Preliminary and Incomplete

Equilibrium in a Production Model with Limited Contract Enforcement

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Abstract

This paper develops a tractable macroeconomic model with production and limited commitment (limited enforceability of contracts). The paper provides a characterization of recursive equilibria that is independent of the (endogenous) wealth distribution, which implies that recursive equilibria can be computed solving a finite-dimensional fixed-point problem. The paper also shows that a calibrated version of the model is consistent with some important empirical facts about individual income and consumption. In particular, the model generates a random walk component in individual labor income and, for households with little financial wealth, a limited amount of individual consumption insurance. The quantitative analysis also shows that equilibrium risk sharing is highly sensitive to the degree of contract enforcement as measured by the amount of capital seized upon default.

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I. Introduction

There is by now a large body of empirical work documenting that individual households face a substantial amount of labor income risk, and that this risk has non-negligible effects on consumption.[1] In other words, the complete-market hypothesis (perfect risk sharing) is strongly rejected by the data. A recent macroeconomic literature has suggested that this failure of the complete-market model may be explained by the limited enforceability of contracts (Alvarez and Jermann (2000), Kehoe and Levine (1993), Kocherlakota (1996), Ligon, Thomas, and Worrall (2002), and Thomas and Worrall (1988)). In this literature, financial contracts can only be enforced through the threat of exclusion from financial market participation in the future. Though this literature has initially received a large amount of attention in macroeconomics, certain theoretical and empirical challenges have hindered progress more recently. In this paper, we develop a simple macroeconomic model with limited commitment (enforceability of contracts) that may help overcome some of these challenges.

The first theoretical problem of models with limited commitment is one of tractability. More specifically, dynamic stochastic models with heterogeneous households are in general difficult to analyze since the endogenous wealth distribution becomes a relevant state variable. Thus, even if we confine attention to Markov shock processes with a finite support, recursive equilibria are in general the solution to a complicated infinite-dimensional fixed point problem. For the model developed in this paper, we show that this tractability problem can be avoided. More specifically, we show that individual decision rules are linear in total wealth (financial plus human), which allows us to prove the equivalence between the complicated infinite-dimensional fixed point problem and a much simpler finite-dimensional

[1] For the estimation of income risk, see, for example, MaCurdy (1982), Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten, Telmer, and Yaron (2004). For the consumption response, see, for example, Cochrane (1990), Flavin (1981), Townsend (1994), and Blundell, Pistaferri, and Preston (2008).

fixed-point problem. We then prove the existence of recursive equilibria by proving the existence of a solution to the finite-dimensional fixed point problem.

A second theoretical challenge in models with limited commitment arises when production is considered. In this case, capital becomes an individual state variable and limited enforcement constraints (participation constraints) may introduce non-convexities into the choice sets of individual households. Thus, individual demand correspondences may not be convex-valued and first-order conditions may not be sufficient. In our model, this non-convexity problem can be avoided. More precisely, we show that after we have transformed the infinite-dimensional fixed point problem into an equivalent finite-dimensional problem, the resulting choice sets are convex.

The final challenge for limited commitment models is an empirical one. Recent work in the literature has shown that realistically calibrated models with physical capital and production yield almost perfect consumption insurance even if labor income shocks are highly persistent (Krueger and Perri, 2006). In other words, the theory fails to explain the very fact is was meant to explain. We show that this finding crucially depends on the assumption that defaulting households are punished by exclusion from financial market in the future *and* confiscation of all their capital. In contrast, when we allow defaulting households to keep their capital, we find that our model implies only a very limited amount of individual consumption insurance. In other words, limited enforcement of financial contracts itself has the potential to explain a substantial part of the observed lack of risk sharing.

In reality, defaulting households keep some of their capital, but not all. We conduct a quantitative analysis that shows that this dimension of contract enforcement is quite important for understanding risk sharing: for a typical household whose financial wealth is equal to the average amount of capital in the US economy, consumption insurance is almost completely absent if no capital is seized upon default, but for the same household perfect consumption smoothing is achieved even if only half of his capital is seized upon default. We go on and present a calibrated version of the model that generates, consistent with the US data, an endogenous wealth distribution that has around 50 percent of households holding almost no financial wealth. In equilibrium, these households have very limited consumption insurance despite even if the bankruptcy code calls for the seizure of all their capital in case of default. More precisely, for these households neither the threat of exclusion from financial market participation nor the punishment of seizing all assets is sufficient to prevent default, so that in equilibrium these household will have no access to credit and therefore will not buy consumption insurance. In short, for a substantial fraction of the US population, limited contract enforceability provides a coherent explanation for the lack of risk sharing observed in the micro data.

II. Model

In this section, we develop the model and define the relevant equilibrium concept. Our model is a production economy with an aggregate constant-returns-to-scale production function using physical and human capital as input factors. There are a large number of individual households who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to idiosyncratic shocks to the stock of human capital. Financial intermediaries provide debt and insurance contracts to individual households in competitive markets. Financial contracts (risk sharing agreements) have to be self-enforcing in the sense that at any point in time it must be in the best interest of households to honor their promises. Following the literature, we assume that a household who defaults will be excluded from participation in financial markets forever. In addition, we allow for the possibility of seizing some of the physical capital upon default (collateral).[2]

[2]With its emphasis on physical and human capital investment within a convex framework, the model is similar to Krebs (2003), but Krebs (2003) assumes exogenous market incompleteness and i.i.d. shocks.

a) Time and Uncertainty

Time is discrete and indexed by $t = 0, 1, \ldots$. Aggregate variables are denoted by upper-case letters and individual-specific variables by lower-case letters. There is no aggregate risk and we confine attention to stationary equilibria. Idiosyncratic risk is represented by a Markov shock process with realizations, s_t, that take on a finite number of possible values. We denote by $s^t = (s_0, s_1, \ldots, s_t)$ the history of idiosyncratic shocks up to period t (date-event, node) and let $\pi(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \ldots \pi(s_1|s_0)$ stand for the probability that s^t occurs. At time $t = 0$, the type of an individual household is characterized by his initial shock, s_0, and initial state, $x_0 = (k_0, h_0)$, consisting of the initial stock of physical capital, k_0, and the initial stock of human capital, h_0. We take as given an initial measure, π_0, over initial types (x_0, s_0).

b) Production

There is one all-purpose good that can be consumed, invested in physical capital, or invested in human capital. Production of this one good is undertaken by one firm (a large number of identical firms) that rents capital and labor in competitive markets and uses these input factors to produce output, Y_t, according to the aggregate production function $Y_t = F(K_t, H_t)$. Here K_t and H_t are the (aggregate) levels of physical and human capital employed by the firm. The production function, F, is a standard neoclassical function, that is, it has constant-returns-to-scale, satisfies a Inada condition, and is continuous, concave, and strictly increasing in each argument. Given these assumptions on F, the derived intensive-form production function, $f(\tilde{K}) = F(\tilde{K}, 1)$, is continuous, strictly increasing, strictly concave, and satisfies a corresponding Inada condition, where we introduced the "capital-to-labor ratio" $\tilde{K} = K/H$. Given the assumption of perfectly competitive labor and capital markets,

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Wright (2003) considers an "AK" model with i.i.d. shocks and limited enforcement that, in a certain sense, is a simplified version of the current set-up.

profit maximization implies:

$$\begin{aligned} r_k &= f'(\tilde{K}) \\ r_h &= f(\tilde{K}) + f'(\tilde{K})\tilde{K} \ , \end{aligned} \tag{1}$$

where r_k is the rental rate of physical capital and r_h is the rental rate of human capital. Note that r_h is simply the wage rate per unit of human capital and that we dropped the time index because of our stationarity assumption. Clearly, (1) defines rental rates as functions of the capital to labor ratio: $r_k = r_k(\tilde{K})$ and $r_h = r_h(\tilde{K})$.

c) Preferences

There are a large number of infinitely-lived, risk-averse households who have well-defined preferences over consumption allocations, $\{c_t\}$, where $\{c_t\}$ denotes a sequence of functions (random variables), c_t, mapping initial types, x_0, and histories, s^t, into consumption levels, $c_t(x_0, s^t)$. Similar notation will be used for investment plans (see below). A consumption allocation, $\{c_t\}$, is a family of consumption plans, $\{c_t(x_0, s_0)\}$, one for each initial type (x_0, s_0). Preferences are individualistic in the sense that a household of type (x_0, s_0) only cares about his own consumption plan, $\{c_t(x_0, s_0)\}$. Moreover, preferences allow for a time-additive expected utility representation:

$$U\left(\{c_t(x_0, s_0)\}\right) \doteq \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(x_0, s^t))\pi(s^t|s_0) \ . \tag{2}$$

We assume that the one-period utility function exhibits constant relative risk aversion: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(c) = lnc$ otherwise. Put differently, we assume that preferences are homothetic.

d) Budget Constraint

Each household can invest in physical capital, k, or human capital, h. In addition, he can buy and sell a complete set of financial contracts (assets) with state-contingent payoffs. More specifically, there is one contract (Arrow security) for each state, and we denote by $a_{t+1}(s_{t+1})$ the quantity bought in period t (sold if negative) of the contract that pays off one unit of the good in period $t+1$ if s_{t+1} occurs. Given his initial type, (s_0, k_0, h_0), a household chooses a plan, $\{c_t(x_0, s_0), k_t(x_0, s_0), h_t(x_0, s_0), \vec{a}_t(x_0, s_0)\}$, where the notation $\vec{a}$ indicates that in each period the household chooses a vector of contract holdings. A budget-feasible plan has to satisfy the sequential budget constraint

$$\begin{aligned} c_t + k_{t+1} + h_{t+1} + \sum_{s_{t+1}} a_{t+1}(s_{t+1})q(s_{t+1}) &= (1 + r_k - \delta_k)k_t + (1 + r_h - \delta_h(s_t))h_t + a_t(s_t) \\ c_t \geq 0 \ , \ \ k_{t+1} \geq 0 \ , \ \ h_{t+1} \geq 0 & \qquad , \end{aligned} \tag{3}$$

where $q(s_{t+1})$ is the price of a financial contract that pays off if s_{t+1} occurs and δ_k and $\delta_h(s_t)$ are the depreciation rates of physical and human capital, respectively. Note that (3) has to hold in realizations, that is, for any household of type (k_0, h_0, s_0), it has to hold for all t and all histories of shocks, $(s_1, \ldots, s_t)$.

The budget constraint (3) assumes that physical capital can be accumulated by investing $k_{t+1} - (1 - \delta_k))k_t$. Similarly, human capital can be accumulated by investing $h_{t+1} - (1 - \delta_h(s_t))h_t$. The budget constraint (3) makes three implicit assumptions about the accumulation of human capital. First, it lumps together general human capital (education, health) and specific human capital (on-the-job training). Second, it neglects the decision of households to allocate a fixed amount of time across different activities. Third, (3) does not impose a non-negativity constraint on human capital investment ($x_{hit} \geq 0$).

The random variable δ_{ht} represents uninsurable idiosyncratic labor income risk. A negative human capital shock, $\delta_h(s_t) < \sum_s \delta_h(s)\pi(s)$, can occur when a worker loses firm- or sector-specific human capital subsequent to job termination (worker displacement). In order to preserve the tractability of the model, the budget constraint (3) rules out extended periods of unemployment because it assumes that the wage payment is received in each period. Thus, the emphasis is on earnings uncertainty, not employment uncertainty. A decline in health (disability) provides a second example for a negative human capital shock. In this case, both general and specific human capital might be lost. Internal promotions and upward movement in the labor market provide two examples of positive human capital shock.

It is convenient to introduce new variables that emphasize that individual households solve a standard inter-temporal portfolio choice problem with additional participation constraints. To this end, introduce the following variables:

$$\begin{aligned} w_t &= k_t + h_t + \sum_{s_t} q_{t-1}(s_t) a_t(s_t) \\ \theta_{kt} &= \frac{k_t}{w_t} \ , \ \ \theta_{ht} = \frac{h_t}{w_t} \ , \ \ \theta_{at}(s_t) = \frac{a_t(s_t)}{w_t} \\ 1 + r(\theta_t, s_t) &= (1 + r_k - \delta_k)\theta_{kt} + (1 + r_h - \delta_h(s_t))\theta_{ht} + \theta_{at}(s_t) \end{aligned} \tag{4}$$

In (4) the variable w_t stands for beginning-of-period wealth consisting of real wealth, $k_t + h_t$, and financial wealth, $\sum_{s_t} q_{t-1}(s_t) a_t(s_t)$. The variable $\theta_t = (\theta_{kt}, \theta_{ht}, \vec{\theta}_{at})$ denotes the vector of portfolio shares and $(1 + r)$ is the gross return to investment. Note that $r(.)$ is a function mapping portfolio choices and shocks into returns, and that the total investment return in period t is $r_t = r(\theta_t, s_t)$ and the total investment return in period $t+1$ is $r(\theta_{t+1}, s_{t+1})$. Using the new notation, the budget constraint (4) reads

$$\begin{aligned} w_{t+1} &= [1 + r(\theta_t, s_t)]\, w_t - c_t \\ 1 &= \theta_{kt} + \theta_{ht} + \sum_{s_t} q_{t-1}(s_t)\theta_{at}(s_t) \\ c_t &\geq 0 \ , \ \ w_t \geq 0 \ , \ \ \theta_{kt} \geq 0 \ , \ \ \theta_{ht} \geq 0 \ . \end{aligned} \tag{5}$$

Clearly, (5) is the budget constraint corresponding to an intertemporal portfolio choice problem with linear investment opportunities and no exogenous source of income. Note that r not only depends on the individual choice of θ, but also on the aggregate variables $\tilde{K}$ and q, but we will suppress this dependence until we turn to the general equilibrium analysis.

So far, we have not imposed any restrictions on trading of financial assets. In this paper, we augment the sequential budget constraint by the following short-sale constraints:

$$\theta_{at}(s_t) \geq -\bar{\theta}_a(s_t) \ , \tag{6}$$

where $\bar{\theta}(s_t)$ is a number that will be chosen large enough so that it will not bind in equilibrium. In this case, (6) is equivalent to a no-Ponzi-scheme condition if $r_f > 0$. However, in contrast to the no-Ponzi-scheme condition, the short-sale constraint (6) has three advantages. First, it allows us to consider equilibria with $r_f < 0$. Second, it nicely fits into a recursive formulation of the problem. Finally, it will be useful for the proof of proposition 1.

To sum up, a household of type (x_0, s_0) with $x_0 = (w, \theta_{k0}, \theta_{h0})$, chooses a plan $\{w_t(x_0, s_0), \theta_t(x_0, s_0), c_t(x_0,$ A household chooses a plan so as to maximize expected lifetime utility (2) subject to the budget constraint (5,6) and an additional participation constraint, to which we turn next.

e) Participation Constraint

In addition to the standard budget constraint, the household has to satisfy a sequential participation constraint, which ensures that at no point in time individual households have an incentive to default on their financial obligations. More precisely, individual consumption plans have to satisfy

$$E[\sum_{n=0}^{\infty} \beta^n u(c_{t+n})|x_0, s^t] \geq V_d(w_t, \theta_{kt}, \theta_{ht}, s_t) \ , \tag{7}$$

where we used the notation $E[\sum_{n=0}^{\infty} \beta^n u(c_{t+n})|x_0, s^t] = \sum_{n=0}^{\infty} \sum_{s^{t+n}|s^t} \beta^n u(c_{t+n}(x_0, s^t))\pi(s^{t+n}|s^t)$ and $V_d(.)$ is the value function of a defaulting household (autarky value).

We assume that for a household who defaults in period t all the short and long positions in financial assets are canceled, $\theta_{at}(s_t) = 0$, which is the reason why θ_a is not an argument in $V_d(.)$. Further, there are two types of punishment for default: defaulting households will be excluded from participation in financial markets forever, $\theta_{a,t+n}(s_{t+n} = 0$, and upon default they lose a fraction ϕ of their physical capital (but keep all their human capital). Finally, with respect to the use of physical and human capital after default, we consider two different scenarios. In the first case, households become self-employed and use their physical and human capital to produce according to the neoclassical production function $y_d = F_d(k_d, h_d)$. In the second case, defaulting households still participate in the market for physical and human capital, that is, they rent out their physical and human capital at the going market rate. In either case, a defaulting household faces the budget constraint

$$\begin{aligned} w_{d,t+n+1} &= [1 + r_d(\theta_{dk,t+n}, \theta_{dh,t+n}, s_{t+n})]\, w_{d,t+n} - c_{d,t+n} \\ 1 &= \theta_{dk,t+n} + \theta_{dh,t+n} \\ c_{d,t+n} &\geq 0 \ , \ \ w_{d,t+n} \geq 0 \ , \ \ \theta_{dk,t+n} \geq 0 \ , \ \ \theta_{dh,t+n} \geq 0 \ , \end{aligned} \tag{8}$$

where $r_d(\theta_{dk}, \theta_{dh}, s) = F_d(\theta_{dk}, \theta_{dh}) - \delta_k\theta_{dk} - \delta_h(s)\theta_{dh}$ in the case of self-employment and $r_d(\theta_{dk}, \theta_{dh}, s) = (r_k - \delta_k)\theta_{dk} + (r_h - \delta_h(s))\theta_{dh}$ in the other case. Note that in the second case, in which households continue to supply physical and human capital in competitive markets, the investment return after default depends on the market returns r_k and r_h, and therefore on the aggregate capital-to-labor ratio, $\tilde{K}$.

After default a household chooses a continuation plan, $\{c_{d,t+n}(x_0, s^t), w_{d,t+n}(x_0, s^t), \theta_{d,t+n}(x_0, s^t)\}$, that maximizes expected lifetime utility (2) subject to the sequential budget constraint ( ). The resulting choice problem of households is a standard utility maximization problem with CRRA-preferences and linear investment opportunities. The subsequent analysis (propositions 1 and 2) or results in Alvarez and Jermann (2000) and Krebs (2006) show that the value function associated with this maximization problem has the functional form[3]

$$V_d(w_t, \theta_t, s_t) \ = \ \begin{cases} \tilde{V}_d(s_t)\,(1 + r(\theta_{kt}, \theta_{ht}, 0, s_t) - \phi\theta_{kt})^{1-\gamma}\, w_t^{1-\gamma} & if\ \gamma \neq 1 \\ \tilde{V}_d(s_t) \ + \ \frac{1}{1-\beta} log(1 + r(\theta_{kt}, \theta_{ht}, 0, s_t) - \phi\theta_{kt}) \ + \ \frac{1}{1-\beta} log w_t & otherwise \end{cases} , \tag{9}$$

[3] In the case of self-employment, we assume that defaulting households still supply their physical and human capital in competitive markets in the period of default.

where the intensive-form value function, $\tilde{V}_d$, is the solution to an, appropriately re-defined, intensive-from Bellman equation (15). Note that for the case in which defaulting households continue to supply their physical and human capital in the capital and labor market, the intensive-form value function also depends on market returns and therefore the aggregate capital-to-labor ratio: $\tilde{V}_d = \tilde{V}_d(s, \tilde{K})$.

f) Equilibrium

In equilibrium, a household of type (x_0, s_0), with $x_0 = (w_0, \theta_0)$, chooses a plan, $\{c_t(x_0, s_0), w_t(x_0, s_0), \theta_t(x_0$ where $\theta_t = (\theta_{kt}, \theta_{ht}, \vec{\theta}_{at})$. The collection of plans for all initial types defines a global plan or allocation $\{c_t, w_t, \theta_t\}$. The aggregate capital stock held by households, K^h, is

$$\begin{aligned} K^h_{t+1} &= \int_{x_0} \sum_{s^t} k_{t+1}(x_0, s^t)\pi(s^t|s_0)d\pi_0(x_0, s_0) \\ &= \int_{x_0} \sum_{s^t} \theta_{k,t+1}(x_0, s^t)w_{t+1}(x_0, s^t)\pi(s^t|s_0)d\pi_0(x_0, s_0) \end{aligned} \quad (10)$$

Note that $\theta_k w$ is simply the physical capital stock of an individual household and $\theta_h w$ the corresponding human capital stock.

In equilibrium, the level of physical and human capital demanded by the firm must be equal to the corresponding aggregate levels supplied by households. Because of the constant-returns-to-scale assumption, only the ratio of physical to human capital is pinned down by this market clearing condition. That is, in equilibrium we must have

$$\tilde{K}_{t+1} = \frac{\int_{x_0} \sum_{s^t} \theta_{k,t+1}(x_0, s^t)w_{t+1}(x_0, s^t)\pi(s^t|s_0)d\pi_0(x_0, s_0)}{\int_{x_0} \sum_{s^t} \theta_{h,t+1}(x_0, s^t)w_{t+1}(x_0, s^t)\pi(s^t|s_0)d\pi_0(x_0, s_0)}\ , \quad (11)$$

where $\tilde{K}$ is the capital-to-labor ratio chosen by the firm.

The second market clearing condition requires that no resources are created or destroyed by trading in financial contracts. In other words, the total value of all financial asset holdings has to sum up to zero:

$$\int_{x_0} \sum_{s^t} \sum_{s_{t+1}} q_t(s_{t+1}, x_0, s^t)\theta_{a,t+1}(s_{t+1}; x_0, s^t) w_{t+1}(x_0, s^t)\pi(s^t|s_0) d\pi_0(x_0, s_0) \ = \ 0 \ . \tag{12}$$

Straightforward calculation shows that the two market clearing conditions in conjunction with the budget constraint (5) imply the standard aggregate resource constraint (goods market clearing).

We assume that there are financial intermediaries that can buy and sell financial assets without any short-sale constraint. In particular, these financial intermediaries can sell insurance contracts to individual households and invest the proceeds in the risk-free asset that can be created from the complete set of financial contracts. Given that financial intermediaries face linear investment opportunities and no quantity restrictions on the trading of financial contracts, equilibrium requires that financial intermediaries make zero profit. Zero profit, in turn, implies that financial assets are priced in a risk-neutral manner:

$$\begin{aligned} q_t(s_{t+1}; x_0, s^t) &= q(s_{t+1}; s_t) \\ &= \frac{\pi(s_{t+1}|s_t)}{1+r_f} \ , \end{aligned} \tag{13}$$

where r_f is the interest rate on financial transactions. Given the pricing condition (13), the asset market clearing condition (12) reads

$$\int_{x_0} \sum_{s^{t+1}} \theta_{a,t+1}(s_{t+1}; x_0, s^t) w_{t+1}(x_0, s^t)\pi(s^{t+1}|s_0) d\pi_0(x_0, s_0) \ = \ 0 \ . \tag{14}$$

To sum up, we have the following definition of equilibrium:

Definition 1. A stationary equilibrium is an allocation, $\{c_t, w_t, \theta_t\}$ and $\tilde{K}$, together with rental rates, (r_k, r_h), and a financial interest rate, r_f, so that

i) Utility maximization of households: for each household type, $(x_0, s_0) = (w_0, \theta_0, s_0)$, the corresponding plan, $\{c_t(x_0, s_0), w_t(x_0, s_0), \theta_t(x_0, s_0)\}$, maximizes expected lifetime utility (2) subject to the sequential budget constraint (5), the short-sale constraint (6), and the sequential participation constraint (7).

ii) Profit maximization of firms: the aggregate capital-to-labor ratio and rental rates satisfy the first-order conditions (1).

iii) Zero profit of financial intermediaries: financial asset prices are given by (13)

iv) Market clearing: equations (11) and (14) hold.

A stationary equilibrium is recursive if $\{c_t, w_t, \theta_t\}$ is generate by a policy function $x_{t+1} = g(x_t, s_t)$ and $c_t = c(x_t, s_t)$.

Note that in a stationary recursive equilibrium, the stochastic process $\{x_t, s_t\}$ is a stationary Markov process and the initial distribution over household types, $\pi_0(x_0, s_0)$, is one of the corresponding stationary distribution.

III. Theoretical Results

In this section, we present the main theoretical results. We begin with the principle of optimality for the individual household problem (propositions 1) and the equivalence between intensive-form Bellman equation and extensive-form Bellman equation (proposition 2). We then show the equivalence between stationary recursive equilibria and stationary intensive-form equilibria, which are the solution to a finite-dimensional fixed-point problem (proposition 3).

a) Principle of Optimality

The budget constraint (5,6) and the participation constraint (7) suggest that the utility maximization problem of an individual household is recursive in the state variable (w, θ, s). More precisely, consider the Bellman equation

$$V(w, \theta, s) = \max_{w', \theta'} \left\{ u\left((1 + r(\theta, s))w - w'\right) + \beta \sum_{s'} V\left(w', \theta', s'\right) \pi(s'|s) \right\} \tag{15}$$

$$
\begin{aligned}
s.t. \quad 1 \ &= \ \theta'_k + \theta'_h + \sum_{s'} \frac{\pi(s')\theta'_a(s')}{1+r_f} \\
0 \ \leq \ w' \ &\leq \ (1+r(\theta,s))w \ \ , \ \ \theta'_k \geq 0 \ \ , \ \ \theta'_h \geq 0 \ \ , \ \ \theta'_a(s') \geq -\bar{\theta}_a(s') \\
V(w',\theta',s') \ &\geq \ V_d(w'\theta',s')
\end{aligned}
$$

and the corresponding Bellman operator defined as

$$
\begin{aligned}
TV(w,\theta,s) \quad &= \quad \max_{w',\theta'} \left\{ u((1+r(\theta,s))w - w') + \beta \sum_{s'} V(w',\theta',s')\pi(s'|s) \right\} \qquad (16) \\
s.t. \quad 1 \ &= \ \theta'_k + \theta'_h + \sum_{s'} \frac{\pi(s')\theta'_a(s')}{1+r_f} \\
0 \leq w' &\leq (1+r(\theta,s))w \ \ , \ \ \theta'_k \geq 0 \ \ , \ \ \theta'_h \geq 0 \ \ , \ \ \theta'_a(s') \geq -\bar{\theta}_a(s') \\
V(w',\theta',s') \ &\geq \ V_d(w',\theta',s') \ ,
\end{aligned}
$$

where the value function after default is specified in (8).

Our first proposition shows that under the condition that for all s

$$
\begin{aligned}
\forall\theta' \ : \ \beta \sum_{s'} (1+r(\theta',s'))^{1-\gamma} \pi(s'|s) \ &< \ 1 \quad if \ 0 \ < \ \gamma \ < \ 1 \qquad (17) \\
\exists\theta' \ : \ \beta \sum_{s'} (1+r(\theta',s'))^{1-\gamma} \pi(s'|s) \ &< \ 1 \quad if \ \gamma \ > \ 1
\end{aligned}
$$

the principle of optimality holds. Note that for the log-utility case, no condition of the type (17) is required. The proposition shows further how the value function (the maximal solution of the Bellman equation) can be found by iterating on the solution to the Bellman equation without participation constraint:

Proposition 1. Suppose that condition (17) is satisfied and let V_0 be the (unique) solution to the Bellman equation (15) without participation constraint. Define the operator T as in (16). Then

i) $\lim_{n\to\infty} T^n V_0 = V_\infty$ exists and is the maximal solution to the Bellman equation (17)

ii) V_∞ is the value function, V^*, of the sequential household maximization problem.

Proof Appendix.

Comment on the proposition; method of proof (monotone operator theorem)

b) Intensive-form Bellman equation

The next proposition is a direct consequence of proposition 1 and is the first step for our transformation of a rather complex equilibrium problem into a very simple problem. More specifically, proposition 2 below shows that the value function, V^*, has the functional form

$$V^*(w,\theta,s) = \begin{cases} \tilde{V}^*(s)(1+r(\theta,s)^{1-\gamma}w^{1-\gamma} & if\ \gamma \neq 1 \\ \tilde{V}^*(s) + \frac{1}{1-\beta}log\,(1+r(\theta,s)) + \frac{1}{1-\beta}w & otherwise \end{cases} \tag{18}$$

and that the corresponding optimal policy functions are

$$c(w,\theta,s) = \begin{cases} \tilde{c}(s)(1+r(\theta,s))w & if\ \gamma \neq 1 \\ (1-\beta)(1+r(\theta,s))w & otherwise \end{cases} \tag{19}$$

$$w'(w,\theta,s) = \begin{cases} (1-\tilde{c}(s))(1+r(\theta,s))w & if\ \gamma \neq 1 \\ \beta(1+r(\theta,s))w & otherwise \end{cases}$$

$$\theta'(w,\theta,s) = \theta'(s)\ .$$

In other words, the value function has the functional form of the underlying utility function, consumption and next-period wealth are linear functions of this-period wealth, and next-period portfolio choices only depend on the current shock. Moreover, proposition 2 also shows that the intensive-form value function, $\tilde{V}^*$, together with the optimal consumption and portfolio choices, $\tilde{c}$ and θ, can be found by solving an intensive-form Bellman equation that reads

$$\tilde{V}(s) = \max_{\tilde{c},\theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c})^{1-\gamma} \sum_{s'} (1+r(\theta',s'))^{1-\gamma}\, \tilde{V}(s')\pi(s'|s) \right\} \tag{20}$$

$$s.t.\quad 1 = \theta'_k + \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1+r_f}$$

$$0 \ \leq \ \tilde{c} \ \leq \ 1 \ \ , \ \ \theta_k' \geq 0 \ \ , \ \ \theta_h' \geq 0 \ \ , \ \ \theta_a'(s') \geq \bar{\theta}_a(s')$$

$$\left(\frac{\tilde{V}(s')}{\tilde{V}_d(s')}\right)^{\frac{1}{1-\gamma}} (1 + r(\theta_k'(s), \theta_h'(s), \theta_a(s'; s))) \ \geq \ 1 + r(\theta_k'(s), \theta_h'(s), 0, s') - \phi\theta_k'(s)$$

for $\gamma \neq 1$ and

$$\tilde{V}(s) \quad = \quad \max_{\theta'} \left\{ log(1-\beta) + \frac{\beta}{1-\beta} log\beta + \frac{\beta}{1-\beta} \sum_{s'} log(1 + r(\theta', s'))\pi(s'|s) + \beta \sum_{s'} \tilde{V}(s)\pi(s'|s) \right\}$$

$$s.t. \quad 1 \ = \ \theta_k' + \theta_h' + \sum_{s'} \frac{\theta_a'(s')\pi(s'|s)}{1 + r_f}$$

$$\theta_k' \geq 0 \ \ , \ \ \theta_h' \geq 0 \ \ , \ \ \theta_a'(s') \geq \bar{\theta}_a(s')$$

$$e^{(1-\beta)\left(\tilde{V}(s') - \tilde{V}_d(s')\right)} (1 + r(\theta_k', \theta_h', \theta_a(s')) \ \geq \ 1 + r(\theta_k', \theta_h', 0, s') - \phi\theta_k'$$

for the log-utility case.

Proposition 2. Suppose that condition (17) is satisfied. Then the value function, V^*, has the functional form (18) and the optimal policy function is given by (19). Moreover, the intensive-form value function, $\tilde{V}^*$, and the corresponding optimal consumption and portfolio choices, $\tilde{c}$ and θ', are the maximal solution to the intensive-form Bellman equation (20). This maximal solution is obtained by iterating on the solution, $\tilde{V}_0$, of the intensive-form Bellman equation (20) without participation constraint:

$$\tilde{V}^* \ = \ \lim_{n \to \infty} T^n \tilde{V}_0 \ .$$

Proof Appendix.

Note that proposition 2 cannot simply be proved by using substitution since there may be multiple solutions to the Bellman equation (15). In other words, the operator (16) associated with the Bellman equation is monotone, but not a contraction. However, proposition 2 ensures that we have indeed found the value function associated with the original utility maximization problem, and also provides us with a iterative method to compute this solution.

Note further that the constraint set in (15) is linear since the return functions are linear in θ. Thus, the constraint set is convex and we have transformed the original utility maximization problem into a convex problem. In other words, the non-convexity problem alluded to in the introduction has been resolved.

c) Intensive-form equilibrium

Proposition 2 shows how to rewrite the maximization problem of individual households into a recursive problem that is wealth-independent. One implication of the intensive-form representation of the individual maximization problem is that optimal portfolio choices are wealth independent. This result, in turn, implies that the market clearing conditions (11) and (14) can be re-written as intensive-form market clearing conditions that are independent of the wealth distribution. To see this, define the relative wealth share of all households with current shock s_t as

$$\omega_{t+1}(s_t) \doteq \frac{\int_{x_0} \sum_{s^{t-1}} w_{t+1}(s^t)\pi(s^t|s_0)d\pi(x_0, s_0)}{\int_{x_0} \sum_{s^t} w_{t+1}(s^t)\pi(s^t|s_0)d\pi(x_0, s_0)} .$$

Note that $\sum_{s_t} \omega(s_t) = 1$. Using this definition and the wealth-independence of θ-choices, the next proposition shows that the market clearing conditions (11) and (14) are equivalent to the following intensive-form market clearing conditions

$$\tilde{K} = \frac{\sum_s \theta_k(s)\omega(s)}{\sum_s \theta_h(s)\omega(s)} \tag{21}$$

$$0 = \sum_{s,s'} \theta_a(s'; s)\pi(s'|s)\omega(s) .$$

Moreover, using the definition of ω and the optimal policy function, we also show that the evolution of the ω-distribution is given by

$$\omega_{t+1}(s') = \frac{\sum_s (1 - \tilde{c}(s'))(1 + r(\theta(s), s'))\pi(s'|s)\omega_t(s)}{\sum_s \sum_{s''} (1 - \tilde{c}(s''))(1 + r(\theta(s), s''))\pi(s''|s)\omega_t(s)} , \tag{22}$$

which leads to the following stationarity condition for the stationary ω-distribution:

$$\omega(s') \;=\; \frac{\sum_s (1-\tilde{c}(s'))(1+r(\theta(s),s'))\pi(s'|s)\omega(s)}{\sum_s \sum_{s''}(1-\tilde{c}(s''))(1+r(\theta(s),s''))\pi(s''|s)\omega(s)}\;. \tag{23}$$

In sum, a stationary recursive equilibrium can be found by solving (20), (21), and (23) if the solution satisfies the condition

$$\beta \sum_{s'} \left(1 + r(\theta', s')\right)^{1-\gamma} \pi(s'|s) \;<\; 1\;. \tag{24}$$

Note that inequality (24) simply ensures that condition (17) is satisfied so that proposition 1 is applicable.

Proposition 3. Suppose that $(\theta, \tilde{c}, \tilde{V}, \tilde{K}, r_f, \omega)$ is a stationary intensive-form equilibrium, that is, $(\theta, \tilde{K}, \omega)$ solves the intensive-form market clearing conditions (21), ω solves the stationarity condition (23), and for given $(r_k(\tilde{K}), r_h(\tilde{K}), r_f)$, the consumption-portfolio choice $(\tilde{c}, \theta)$ together with the intensive-form value function $\tilde{V}$ are the maximal solution to the intensive-form Bellman equation (20) satisfying condition (24). Then the corresponding allocation $\{c_t, w_t, \theta_t\}$ together with prices $(r_k(\tilde{K}), r_h(\tilde{K}), r_f)$ form a stationary recursive equilibrium.

Comment on tractability – it's a static problem and only a lower-dimensional wealth distribution matters. Also, short discussion of i.i.d. case.

Proof. See appendix.

IV. Quantitative Results

We now discuss the quantitative implications of the model. First, we discuss how the model matches some of the micro-level evidence on income and consumption inequality when human capital shocks are i.i.d. After this, we analyze a calibrated version of the model economy with i.i.d. shocks. In particular, we report the degree of consumption insurance in equilibrium and how it depends on the level of contract enforcement measured by the amount of capital that is seized after default. Finally, we discuss the quantitative implication of a version of the model with two types of households, the young who face high returns to human capital investment and the old who face low returns to human capital investment.

a) Income and Consumption Inequality

Suppose that the process of human capital shocks, $\{\delta_h(s_t)\}$, is i.i.d. In this case, the equilibrium formula (19) implies that portfolio choices are constant, $\theta(s_t) = \theta$, and that the consumption-to-wealth ratio is constant, $\tilde{c}(s_t) = \tilde{c}$. We now discuss the implications of this assumption for individual income and consumption in equilibrium. To simplify this discussion, we write the individual depreciation shocks as $\delta_h(s_t) = \delta_h + s_t$, where s_t is a mean zero random variable with distribution to ber specified below.

In the model, labor income of an individual household in period t is given by $y_{ht} = r_h h_t$. Thus, the growth rate of labor income is equal to the growth rate of human capital: $y_{h,t+1}/y_{ht} = h_{t+1}/h_t$. From the accumulation equation for human capital, $h_{t+1} = (1 - \delta_h + s_t)h_t + x_{ht}$, we therefore conclude that labor income can change stochastically for two reasons: either a direct shock to human capital, s_t, or a change in the human capital investment level, x_{ht}. In the model, any increase in human capital brought about by additional investment translates immediately into an increase in labor income, but in reality this process is likely to take several years. Thus, if we use annual or quarterly data to calibrate the income process, we have to take into account that the model overpredicts the importance of the second channel. To correct for this when matching the model's prediction with the data, we replace the actual amount of human capital investment, x_{ht}, with something that is less volatile, namely its mean value, $E[x_{ht}]$. In this case, we find

$$\frac{y_{h,t+1}}{y_{ht}} = \frac{h_{t+1}}{h_t} \tag{25}$$

$$\approx \quad (1-\delta_h+s_t)+\frac{E[x_{ht}]}{h_t}$$
$$= \quad \beta(1+E[r_t]+s_t)\ ,$$

where we used in the last line that in equilibrium $E[x_{ht}]/h_t = \beta(1+E[r_t]) - (1-\delta_h)$ (see also equation (19)). Taking logs and re-arranging terms in (25) yields:

$$\begin{aligned} lny_{h,t+1} &= lny_{ht} + ln\left(\beta(1+E[r_t]+s_t)\right) \\ &\approx lny_{ht} + d + s_t\ , \end{aligned} \tag{26}$$

where $d = ln(1-\tilde{c}) + E[r_t]$ is a constant. Thus, if the process of human capital shocks, $\{s_t\}$, is i.i.d., then the logarithm of labor income follows a random walk with drift d and innovation term s_t.[4] Moreover, if we make the distributional assumption $s_t \sim N(0,\sigma_s^2)$, then (26) is exactly the specification often used by the empirical literature to model the permanent component of labor income risk (Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten et al. (2004)). In our quantitative analysis conducted in the next section we will assume i.i.d. shocks and use the estimate of the variance of the innovation term for the random walk component obtained by this empirical literature to assign a value for σ_s^2.

Turning to consumption, we notice that (19) implies that consumption growth is i.i.d. and given by

$$\frac{c_{t+1}}{c_t} = (1-\tilde{c})\left(1+E[r_t]+\theta_a(s_{t+1})+\theta_h s_{t+1}\right)\ . \tag{27}$$

Taking logs and using again the approximation $ln(1+x)\approx x$, we find

$$lnc_{t+1} = lnc_t + d + \epsilon_{t+1}\ , \tag{28}$$

where d is the same constant as in (26) and $\epsilon_{t+1} = \theta_a(s_{t+1}) + \theta_h s_{t+1}$. Hence, if the process of human capital shocks, $\{s_t\}$, is i.i.d., consumption follows a logarithmic random walk. Fur-

[4]We have s_t instead of s_{t+1} in equation (26), and the latter is the common specification for a random walk. However, this is not a problem if the econometrician observes the idiosyncratic depreciation shocks with a one-period lag. In this case, (26) is the correct equation from the household's point of view, but a modified version of (26) with s_{t+1} replacing s_t is the specification estimated by the econometrician.

ther, the effect of human capital shocks (permanent labor income shocks), s_t on consumption is weakened for two reasons. First, households can self-insure through their own savings, and in equation (28) the shock s_{t+1} is therefor multiplied by labor's share in income, θ_h. Second, households buy insurance contracts, and this is represented by the term $\theta_a(s)$.

b) Calibration

The quantitative analysis is based on an economy where workers have logarithmic utility functions: $u(c) = logc$ and the production function is Cobb-Douglas: $f(\tilde{k}) = A\tilde{k}^{\alpha}$. We use $\alpha = .36$ to match capital's share in income and $\delta = .06$ (annually) as a compromise between the higher depreciation rate of physical capital used in the literature (but see also Cooley and Prescott (1995) for an argument that $\delta_k = .05$) and the probably lower depreciation rate of human capital. The values of the fundamental parameters A, σ_η^2, and β are chosen so that the model is roughly consistent with the US evidence on (physical capital) saving and growth. More specifically, we require that per capita consumption growth satisfies $\mu_g = E[c_{i,t+1}/c_{it}] - 1 = .02$ and that the implied saving rate is $s_k = x_{kt}/y_t = .20$. For the annual US data on saving and growth, see Summers and Heston (1991). This approach yields $A = .2674$ and $\beta = .946$.

As in section a), we assume that human capital shocks are i.i.d. and normally distributed, $s_t \sim N(0, \sigma_s^2)$,[5] so that labor income follows (approximately) a logarithmic random walk with variance of the innovation term σ_s^2. This variance has been estimate by the empirical labor literature. Carroll and Samwick (1997) find .15, Meghir and Pistaferri (2004) estimate .19, and Storesletten et al. (2004) have .25 (averaged over age-groups and, if applicable, over business cycle conditions). All these studies use labor income before transfer payments, which is the relevant variable from our point of view. In our baseline economy, we choose $\sigma_s = .20$.

[5]We approximate this normal distribution by a two-state distribution.

There are at least two reasons why the above approach might underestimate human capital risk. First, a constant $\sigma_s = .20$ represents less uncertainty than a σ_s that fluctuates with business cycle conditions and has a mean of .20. Second, the assumption of normally distributed innovations understates the amount of idiosyncratic risk households face if the actual distribution has a fat lower tail. For strong evidence for such a deviation from the normal-distribution framework, see Geweke and Keane (2000). Further, the literature on the long-term consequences of job displacement (Jacobson, LaLonde, and D. Sullivan (1993)) has found wage losses of displaced workers that are somewhat larger than suggested by our mean-variance framework.

There are, however, also arguments that the current approach might overestimate human capital risk. First, we assumes that all of labor income is return to human capital investment. However, if some component of labor income is independent of human capital investment and if this component is random (random endowment of genetic skills), then some part of the variance of labor income is not human capital risk. Second, by ignoring job mobility the empirical literature cited above attributes wage hikes due to improved firm-worker matches to income risk, a point that has been emphasized by Low, Meghir, and Pistaferri (2008).

The above calibration procedure ensures that the model economy matches as many features of the US economy as there are free parameters. It is also interesting to investigate how the calibrated model performs in matching additional features of the U.S. economy. For example, if there is no consumption insurance beyond self-insurance ($\theta_a(s) = 0$), the implied values for the average return on physical and human capital are $r_k = 5.52\%$ and $r_h = 9.47\%$, respectively. The return $r_k = 5.52\%$ is higher than the observed real interest rate on short-term U.S. government bonds (1%), but lower than the observed real return on US equity (8%). Given that there is no aggregate risk, and therefore no equity premium, in the model, it is not clear which one of the many financial return variables should be used as a basis for calibration, and we therefore conclude that the implied value is within the range of reasonable values.[6] The implied average return on investment in human capital, $r_h = 9.47\%$, is in line with the estimates of rate of returns to schooling.[7] Notice that the implied excess return on human capital investment is $r_h - r_k = 3.95\%$. Thus, the model generates a substantial "human capital premium".

c) Quantitative Results

To be written.

[6] The RBC literature usually strikes a compromise and chooses the parameter values so that the implied return on capital is 4%, which is somewhat lower than the value used here.

[7] The estimates vary considerably across households and studies, with an average of about 10% (Krueger and Lindhal 2001).

Appendix A

Proof of Proposition 1

Define the endogenous state vector as $x_t = (w_t, \theta_t)$, the payoff function as $F(x_t, s_t, x_{t+1}) = u((1 + r(\theta_t, s_t))w_t - w_{t+1})$, and the feasibility correspondence as

$$\begin{aligned} \Gamma(x_t, s_t) \quad = \quad & \Big\{ x_{t+1} \in \mathbf{X} \mid \theta_{k,t+1}(s_t) + \theta_{h,t+1}(s_t) + \sum_{s_{t+1}} \frac{\theta_{a,t+1}(s_{t+1}; s_t)\pi(s_{t+1}|s_t)}{1 + r_f} = 1 \ , \\ & 0 \ \leq \ w_{t+1} \ \leq \ (1 + r(\theta_{t+1}, s_t))w_t \ , \\ & \theta_{k,t+1} \geq 0 \ \ , \ \ \theta_{h,t+1} \geq 0 \ \ , \ \ \theta_{a,t+1}(s_{t+1}; s_t) \geq \bar{\theta}_a(s_{t+1}; s_t) \ . \Big\} \end{aligned} \qquad (A1)$$

Using this notation, the household maximization problem reads

$$\begin{aligned} max \quad & E\left[\sum_{t=0}^{\infty} \beta^t F(x_t, s_t, x_{t+1})|x_0, s_0\right] \qquad (A2) \\ s.t. \quad & x_{t+1} \ \in \ \Gamma(x_t, s_t) \\ & E\left[\sum_{n=0}^{\infty} \beta^n F(x_{t+n}, s_{t+n}, x_{t+n+1})|x_0, s^t\right] \ \geq \ V_d(x_t, s_t) \end{aligned}$$

The corresponding Bellman equation reads:

$$\begin{aligned} V(x, s) \quad = \quad & \max_{x'} \left\{ F(x, s, x') + \beta \sum s' V(x', s')\pi(s'|s) \right\} \qquad (A3) \\ s.t. \quad & x' \ \in \ \Gamma(x, s) \\ & V(x', s') \ \geq \ V_d(x', s') \end{aligned}$$

Define an operator, T, that maps semi-continuous functions into semi-continuous functions as

$$\begin{aligned} TV(x, s) \quad = \quad & \max_{x'} \{ F(x, s, x') + \beta E[V(x', s')|s] \} \qquad (A4) \\ s.t. \quad & x' \in \Gamma(x, s) \\ & V(x's') \ \geq \ V_d(x', s') \ . \end{aligned}$$

Let V_0 be the solution of the Bellman equation (A3) without the participation constraint. A standard contraction mapping argument shows that this function exists if F is continuous, ii) Γ is compact-valued and continuous, and (17) holds.[8] In appendix B, we show that $V_\infty = \lim_{n\to\infty} T^n V_0$ exists, is equal to the maximal solution of the Bellman equation (A), and is also the value function of the sequential maximization problem (A3) if the following four conditions hold: i) F is continuous, ii) Γ is compact-valued and continuous, iii) for all states, (x, s), there exists a feasible plan, π, for the sequential problem (A2) so that the corresponding expected lifetime utility (payoff) is greater than $-\infty$, and iv) for any given state, (x, s), the value function of the sup-problem without participation constraints satisfies $V_0^*(x, s) < +\infty$. Thus, to prove proposition 1 it suffices to show that conditions i)-iv) hold.[9]

The continuity of the payoff function, F, is obvious. The correspondence, Γ, is compact-valued since portfolio-choices, θ', are elements of a closed and bounded subset of $\mathbb{R}^m$. Closedness follows from the fact that the set is defined by equalities and weak inequalities. The set is bounded from below because of the short-sale constraints (6) and it is then bounded from above because of the budget constraint. Continuity of the correspondence Γ is also straightforward to show. A standard argument shows that conditions iii) and iv) hold if condition (17) is satisfied. This proves proposition 1.

Proof of Proposition 2

As before, let V_0 be the solution of the Bellman equation (A3) without the participation

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[8]See, for example, Alvarez and Jermann (1998) or Le Van and Vailakis (2005).

[9]Rustichini (1998) consider a class of dynamic programming problems with participation constraint (incentive compatability constraint) and possibly unbounded utility. However, he requires bi-convergence, which is always satisfied if lifetime-utility is bounded for all feasible pathes (Streufert, 1990). Unfortunately, in our problem with $\gamma \leq 1$ the requirement of lower convergence is not satisfied, so that Rustichini (1998) is not applicable.

constraint. Simple guess-and-verify shows that V_0 has the following functional form:

$$V_0(w,\theta,s) \;=\; \begin{cases} \tilde{V}_0(s)\,(1+r(\theta,s))^{1-\gamma}\,w^{1-\gamma} & if\ \gamma\ \neq\ 1 \\ \tilde{V}_0(s)\ +\ \frac{1}{1-\beta}log(1+r(\theta,s))\ +\ \frac{1}{1-\beta}w & otherwise \end{cases} \qquad (A5)$$

where $\tilde{V}_0$ is the solution to the intensive-form Bellman equation (20) without participation constraint. Let the operator T be defined as in (A4). We show by induction that if $V_n = T^n V_0$ has the functional form, then $V_{n+1} = T^{n+1}V_)$ has the functional form. For $n = 0$ the claim is true because V_0 has the functional form. Suppose now V_n has the functional form. We then have

$$\begin{aligned} V_{n+1}(w,\theta,s) \;&=\; TV_n(w,\theta,s) \\ &=\; \max_{w',\theta'}\left\{\frac{((1+r(\theta,s))w-w')^{1-\gamma}}{1-\gamma}\ +\ \sum_{s'}\tilde{V}_n(s')(1+r(\theta',s'))^{1-\gamma}(w')^{1-\gamma}\pi(s'|s)\right\} \\ s.t.\ \ 1\ &=\ \theta'_k+\theta'_h+\sum_{s'}\frac{\theta'_a(s')\pi(s'|s)}{1+r_f} \qquad\qquad (A6) \\ &0\ \leq\ w'\ \leq\ (1+r(\theta,s))w\ \ ,\ \ \theta'_k\geq 0\ \ ,\ \ \theta'_h\geq 0\ \ ,\ \ \theta'_a(s')\geq\bar{\theta}_a(s') \\ &\tilde{V}_n(s')\,(1+r(\theta'_k,\theta'_h,\theta_a(s')))^{1-\gamma}\,(w')^{1-\gamma} \\ &\geq\ \tilde{V}_d(s')(1+r(\theta'_k,\theta'_h,0,s')-\phi\theta'_k(s))^{1-\gamma}(w')^{1-\gamma} \end{aligned}$$

for $\gamma \neq 1$ and

$$\begin{aligned} V_{n+1}(w,\theta,s) \;&=\; TV_n(w,\theta,s) \\ &=\; \max_{w',\theta'}\Bigg\{log\,(1+r(\theta,s))w-w')\ +\ \beta\sum_{s'}\tilde{V}_n(s')\pi(s'|s) \\ &\qquad +\frac{\beta}{1-\beta}\sum_{s'}log(1+r(\theta',s'))\pi(s'|s)\ +\ \frac{\beta}{1-\beta}log\,w'\Bigg\} \\ s.t.\ \ 1\ &=\ \theta'_k+\theta'_h+\sum_{s'}\frac{\theta'_a(s')\pi(s'|s)}{1+r_f} \\ &0\ \leq\ w'\ \leq\ (1+r(\theta,s))w\ \ ,\ \ \theta'_k\geq 0\ \ ,\ \ \theta'_h\geq 0\ \ ,\ \ \theta'_a(s')\geq\bar{\theta}_a(s') \\ &\tilde{V}_n(s')\ +\ \frac{1}{1-\beta}log(1+r(\theta'_k,\theta'_h,\theta'_a(s'),s'))\ +\ \frac{1}{1-\beta}log\,w' \\ &\geq\ \tilde{V}_d(s')\ +\ \frac{1}{1-\beta}log(1+r(\theta'_k,\theta'_h,0,s'))\ +\ \frac{1}{1-\beta}log\,w' \end{aligned}$$

for the log-utility case. Clearly, the solution to the maximization problem defined by the right-hand-side of (A6) has the form

$$\begin{aligned} w'_{n+1} &= (1-\tilde{c}_{n+1}(s))(1+r(\theta_{n+1},s))w \qquad (A7) \\ \theta'_{n+1} &= \theta'_{n+1}(s) \ , \end{aligned}$$

where the subscript $n+1$ indicates step $n+1$ in the iteration. Thus, we have

$$V_{n+1}(w,\theta,s) = \begin{cases} \tilde{V}_{n+1}(s)\left(1+r(\theta,s)\right)^{1-\gamma} w^{1-\gamma} & if\ \gamma \neq 1 \\ \tilde{V}_{n+1}(s) + \frac{1}{1-\beta} log(1+r(\theta,s)) + \frac{1}{1-\beta} w & otherwise \end{cases} \qquad (A8) \quad ,$$

where $\tilde{V}_{n+1}$ is defined accordingly.

From proposition 1 we know that $V_\infty = \lim_{n\to\infty} T^n V_0$ exists and that it is the maximal solution to the Bellman equation (A6) as well as the value function of the corresponding sequential maximization problem. Since the set of functions with functional form given by (A5) is a closed subset of the set of semi-continuous functions, we know that V_∞ has the functional form. This prove proposition 2.

Proof of Proposition 3

From proposition 2 we know that individual households maximize utility subject to the budget constraint and participation constraint if condition (17) is satisfied. It is easy to see that condition (17) is satisfied if the proposed portfolio choice, θ', satisfies condition (24). Thus, it remains to show that market clearing conditions (11) and (14) hold and that the law of motion (22) describes the equilibrium evolution of the relative wealth distribution.

Using the wealth-independence of optimal portfolio choices in conjunction with the definition of ω, we find the following expression for the aggregate capital stock held by households:

$$\begin{aligned} K_{t+1} &= \int_{x_0} \sum_{s^t} k_{t+1}(x_0,s^t)\pi(s^t|x_0)d\pi(x_0) \qquad (A9) \\ &= \int_{x_0} \sum_{s^t} \theta_k(s_t) w_{t+1}(x_0,s^t)\pi(s^t|x_0)d\pi(x_0) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s_t} \theta_k(s_t) \int_{x_0} \sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^t|x_0)d\pi(x_0) \\
&= W_{t+1} \sum_{s_t} \theta_k(s_t)\omega(s_t) \ .
\end{aligned}
$$

A similar expression holds for the aggregate stock of human capital, H_{t+1}, held by households. This proves the equivalence of the first intensive-form market clearing condition in (21) with the market clearing condition (11).

For the aggregate value of all financial asset holdings we find:

$$
\begin{aligned}
E[a_{t+1}q_t] &= \int_{x_0} \sum_{s^{t+1}} a_{t+1}(s_{t+1}; x_0, s^t) q_t(s_{t+1}; x_0, s^t)\pi(s^t|x_0)d\pi(x_0) \qquad (A10) \\
&= \frac{1}{1+r_f} \int_{x_0} \sum_{s^{t+1}} \theta_a(s_{t+1}; s_t) w_{t+1}(x_0, s^t)\pi(s_{t+1}|s_t)\pi(s^t|x_0)d\pi(x_0) \\
&= \frac{1}{1+r_f} \sum_{s_t, s_{t+1}} \theta_a(s_{t+1}; s_t)\pi(s_{t+1}|s_t) \int_{x_0} \sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^t|x_0)d\pi(x_0) \\
&= \frac{W_{t+1}}{1+r_f} \sum_{s_t, s_{t+1}} \theta_a(s_{t+1}; s_t)\pi(s_{t+1}|s_t)\omega(s_t) \ .
\end{aligned}
$$

This proves the equivalence of the second intensive-form market clearing condition in (21) with the market clearing condition (14).

Finally, we derive the law of motion for ω. Using the definition and the optimal policy functions (19), we find that the growth rate of aggregate wealth is

$$
\begin{aligned}
\frac{W_{t+1}}{W_t} &= \frac{\int_{x_0} \sum_{s^t} w_{t+1}(x_0, s^t)\pi(s^t|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} \qquad (A11) \\
&= \frac{\int_{x_0} \sum_{s^t} (1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t)) w_t(x_0, s^{t-1})\pi(s^t|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} \\
&= \frac{\sum_{s_{t-1}} \sum_{s_t} (1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t))\pi(s_t|s_{t-1}) \int_{x_0} \sum_{s^{t-2}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} \\
&= \sum_{s_{t-1}} \sum_{s_t} (1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t))\pi(s_t|s_{t-1})\omega_t(s_{t-1}) \ .
\end{aligned}
$$

A similar argument shows

$$
\frac{\sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} = \sum_{s_{t-1}} (1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t))\pi(s_t|s_{t-1})\omega_t(s_{t-1}) \ . \qquad (A12)
$$

Thus, the law of motion for ω reads:

$$\begin{aligned}
\omega_{t+1}(s_t) &= \frac{\int_{x_0} \sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^t} w_{t+1}(x_0, s^t)\pi(s^t|x_0)d\pi(x_0)} \qquad (A13) \\
&= \frac{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^t} w_{t+1}(x_0, s^t)\pi(s^t|x_0)d\pi(x_0)} \frac{\sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} \\
&= \frac{W_t}{W_{t+1}} \frac{\sum_{s^{t-1}} w_{t+1}(x_0, s^t)\pi(s^{t-1}|x_0)d\pi(x_0)}{\int_{x_0} \sum_{s^{t-1}} w_t(x_0, s^{t-1})\pi(s^{t-1}|x_0)d\pi(x_0)} \\
&= \frac{\sum_{s_{t-1}}(1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t))\pi(s_t|s_{t-1})\omega_t(s_{t-1})}{\sum_{s_{t-1}}\sum_{s_t}(1-\tilde{c}(s_t))(1+r(\theta(s_{t-1}), s_t))\pi(s_t|s_{t-1})\omega_t(s_{t-1})}
\end{aligned}$$

which proves the last part of proposition 3.

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