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Solving discretely constrained, mixed linear complementarity problems with applications in energy

Steven A. Gabriel [a,*], Antonio J. Conejo [b], Carlos Ruiz [d], Sauleh Siddiqui [c]

[a] Civil Systems Program, Department of Civil and Environmental Engineering, and Applied Mathematics & Statistics, and Scientific Computation Program, University of Maryland, College Park, MD 20742-3021, USA
[b] Department of Electrical Engineering, University of Castilla – La Mancha, Campus Universitario s/n, 13071 Ciudad Real, Spain
[c] Department of Civil Engineering and the Johns Hopkins Systems Institute, Johns Hopkins University, Baltimore, MD 21218, USA
[d] Department of Statistics, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés, Spain

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ABSTRACT

This paper presents an approach to solving discretely constrained, mixed linear complementarity problems (DC-MLCPs). Such formulations include a variety of interesting and realistic models of which two are highlighted: a market-clearing auction typical in electric power markets but suitable in other more general contexts, and a network equilibrium suitable to energy markets as well as other grid-based industries. A mixed-integer, linear program is used to solve the DC-MLCP in which both complementarity as well as integrality are allowed to be relaxed. Theoretical and numerical results are provided to validate the approach.

1. Introduction

In this paper, we present a new approach to solve discretely constrained, mixed linear complementarity problems (DC-MLCPs) in which some of the variables are constrained to be integer-valued and some can take on continuous values. This is an important extension of the general MLCP in which all variables are assumed to be continuous and relates for example to Nash–Cournot games in which some of the players' variables are discrete and some are continuous. A mixed-integer linear program (MILP) is presented which solves DC-MLCPs with complementarity and integrality suitably relaxed. As an example in Section 2 shows, enforcing exact complementarity and exact integrality may not be feasible. From a compromise perspective, the MILP that relaxes both of these conditions is somewhat related to the notion of bounded rationality in equilibrium systems as discussed in [22].

This focus on integer variables (and/or related techniques) and one [8] or two-level equilibria, e.g., mathematical programs with equilibrium constraints [21] has seen some research efforts over the years in both modeling and methods (e.g., [5,20,1,23,24,15,13,3]) and joins two important fields of operations research. This work also has relevance to both energy market modeling [25] and network optimization [17].

Section 2 presents a general formulation for an MLCP with a mixture of discrete and continuous variables and introduces two relaxations: σ-complementarity and ε-integrality. Depending on the particular application one or both of these relaxations may be useful. Theorem 1 provides justification for one of the disjunctive constants (M_2) used in this MILP. Theorem 2 shows under reasonable conditions, when there exists a solution to this MILP. Section 2 also discusses some practical aspects of solving the aforementioned DC-MLCP including a heuristic for how to estimate the key complementarity relaxation constant M_1 in a general context. Note that these constants are problem specific and change with the type of application. In Theorem 4 we show how to calculate M_1 for an illustrative network example.

In Section 3, the general DC-MLCP is specialized to a market-clearing problem. Such a problem is an auction mechanism to determine which production offers and consumption bids are accepted by the market operator, which has the target of maximizing social welfare. Social welfare is computed based on producer-declared offer prices and consumers' declared bid prices, and therefore it is called "declared" social welfare. The clearing algorithm that is considered is intended for electricity markets and specifically represents the transmission (transportation) network and its capacity. The algorithm is also multi-period as it considers simultaneously the clearing of the market at several time periods (e.g., the hours of the day). Market-clearing algorithms of this type are commonly used by electricity market operators across the East Coast of the United States (http://www.pjm.com, http://www.iso-ne.com). The algorithm that we propose provides clearing prices that support market outcomes in the sense the producers that are actually producing have no

* Corresponding author.
E-mail addresses: sgabriel@umd.edu (S.A. Gabriel), Antonio.Conejo@uclm.es (A.J. Conejo), caruizm@est-econ.uc3m.es (C. Ruiz), siddiqui@jhu.edu (S. Siddiqui).

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incentive to leave the market. Note that this is so even though the proposed market-clearing formulation is non-convex and represents a new approach for this previously studied uplift problem. In other words, we propose a consistent price mechanism within a non-convex market clearing formulation.

The distinguishing features of the proposed pricing technique with respect to other procedures reported in the technical literature (e.g., [16,18,4]) are two-fold. First, the initial market-clearing problem is not manipulated to achieve prices that support market outcomes. Instead, optimality conditions of the original problem with integrality conditions relaxed are formulated and incorporated into a relaxation problem that allows realizing the tradeoff of integrality vs. complementarity, and obtaining via uplifts, prices that support market outcomes. Second, instead of using a two-step procedure (first solving the original MILP and then formulating and solving a modified LP), as indicated above, the proposed technique is single-step, and does not require altering the original problem by fixing integer variables to their optimal values to formulate a continuous problem from which prices (that support market outcomes) can be derived.

In Section 4, a stylized network equilibrium problem with multiple players is presented from [14] based on the earlier works [11,12]. This application is suitable to energy and other grid-based industries involving multiple players and a system operator. For this problem, two theoretical results are presented. In Theorem 3, under a very mild condition on the demand function, it is shown that there exists a valid bound M_1 that does not cut off any solutions. Then, in Theorem 4, given linear demand functions, a specific valid disjunctive constraints value for M_1 is presented. Both these sorts of results are application-specific but presented in a rather general network context to show how they might be done for other related problems as well as give guidance for this specific one.

After these motivating examples, in Section 5, we provide numerical experiments that validate the proposed approaches followed by conclusions and extensions in Section 6.

2. Discretely constrained mixed linear complementarity problems

2.1. Problem formulation

We consider a general, discretely constrained mixed linear complementarity problem. The formulation is as follows: given the vector $q=(q_1^T\ q_2^T)^T$ and matrix $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, find $z=(z_1^T,z_2^T)^T \in R^{n_1}\times R^{n_2}$ such that

$$0 \le q_1+(A_{11}\ A_{12})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \perp z_1 \ge 0 \tag{1a}$$

$$0 = q_2+(A_{21}\ A_{22})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad z_2 \text{ free} \tag{1b}$$

$$(z_1)_c \in R_+, \quad c \in I_1^C, \quad (z_1)_d \in D_1 \subseteq Z_+, \quad d \in I_1^D \tag{1c}$$

$$(z_2)_c \in R, \quad c \in I_2^C, \quad (z_2)_d \in D_2 \subseteq Z, \quad d \in I_2^D \tag{1d}$$

where D_1, D_2 are given discrete sets of values. Also, $I_1^C \cup I_1^D$ is a partition of the indices $\{1,\ldots,n_1\}$ for z_1 and $I_2^C \cup I_2^D$ a partition of the indices $\{1,\ldots,n_2\}$ for z_2, i.e., $z_k=((z_k)_{I_k^C}^T\ (z_k)_{I_k^D}^T)^T, k=1,2$ with the continuous variables shown first, without loss of generality. As an example, suppose that the nonnegative vector z_1 has five components, i.e., $z_1=(z_{11},z_{12},z_{13},z_{14},z_{15})^T$ with the first and third constrained to be discrete and the second, fourth, and fifth continuous. In that case $I_1^D=\{1,3\}, I_1^C=\{2,4,5\}$, and if $D_1=Z_+, z_{11},z_{13}\in\{0,1,2,\ldots\}$, $z_{12},z_{14},z_{15}\in R_+$. Also note that the notation $0 \le w \perp v \ge 0$ is standard shorthand in complementarity modeling to indicate that the vectors w and v are both nonnegative and their inner product is zero, i.e., $w^T v=0$.

From here on for specificity, unless otherwise indicated, the discrete sets, $D_1=\{0,1,\ldots,N\}$ and $D_2=\{-N_1,\ldots,-1,0,1,\ldots,N_2\}$ will be assumed with N,N_1,N_2 nonnegative integers. Note that all the problem formulations in this paper assume that the problems are bounded. This is not a restrictive assumption for most real-world engineering problems where quantities are bounded by physical limits and shadow prices are often bounded by demand curves or other economic mechanisms (see Theorems 3 and 4 for a demonstration of these concepts).

First, the complementarity relationship and nonnegativity for z_1 (1a) can be recast as the following disjunctive constraints [9]:

$$0 \le q_1+(A_{11}\ A_{12})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \le M_1(u) \tag{2a}$$

$$0 \le z_1 \le M_1(1-u), \quad u_j \in \{0,1\}, \quad \forall j \tag{2b}$$

where M_1 is a suitably large, positive constant and u is a vector of binary variables. The other constraints (1b) can be used as is and taking (1b) with (2) would represent a reformulation of (1) with just continuous variables z_1,z_2 allowed. If we assume that there were a solution to this version of the original problem, the existence of a solution would not necessarily be guaranteed if we imposed the discrete restrictions from (1c) and (1d). To be specific, consider the following counter-example with $A=\begin{pmatrix} 1 & 0 \\ -1 & 0\end{pmatrix}$, $q=\begin{pmatrix} -0.2 \\ 0.2\end{pmatrix}$. For z with real components, this LCP is feasible. For example, $z=(0.2,\ 2)^{\mathrm{T}}$ is a solution. However, if the first component of z must be integer, i.e., $z=(z_1,z_2)^T \in \mathbb{Z}_+ \times \mathbb{R}_+$, then this LCP is infeasible. For the LCP to be feasible, there must be an integer-valued z_1 such that $\begin{pmatrix} -0.2 \\ 0.2\end{pmatrix}+\begin{pmatrix} 1 & 0 \\ -1 & 0\end{pmatrix}\begin{pmatrix} z_1 \\ z_2\end{pmatrix}=\begin{pmatrix} -0.2+z_1 \\ 0.2-z_1\end{pmatrix}\ge 0$ which can only be true if $0.2 \le z_1 \le 0.2$. Hence, there are no integer values of z_1 for which this LCP is feasible.

2.2. Relaxation of the complementarity conditions

To protect against infeasibility, relaxations on both complementarity as well as integrality are used. First, we assume that the associated LCP is at least feasible, that is, there exists (z_1,z_2) such that

$$0 \le q_1+(A_{11}\ A_{12})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad z_1 \ge 0 \tag{3a}$$

$$0 = q_2+(A_{21}\ A_{22})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad z_2 \text{ free} \tag{3b}$$

Then, to relax complementarity, we introduce a nonnegative vector σ of deviations combined with the disjunctive form of the complementarity conditions to get

$$0 \le q_1+(A_{11}\ A_{12})\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \le M_1(u)+M_1\sigma \tag{4a}$$

$$0 \le z_1 \le M_1(1-u)+M_1\sigma \tag{4b}$$

with $u_j \in \{0,1\}, \forall j$. It is not hard to see that by adding the term $M_1\sigma$, the above complementarity is relaxed and these set of conditions are always feasible assuming that $\sigma \ge 0$ is allowed to vary. The constant M_1 can vary by constraint. Clearly, $\sigma=0$ means exact complementarity is enforced. In principle, one could also add a relaxation to always ensure that the relaxed LCP is feasible by including a term $-M_1\sigma$ instead of zero as the lower bound on the left-hand side of (4a) and (4b). In what follows, we do not consider this extra relaxation as it is assumed that the MLCP has been formulated to be at least feasible. This is reasonable given practical considerations.

2.3. Relaxation of the integrality conditions

The next step is to target one of the integers $\{0,1,\ldots,N\}$ and $\{-N_1,\ldots,-1,0,1,\ldots,N_2\}$, that the elements of the discrete vectors $(z_1)_{I_1^D}$ and $(z_2)_{I_2^D}$, respectively, are assumed to be able to take and get as close to those targeted values as possible. (Note: each discrete variable can have a separate range of values and a separate target. For ease of presentation and without loss of generality though we have assumed the same range for each variable but do allow for separate targeted values.) Here, closeness is measured in terms of the absolute deviation between a solution and one of the integer values. Consider the following "targeting" constraints for the discrete z_1 variables:

$$-M_2(1-w_{1ri}) \le (z_1)_r - i - \varepsilon_{1ri} \le M_2(1-w_{1ri}), \quad i=0,1,\ldots,N, \ \forall r \in I_1^D,$$

$$\sum_{i=0}^{N} w_{1ri} = 1, \quad w_{1ri} \in \{0,1\}, \quad i=0,1,\ldots,N, \ \forall r \in I_1^D.$$

Here M_2 is a suitably large, positive constant and the index i represents the targeted integer variable. We see that if the binary variable $w_{1ri}=1$, we have $(z_1)_r - i = \varepsilon_{1ri}$ for $r \in I_1^D$ with ε_{1ri} measuring the deviation (positive or negative) of $(z_1)_r$ from the targeted integer value i. If $w_{1ri}=0$, then $-M_2 \le (z_1)_r - i - \varepsilon_{1ri} \le M_2$ which if M_2 was chosen large enough, represents no realistic restriction on $(z_1)_r - i - \varepsilon_{1ri}$. (See Theorem 1 for how to calculate valid values for M_2.) Of course, for the case when $w_{1ri}=0$, we would like to "turn off" those variables ε_{1ri} since only one integer i is to be targeted. The constraint $\sum_{i=0}^{N} w_{1ri}=1$ makes sure that only one of the w_{1ri} can equal 1 but we need another device to force the corresponding ε_{1ri} values to zero. This is accomplished if we define $\varepsilon_{1ri} = (\varepsilon_{1ri})^+ - (\varepsilon_{1ri})^-$, i.e., the difference of two nonnegative deviations and we minimize

$$\sum_{r \in I_1^D} \sum_{i=0}^{N} |\varepsilon_{1ri}| = \sum_{r \in I_1^D} \sum_{i=0}^{N} (\varepsilon_{1ri})^+ + (\varepsilon_{1ri})^-$$

We can of course add the same type of variables and constraints for the discrete z_2 variables but with the adjustment that the free variables z_2 can take any values in $\{-N_1,\ldots,-1,0,1,\ldots N_2\}$ where N_1, N_2 are nonnegative integers.

2.4. The resulting relaxed problem

Thus, putting all these reformulations together gives a mixed integer linear program shown below in which both complementarity and integrality are relaxed for the problem (1). The objective of the relaxed (5) (shown just below) is to minimize the deviations from complementarity and integrality with each relaxation weighted by a positive constant $\omega_i, i=1,2$. It is important to note that this problem is always feasible (assuming the LCP was feasible) and has an optimal solution as long as a certain feasibility assumption on the relaxed complementarity problem is in force (see the assumption and related theorem below). As these positive weights ω_1, ω_2, are varied, elements of a Pareto curve [6] can be determined allowing for tradeoffs between inexact complementarity or inexact integrality. The particular application would dictate which relaxation was preferred and how much deviation from exact complementarity or integrality is allowed. Of course, if the objective function to (5) were equal to zero, corresponding to finding a solution to the original problem (1), this solution would dominate any that contained inexact complementarity and/or integrality. Note that $1=(1,\ldots,1)^T$ and $\sigma=(\sigma_1,\ldots,\sigma_{n_1})^T$ and that in all the examples, different σ were used.

Before presenting a result showing existence of a solution to (5), we indicate a valid value for the constant M_2 in the next theorem.

Theorem 1. *Let $M_2 \ge \max\{N, N_1+N_2\}$. Then, this value will be valid for the constraints (5f) and (5g).*

$$\min \ \omega_1 \left[\sum_{r \in I_1^D} \sum_{i=0}^{N} (\varepsilon_{1ri})^+ + (\varepsilon_{1ri})^- + \sum_{r \in I_2^D} \sum_{i=0}^{N} (\varepsilon_{2ri})^+ + (\varepsilon_{2ri})^- \right] + \omega_2 [1^T \sigma] \tag{5a}$$

$$0 \le q_1 + (A_{11} \ A_{12}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \le M_1(u) + M_1 \sigma \tag{5b}$$

$$0 \le z_1 \le M_1(1-u) + M_1 \sigma \tag{5c}$$

$$0 = q_2 + (A_{21} \ A_{22}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{5d}$$

$$u_j \in \{0,1\}, \quad \forall j \tag{5e}$$

$$-M_2(1-w_{1ri}) \le (z_1)_r - i - \varepsilon_{1ri} \le M_2(1-w_{1ri}), \quad i=0,1,\ldots,N, \ \forall r \in I_1^D \tag{5f}$$

$$-M_2(1-w_{2ri}) \le (z_2)_r - i - \varepsilon_{2ri} \le M_2(1-w_{2ri}), \quad i=-N_1,\ldots,-1,0,1,\ldots,N_2, \quad \forall r \in I_2^D \tag{5g}$$

$$\varepsilon_{1ri} = (\varepsilon_{1ri})^+ - (\varepsilon_{1ri})^-, \quad i=0,1,\ldots,N, \ \forall r \in I_1^D \tag{5h}$$

$$\varepsilon_{2ri} = (\varepsilon_{2ri})^+ - (\varepsilon_{2ri})^-, \quad i=-N_1,\ldots,0,\ldots,N_2, \ \forall r \in I_2^D \tag{5i}$$

$$\sum_{i=0}^{N} w_{1ri} = 1, \qquad \sum_{i=0}^{N} w_{2ri} = 1 \tag{5j}$$

$$w_{1ri} \in \{0,1\}, \quad i=0,1,\ldots,N, \ \forall r \in I_1^D \tag{5k}$$

$$w_{2ri} \in \{0,1\}, \quad i=0,1,\ldots,N, \ \forall r \in I_2^D \tag{5l}$$

$$\sigma \ge 0 \tag{5m}$$

$$(\varepsilon_{1ri})^+, (\varepsilon_{1ri})^- \ge 0, \quad i=0,1,\ldots,N, \ \forall r \in I_1^D \tag{5n}$$

$$(\varepsilon_{2ri})^+, (\varepsilon_{2ri})^- \ge 0, \quad i=-N_1,\ldots,0,\ldots,N_2, \ \forall r \in I_2^D \tag{5o}$$

Proof. In Appendix.

In order to prove the next result, we make an assumption on a relaxation of the original problem (1) in which the discrete restrictions are dropped.

Assumption 1. Define the set

$$S = \left\{ \begin{array}{l} (z_1, z_2) \,|\, 0 \le q_1 + (A_{11} \ A_{12}) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \\ 0 = q_2 + \begin{pmatrix} A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, z_1 \ge 0 \end{array} \right\}$$

Then, assume that S is nonempty and there exists a constant M^* such that

$$M^* \ge \max\{\|z_1\|_\infty, \|z_2\|_\infty\} = \left| \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right|_\infty \quad \text{for all } (z_1, z_2) \in S$$

The first assumption is quite reasonable since if this relaxed version of the problem is not even feasible, there is no hope to have a solution for the integer-constrained version with the complementarity restriction as was mentioned previously. Moreover, for certain classes of matrices this first assumption is automatically guaranteed. For example if A_{22} is invertible, then we can solve for z_2 and get the reduced conditions:

$$S=\{(z_1)\,|\,0\le(q_1-A_{12}A_{22}^{-1}q_2)+(A_{11}-A_{12}A_{22}^{-1}A_{21})z_1, z_1\ge 0\}$$

Then, this first assumption states that the LCP $((A_{11}-A_{12}A_{22}^{-1}A_{21}), (q_1-A_{12}A_{22}^{-1}q_2))$ needs to be feasible. A sufficient (and stronger condition) is that $(A_{11}-A_{12}A_{22}^{-1}A_{21})$ be an S-matrix [7]. The second assumption is also reasonable for this setting as we are assuming that the discretely constrained variables $(z_1)_d\in Z_+, d\in I_1^D, (z_2)_d\in Z, d\in I_2^D$ can only take on a finite set of integer values $\{0,1,\ldots,N\}$ and therefore it is not unreasonable that the continuous components $(z_1)_c\in R_+,\ c\in I_1^C, (z_2)_c\in R,\ c\in I_2^C$ also be bounded.

Note that relative to (5b) and (5c) it is sufficient to just require that the variables are bounded as stated in the second part of the assumption above. The reason is that if there is an $M^*\ge\max\{\|z_1\|_\infty,\|z_2\|_\infty\}$, then we can let $A_{11}=\begin{bmatrix}A_{11}^1\\ A_{11}^p\end{bmatrix}, A_{12}=\begin{bmatrix}A_{12}^1\\ A_{12}^p\end{bmatrix}$ where A_{11}^i, A_{12}^j are, respectively, the ith and jth rows of A_{11} and A_{12},

$$q_1+(A_{11}\ A_{12})\begin{pmatrix}z_1\\ z_2\end{pmatrix}\le q_1+\begin{pmatrix}\|A_{11}^1\|_1\|z_1\|_\infty\\ \vdots\\ \|A_{11}^p\|_1\|z_1\|_\infty\end{pmatrix}+\begin{pmatrix}\|A_{12}^1\|_1\|z_2\|_\infty\\ \vdots\\ \|A_{12}^p\|_1\|z_2\|_\infty\end{pmatrix}\le\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}M_1$$

where

$$M_1\ge\max\left\{M^*,\max_i\{(q_1)_i\}+\max_j\{\|A_{11}^j\|_1+\|A_{12}^j\|_1\}M^*\right\} \tag{6}$$

and use the fact that for all $x,y\in R^n, |x^Ty|\le\|x\|_1\|y\|_\infty$ which is a special case of Hölder's inequality [19]. Additionally, if a specific value for M^* is known, then computing M_1 as shown in (6) is straightforward as it only involves input data in the problem, namely, q_1, A_{11}, and A_{12}.

With this first assumption stated, we have the following theorem.

Theorem 2. *If Assumption 1 holds, and $M_2\ge\max\{N,N_1+N_2\}$ then problem (5) always has a solution.*

Proof. In Appendix.

In (5), there was a disjunctive constant M_1 to be determined. This value is generally problem-specific but in this section we provide some guidance on how to compute it as well as an algorithm to solve the DC-MLCP.

First note that M_1 is needed to convert complementarity conditions to disjunctive constraints. As such, too low a value of M_1 could cut off solutions to the complementarity conditions. Too high a value may result in computational problems from ill-conditioning. An important but somewhat obvious observation is that the discretely constrained MLCP will not have a solution if the continuous version does not. Thus, a first logical step is to solve the continuous, relaxed version of the problem (1) where all the discrete variables indexed by I_1^D and I_2^D are taken to be continuous. Assuming that this MLCP has a solution denoted as $z^*=(z_1^{*\mathrm{T}}\ z_2^{*\mathrm{T}})^T$ a reasonable choice would be to take $M^*\ge\|z^*\|_\infty$. If there were other solutions to the continuous MLCP and it was desired to not potentially exclude them with the choice of M^* just stated, using different starting points or MLCP algorithms, the relaxed form of (1) could be solved resulting in a finite subset of the solution set given as $\{(z^*)^i\,|\,i=1,\ldots,K\}$. An improved choice for M^* would then be

$$M^*\ge\max_{i=1,\ldots,K}\{\|(z^*)^i\|_\infty\} \tag{7}$$

Next consider an algorithm to solve DC-MLCP.

Step 1: Solve the relaxed version of (1) with the variables $(z_1)_r, r\in I_1^D, (z_2)_r, r\in I_2^D$ taken to be continuous. If there is no solution to this problem, then STOP. DC-MLCP has no solution. Otherwise, let $(z^*)^1=\binom{(z_1^*)^1}{(z_2^*)^1}$ be a solution. If desirable, solve the relaxed DC-MLCP using different starting points/methods to generate additional solutions: $\{(z^*)^i\,|\,i=1,\ldots,K\}$.

Step 2: Take $M^*\ge\max_{i=1,\ldots,K}\{\|(z^*)^i\|_\infty\}$, M_1 according to (6), $M_2\ge\max\{N,N_1+N_2\}$ and solve (5).

Note that from the above discussion, assuming that Assumption 1 is in force, (5) solved in Step 2 will always yield a solution which represents a tradeoff between complementarity and integrality. The key idea is that in Step 1, it is necessary to show that the relaxed form of the problem can always be solved.

3. Market-clearing problem expressed as a DC-MLCP

We consider below a multi-period, network-constrained, market-clearing model. The formulation considered corresponds to a multi-period auction in which producers submit production offers consisting of energy blocks and their corresponding selling prices as well as startup and shutdown costs. In addition, consumers submit consumption bids consisting of energy blocks and their corresponding buying prices. This model clears the market maximizing social welfare which is computed using producer offers and consumer bids (i.e., "declared" social welfare).

The corresponding problem is a mixed-integer, linear program. If integrality is relaxed and the KKT conditions are written for the resulting continuous, convex problem, an MLCP is obtained. If integrality conditions are added back to this MLCP, an infeasible DC-MLCP may be obtained. However, this infeasible DC-MLCP can be analyzed through the relaxed optimization (5), which is proposed in this paper.

In particular, we study the following two relevant issues. First, how optimal values for primal variables (production and consumption) and dual variables (prices) vary across formulations: MILP, MLCP and relaxed-DC-MLCP. To compute prices for the MILP formulation, an LP is solved with binary variables fixed to their optimal values. Second, how to include an appropriate uplift on prices to ensure nonnegative producers' profits. The nonnegative profit constraints (assuming that the production costs of the producers are known) include bilinear terms (price times production quantities) but can be effectively linearized through a binary expansion approach rendering a mixed-integer linear programming problem solvable by conventional branch-and-cut solvers. Observe that nonnegative profit constraints result in clearing prices that support the market in the sense that neither producers nor consumers have incentives to leave the market.

In this problem, the producers are indexed by i, the generation blocks by b, the demand by j, the demand blocks by k and time by t. Additionally, the nodes of the networks are given by n,m with the set of nodes connected through lines to n shown by Θ_n and the set of producers and demands at node n represented by Ψ_n.

The parameters include: λ_{tib}^G, the marginal cost of block b of unit i in period t; λ_{tjk}^D, the marginal utility of block k of demand j in period t; $P_{tib}^{G\max}$, the upper limit of block b of unit i in period

t; $P^{D\max}_{tdk}$, the upper limit of block k of demand j in period t; B_{nm}, the susceptance of line $n-m$; $P^{\max}_{nm}$, the transmission capacity of line $n-m$; K^U_i and K^D_i, respectively, the startup and shutdown costs of unit i; $P^{G\min}_{ti}$ the minimum power output of unit i in period t and $\Delta\hat{P}^G_{ti}$ the parameter used to discretize the uplift constraints for unit i in period t.

The variables include: P^G_{tib}, the power produced by block b of unit i in period t; P^D_{tjk}, the power consumed by block k of demand d in period t; δ_{tn}, the voltage angle of node n in period t; ν_{ti}, a binary variable describing the on/off status of unit i in period t; P_{tnm}, the power flow through line $n-m$ in period t; C^U_{ti} and C^D_{ti}, respectively, the nonnegative startup and shutdown cost of unit i in period t; λ_{tn}, the locational marginal price (LMP) at node n in period t and the dual variable of the power balance equation (8b); τ^{up}_{ti}, the price uplift applied to producer i in period t. The resulting model is shown below and referred to as the "MILP formulation".

$$\min_{P^G_{tib},P^D_{tjk},\delta_{tn},\nu_{ti},C^U_{ti},C^D_{ti}} \quad \sum_{tib}\lambda^G_{tib}P^G_{tib}+\sum_{ti}C^U_{ti}+\sum_{ti}C^D_{ti}-\sum_{tjk}\lambda^D_{tjk}P^D_{tjk} \tag{8a}$$

$$\text{s.t.} \quad \sum_{(i\in\Psi_n)b}P^G_{tib}-\sum_{(j\in\Psi_n)k}P^D_{tjk}=\sum_{m\in\Theta_n}B_{nm}(\delta_{tn}-\delta_{tm}), \quad (\lambda_{tn}) \quad \forall t,\forall n \tag{8b}$$

$$0\le P^G_{tib}\le \nu_{ti}P^{G\max}_{tib}, \quad (\mu^{G\min}_{tib},\mu^{G\max}_{tib}) \quad \forall t,\forall i,\forall b \tag{8c}$$

$$0\le P^D_{tjk}\le P^{D\max}_{tjk}, \quad (\mu^{D\min}_{tjk},\mu^{D\max}_{tjk}) \quad \forall t,\forall j,\forall k \tag{8d}$$

$$\nu_{ti}P^{G\min}_{ti}\le\sum_b P^G_{tib}, \quad (\mu^{\min}_{ti}) \quad \forall t,\forall i \tag{8e}$$

$$-P^{\max}_{nm}\le B_{nm}(\delta_{tn}-\delta_{tm})\le P^{\max}_{nm}, \quad (\nu^{\min}_{tnm},\nu^{\max}_{tnm}) \quad \forall t,\forall n,\forall m\in\Theta_n \tag{8f}$$

$$-\pi\le\delta_{tn}\le\pi, \quad (\xi^{\min}_{tn},\xi^{\max}_{tn}) \quad \forall t,\forall n \tag{8g}$$

$$\delta_{tn}=0, \quad (\xi^1_t) \quad \forall t, n=1 \tag{8h}$$

$$C^U_{ti}\ge(\nu_{ti}-\nu_{(t-1)i})K^U_i, \quad (\eta^U_{ti}) \quad \forall t,\forall i \tag{8i}$$

$$C^U_{ti}\ge 0, \quad (\eta^{U0}_{ti}) \quad \forall t,\forall i \tag{8j}$$

$$C^D_{ti}\ge(\nu_{(t-1)i}-\nu_{ti})K^D_i, \quad (\eta^D_{ti}) \quad \forall t,\forall i \tag{8k}$$

$$C^D_{ti}\ge 0, \quad (\eta^{D0}_{ti}) \quad \forall t,\forall i \tag{8l}$$

$$\nu_{ti}\in\{0,1\},\forall t,\forall i \tag{8m}$$

Problem (8) is the mixed-integer linear formulation of the multi-period market clearing with the target of maximizing the social welfare, as expressed by (8a). A dc linear model of the network is used to represent the power balance at each node as well as the line capacity limits. Eq. (8b) enforces the power balance at every node. Eqs. (8c) and (8d) are power bounds for blocks of both generation and demand, respectively. Constraints (8e) set the minimum power output for each unit. Constraints (8f) enforce the transmission capacity limits of each line. Constraints (8g) fix phase angle bounds for each node. By considering (8g), the phase angle of any two nodes in the network will not differ by more than 2π radians (360°). Constraints (8h) impose $n=1$ to be the reference node. For all $m\in\Theta_n$ identifies the nodes m connected to node n in all periods. Eqs. (8i)–(8l) define the startup and shutdown cost as a function of the on/off status of each generating unit. Note that ν_{ti} is a binary variable as stated in (8m). In order to formulate the MLCP version of (8), Eq. (8m) is relaxed as $0\le\nu_{ti}\le 1$ with its corresponding dual variables being $\beta^{\min}_{ti}$ and $\beta^{\max}_{ti}$, respectively. The rest of the dual variables resulting from this relaxation are indicated in (8) in parentheses next to their corresponding equations. The (8) is as follows:

$$0\le P^G_{tib}\perp\lambda^G_{tib}-\lambda_{tn}+\mu^{G\max}_{tib}-\mu^{\min}_{ti}\ge 0, \quad \forall t,\forall i\in\Psi_n,\forall b \tag{9a}$$

$$0\le P^D_{tjk}\perp -\lambda^D_{tjk}+\lambda_{tn}+\mu^{D\max}_{tjk}\ge 0, \quad \forall t,\forall j\in\Psi_n,\forall k \tag{9b}$$

$$0\le \mu^{G\max}_{tib}\perp \nu_{ti}P^{G\max}_{tib}-P^G_{tib}\ge 0, \quad \forall t,\forall i,\forall b \tag{9c}$$

$$0\le \mu^{D\max}_{tjk}\perp P^{D\max}_{tjk}-P^D_{tjk}\ge 0, \quad \forall t,\forall j,\forall k \tag{9d}$$

$$0\le \mu^{\min}_{ti}\perp \sum_b P^G_{tib}-\nu_{ti}P^{G\min}_{ti}\ge 0, \quad \forall t,\forall i \tag{9e}$$

$$0\le \nu^{\min}_{tnm}\perp B_{nm}(\delta_{tn}-\delta_{tm})+P^{\max}_{nm}\ge 0, \quad \forall t,\forall n,\forall m\in\Theta_n \tag{9f}$$

$$0\le \nu^{\max}_{tnm}\perp P^{\max}_{nm}-B_{nm}(\delta_{tn}-\delta_{tm})\ge 0, \quad \forall t,\forall n,\forall m\in\Theta_n \tag{9g}$$

$$\begin{aligned}0\le \delta_{tn}+\pi\perp &\sum_{m\in\Theta_n}B_{nm}(\lambda_{tn}-\lambda_{tm})+\sum_{m\in\Theta_n}B_{nm}(\nu^{\max}_{tnm}-\nu^{\max}_{tmn})\\ &+\sum_{m\in\Theta_n}B_{nm}(\nu^{\min}_{tmn}-\nu^{\min}_{tnm})+\xi^{\max}_{tn}+(\xi^1_{t(n=1)})\ge 0, \quad \forall t,\forall n\end{aligned} \tag{9h}$$

$$0\le \pi-\delta_{tn}\perp \xi^{\max}_{tn}\ge 0, \quad \forall t,\forall n \tag{9i}$$

$$0\le \eta^U_{ti}\perp C^U_{ti}-(\nu_{ti}-\nu_{(t-1)i})K^U_i\ge 0, \quad \forall t,\forall i \tag{9j}$$

$$0\le 1-\eta^U_{ti}\perp C^U_{ti}\ge 0, \quad \forall t,\forall i \tag{9k}$$

$$0\le \eta^D_{ti}\perp C^D_{ti}-(\nu_{(t-1)i}-\nu_{ti})K^D_i\ge 0, \quad \forall t,\forall i \tag{9l}$$

$$0\le 1-\eta^D_{ti}\perp C^D_{ti}\ge 0, \quad \forall t,\forall i \tag{9m}$$

$$\begin{aligned}0\le \nu_{ti}\perp\ &\mu^{\min}_{ti}P^{G\min}_{ti}-\sum_b\mu^{G\max}_{tib}P^{G\max}_{tib}+K^U_i(\eta^U_{ti}-\eta^U_{(t+1)i})\\ &-K^D_i(\eta^D_{ti}-\eta^D_{(t+1)i})+\beta^{\max}_{ti}\ge 0, \quad \forall t<T,\forall i\end{aligned} \tag{9n}$$

$$\begin{aligned}0\le \nu_{ti}\perp\ &\mu^{\min}_{ti}P^{G\min}_{ti}-\sum_b\mu^{G\max}_{tib}P^{G\max}_{tib}+K^U_i\eta^U_{ti}\\ &-K^D_i\eta^D_{ti}+\beta^{\max}_{ti}\ge 0,\ t=T,\forall i\end{aligned} \tag{9o}$$

$$0\le 1-\nu_{ti}\perp\beta^{\max}_{ti}\ge 0, \quad \forall t,\forall i \tag{9p}$$

$$\sum_{(i\in\Psi_n)b}P^G_{tib}-\sum_{(j\in\Psi_n)k}P^D_{tjk}=\sum_{m\in\Theta_n}B_{nm}(\delta_{tn}-\delta_{tm}), \quad \forall t,\forall n \tag{9q}$$

$$\delta_{tn}=0, \quad \forall t, n=1 \tag{9r}$$

To analyze the tradeoff of integrality vs. complementarity (previously dropped) integrality conditions need to be added to MLCP (9) to obtain a DC-MLCP. Taking into account (5) and including a price uplift to ensure the nonnegativity of profits, renders the following problem:

$$\begin{aligned}\min\quad \omega_1\Bigg\{&\sum_{tib}(\sigma^G_{tib}+\sigma^{G\max}_{tib})+\sum_{tjk}(\sigma^D_{tjk}+\sigma^{D\max}_{tjk})\\ &+\sum_{ti}(\sigma^{\min}_{ti}+\sigma^U_{ti}+\sigma^{U0}_{ti}+\sigma^D_{ti}+\sigma^{D0}_{ti}+\sigma^{\nu\min}_{ti}+\sigma^{\nu\max}_{ti})\\ &+\sum_{tnm}(\sigma^{L\min}_{tnm}+\sigma^{L\max}_{tnm})+\sum_{tn}(\sigma^{\delta\min}_{tn}+\sigma^{\delta\max}_{tn})\Bigg\}\\ &+\omega_2\Bigg\{\sum_{ti}(\varepsilon^{0-}_{ti}+\varepsilon^{0+}_{ti}+\varepsilon^{1+}_{ti}+\varepsilon^{1-}_{ti})\Bigg\}+\alpha\sum_{ti}\tau^{\text{up}}_{ti}\end{aligned} \tag{10a}$$

$$\begin{aligned}\text{s.t. Appropriate constraints of the form of (5b)–(5o)}\\ \text{corresponding to the relaxation of constraints (9)}\end{aligned} \tag{10b}$$

$$\sum_{tb}(\lambda_{tn}+\tau^{\text{up}}_{ti})P^G_{tib}-\sum_{tb}\lambda^G_{tib}P^G_{tib}$$

$$-\sum_t C_{ti}^U - \sum_t C_{ti}^D \geq 0 \quad \forall i, \forall n : i \in \Psi_n \tag{10c}$$

where ω_1, ω_2 and α are positive weighting factors. Note that τ_{ti}^{up} represents the price uplift applied to producer i in period t. The term $\alpha_3 \sum_{ti} \tau_{ti}^{\text{up}}$ is included in the objective function (10a) so that the sum of uplifts is minimized. Constraints (10c) impose the nonnegativity of producer's profits (including the price uplifts τ_{ti}^{up}). Problem (10) is an MINLP since constraints (10c) include products of variables ($\lambda_{tn} P_{tib}^G$ and $\tau_{ti}^{\text{up}} P_{tib}^G$). To avoid the drawbacks associated to MINLP solvers, Eqs. (10c) are replaced by the following mixed-integer, linear set of equations:

$$\sum_b P_{tib}^G - \Delta \hat{P}_{ti}^G \leq \sum_q \hat{P}_{tiq} x_{tiq} \leq \sum_b P_{tib}^G \quad \forall t, \forall i \tag{11a}$$

$$\sum_q x_{tiq} = 1 \quad \forall t, \forall i \tag{11b}$$

$$0 \leq \lambda_{tn} - z_{tiq} \leq G(1 - x_{tiq}) \quad \forall t, \forall i, \forall n : i \in \Psi_n, \forall q \tag{11c}$$

$$0 \leq z_{tiq} \leq G x_{tiq} \quad \forall t, \forall i, \forall q \tag{11d}$$

$$0 \leq \tau_{ti}^{UP} - y_{tiq} \leq G(1 - x_{tiq}) \quad \forall t, \forall i, \forall q \tag{11e}$$

$$0 \leq y_{tiq} \leq G x_{tiq} \quad \forall t, \forall i, \forall q \tag{11f}$$

$$\sum_{tq} (z_{tiq} + y_{tiq}) \hat{P}_{tiq}^G - \sum_{tb} \lambda_{tib}^G P_{tib}^G \geq 0 \quad \forall i \tag{11g}$$

$$x_{tiq} \in \{0,1\} \quad \forall t, \forall i, \forall q \tag{11h}$$

where G is a suitably large, positive constant. The aim of (11) is to substitute the production $\sum_b P_{tib}^G$ in (10c) by a discrete value $\sum_q \hat{P}_{tiq}^G x_{tiq}$ that is as close as possible to the continuous one, i.e., $\sum_b P_{tib}^G \cong \sum_q \hat{P}_{tiq}^G x_{tiq}$ (note that index q refers to the discretized generation blocks running from 1 to Q). This is achieved by equation (11a) which enforces that the distance between the actual production value $\sum_b P_{tib}^G$ and the discrete one $\sum_q \hat{P}_{tiq}^G x_{tiq}$ does not exceed $\Delta \hat{P}_{ti}^G$, which is considered constant for each producer i and period t, and is defined as $\Delta \hat{P}_{ti}^G = \hat{P}_{ti(q+1)}^G - \hat{P}_{tiq}^G$. Eq. (11b) ensures that the actual production of producer i at period t is approximated by only one discrete value. Eqs. (11c)–(11f) linearize the products $\lambda_{tn} x_{tiq}$ and $\tau_{tiq}^{UP} x_{tiq}$ that results in (10c) from replacing $\sum_b P_{tib}^G$ by $\sum_q \hat{P}_{tiq}^G x_{tiq}$, where z_{tiq} and y_{tiq} are two auxiliary continuous variables. The approximate linear version of the nonnegative profit constraints is finally formulated in (11g). Numerical results in Section 5 using the linearized uplift constraints (11) show that this approach is viable.

4. Energy network equilibrium with multiple players

The next example is from [14] and depicts an equilibrium in an energy network (e.g., natural gas, electricity) where production, consumption, and transmission of the energy product is analyzed. The model is a simplified form of the natural gas market equilibrium problems described in [11,12] but also applicable to power markets. For ease of presentation, only two nodes are considered but larger realistic examples would also have the same structure. There are four energy price-taking producers denoted as: A, B, C, D with the first two located at node 1 and the latter two at node 2. The production levels are denoted as q_n^p where node $n \in \{1,2\}$ and producer $p \in \{A,B,C,D\}$. Similarly, the sales levels are denoted as s_n^p. Lastly, at node 1, the two producers A and B have the additional option of sending energy to node 2 and f_{12}^A, f_{12}^B represents the associated amounts of flow. (Note that the producers at node 2 are not allowed to ship their product to node 1.)

Both producers A and B at node 1 have structurally a similar optimization problem shown below just for producer A. For node 2, the producers have an optimization that is almost the same as at node 1 with the exception that no flow variables (nor related terms) are included.

$$\max_{s_1^A, q_1^A, f_{12}^A} \quad \pi_1 s_1^A + \pi_2 f_{12}^A - c_1^A(q_1^A) - (\tau_{12}^{\text{Reg}} + \tau_{12}) f_{12}^A \tag{12a}$$

$$\text{s.t.} \quad q_1^A \leq \overline{q}_1^A \quad (\lambda_1^A) \tag{12b}$$

$$s_1^A = q_1^A - f_{12}^A \quad (\delta_1^A) \tag{12c}$$

$$s_1^A, q_1^A, f_{12}^A \geq 0 \tag{12d}$$

where π_n is the producer price at node $n \in \{1,2\}$, $c_1^A(q_1^A)$ is the (marginal) production cost function assumed to be linear, i.e., $c_1^A(q_1^A) = \gamma_1^A q_1^A, \gamma_1^A > 0, \tau_{12}^{\text{Reg}}$ represents the nonnegative, regulated tariff for using the network from node 1 to node 2; τ_{12}^{Reg} is a fixed parameter, τ_{12} is the congestion tariff for using the network from node 1 to node 2 and a variable from another part of the equilibrium model,[1] $\overline{q}_1^A$ is the maximum production quantity.[2]

Each producer is maximizing their profit (12a) by choosing appropriate nonnegative levels of production, sales, and flow variables subject to not exceeding production limits (12b), and consistency between sales, production, and flow (12c). The KKT conditions for each of the producers' problems are both necessary and sufficient [2] given the functions chosen and these conditions for each of the producers is as follows:

Producer $X = A,B$, node 1

$$0 \leq -\pi_1 + \delta_1^X \perp s_1^X \geq 0 \tag{13a}$$

$$0 \leq \gamma_1^X + \lambda_1^X - \delta_1^X \perp q_1^X \geq 0 \tag{13b}$$

$$0 \leq -\pi_2 + (\tau_{12}^{\text{Reg}} + \tau_{12}) + \delta_1^X \perp f_{12}^X \geq 0 \tag{13c}$$

$$0 \leq \overline{q}_1^X - q_1^X \perp \lambda_1^X \geq 0 \tag{13d}$$

$$0 = s_1^X - q_1^X + f_{12}^X, \quad \delta_1^X \text{ free} \tag{13e}$$

Producer $Y = C,D$ node 2

$$0 \leq -\pi_2 + \delta_2^Y \perp s_2^Y \geq 0 \tag{14a}$$

$$0 \leq \gamma_2^Y + \lambda_2^Y - \delta_2^Y \perp q_2^Y \geq 0 \tag{14b}$$

$$0 \leq \overline{q}_2^Y - q_2^Y \perp \lambda_2^Y \geq 0 \tag{14c}$$

$$0 = s_2^Y - q_2^Y, \quad \delta_2^Y \text{ free} \tag{14d}$$

In addition to the KKT conditions for the four producer problems, there are market-clearing conditions that force supply to equal demand:

$$0 = [s_1^A + s_1^B] - D_1(\pi_1) \qquad \pi_1 \text{ free} \tag{15a}$$

[1] Note that the parameter τ_{12}^{Reg} and variable τ_{12} represent tariffs or extra payments/costs similar to the τ used in the uplift computations shown earlier. However, τ_{12}^{Reg} and τ_{12} are otherwise not related to the previous τ.

[2] All maximum values for primal variables denoted by an overbar are assumed to be positive as are cost coefficients $\gamma_1^A, \gamma_1^B, \gamma_2^C, \gamma_2^D$.

$$0=[s_2^C+s_2^D+f_{12}^A+f_{12}^B]-D_2\,(\pi_2) \quad \pi_2 \text{ free} \tag{15b}$$

Note that the terms in square brackets are the net supply at each node (assuming no losses) and $D_n(\pi_n), n=1,2$ are the nodal demands as a function of the price π_n. While the producers depicted above operate using the network, there is additionally a transportation system operator (TSO) who manages the congestion and flows. The TSO's linear program is as follows (where other objectives are also possible):

$$\max_{g_{12},g_{21}} \quad (\tau_{12}^{\text{Reg}}+\tau_{12})g_{12}-c^{TSO}(g_{12}) \tag{16a}$$

$$\text{s.t.} \quad g_{12} \le \overline{g}_{12} \quad (\varepsilon_{12}) \tag{16b}$$

$$g_{12} \ge 0 \tag{16c}$$

Here, the TSO controls the variable g_{12} which is the flow from node 1 to node 2, $c^{TSO}(g_{12})$ is a network operations cost function (assumed linear, i.e., $c^{TSO}(g_{12})=\gamma^{TSO}g_{12}$ where $\gamma^{TSO}>0$) and $\overline{g}_{12}$ is the capacity of the link between nodes 1 and 2. The KKT conditions for this problem are both necessary and sufficient and since it is a linear program and these conditions are the following:

$$0 \le -\tau_{12}^{\text{Reg}}-\tau_{12}+\gamma^{TSO}+\varepsilon_{12} \perp g_{12} \ge 0 \tag{17a}$$

$$0 \le \overline{g}_{12}-g_{12} \perp \varepsilon_{12} \ge 0 \tag{17b}$$

The last part of the equilibrium are the market-clearing conditions that balance the flow controlled by the network operator and thus by Producers A and B:

$$0=g_{12}-[f_{12}^A+f_{12}^B] \quad \tau_{12} \text{ free} \tag{18}$$

The LCP for this energy network problem is thus the KKT conditions of the producers: (13), (14), the nodal market-clearing (15), the KKT conditions of the TSO (17) and the market-clearing conditions of the transportation market (18).

Theorem 3. *Assume that the demand functions $D_n(\pi_n)$ satisfy the following condition*:

$[|D_n(\pi_n)|$ *is bounded* $\Rightarrow |\pi_n|$*is bounded*$], \quad n=1,2$

Then, there exists a constant M^* for the network equilibrium problem that satisfies Assumption 1.

Proof. In Appendix.

Remark. Depending on the form of the demand function, the condition that $D_n(\pi_n)$ bounded $\Rightarrow \pi_n$ is bounded in the previous theorem can be verified. Note that this condition is the contrapositive, hence equivalent form of coercivity of $D_n(\pi_n)$, i.e., $\|\pi_n\| \to \infty \Rightarrow D_n(\pi_n) \to \infty$.

Remark. More generally, if $D_n(\pi_n)$ is invertible, this boundedness condition is not necessarily satisfied. To see this take for example $D_n(\pi_n)=\max\{1,e^{-\pi_n}\}$ which is bounded below by 0 and above by 1 for $\pi \ge 0$. However,the boundedness premise in the previous theorem does not hold as the sequence $\{\pi_n^k\} \to \infty$ as $k \to \infty$ will not violate the bounds on the demand function.

In the case of linear demand, then a specific value for M_1 can be determined and is shown in the next result. This is directly useful for the relaxation of complementarity discussed earlier.

Theorem 4. *Suppose that $D_n(\pi_n)=a_n-b_n\pi_n$ for $a_n,b_n>0$. Then, the following bounds are valid with*

1. $0 \le \{s_1^A,q_1^A,f_{12}^A,s_1^B,q_1^B,f_{12}^B,s_2^C,q_2^C,s_2^D,q_2^D\} \le q^{\max}$
2. $0 \le f_{12}^A,f_{12}^B \le \overline{g}_{12}$
3. $(a_n-3\kappa_1)/b_n \le \pi_n \le a_n/b_n,\ n=1,2$
4. $0 \le \{\lambda_1^A,\lambda_1^B,\lambda_2^C,\lambda_2^D\} \le (a_1/b_1+a_2/b_2)$
5. $(a_1-3\kappa_1)/b_1 \le \delta_1^A \le \gamma_1^A+(a_1/b_1+a_2/b_2)$
6. $(a_1-3\kappa_1)/b_1 \le \delta_1^B \le \gamma_1^B+(a_1/b_1+a_2/b_2)$
7. $(a_2-3\kappa_1)/b_2 \le \delta_2^C \le \gamma_2^C+(a_1/b_1+a_2/b_2)$
8. $(a_2-3\kappa_1)/b_2 \le \delta_2^D \le \gamma_2^D+(a_1/b_1+a_2/b_2)$
9. $-3\kappa 1/b_2-\gamma_1^A-a_1/b_1-\tau_{12}^{\text{Reg}} \le \tau_{12} \le \max\{\gamma^{TSO}-\tau_{12}^{\text{Reg}},$ $a_2/b_2-\tau_{12}^{\text{Reg}}-(a_1-3\kappa_1)/b_1\}$
10. $0 \le \varepsilon_{12} \le \tau_{12}^{\text{Reg}}-\gamma^{TSO}+\max\{\gamma^{TSO}-\tau_{12}^{\text{Reg}},$ $a_2/b_2-\tau_{12}^{\text{Reg}}-(a_1-3\kappa_1)/b_1\}$

where $\kappa_1 := \max\{\overline{q}_1^A,\overline{q}_1^B,\overline{q}_2^C,\overline{q}_2^D,\overline{g}_{12}\}$. Thus, a valid bound M^* for all the variables for the network equilibrium problem is greater than or equal to the maximum of all the right-hand sides in 1–10 with associated M_1 that does not cut off any solutions given by $M_1 \ge \max\{M^*,\max_i\{(q_1)_i\}+\max_j\{\|A_{11}^j\|_1+\|A_{12}^j\|_1\}M^*\}$ where A_{11}^j, A_{12}^j are defined just after Assumption 1.

Proof. In Appendix.

5. Numerical experiments

In this section, numerical results for the electric power market model and an energy network equilibrium are presented. These results show that the proposed approach is viable for equilibrium problems with integer constraints. All results were obtained by using GAMS for solving the appropriate models.

5.1. Numerical results: multi-period network-constrained electric power market model

In this numerical example we apply the market-clearing procedure presented in (8) and its DC-MLCP version derived in (10) to a multi-period network-constrained electric power market. The test system considered is a power market made up of eight producers and six demands that are distributed in a six-node power network. The purpose of this example is to verify that the market-clearing outcomes (prices, uplifts, production, on/off status of the units and profits) are adequate and reasonable when the DC-MLCP approach derived in this paper is applied.

5.1.1. Data

The network topology used for this case study is depicted in Fig. 1. There are two separated areas interconnected by two tie-lines. Note that most of the generating units (and thus most of the capacity) are located on the left-hand side of the network whereas most of the demands are located on the right-hand side. All the lines are considered to have the same susceptance equal to $B_{nm}=100$ p.u.

Table 1 provides data for the units considered in this example. The second column contains the marginal cost of each unit. Note that a two-period study horizon is considered with a single block marginal offer. Moreover, we assume that the marginal cost does not vary over the time horizon. The third and fourth columns indicate, respectively, the lower and upper bounds for the power production. Columns five and six show the startup and shutdown costs of each generating unit. The last column displays the on/off status of each unit at time $t=0$.

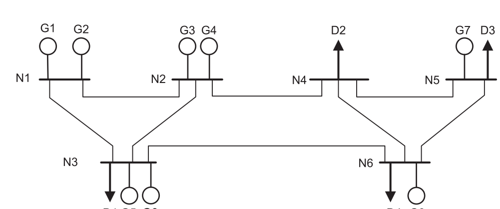

Fig. 1. Six-bus test system.

Table 1
Data for the generating units.

Generator	λ^G_{ti1} ($/MWh)	$P^{G\min}_{ti1}$ (MWh)	$P^{G\max}_{ti1}$ (MWh)	K^U_{ti1} ($)	K^D_{ti1}($)	v_{0i}
G_1	24	25	50	100	500	0
G_2	22	25	50	140	350	0
G_3	20	25	50	180	300	1
G_4	18	25	50	220	250	1
G_5	16	25	50	250	220	1
G_6	14	25	50	300	180	1
G_7	12	25	50	350	140	0
G_8	10	25	50	500	100	0

Table 2
Demand blocks of each period of time.

Marginal Cost of Gen. Unit	λ^D_{1j1} ($/MWh)	λ^D_{2j1} ($/MWh)	$P^{D\max}_{1j1}$ (MWh)	$P^{D\max}_{2j1}$ (MWh)
D_1	25	20	100	50
D_2	26	20	100	50
D_3	26	21	100	50
D_4	27	21	100	50

Table 2 provides demand bids (energy and price) for each period of time and for each demand. The second and third columns give the marginal utility for the time periods $t=1$ and $t=2$, respectively. The last two columns correspond to the upper limit of each block for time periods $t=1$ and $t=2$. Note that we assume that each demand offer is characterized by a single block. Observe also that the demand is reduced from $t=1$ to $t=2$.

Other relevant parameters considered in this case study are $M_1=M_2=G=1000$, $\omega_1=\omega_2=10{,}000$, $\alpha=0.0001$ and $Q=16$.

5.1.2. *Results*

Two case studies are analyzed, the first one does not impose any limits on any line and the second one forces congestion on the two tie-lines.

The results are given in Table 3 and 4. These tables provide results for the locational marginal prices for each node (rows 3–8), on/off status of the units (rows 9–16), profits of the producers (rows 17–24), total profit of the producers (row 25) and social welfare (SW) of the market (row 26). These market outcomes are computed for four different cases: MILP (columns 2 and 3) corresponds to the formulation provided in (8), MLCP (columns 4 and 5) is the complementarity formulation provided in (9) where the binary variables (v_{ti}) have been relaxed, Relaxed both (columns 6 and 7) corresponds to the model derived in (10) without including (10c) and R.B. + $\pi_i \geq 0$ (columns 8 and 9) includes (10) and the mixed-integer, linear nonnegative profits conditions stated in (11).

It should be noted that the prices provided by the MILP model are obtained as dual variables of the balance equation (8b). This is achieved by first solving the MILP (8) and then fixing the binary variables to their optimal values. The resulting problem is linear and therefore its sensitivities can be easily computed.

Results for the non-congested case are reported in Table 3. Note that when there is no congestion the locational marginal prices are identical throughout the network. Prices decrease from $t=1$ to $t=2$ since the demand bid prices also decrease. Prices and profits are similar among the models and slightly higher for the case where both integrality and complementarity are relaxed. Units 1 and 2 are off at all time periods since they are the most expensive ones.

All the uplifts (τ^{UP}_{ti}) obtained for the R.B. + $\pi_i \geq 0$ case are equal to 0. Note also that although all profits are nonnegative in the Relaxed both case, imposing (11) alters the markets outcomes by shutting down units 3 and 4 from $t=1$ to $t=2$. This effect is caused by the slight difference between the actual production and its discretization, i.e., $\Delta\hat{P}^G_{ti}$. Finally, note that the complementarity gap is slightly higher than zero.

The results for the congested case are presented in Table 4. Model MLCP does not provide adequate results since the on/off

Table 3
Case study #1 (no congestion).

Lower Bound on Power Prod.	MILP		MLCP		Relaxed both		R.B. + $\pi_i \geq 0$	
	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$
λ_{t1} ($/MWh)	26	14	25	12.2	27	14	26	16
λ_{t2} ($/MWh)	26	14	25	12.2	27	14	26	16
λ_{t3} ($/MWh)	26	14	25	12.2	27	14	26	16
λ_{t4} ($/MWh)	26	14	25	12.2	27	14	26	16
λ_{t5} ($/MWh)	26	14	25	12.2	27	14	26	16
λ_{t6} ($/MWh)	26	14	25	12.2	27	14	26	16
v_{t1}	0	0	0	0	0	0	0	0
v_{t2}	0	0	0	0	0	0	0	0
v_{t3}	1	1	1	1	1	1	1	0
v_{t4}	1	1	1	1	1	1	1	0
v_{t5}	1	1	1	1	1	1	1	1
v_{t6}	1	1	1	1	1	1	1	1
v_{t7}	1	1	1	1	1	1	1	1
v_{t8}	1	1	1	1	1	1	1	1
π_1 ($)	0		0		0		0	
π_2 ($)	0		0		0		0	
π_3 ($)	150		55.8		200		0	
π_4 ($)	300		205.8		350		150	
π_5 ($)	450		355.8		500		500	
π_6 ($)	600		505.8		650		700	
π_7 ($)	450		311.7		500		550	
π_8 ($)	500		361.7		550		600	
π_T ($)	2450		1797		2750		2500	
SW	3850		3850		3850		3500	
GAP compl	–		–		0		0.0015	
GAP integr	–		–		0		0	

Table 4
Case study #1 (lines 2–4 and 3–6 limited to 20 MW).

Upper Bound on Power Prod.	MILP		MLCP		Relaxed both		R.B. + $\pi_i \geq 0$	
	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$
λ_{t1} ($/MWh)	18	12.8	18	11.4	18	11.5	18	20
λ_{t2} ($/MWh)	18	11.6	18	10	18	10	18	20
λ_{t3} ($/MWh)	18	14	18	12.8	18	13	18	20
λ_{t4} ($/MWh)	26	20	26	20	26	20.5	26	20
λ_{t5} ($/MWh)	26	18.8	26	18.5	26	19	26	20
λ_{t6} ($/MWh)	26	17.6	26	17.1	26	17.5	26	20
v_{t1}	0	0	0	0	0	0	0	0
v_{t2}	0	0	0	0	0	0	0	0
v_{t3}	0	0	0.1	0.1	0	0	0	1
v_{t4}	1	1	1	1	1	1	1	0
v_{t5}	1	1	1	1	1	1	1	1
v_{t6}	1	1	1	1	1	1	1	1
v_{t7}	1	1	1	1	1	1	1	1
v_{t8}	1	1	1	1	1	1	1	1
π_1 ($)	0		0		0		0	
π_2 ($)	0		0		0		0	
π_3 ($)	−300		−300		−300		0	
π_4 ($)	−160		−200		−200		0	
π_5 ($)	50		21.4		25		300	
π_6 ($)	200		171.4		170		20	
π_7 ($)	690		678.5		700		750	
π_8 ($)	680		657.1		675		800	
π_T ($)	1160		1028.4		1070		1870	
SW	3100		3105.7		3090		2360	
GAP compl	–		–		0		0.0034	
GAP integr	–		–		0		0	

Table 5
Dataset used.

Parameter	τ_{12}^{Reg}	γ_1^A	γ_1^B	γ_2^C	γ_2^D	a_1	b_1	a_2	b_2	$\overline{q}_1^A$	$\overline{q}_1^B$	$\overline{q}_2^C$	$\overline{q}_2^D$	$\overline{g_{12}}$	γ^{TSO}
Value	0.5	10	12	15	18	20	1	40	2	10	10	4.5	5	15	1

Table 6
Description of problem variations.

Variation	σ-Complementarity?	ϵ-Integrality?	Description
1	No	No	MLCP
2	No	No	Integer variables
3	Yes	No	Integer variables
4	Yes	No	Continuous variables
5	No	Yes	Continuous variables
6	Yes	Yes	Continuous variables

Table 7
Description of results.

Variations	1	2	3	4	5	6
s_1^A	7.440	Infeasible	8.000	8.000	8.000	8.000
s_1^B	0.560	Infeasible	0	0	0	0
s_2^C	4.500	Infeasible	4.000	4.500	4.500	4.000
s_2^D	0	Infeasible	0	0	0	0
q_1^A	10.000	Infeasible	10.000	10.000	10.000	10.000
q_1^B	3.000	Infeasible	3.000	3.000	3.000	3.000
q_2^C	4.500	Infeasible	4.000	4.500	4.500	4.000
q_2^D	0	Infeasible	0	0	0	0
f_{12}^A	2.560	Infeasible	2.000	2.000	2.000	2.000
f_{12}^B	2.440	Infeasible	3.000	3.000	3.000	3.000
λ_1^A	2.000	Infeasible	2.000	2.000	2.000	2.000
λ_1^B	0	Infeasible	0	0	0	0
λ_2^C	0.250	Infeasible	0.500	0.250	0.250	0.500
λ_2^D	0	Infeasible	0	0	0	0
g_{12}	5.000	Infeasible	5.000	5.000	5.000	5.000
ϵ_{12}	2.250	Infeasible	2.500	2.250	2.250	2.500
δ_1^A	12.000	Infeasible	12.000	12.000	12.000	12.000
δ_2^B	12.000	Infeasible	12.000	12.000	12.000	12.000
δ_2^C	15.250	Infeasible	15.500	15.250	15.250	15.500
δ_2^D	17.581	Infeasible	18.000	18.000	18.000	18.000
π_1	12.000	Infeasible	12.000	12.000	12.000	12.000
π_2	15.250	Infeasible	15.500	15.250	15.250	15.500
τ_{12}	2.750	Infeasible	3.000	2.750	2.750	3.000
Sum σ	n/a	n/a	0.005	0	n/a	0.005
Sum ϵ	n/a	n/a	n/a	n/a	1.000	0

status of unit 3 is fixed to a non-integer value (0.1). However, the Relaxing both complementarity and integrality model overcomes this problem and provides feasible results (integer on/off status of the units). Note that the market outcomes are very similar when comparing models MILP and Relaxing both complementarity and integrability.

Due to the congestion, prices are higher in the right-hand-side of the network, that is where the consumption prevails. Prices are again higher at $t=1$ because of the higher demand's marginal cost.

Units 3 and 4 incur looses in the MILP and Relaxed both cases. Unit 3 is initially on and it is forced to shut down because of its high production cost. Unit 4 is also expensive although is kept working due to its higher shutdown cost.

All profits are nonnegative for the R.B. + $\pi_i \geq 0$ model. For this case the price uplifts are equal to 0 except for $\tau_{14}^{up}=6.25$ and $\tau_{23}^{up}=15.45$ that are used to compensate the startup and shutdown cost of units 3 and 4, respectively. Note that the social welfare decreases if compared with the MILP and Relaxed both cases.

As in the non-congested case, the complementarity gap is different from zero.

We can conclude that the DC-MLCP approach described in this paper behaves properly and provides an adequate framework to incorporate nonnegative profit constraints to obtain prices that support a multi-period network-constrained electric power market. This is shown for congested and non-congested network examples.

5.2. *Numerical results: energy network equilibrium with multiple players*

In this problem, $s_1^A,s_1^B,s_2^C,s_2^D,q_1^A,q_1^B,q_2^C,q_2^D$ are the variables that are integer-constrained in variations 2 and 3. The goal is to find a solution which has these variables as integers. Note that there are multiple integer solutions. In what is described below, the values of M_1, and M_2 were equal to 100. Given the comment about multiple solutions, different values of M_1, and M_2 may lead to different solutions. The values for the input parameters as well as the six variations that were tested are shown in Tables 5 and 6.

Several numerical variations were done to see the change in solutions. Variation 1 was a mixed-complementary problem (MCP) without imposing integer restrictions. Variation 2 involved converting the MCP to a formulation with disjunctive constraints but restricting the variables of production and sales to be integer. The rest of the variations then go through the different combinations. First, variations 3 and 6 give an integer solution. However, due to the presence of multiple equilibria, these solutions need not be unique. Multiple starting points were chosen, and, according to the numerical tests, the reported solution had the highest objective function value (along with some other equilibria not reported) when a feasible integer solution was desired. Hence, variations 3 and 6 can be used to obtain optimal, integer solutions that are feasible. Note that variation 6 targets integers through ε-complementarity, while variation 3 actually restricts solutions to integer values. Variation 1 yielded a non-integer but optimal and feasible solution while variation 2 was infeasible. Again, this shows that σ-complementarity is essential to obtain a feasible integer solution (as in variations 3 and 6). However, only σ-complementarity is not enough to obtain integer solutions (variation 4) nor is only ε-complementarity (variation 5). The extra advantage of using variations 3 and 6 is that values of dual variables can be obtained and interpreted. It is interesting to note that the dual variables change from the continuous to the integer case, which is what was expected. However, it also shows the differences in solutions with relaxation of integer variables to solve a problem and how it leads to solutions that can be very different from the market dynamics of an integer-constrained problem (Table 7).

Remark. Using Theorem 4 (since linear demand functions are used), the values of M^* and M_1 can be figured out for the network example. In this case, M^* is calculated to be 58 while M_1 is 192. Note that these upper bounds are not tight. Sensitivity analysis was performed on the problem, for which it was shown that $M_1=27$ is actually the numerically evident minimum value for which no solutions are missed and the formulation is still valid.

To allow easy interpretation of the results, a value of 100 was used for M_1 and M_2. Both M_1 and M_2 were tested for values from 27 to 10^6 and gave the same solution (profit function optimal value).

6. Conclusions and extensions

This paper proposes a methodology to analyze the way integrality conditions affect complementarity problems, in particular, MLCPs. From the theoretical analysis carried out and the examples considered, the following conclusions can be drawn. First, relaxing both integrality and complementarity allows building meaningful feasible problems. Second, the relaxed problem formulated in (5) allows analyzing the tradeoff between complementarity and integrality. This is done by actually computing the cost of integrality in terms of complementarity and, conversely, the cost of complementarity in terms of integrality. Third, the analysis involves the study of enforcing integrality/complementarity on the optimal values of both primal and dual variables. Dual variables are particularly relevant as they often represent prices. Fourth, a set of relevant case examples is used to illustrate the interest of the technique proposed and its practical relevance.

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Appendix

Proof of Theorem 1. First note that ε_{1ri} and ε_{2ri} can be positive, negative, or zero as they are defined as the difference of two nonnegative terms. When $w_{1ri}=1$ or $w_{2ri}=1$, the bounds M_2 do not come into play so it suffices to only consider the cases when $w_{1ri}=0$ or $w_{2ri}=0$ for a given integer value i. When $w_{1ri}=0$, if M_2 is chosen greater than or equal to N, then $\varepsilon_{1ri}=0$ is optimal since it is feasible, i.e., from (5f), $-M_2 \le -N \le (z_1)_r - i = (z_1)_r - i - \varepsilon_{1ri} = (z_1)_r - i \le N \le M_2$ and any other feasible choice of $\varepsilon_{1ri} \neq 0$ will have a worse objective function value in light of terms of the form $(\varepsilon_{1ri})^+ + (\varepsilon_{1ri})^-$. Thus, $M_2 \ge N$ is valid. When $w_{2ri}=0$, (5g) is just $-M_2 \le (z_2)_r - i - \varepsilon_{2ri} \le M_2$. Letting $M_2 \ge \max\{N_1+N_2, N_1-1\} = N_1+N_2$, shows that $\varepsilon_{2ri}=0$ is optimal since it is feasible, i.e.,

$$-M_2 \le \begin{cases} -N_1-N_2 & \text{for } i=0,\ldots,N_2 \\ -N_1+1 & \text{for } i=-N_1,\ldots,-1 \end{cases} \le (z_2)_r - i = (z_2)_r - i - \varepsilon_{2ri}$$

and

$$(z_2)_r - i - \varepsilon_{2ri} = (z_2)_r - i \le \begin{cases} N_2 & \text{for } i=0,\ldots,N_2 \\ N_1+N_2 & \text{for } i=-N_1,\ldots,-1 \end{cases} \le M_2$$

and any other feasible choice of $\varepsilon_{2ri} \neq 0$ will have a worse objective function value in light of terms of the form $(\varepsilon_{2ri})^+ + (\varepsilon_{2ri})^-$. So a valid bound is to take $M_2 \ge \max\{N_1+N_2\}$. Thus, collecting all the conditions on M_2 gives the desired result. □

Proof of Theorem 2. Problem (5) is a linear, hence quadratic program whose objective function is bounded below by zero since it is a positively weighted sum of nonnegative variables. If it can be shown that the feasible region is a nonempty polyhedron, then by the Frank–Wolfe Theorem [10] the result will be proven. First note that if the binary variables u, w_{1ri}, w_{2ri} are fixed, then the resulting feasible region is a polyhedron given the linear equations and inequalities. We first consider one specific set of values of these binary variables, namely $u_j=1$ for all j and $w_{1ri}, w_{2ri}=1$ if $i=0$ and equal to 0 otherwise satisfying (5e), (5k) and (5l). By Assumption 1, the left-most inequalities of constraints (5b) and (5c) as well as constraint (5d) are satisfied. Also, apart from the nonnegativity constraints, since $(\varepsilon_{1ri})^+, (\varepsilon_{1ri})^-$ only appear in (5h) and $(\varepsilon_{2ri})^+, (\varepsilon_{2ri})^-$ only in (5i), these two constraints plus the nonnegativity restrictions (5m)–(5o) will always be feasible. By choice of u and selection of M_1 (e.g., using (6)), the right-hand inequality of (5b) is met. The right-most inequality of (5c) is satisfied given that σ is unconstrained and M_1 is sufficiently large given the second part of Assumption 1. Thus, (5b)–(5e) are feasible. Now by virtue of the choice of values for w_{1ri}, w_{2ri}, (5j) is satisfied. With this choice for w_{1ri}, (5f) reduces to

$$0 \le (z_1)_r - \varepsilon_{1r0} \le 0, \quad \forall r \in I_1^D \tag{19a}$$

$$-M_2 \le (z_1)_r - i - \varepsilon_{1ri} \le M_2, \quad \forall i=1,\ldots,N, \ r \in I_1^D \tag{19b}$$

Eq. (19a) means that $(z_1)_r = \varepsilon_{1r0}$ which is valid as $(z_1)_r \ge 0$ and ε_{1r0} is a free variable. Eq. (19b) is feasible given the choice of M_2 and Theorem 1. As for (5g), we have

$$0 \le (z_2)_r - \varepsilon_{2r0} \le 0, \quad \forall r \in I_2^D \tag{20a}$$

$$-M_2 \le (z_2)_r - i - \varepsilon_{2ri} \le M_2, \quad \forall i=-N_1,\ldots,-1,1,\ldots,N_2, \ \forall r \in I_2^D \tag{20b}$$

But (20a) is valid as it just states that $(z_2)_r = \varepsilon_{2r0}$ and both variables are free. Eq. (20b) holds by virtue of Theorem 1 and the choice of M_2. In summary, the feasible region to (5) for this particular choice of binary values has been shown to be a nonempty polyhedron so that the Frank–Wolfe theorem applies. For the remaining finite number of possible choices for the binary variables, either fixing them leads to an infeasible set of constraints or the Frank–Wolfe theorem applies. For all those finite number of choices that lead to a feasible region (including the one just shown), we can take the minimum objective function value and the desired result is shown. □

Proof of Theorem 3. First, note that all "primal" variables

$$\{s_1^A, q_1^A, f_{12}^A, s_1^B, q_1^B, f_{12}^B, s_2^C, q_2^C, s_2^D, q_2^D\}$$

are bounded given that they are nonnegative, the (positive) maximum production levels $\overline{q}_1^A$, $\overline{q}_1^B$, $\overline{q}_2^C$, $\overline{q}_2^D$, the maximum flow amount $\overline{g}_{12}$, the market-clearing conditions (18), and the constraints linking sales, production, and flow. In particular, by the constraints in producer A's problem: $s_1^A + f_{12}^A = q_1^A \le \overline{q}_1^A$ given that s_1^A, f_{12}^A are nonnegative and likewise for the other producers, $s_1^B + f_{12}^B = q_1^B \le \overline{q}_1^B, s_2^C = q_2^C \le \overline{q}_2^C, s_2^D = q_2^D \le \overline{q}_2^D$. So,

$$\{s_1^A, q_1^A, f_{12}^A, s_1^B, q_1^B, f_{12}^B, s_2^C, q_2^C, s_2^D, q_2^D\} \le q^{\max} := \max\{\overline{q}_1^A, \overline{q}_1^B, \overline{q}_2^C, \overline{q}_2^D\}$$

From the TSO's (16) and related market-clearing conditions (18), the following conditions hold:

$$0 \le f_{12}^A + f_{12}^B = g_{12} \le \overline{g}_{12} \Rightarrow \{f_{12}^A, f_{12}^B\} \le \overline{g}_{12}$$

Letting $\kappa_1 : \max\{q^{\max}, \overline{g}_{12}\}$, we see that this value is a valid upper bound on all the primal variables with zero as a lower bound. As for the other variables, first consider (15) as well as the above reasoning to get

$$0 \le D_1(\pi_1) = s_1^A + s_1^B \le 2\kappa_1 \tag{21a}$$

$$0 \le D_2(\pi_2) = s_2^C + s_2^D + f_{12}^A + f_{12}^B \le 3\kappa_1 \tag{21b}$$

so that the demand functions are bounded (the lower bound of zero is from the fact that the sales and flow variables are nonnegative). By the premise, this means that $\pi_n, n=1,2$ are also bounded. Combining (13a) and (13b) for Producer A shows that $\pi_1 \le \delta_1^A \le \gamma_1^A + \lambda_1^A$ so if λ_1^A can be shown to be bounded, then δ_1^A will also be bounded. But Producer A's problem is a linear program given the choice of a linear cost function. A feasible solution is to take all primal variables equal to zero. The feasible region is compact so by the Weierstrass theorem [2], an optimal solution exists to this problem and its dual by strong duality [2]. The dual of Producer A's problem is

$$\begin{aligned}\min_{\lambda_1^A,\delta_1^A} \quad & \overline{q}_1^A\lambda_1^A \\ \text{s.t. } & \delta_1^A \geq \pi_1 \\ & \lambda_1^A - \delta_1^A \geq -\gamma_1^A \\ & \delta_1^A \geq \pi_2 - \tau_{12}^{\text{Reg}} - \tau_{12} \\ & \lambda_1^A \geq 0,\ \delta_1^A \text{ free}\end{aligned}$$

By strong duality for linear programming [2], using the optimal values for $\lambda_1^A, s_1^A, f_{12}^A, q_1^A$:

$$\lambda_1^A = \frac{\pi_1 s_1^A + \pi_2 f_{12}^A - c_1^A(q_1^A) - (\tau_{12}^{\text{Reg}} + \tau_{12})f_{12}^A}{\overline{q}_1^A} \tag{22}$$

where given the previous arguments, all the terms on the right-hand side are bounded except possibly $-\tau_{12}f_{12}^A$. If $f_{12}^A = 0$ then we are done. Assume for sake of contradiction that $f_{12}^A > 0$ and that there is a sequence of values $\{\tau_{12}^k\} \to \infty$ as $k \to \infty$. Since all the other terms on the right-hand side of (22) are bounded, after some large enough value of k, this would mean that $\lambda_1^A < 0$ which is a contradiction to being an optimal solution to the dual due to the nonnegativity constraints. Since τ_{12} is a free variable, also consider a sequence of values $\{\tau_{12}^k\} \to -\infty$ as $k \to \infty$. But since $f_{12}^A > 0 \Rightarrow g_{12} > 0$ by (18)$\Rightarrow \tau_{12}^{\text{Reg}} + \tau_{12} = \gamma^{TSO} + \varepsilon_{12}$ by (17a). If $\{\tau_{12}^k\} \to -\infty$ as $k \to \infty$ then for some large enough value of k, the left-hand side of this equation would be negative contradicting the nonnegativity of the right-hand side. Consequently, τ_{12} and hence λ_1^A are also bounded which implies δ_1^A is too. A similar line of reasoning for the three other producers' problems shows that their dual variables $\lambda_1^B, \lambda_2^C, \lambda_2^D, \delta_1^B, \delta_2^C, \delta_2^D$ are also bounded. It remains to establish the boundedness of ε_{12}. Note that the TSO's problem always has a feasible solution since $g_{12} = 0$ so the feasible region is nonempty and compact. Thus by the Weierstrass theorem [2], an optimal solution exists to the TSO's problem and its dual by strong duality. But the dual of the TSO's problem is

$$\begin{aligned}\min_{\varepsilon_{12}} \quad & \overline{g}_{12}\varepsilon_{12} \\ \text{s.t.} \quad & \varepsilon_{12} \geq \tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO} \\ & \varepsilon_{12} \geq 0\end{aligned}$$

By strong duality we have

$$\varepsilon_{12} = \frac{(\tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO)})g_{12}}{\overline{g}_{12}} \tag{23}$$

for optimal ε_{12}, g_{12}. Given the boundedness of g_{12} and τ_{12}, this establishes the boundeness of ε_{12}. □

Proof of Theorem 4. From the proof of the previous theorem, it was shown that the primal variables satisfied

$$0 \leq \{s_1^A, q_1^A, f_{12}^A, s_1^B, q_1^B, f_{12}^B, s_2^C, q_2^C, s_2^D, q_2^D\} \leq q^{\max} := \max\{\overline{q}_1^A, \overline{q}_1^B, \overline{q}_2^C, \overline{q}_2^D\}$$
$$0 \leq f_{12}^A, f_{12}^B \leq \overline{g}_{12}$$

Under the assumption of linear demand, we see that

$$\frac{a_n - 3\kappa_1}{b_n} \leq \pi_n \leq \frac{a_n}{b_n}, \quad n = 1,2$$

The lower bound follows since,

$$a_1 - b_1\pi_1 = D_1(\pi_1) = s_1^A + s_1^B \leq 2\kappa_1$$

from the analysis in the proof of Theorem 3 and the market-clearing conditions for node 1. Solving for π_1 then gives

$$\pi_1 \geq \frac{a_1 - 2\kappa_1}{b_1} \geq \frac{a_1 - 3\kappa_1}{b_1}$$

A similar analysis for node 2 yields

$$a_2 - b_2\pi_2 = D_2(\pi_2) = s_2^C + s_2^D + f_{12}^A + f_{12}^B \leq 3\kappa_1 \Rightarrow \pi_2 \geq \frac{a_2 - 3\kappa_2}{b_2}$$

The upper bound on π_n follows since the demand functions are nonnegative (being equal to nonnegative sales and flows in the market-clearing conditions). Thus,

$$0 \leq D_n(\pi_n) = a_n - b_n\pi_n \Rightarrow \pi_n \leq \frac{a_n}{b_n}$$

Thus, it suffices to consider the bounds on the remaining "dual" variables: $\lambda_1^A, \delta_1^A, \lambda_1^B, \delta_1^B, \lambda_2^C, \delta_2^C, \lambda_2^D, \delta_2^D, \tau_{12}, \varepsilon_{12}$. First consider λ_1^A. From (22) we have

$$\begin{aligned}\lambda_1^A &= \frac{\pi_1 s_1^A + (\pi_2 - \tau_{12}^{\text{Reg}} - \tau_{12})f_{12}^A - \gamma_1^A q_1^A}{\overline{q}_1^A} \\ &= \frac{\pi_1 s_1^A + (\pi_2 - \tau_{12}^{\text{Reg}} - \tau_{12})f_{12}^A - \gamma_1^A\left(s_1^A + f_{12}^A\right)}{\overline{q}_1^A} \\ &= \frac{(\pi_1 - \gamma_1^A)s_1^A + (\pi_2 - \tau_{12}^{\text{Reg}} - \tau_{12} - \gamma_1^A)f_{12}^A}{\overline{q}_1^A} \\ &\leq \frac{(\pi_1)s_1^A + (\pi_2)f_{12}^A - \left(\tau_{12}^{\text{Reg}} + \tau_{12}\right)f_{12}^A}{\overline{q}_1^A} \leq \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) - \frac{(\tau_{12}^{\text{Reg}} + \tau_{12})f_{12}^A}{\overline{q}_1^A} \\ &\leq \begin{cases}\left(\dfrac{a_1}{b_1} + \dfrac{a_2}{b_2}\right) & \text{if } f_{12}^A = 0, \text{ or } [f_{12}^A > 0 \text{ and } \tau_{12}^{\text{Reg}} + \tau_{12} \geq 0] \\ \left(\dfrac{a_1}{b_1} + \dfrac{a_2}{b_2}\right) - (\tau_{12}^{\text{Reg}} + \tau_{12}) & \text{if } [f_{12}^A > 0 \text{ and } \tau_{12}^{\text{Reg}} + \tau_{12} < 0]\end{cases}\end{aligned}$$

since $\overline{q}_1^A \geq \{s_1^A, f_{12}^A\}$. But $[f_{12}^A > 0$ and $\tau_{12}^{\text{Reg}} + \tau_{12} < 0]$ cannot happen since $\tau_{12}^{\text{Reg}} + \tau_{12} < 0 \Rightarrow \varepsilon_{12} = 0$ is the unique solution to the dual of the TSO's problem which then makes the objective function also equal to zero. By strong duality, the TSO's problem must also have a zero optimal objective function value which is only achieved for $g_{12} = 0$. By (18) and noting that both f_{12}^A and f_{12}^B are nonnegative, this means that $f_{12}^A = f_{12}^B = 0$ which is a contradiction. Thus,

$$\lambda_1^A \leq \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

A similar line of reasoning holds for establishing the bounds on λ_1^B, λ_2^C, and λ_2^D. Note that it can be shown that λ_2^C, $\lambda_2^D \leq a_2/b_2$ but since $a_1, b_1 > 0$, the common bound of

$$\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

for all the λ's is also valid.

To establish bounds on $\delta_1^A, \delta_1^B, \delta_2^C$, and δ_2^D, due to nearly similar reasoning, it suffices to consider just δ_1^A. By (13b)

$$\delta_1^A \leq \gamma_1^A + \lambda_1^A \leq \gamma_1^A + \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

and by (13a)

$$\frac{a_1 - 3\kappa_1}{b_1} \leq \pi_1 \leq \delta_1^A$$

Likewise, it can be shown that:

$$\frac{a_1 - 3\kappa_1}{b_1} \leq \delta_1^B \leq \gamma_1^B + \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

$$\frac{a_2 - 3\kappa_1}{b_2} \leq \delta_2^C \leq \gamma_2^C + \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

$$\frac{a_2 - 3\kappa_1}{b_2} \le \delta_2^D \le \gamma_2^D + \left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)$$

Next consider τ_{12}. The lower bound is necessarily $-\tau_{12}^{\text{Reg}}$ since from strong duality of the TSO's problem combined with dual feasibility ($\varepsilon_{12} \ge 0$)

$$0 \le \varepsilon_{12} = \frac{\left(\tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO)}\right) g_{12}}{\overline{g}_{12}} \le \frac{\left(\tau_{12}^{\text{Reg}} + \tau_{12}\right) g_{12}}{\overline{g}_{12}}$$

$$\Rightarrow -\tau_{12}^{\text{Reg}} \frac{g_{12}}{\overline{g}_{12}} \le \tau_{12} \frac{g_{12}}{\overline{g}_{12}}$$

or that $-\tau_{12}^{\text{Reg}} \le \tau_{12}$ as long as the nonnegative variable $g_{12} > 0$. In the case when $g_{12} = 0$, from (13c) we see that

$$\pi_2 - \delta_1^A - \tau_{12}^{\text{Reg}} \le \tau_{12}$$

But taking into account that

$$\pi_2 \ge \frac{a_2 - 3\kappa 1}{b_2}, \quad -\delta_1^A \ge -\gamma_1^A - \frac{a_1}{b_1} - \frac{a_2}{b_2}$$

shows

$$\frac{a_2 - 3\kappa 1}{b_2} - \gamma_1^A - \frac{a_1}{b_1} - \frac{a_2}{b_2} - \tau_{12}^{\text{Reg}} = \frac{-3\kappa 1}{b_2} - \gamma_1^A - \frac{a_1}{b_1} - \tau_{12}^{\text{Reg}} \le \tau_{12}$$

Since

$$\frac{-3\kappa 1}{b_2} - \gamma_1^A - \frac{a_1}{b_1} < 0$$

the lower bound is shown. The upper bound can be established by looking at the following two cases: Case 1: $g_{12} = 0$, Case 2: $g_{12} > 0$, i.e., $f_{12}^A > 0$ or $f_{12}^B > 0$. Under Case 1, by (17b) and the fact that $\overline{g}_{12} > 0$, $\varepsilon_{12} = 0$. From the first constraint of the dual to the TSO problem we have $\varepsilon_{12} \ge \tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO}$ or $\gamma^{TSO} - \tau_{12}^{\text{Reg}} \ge \tau_{12}$. Under Case 2, without loss of generality assume $f_{12}^A > 0$. Then, either $0 < g_{12} < \overline{g}_{12}$ but with $\varepsilon_{12} = 0$ so we have the same result as just stated or $0 < g_{12} = \overline{g}_{12}$ with $\varepsilon_{12} > 0$. In this latter case, since $f_{12}^A > 0$, by (13c),

$$\delta_1^A + \tau_{12} = \pi_2 - \tau_{12}^{\text{Reg}} \le \frac{a_2}{b_2} - \tau_{12}^{\text{Reg}}$$

$$\Rightarrow \tau_{12} \le \frac{a_2}{b_2} - \tau_{12}^{\text{Reg}} - \delta_1^A$$

$$\Rightarrow \tau_{12} \le \frac{a_2}{b_2} - \tau_{12}^{\text{Reg}} - \frac{a_1 - 3\kappa_1}{b_1}$$

since

$$\delta_1^A \ge \frac{a_1 - 3\kappa_1}{b_1}$$

With

$$\kappa_2 := \max\left\{\gamma^{TSO} - \tau_{12}^{\text{Reg}}, \frac{a_2}{b_2} - \tau_{12}^{\text{Reg}} - \frac{a_1 - 3\kappa_1}{b_1}\right\}$$

then we see that

$$\frac{-3\kappa 1}{b_2} - \gamma_1^A - \frac{a_1}{b_1} - \tau_{12}^{\text{Reg}} \le \tau_{12} \le \kappa_2$$

Lastly, to establish upper bounds on ε_{12}, note that in the dual to the TSO problem $\varepsilon_{12} = \max\{0, \tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO}\}$. But

$$\varepsilon_{12} = \tau_{12}^{\text{Reg}} + \tau_{12} - \gamma^{TSO} \le \tau_{12}^{\text{Reg}} - \gamma^{TSO} + \kappa_2$$

with the lower bound being zero as this variable is always nonnegative. The desired bound M^* follows from considering all the individual upper and lower bounds described above and the bound M_1 follows from (6). □

References

[1] Bard JF, Plummer J, Sourie JC. A bilevel programming approach to determining tax credits for biofuel production. European Journal of Operational Research 2000;120:30–46.
[2] Bazaraa MS, Sherali HD, Shetty CM. Nonlinear programming theory and algorithms. New York: John Wiley & Sons, Inc.; 1993.
[3] Beraldi P, Conforti D, Violi A. A two-stage stochastic programming model for electric energy producers. Computers & Operations Research 2008;35(10): 3360–70.
[4] Bjørndal M, Jörnsten K. Equilibrium prices supported by dual price functions in markets with non-convexities. European Journal of Operational Research 2008;190(3):768–89.
[5] Chandrasekaran R, Kabadi SN, Sridhar R. Integer solution for linear complementary problem. Mathematics of Operations Research 1998;23(2):390–402.
[6] Cohon JL. Multiobjective programming and planning. New York: Academic Press; 1978.
[7] Cottle RW, Pang J-S, Stone RE. The linear complementarity problem. New York: Academic Press; 1992.
[8] Facchinei F, Pang J-S. Finite-dimensional variational inequalities and complementarity problems, vols. I and II. New York: Springer; 2003.
[9] Fortuny-Amat J, McCarl B. A representation and economic interpretation of a two-level programming problem. The Journal of the Operational Research Society 1981;32(9):783–92.
[10] Frank M, Wolfe P. An algorithm for quadratic programming. Naval Research Logistics Quarterly 1956;3:95–110.
[11] Gabriel SA, Kiet S, Zhuang J. A mixed complementarity-based equilibrium model of natural gas markets. Operations Research 2005;53(5):799–818.
[12] Gabriel SA, Zhuang J, Kiet S. A large-scale complementarity model of the North American natural gas market. Energy Economics 2005;27:639–65.
[13] Gabriel SA, Shim Y, Conejo AJ, de la Torre S, García-Bertrand R. A benders decomposition method for discretely-constrained mathematical programs with equilibrium constraints. Journal of the Operational Research Society 2010;61:1404–19.
[14] Gabriel SA, Conejo AJ, Fuller JD, Hobbs BF, Ruiz C. Complementarity modeling in energy markets. Springer; 2012.
[15] Gabriel SA, Leuthold FU. Solving discretely-constrained mpec problems with applications in electric power markets. Energy Economics 2010;32:3–14.
[16] Galiana FD, Motto AL, Bouffard F. Reconciling social welfare, agent profits, and consumer payments in electricity pools. IEEE Transactions on Power Systems 2003;18:452–9.
[17] Golden B, Ball M, Bodin L. Current and future research directions in network optimization. Computers & Operations Research 1981;8(2):71–81.
[18] Hogan WW, Ring BJ. On minimum-uplift pricing for electricity markets. Working Paper, John F. Kennedy School of Government, Harvard University; 2003.
[19] Horn RA, Johnson CR. Matrix analysis. New York: Cambridge University Press; 1985.
[20] Labbé M, Marcotte P, Savard G. A bilevel model of taxation and its application to optimal highway pricing. Management Science 1998;44(12):1608–22.
[21] Luo ZQ, Pang JS, Ralph D. Mathematical programs with equilibrium constraints. Cambridge, United Kingdom: Cambridge University Press; 1996.
[22] Mahmassani HS, Chang G-L. On boundedly rational user equilibrium in transportation systems. Transportation Science 1987;21(2):89–99.
[23] Marcotte P, Savard G, Zhu DL. A trust region algorithm for nonlinear bilevel programming. Operations Research Letters 2001;29:171–9.
[24] P O'Neill R, Sotkiewicz PM, Hobbs BF, Rothkopf MH, Stewart Jr. WR. Efficient market-clearing prices in markets with nonconvexities. European Journal of Operational Research 2005;164(1):269–85.
[25] Shapiro J. OR models for energy planning. Computers & Operations Research 1975;2(3–4):145–52.