)$ and $(b, c) \in P_2(S)$. By definition $(a, b), (b, a) \notin P_1(S) \cup P_2(S)$. a) Assume that $C(\{a, b\}) = b$. $(b, c) \in P_2(S)$ implies that $\exists S'' \in \mathcal{X}$ such that $S \subseteq S''$ and $b = C(S'')$. Hence, $[(a, b), (b, a) \notin P_1(S)$ and $\exists S', S'' \in \mathcal{X}$ such that $a, b \in S' \subseteq S \subseteq S''$, $b = C(S')$ and $b = C(S'')]$ which implies that $(b, a) \in P_2(S)$. This contradicts the assumptions. b) Assume that $C(\{a, b\}) = a$. $(b, c) \in P_2(S)$ implies that $\exists S'' \in \mathcal{X}$ such that $S \subseteq S''$ and $b = C(S'')$. Hence, $\exists S', S'' \in \mathcal{X}$ such that $a, b \in S' \subseteq S'' \subseteq X$, $a = C(S')$ and $b = C(S'')$. Then, $(b, a) \in P_1(X)$. Hence, $(b, a) \in P_1(X) \subseteq \overline{P}$,

$(a, b) \in P_3(S) \subseteq \overline{P}$. This contradicts the fact that $\overline{P}$ is a linear order. For the same reasons, $\neg((j, k) = (3, 2))$, $\neg((k, i) = (3, 2))$. i), ii), iii) and iv) show that $P_1(S) \cup P_2(S) \cup P_3(S)$ is transitive.

Hence, we can define the linear order function π as $\forall S \in \mathcal{X}, \pi(S) = P_1(S) \cup P_2(S) \cup P_3(S)$.

III. Let us show that π is monotonous. Let $S, S', S'' \in \mathcal{X}$ such that $S \subseteq S' \subseteq S''$ and $a, b \in S$ such that $(a, b) \in \pi(S) \cap \pi(S'')$. Let us show that $(a, b) \in \pi(S')$. Assume on the contrary that $(a, b) \notin \pi(S')$. Since $\pi(S')$ is a linear order, $(b, a) \in \pi(S')$. 1) Assume that $(b, a) \in P_1(S')$. By definition, $(b, a) \in P_1(S'')$ and then, $(a, b) \notin \pi(S'')$ which contradicts the assumptions. 2) Assume that $(b, a) \in P_2(S')$. With the same proof as II.4)ii), we have $b = C(\{a, b\})$. Moreover, if we had $(a, b) \in P_1(S)$ or $(b, a) \in P_1(S)$, by definition, we would have $(a, b) \in P_1(S')$ or $(b, a) \in P_1(S')$ which would contradict $(b, a) \in P_2(S')$. Then, by definition, $(b, a) \in P_2(S)$ which contradicts the assumptions. 3) For the same reason, $(a, b) \in P_2(S'')$ would imply $(a, b) \in P_2(S')$ which would be a contradiction. 4) Assume that $((a, b) \in P_1(S'')$ or $(a, b) \in P_3(S''))$ and $(b, a) \in P_3(S')$. Then, by definition, $(a, b), (b, a) \in \overline{P}$ which contradicts the fact that $\overline{P}$ is a linear order.

IV. Let us show that C is rationalized by π *i.e.* $\forall S \in \mathcal{X}, C(S) = a$ if and only if $\forall b \in S, (b, a) \notin \pi(S)$.

Let $S \in \mathcal{X}$. Let $C(S) = a$. Let $b \in S$. 1) Assume $C(\{a, b\}) = b$. Then, by definition, $(a, b) \in P_1(S)$ which implies $(b, a) \notin \pi(S)$ by definition and asymmetry of $P_1(S)$. 2) Assume that $C(\{a, b\}) = a$. By definition, if $\exists S' \in \mathcal{X}$ with $a, b \in S' \subseteq S$ such that $b = C(S')$, then by definition, $(b, a) \in P_1(S') \subseteq P_1(X)$ and $(a, b) \in P_1(S) \subseteq P_1(X)$. This contradicts the asymmetry of $P_1(X)$ that has been shown above. Hence, by definition, $(b, a), (a, b) \notin P_1(S)$.

Then, by definition, $(a,b) \in P_2(S)$ which implies $(b,a) \notin \pi(S)$ by definition and asymmetry of $P_2(S)$.

This shows that $\forall S \in \mathcal{X}, C(S) = a$ implies $\forall b \in S, (b,a) \notin \pi(S)$.

Now let us have $C(S) \neq a$. Since C is a choice function, $\exists b \in S \setminus \{a\}, C(S) = b$. By what we showed above, $(b,a) \in \pi(S)$. This shows that $\forall S \in \mathcal{X}$, $\forall b \in S, (b,a) \notin \pi(S)$ implies $C(S) = a$.

A.2 If:

Let C be a choice function rationalized by π, monotonic linear order function.

Assume that P_C^r is cyclic. Then, there exists $n \in \mathbb{N} \setminus \{1\}$, there exist $a_1, ..., a_n \in X$ such that $\forall i \in \{1, ..., n-1\}$, $(a_i, a_{i+1}) \in P_C^r$ and $a_n = a_1$. Then, by definition, $\forall i \in \{1, ..., n-1\}$, $\exists S_i', S_i'' \in \mathcal{X}$ such that $a_i, a_{i+1} \in S_i' \subseteq S_i''$, $a_{i+1} = C(S_i')$ and $a_i = C(S_i'')$. Since π is a linear order function rationalizing C, $\forall i \in \{1, ..., n-1\}$, $(a_{i+1}, a_i) \in \pi(S_i')$ and $(a_i, a_{i+1}) \in \pi(S_i'')$. Hence, by monotonicity of π, $\forall i \in \{1, ..., n-1\}$, $(a_i, a_{i+1}) \in \pi(X)$. Then, $\pi(X)$ is cyclic which contradicts the fact that π is a linear order function.

B Proof of Theorem 2

B.1 Only if:

Let C be a choice function such that $P_C^r = \emptyset$.

Let us define P as: $\forall a, b \in X$, $(a,b) \in P$ if and only if $a \neq b$ and $\{a\} = C(\{a,b\})$. Let us define $P(S) = \{(a,b) \in P, a, b \in S\}$.

I. By definition of a choice function, $P(S)$ is complete and asymmetric. Let us show that P is transitive. Let us have $a, b, c \in S$ mutually different (in case they are not mutually different, the proof is straightforward) with $(a,b), (b,c) \in P$. Then, by definition, $\{a\} = C(\{a,b\})$ and

$\{b\} = C(\{b, c\})$. Assume that $C(\{a, b, c\}) = \{b\}$. Then, $C(\{a, b, c\}) = \{b\}$ and $\{a\} = C(\{a, b\})$ imply $(b, a) \in P^r_C$ which contradicts $P^r_C = \emptyset$. For the same reason we cannot have $\{c\} = C(\{a, b, c\})$. Hence, $\{a\} = C(\{a, b, c\})$. Now assume $\{c\} = C(\{a, c\})$. Then, $\{c\} = C(\{a, c\})$ and $\{a\} = C(\{a, b, c\})$ imply $(a, c) \in P^r_C$ which contradicts $P^r_C = \emptyset$. Hence, $\{a\} = C(\{a, c\})$ which implies $(a, c) \in P$ and then, $(a, c) \in P(S)$.

Now we can define the linear order function π as $\forall S \in \mathcal{X}, \pi(S) = P(S)$. II. By definition, π is choice set independent.

III. Let us show that π rationalizes C. 1) Let $S \in \mathcal{X}$ and $a \in S$ be such that $\{a\} = C(S)$. Let $b \in S \setminus \{a\}$. Assume $(b, a) \in \pi(S)$. Then, by definition, $\{b\} = C(\{a, b\})$. However, $\{b\} = C(\{a, b\})$ and $\{a\} = C(S)$ imply $(a, b) \in P^r_C = \emptyset$ which contradicts the assumption. Hence, $a \in C(S)$ implies $\forall b \in S \setminus \{a\}$, $(a, b) \in \pi(S)$. 2) This also shows in a straightforward manner that if $b \in S$ is such that $\{b\} \neq C(S)$, then, $\exists a \in S, (a, b) \in \pi(S)$.

B.2 If:

Assume that C is a choice function such that $P^r_C \neq \emptyset$. Then, by definition, there exist $a, b \in X$ such that $(a, b) \in P^r_C$. Hence, $\exists S, S' \in \mathcal{X}$ such that $a, b \in S \subseteq S'$, $b = C(S)$ and $a = C(S')$. Then, if C is rationalized by the linear order function π, we necessarily have $(a, b) \in \pi(S')$ and $(b, a) \in \pi(S)$ which implies that π is not choice set independent.

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