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Existence and Stability of Overconsumption Equilibria

Gregory Ponthiere*

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Abstract

Growth models with endogenous mortality assume generally that life expectancy is increasing with output per capita, and, thus, with individual consumption, whatever the consumption level is. However, empirical evidence on the effect of overconsumption and obesity on mortality tends to question that postulate. This paper develops a two-period OLG model where life expectancy is a non-monotonic function of consumption. The existence, uniqueness and stability of steady-state equilibria are studied. It is shown that overconsumption equilibria - i.e. equilibria at which consumption exceeds the level maximizing life expectancy - exist in highly productive economies with a low impatience. Stability analysis highlights conditions under which there exist non-converging cycles in output and longevity around overconsumption equilibria.

Keywords: longevity, growth, overconsumption, obesity, OLG model.

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1 Introduction

Growth theory has recently paid a particular attention to the study of the relationship between economic development and survival conditions. Following the pioneer works of Blackburn and Cipriani (2002), Chakraborty (2004) and Galor and Moav (2005), various models have been built to explore how accumulation mechanisms interact with survival conditions.[1]

Although those models differ on significant grounds, a major common feature concerns the modelling of the two-directional relation between economic growth and survival conditions. In those models, longevity affects accumulation decisions (e.g. savings, schooling) through a horizon effect: better survival prospects make agents accumulate more, which is beneficial for the long-run equilibrium of the economy. But those models highlight also the existence of a feedback effect, from economic development to survival conditions. In general, that feedback mechanism takes a simple, *monotonic* form: economic growth is assumed to raise longevity through a survival function that is increasing in human capital

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[1] See Bhattacharya and Qiao (2005), Cervallati and Sunde (2005), Chakraborty and Das (2005), Zhang *et al* (2006), and de la Croix and Licandro (2007).

or in physical capital (through public and/or private health spending). Hence, in those models, a higher output leads necessarily to a higher life expectancy.[2]

The assumption of a monotonic influence of output on survival conditions has two major advantages. On the one hand, it is an analytically convenient way to allow for the endogenization of mortality. On the other hand, this postulate is, at least as a first approximation, in line with the available historical trends showing a positive correlation between consumption and longevity over time.[3]

However, there are some reasons to question the monotonicity postulate. Those reasons have to do with one of its corollaries, concerning the longevity / consumption relation. Actually, existing models predict that longevity must be increasing monotonically with consumption. The problem is that this is not fully compatible with the data. If one adopts a cross-sectional perspective and plots the levels of consumption and life expectancy across countries, it is easy to see that the relationship between the two variables is far from monotonic. As shown by the tendency curve drawn on Figure 1, life expectancy at birth is increasing with consumption per capita only up to some level of consumption, but, beyond that level, life expectancy is declining in consumption.[4]

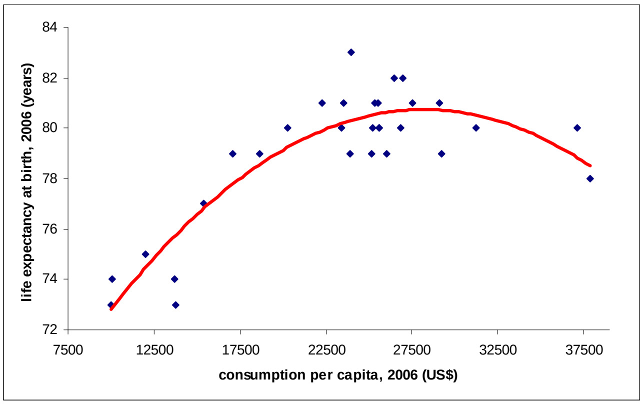

Figure 1: Consumption per head and life expectancy at birth, 2006.

Obviously, Figure 1 is not a proof of the non-monotonicity of the relation under study: the non-linear relationship may actually hide the influence of omitted variables correlated with consumption and influencing life expectancy in a non-monotonic way. Nevertheless, Figure 1 has a strong corollary for the modelling of longevity in models where survival functions have a *single* input. Clearly, Figure 1 suggests that if one wants, on the grounds of analytical tractability,

[2] One exception is Jouvet *et al* (2007), where production-related pollution reduces longevity.

[3] In particular, economic historians emphasized the link between survival conditions and economic development through a larger food consumption (see Fogel, 1994, 2004).

[4] The countries of Figure 1 are: Australia, Austria, Belgium, Canada, Czech Republic, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Japan, Luxemburg, Mexico, New Zealand, Netherlands, Norway, Poland, Portugal, Slovakia, South Korea, Spain, Sweden, Switzerland, Turkey, the United Kingdom, and the United States. Consumption statistics (in US$ with current PPPs) are from the OECD (2009). Period life expectancy statistics (average for both sexes) are from the World Health Organization (2009).

to keep a survival function with a unique input (either consumption or another variable correlated with it), the monotonicity postulate is hard to justify.

Beyond the incapacity of monotonic survival functions to fit aggregate data, the monotonicity postulate can also be questionned in the light of the large epidemiological literature on the negative effects of overconsumption on survival.[5] Epidemiological studies emphasized a non-monotonic relationship between the body mass index and mortality risks. Under the - quite mild - postulate of a link between food consumption and the body mass index, it is straigthfoward to deduce that more consumption is not necessarily better for health and survival. Although more consumption is good for health up to some level, excessive consumption becomes health deteriorating, and may undermine survival prospects.

As a reaction to the growing literature on the effects of obesity and overconsumption, the economics of obesity has developed rapidly.[6] That emerging field has, among other things, studied the economic determinants of obesity, such as the secular fall of food prices induced by technological progress (see Lakdawalla and Philipson, 2002). Moreover, it has also questionned the capacity of agents to anticipate the effects of their consumption choices on future health, and proposed several behavioural theories aimed at explaining obesity.[7]

Despite the expanding literature on overconsumption, there has been so far no attempt, within growth theory, to account for the non-monotonic relationship between consumption and survival prospects. Existing models made survival depend monotonically on a variable correlated positively with consumption, e.g. (physical or human) capital. But such a simplification may not be benign for the dynamics of output and mortality. The monotonicity assumption, although analytically convenient, may truncate the long-run dynamics of the economy.

The goal of this paper is precisely to study the dynamics of longevity and production in an economy where survival prospects depend on consumption in a non-monotonic manner. For that purpose, we develop a two-period OLG model with physical capital accumulation, largely in line with Chakraborty's (2004) seminal paper. However, a major difference with respect to Chakraborty's model is that, in our economy, life expectancy is increasing with first-period consumption up to some "healthy" consumption level, but starts declining for higher consumption levels. As a consequence of this, life expectancy is no longer increasing monotonically with physical capital (unlike in Chakraborty's model).

In economies with a large production capacity, the long-run equilibrium may involve some overconsumption, i.e. a consumption exceeding the level that maximizes survival prospects. That possibility was excluded in existing models, which relied on the "more is always better" postulate. In contrast, this paper pays a particular attention to the conditions guaranteeing the existence of overconsumption equilibria. A strong emphasis will also be laid on the stability of those equilibria. Can there be stable overconsumption equilibria? Or should we expect non-converging cycles to occur? This paper proposes to cast a new light on the existence and stability of overconsumption equilibria.

[5] See Solomon and Manson (1997), Bender *et al* (1998), Fontaine *et al* (2003), Breeze *et al* (2005) and Adams *et al* (2006). The latter study, which focused on a sample of more than 61,000 subjects, estimated that overweight persons exhibit a risk of death that is between 20 and 40 % higher than the one faced by normal persons. Moreover, obese persons exhibit a risk of death that is between two and three times higher than the norm.

[6] See the survey of Philipson and Posner (2008).

[7] See, for instance, Cutler *et al* (2003).

The rest of the paper is organized as follows. The model is presented in Section 2. Section 3 examines the existence, uniqueness and stability of steady-state equilibria. Section 4 illustrates the dynamics of production and longevity on the basis of numerical simulations. Section 5 concludes.

2 The model

Let us consider a two-period OLG model. For simplicity, we assume that reproduction is monosexual, and that each person gives birth to exactly one child.

The first period is a period of young adulthood, during which the adult gives birth to a child, supplies his labour inelastically and saves some resources for the old age. The second period is a period of retirement.

Survival conditions Only a fraction π_{t+1} of a cohort born at time t reaches the retirement age.[8] The fraction π_{t+1} depends on the consumption when being young c_t in a non-monotonic way. There exists a level of consumption c^* that brings the maximum survival probability, i.e. 1. Any departure from c^*, either from below or from above, generates a lower survival probability.

For simplicity, the probability of survival to the old age is determined by the following survival function

$$\pi_{t+1} = \frac{1}{1 + \eta \left(c^* - c_t\right)^2} \tag{1}$$

where $c^* > 0$ is the "healthy" consumption level, that is, the consumption level that yields the maximum life expectancy. The parameter $\eta \geq 0$ captures the impact of consumption on survival prospects. Note that $\lim_{c_t \to 0} \pi_{t+1} = \frac{1}{1+\eta c^{*2}} \equiv \bar{\pi}$, which is close to zero when the healthy consumption level c^* is high. Moreover, we have that $\lim_{c_t \to c^*} \pi_{t+1} = 1$ and $\lim_{c_t \to \infty} \pi_{t+1} = 0$.

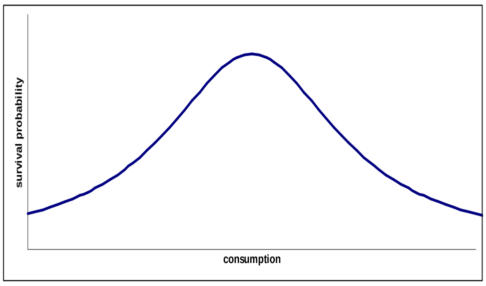

Figure 2: π_{t+1} as a function of c_t

Figure 2 illustrates the relationship between π_{t+1} and c_t.[9] When consumption is inferior to the healthy consumption level c^*, a higher consumption raises survival prospects. On the contrary, if $c_t > c^*$, the opposite prevails: a lower consumption would raise life expectancy.

[8] Hence the life expectancy at birth of cohort t is here equal to $1 + \pi_{t+1}$.

[9] On Figure 1, we have $c^* = 30$, and $\eta = 0.005$.

Production Firms at time t produce some output Y_t according to the production function $Y_t = F(K_t, L_t)$, where K_t denotes the total capital stock, and L_t denotes the labour force. For simplicity, $F(.)$ takes the Cobb-Douglas form:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} \tag{2}$$

where $0 < \alpha < 1$ and $A > 0$. We assume also a full depreciation of capital after one period of use.

In intensive terms, the production process can be rewritten as

$$y_t = Ak_t^{\alpha} \tag{3}$$

where k_t is the capital per worker.

Factors are paid at their marginal productivities:

$$R_t = \alpha A k_t^{\alpha - 1} \tag{4}$$

$$w_t = (1-\alpha) A k_t^{\alpha} \tag{5}$$

where R_t is 1 *plus* the interest rate, while w_t is the wage rate.

The savings decision Each young adult at time t makes a single decision: the amount he saves for his old days (i.e. s_t), and, as a consequence of his budget constraint, his consumption when being young c_t.

For analytical convenience, it is also assumed, as in Chakraborty (2004), that a perfect annuity market exists, which yields an actuarially fair return.

Under logarithmic temporal utility, and provided the utility of being dead is normalized to zero, the problem of each young adult is to maximize

$$\log(w_t - s_t) + \beta \pi_{t+1}^e \log\left(\frac{R_{t+1} s_t}{\pi_{t+1}^e}\right) \tag{6}$$

where β is a time preference factor ($0 \leq \beta \leq 1$), while π_{t+1}^e is the subjective probability of survival to the second period.

The actual survival probability to the old age depends on the consumption when being young. However, to be in line with the microeconomics of obesity and overconsumption, it is assumed here that the agent does not, when he chooses his consumption pattern, internalize the impact of his choice on his survival prospects. On the contrary, the agent takes the future survival prospects *as given*, and thus independent from his behaviour. Agents' decisions are assumed to be based on myopic anticipations about π_{t+1} (i.e. $\pi_{t+1}^e = \pi_t$).

The first-order condition for optimal savings yields

$$s_t = \frac{\beta \pi_t}{1 + \beta \pi_t} w_t \tag{7}$$

As usual, savings is increasing with the expected lifetime of the agent.[10]

[10] Under a full anticipation of the impact of consumption on survival, we would have:

$$s_t \left[1 + \frac{\beta \log\left(\frac{R_{t+1} s_t}{\pi_{t+1}}\right) \pi'(c_t)\, c_t}{1 + \beta \pi_{t+1}}\right] = \frac{\beta \pi_{t+1}}{1 + \beta \pi_{t+1}} w_t$$

where $\pi'(c_t)$ is the derivative of the survival function with respect to consumption. This FOC

3 Steady-state equilibria

Let us now study the long-run dynamics of the economy. Given the replacement fertility and the full depreciation of capital, the capital market equilibrium condition is

$$k_{t+1} = s_t \tag{8}$$

Hence, by substituting for the wage into optimal savings, one has:

$$k_{t+1} = \frac{\beta \pi_t}{1 + \beta \pi_t} A(1-\alpha) k_t^{\alpha}$$

Obviously, if $k_t = 0$, we have $k_{t+1} = 0$, whatever π_t is. Hence, in the (π_t, k_t) space, the kk locus, that is, the combinaisons of k_t and π_t such that k_t is constant over time, includes the horizontal axis, i.e., all points $(\pi_t, 0)$.

Imposing $k_{t+1} = k_t \neq 0$ gives the other part of the kk locus:

$$k_t = \left[\frac{\beta \pi_t A(1-\alpha)}{1 + \beta \pi_t} \right]^{\frac{1}{1-\alpha}} \equiv G(\pi_t) \tag{9}$$

We have $G(0) = 0$, $G'(\pi_t) > 0$ and $G(1) < \infty$. Figure 3 illustrates the kk locus in the (π_t, k_t) space.[11] Under $k_t > 0$, the sustainable level of capital is unique, and is increasing in π_t: the higher the life expectancy $1 + \pi_t$ is, the higher the sustainable level of capital is. Levels of k_t higher than the kk locus cannot, given the prevailing survival conditions, be reproduced over time. Inversely, levels of capital lower than the kk locus lead to a larger capital at the next period (as a high life expectancy implies here a large propensity to save).

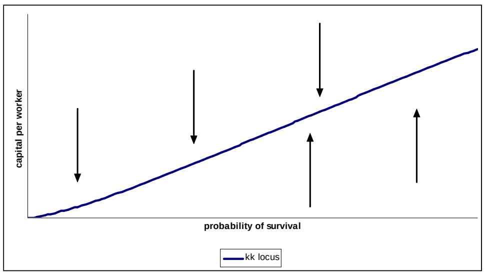

Figure 3: The kk locus

Figure 3 illustrates the relation between the time horizon of agents and the sustainable level of capital. In particular, for bad survival conditions (i.e. a low π_t), only extremely low capital levels can be reproduced over time.

is equal to the previous one only if consumption does not affect survival prospects: $\pi'(c_t) = 0$. However, if π_{t+1} depends on consumption, that derivative is not zero. Clearly, if c_t is below (resp. above) c^*, we have $\pi'(c_t) > 0$ (resp. $\pi'(c_t) < 0$), so that the agent tends to save too much (resp. too little) - and to consume too little (resp. too much) in the first period - with respect to what maximizes his lifetime welfare.

[11] We have $A = 25$, $\alpha = 0.30$, $\beta = 0.40$, $c^* = 20$ and $\eta = 0.01$.

Let us now derive the $\pi\pi$ locus, that is, the combinaisons of k_t and π_t such that π_t is constant over time. From the survival function, we have

$$\pi_{t+1} = \frac{1}{1+\eta\left(c^* - \left(\frac{1}{1+\beta\pi_t}A(1-\alpha)k_t^\alpha\right)\right)^2}$$

Imposing $\pi_{t+1} = \pi_t$ gives the $\pi\pi$ locus. Because of the squared bracket at the denominator of the survival function, there exists, in general, not *one*, but *two* levels of capital that maintain π_t at a constant level. These are given by:

$$k_t = \left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{1+\beta\pi_t}{A(1-\alpha)}\right]^{\frac{1}{\alpha}} \equiv H_1(\pi_t) \quad (10)$$

$$k_t = \left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{1+\beta\pi_t}{A(1-\alpha)}\right]^{\frac{1}{\alpha}} \equiv H_2(\pi_t) \quad (11)$$

If $\pi_t = 1$, then $H_1(1) = H_2(1)$, as there is a unique level of capital that makes π_t constant at its maximal level, and that level of k_t is such that consumption equals c^*. For $0 \leq \pi_t < 1$, we have $H_2(\pi_t) > H_1(\pi_t)$. Hence, we shall call $H_1(\pi_t)$ the low branch of the $\pi\pi$ locus, and $H_2(\pi_t)$ the high branch of the $\pi\pi$ locus. Those two branches intersect only at $\pi_t = 1$.

Regarding the shape of $H_1(\pi_t)$, we have $\lim_{\pi_t\to 0} H_1(\pi_t) = -\infty$ if $1/\alpha$ is odd, and $\lim_{\pi_t\to 0} H_1(\pi_t) = +\infty$ if $1/\alpha$ is even.[12] Moreover, if $1/\alpha$ is odd, we have $H_1'(\pi_t) > 0$ for any π_t. On the contrary, for $1/\alpha$ even, we have $H_1'(\pi_t) < 0$ for $\pi_t < \bar{\pi}$ and $H_1'(\pi_t) > 0$ for $\pi_t > \bar{\pi}$, where $\bar{\pi} = \frac{1}{1+\eta c^{*2}}$. Note also that, if $\pi_t = \frac{1}{1+\eta c^{*2}} \equiv \bar{\pi}$, then $H_1(\pi_t) = 0$.

As far as $H_2(\pi_t)$ is concerned, we have $\lim_{\pi_t\to 0} H_2(\pi_t) = +\infty$. Moreover, we have $H_2'(\pi_t) < 0$ for any π_t such that

$$\left(\frac{1}{\eta}\right)^{\frac{1}{2}}\left(\frac{1-\pi_t}{\pi_t}\right)^{\frac{1}{2}}\left[\frac{1}{2}\frac{(1+\beta\pi_t)}{\pi_t(1-\pi_t)} - \beta\right] > c^*$$

If $\pi_t < \bar{\pi}$, we have $\left(\frac{1}{\eta}\right)^{\frac{1}{2}}\left(\frac{1-\pi_t}{\pi_t}\right)^{\frac{1}{2}} > c^*$. Hence, given that the term in brackets lies in the interval $[2, +\infty]$, it must be the case that $H_2'(\pi_t) < 0$. Alternatively, if $\pi_t > \bar{\pi}$, we have $\left(\frac{1}{\eta}\right)^{\frac{1}{2}}\left(\frac{1-\pi_t}{\pi_t}\right)^{\frac{1}{2}} < c^*$, and so it may be the case that $H_2'(\pi_t) > 0$.

Figures 4a and 4b illustrate the form of the $\pi\pi$ locus in the (π_t, k_t) space.[13] For $\pi_t < 1$, there exists an interval of levels for capital per worker that allow a growth of life expectancy over time. That interval lies between the two branches of the $\pi\pi$ locus. Levels of k_t outside that interval lead to a fall of life expectancy. Such a fall can arise either because k_t is too low (bottom of the phase diagram), or because k_t is too high (top of the phase diagram). In the former case, consumption is lower than the level that would maintain life expectancy unchanged. As a consequence, π_t must fall, as only a lower survival probability can be sustained for such a low consumption level. In the latter

[12] See the Appendix.

[13] On Figure 4a, we have $A = 25$, $\alpha = 0.30$, $\beta = 0.40$, $c^* = 20$ and $\eta = 0.01$. Same values on Figure 4b, except $\alpha = 0.50$.

case, a fall of π_t arises because k_t is too high: consumption exceeds the level that would maintain π_t constant, explaining the fall of π_t.

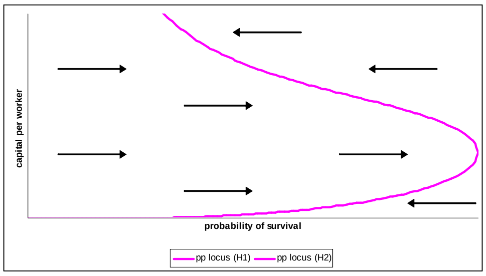

Figure 4a: The $\pi\pi$ locus ($1/\alpha$ is odd)

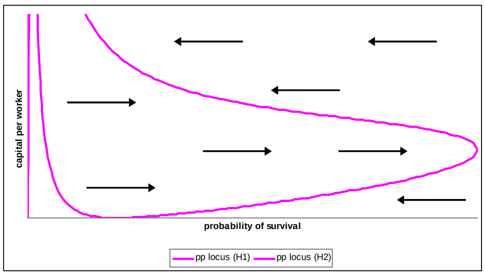

Figure 4b: The $\pi\pi$ locus ($1/\alpha$ is even)

The difference between Figures 4a and 4b concerns the shape of the $\pi\pi$ locus in the neighbourhood of $\pi_t = 0$. In Figure 4a, where $1/\alpha$ is odd, the low branch of the $\pi\pi$ locus tends to $-\infty$ as π_t tends to 0, whereas, in Figure 4b, where $1/\alpha$ is even, the low branch of the $\pi\pi$ locus tends to $+\infty$ as π_t tends to 0.

There can be three distinct kinds of stationary equilibria in the economy under study, as the intersections of the two loci can occur either on the low branch of the $\pi\pi$ locus, or on the high branch of the $\pi\pi$ locus, or at the intersection of the two branches, i.e. at $\pi_t = 1$. In the rest of this paper, we will coin the different types of equilibria as follows. The first type of intersection will be called an *underconsumption equilibrium*, as consumption at such an equilibrium is below the healthy consumption level c^*. The second type of equilibrium will be referred to as an *overconsumption equilibrium*, as consumption exceeds c^* at that equilibrium. The third type of intersection will be called a *healthy consumption equilibrium*, as we have $c_t = c^*$ at that equilibrium.

Let us now use the properties of the kk locus and the $\pi\pi$ locus to study the existence of steady-state equilibria. Proposition 1 summarizes our results.

Proposition 1 *Let us denote* $G(1)$ *by* $k^* \equiv \left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}$ *and* $H_1(1) = H_2(1)$ *by* $\Omega \equiv \left[c^* \frac{1+\beta}{A(1-\alpha)}\right]^{\frac{1}{\alpha}}$.

(1) If $\frac{1}{\alpha}$ *is odd and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} < \Omega$, *there exist at least two underconsumption steady-state equilibria:* $(\bar{\pi}, 0)$ *and* (π_1, k_1), *with* $0 < \bar{\pi} < \pi_1 < 1$ *and* $0 < k_1 < k^*$.

(2) If $\frac{1}{\alpha}$ *is odd and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} > \Omega$, *there exist at least one underconsumption steady-state equilibrium* $(\bar{\pi}, 0)$ *and one overconsumption steady-state equilibrium* (π_2, k_2), *where* $0 < \bar{\pi} \lesseqgtr \pi_2 < 1$ *and* $0 < k^* < k_2$.

(3) If $\frac{1}{\alpha}$ *is odd and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} = \Omega$, *there exist at least one underconsumption steady-state equilibrium* $(\bar{\pi}, 0)$ *and exactly one healthy consumption steady-state equilibrium* $(1, k^*)$.

(4) If $\frac{1}{\alpha}$ *is even and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} < \Omega$*, there exists at least three underconsumption steady-state equilibria* $(\bar{\pi}, 0)$*,* (π_3, k_3) *and* (π_4, k_4)*, with* $0 < \pi_3 < \bar{\pi} < \pi_4 < 1$ *and* $0 < k_3 < k_4 < k^*$*.*

(5) If $\frac{1}{\alpha}$ *is even and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} > \Omega$*, there exists at least two underconsumption steady-state equilibria* $(\bar{\pi}, 0)$ *and* (π_5, k_5)*, and at least one overconsumption equilibrium* (π_6, k_6)*, with* $0 < \pi_5 < \bar{\pi}$*,* $\bar{\pi} \lesseqgtr \pi_6 < 1$ *and* $0 < k_5 < k^* < k_6$*.*

(6) If $\frac{1}{\alpha}$ *is even and if* $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} = \Omega$*, there exists at least two underconsumption steady-state equilibria* $(\bar{\pi}, 0)$ *and* (π_7, k_7)*, and exactly one healthy consumption equilibrium* $(1, k^*)$*, with* $0 < \pi_7 < \bar{\pi} < 1$ *and* $0 < k_7 < k^*$*.*

Proof. See the Appendix. ■

Proposition 1 states that the *minimal* number of steady-state equilibria depends on the structural parameters of the economy.[14] In cases (4) to (6), where $1/\alpha$ is even, there must exist a stationary equilibrium in the left corner of the (π_t, k_t) space, because the low branch of the $\pi\pi$ locus is decreasing for levels of π_t inferior to $\bar{\pi}$, unlike what prevails under cases (1) to (3), where $1/\alpha$ is odd. This is the reason why cases (4) to (6) admit a higher minimal number of steady-state equilibria than cases (1) to (3).

Regarding the distinction between, on the one hand, cases (1) and (4), and, on the other hand, cases (2) and (5), this lies in the types of steady-state equilibria that exist in each case. The stationary equilibria $(\bar{\pi}, 0)$, (π_1, k_1), (π_3, k_3), (π_4, k_4) and (π_5, k_5) are *underconsumption* equilibria, i.e. equilibria located on the low branch of the $\pi\pi$ locus. On the contrary, (π_2, k_2) and (π_6, k_6) are *overconsumption* equilibria, i.e. equilibria located on the high branch of the $\pi\pi$ locus. While this constitutes a significant difference, this does not allow us, however, to deduce whether life expectancy is higher at an underconsumption equilibrium or at an overconsumption equilibrium, as the level of life expectancy depends on the distance between consumption and healthy consumption.

Overconsumption equilibria (π_2, k_2) and (π_6, k_6) are, *ceteris paribus*, more likely to prevail in economies with a higher productivity (i.e. with a high A). Similarly, it is clear, from Proposition 1, that overconsumption equilibria are more plausible in economies where the time preference factor β is close to unity. Under a low impatience, we have $G(1) > H_1(1) = H_2(1)$, which leads to an overconsumption equilibrium. Finally, economies where the healthy consumption level c^* is, because of external reasons, lower are also more likely to exhibit an overconsumption equilibrium.

Cases (1), (2), (4) and (5) of Proposition 1 are illustrated on Figures 5a to 5d, which show the kk locus and the $\pi\pi$ locus in the (π_t, k_t) space.[15] In order to give us some clues regarding the local stability of stationary equilibria, those figures exhibit dynamic arrows, showing the direction of change over time. In

[14] Proposition 1 concerns only the minimal number of equilibria because the existence proof relies on the limits of the functions $G(\pi_t)$, $H_1(\pi_t)$ and $H_2(\pi_t)$ for π_t tending towards 0 and 1. Additional assumptions on the second-order derivatives of the loci are needed to be able to make statements about the actual number of intersections of the two loci (see the Appendix).

[15] On Figure 5a, we have $A = 20$, $\alpha = 0.30$, $\beta = 0.40$, $c^* = 20$, and $\eta = 0.01$. On Figure 5b, we have $A = 25$, $\alpha = 0.30$, $\beta = 0.40$, $c^* = 20$, and $\eta = 0.01$. On Figure 5c, we have $A = 25$, $\alpha = 0.50$, $\beta = 0.40$, $c^* = 20$, and $\eta = 0.001$. On Figure 5d, we have $A = 20$, $\alpha = 0.50$, $\beta = 0.40$, $c^* = 15$, and $\eta = 0.001$.

each case, there exists a vast area of the (π_t, k_t) space where both capital per worker and life expectancy are growing. Note, however, that the size of that area varies strongly across the cases. More importantly, the extent to which an economy with initial conditions (π_0, k_0) can end up in that area varies across cases. Under cases (4) and (5), there exists a large area, in the bottom left corner of the phase diagram, where both capital per worker and life expectancy are falling. In those cases, the intermediate equilibrium is unstable, and acts like a threshold, below which economies are condamned to stagnate, with low output, consumption and life expectancy.

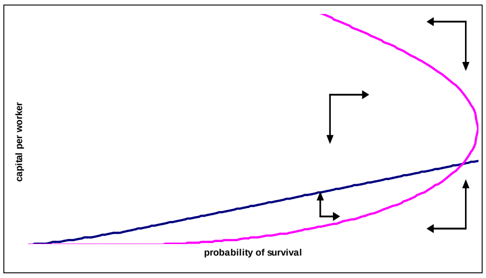

Figure 5a: Case (1)

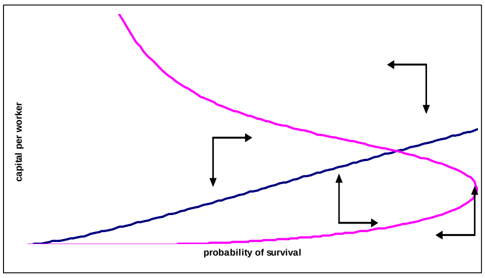

Figure 5b: Case (2)

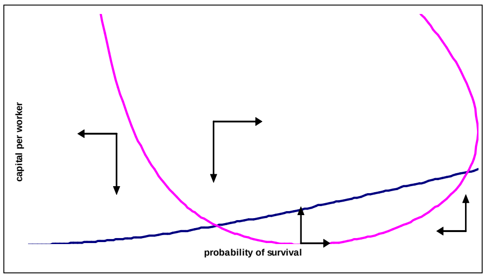

Figure 5c: Case (4)

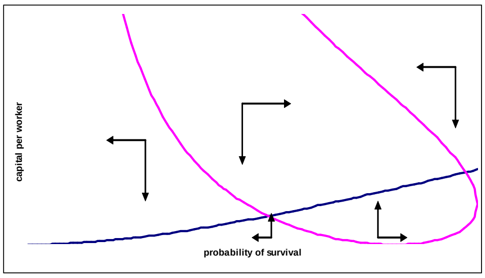

Figure 5d: Case (5)

Whereas this discussion gives us some clues about the instability of some equilibria, it should be reminded, however, that a mere look at a phasis diagram does not suffice to provide accurate results on the stability of equilibria. For instance, on Figure 5b, it is not possible to see, on the mere basis of graphical analysis, whether the equilibrium is stable or not: even though the dynamic arrows seem to point to the equilibrium, nothing insures the actual convergence to that equilibrium. Hence a formal analysis of stability is required.

Such an analysis is carried out in the Appendix of this paper. Proposition 2 summarizes our results.

Proposition 2 *Suppose the economy admits a steady-state equilibrium* (π, k).

The conditions

$$\frac{(1+\alpha)\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}-\left[2\eta\left(c^{*}-c\right)\left(\frac{\beta c}{1+\beta\pi}\right)\right]\left[1+\alpha\frac{1+\beta\pi}{\beta\pi}\right]}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}>0$$

$$\frac{(1-\alpha)\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}+\left[2\eta\left(c^{*}-c\right)\left(\frac{\beta c}{1+\beta\pi}\right)\right]\left[1-\alpha\frac{1+\beta\pi}{\beta\pi}\right]}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}>0$$

$$\frac{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}+2\alpha\eta\left(c^{*}-c\right)\frac{c}{\pi}}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}>0$$

are necessary and sufficient for the local stability of that equilibrium.

Proof. See the Appendix. ■

Those conditions concern the *local* stability only, and not the *global* stability, because, as this was shown above, there exist, in each of the cases (1) to (6), *at least* two steady-state equilibria, so that no condition can guarantee an unconditional convergence towards an equilibrium for *any* initial conditions (π_0, k_0).

The stability conditions stated in Proposition 2 depend on the magnitude of the parameter η, which captures the sensitivity of life expectancy to the consumption behaviour. Clearly, if η tends to 0, the stability conditions of Proposition 2 become respectively $1+\alpha>0$, $1-\alpha>0$ and $1>0$, which are all true given our assumption $0<\alpha<1$. However, larger values of η make the local stability of the stationary equilibrium less likely.

The stability conditions of Proposition 2 are general, and, as such, are not simple to interpret.[16] In order to have a more concrete idea of the conditions under which stability prevails, let us now focus on the case of an overconsumption equilibrium. It can be shown that, for overconsumption equilibria, such as (π_2, k_2) and (π_6, k_6), a simple condition guarantees local stability.

Proposition 3 *The condition*

$$\alpha-\frac{2\eta\left(c^{*}-c\right)\left(c\frac{\beta}{1+\beta\pi}\right)}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}<1$$

is sufficient for the local stability of equilibria (π_2, k_2) *and* (π_6, k_6)*.*

Proof. See the Appendix. ■

A lower elasticity of output with respect to capital α favours, *ceteris paribus*, the stability of the equilibrium. Moreover, the closer the equilibrium consumption is to the healthy consumption c^*, the lower the second term is, making local stability more plausible. Here again, larger values of η make the local stability

[16] One exception concerns the stability of the steady-state equilibrium $(\bar{\pi}, 0)$. Indeed in that case $c = 0$ and it is straigthforward to see that the three conditions of Proposition 2 are satisfied, so that local stability prevails.

of the equilibrium less likely. Finally, the time preference parameter β is also present in the stability condition. The more impatient agents are (i.e. a lower β), the lower the second term is, making stability more plausible.[17]

Let us conclude this stability analysis by considering the possibility of long-run cycles in the (π_t, k_t) space. That question can be formulated as follows: will economies converge, in the long-run, towards *unique* levels of output and life expectancy, or, on the contrary, will economies exhibit cycles around those steady-states?

As stated in Proposition 4, it is only in the presence of an overconsumption equilibrium, that is, in cases (2) and (5) of Proposition 1, that long-run cycles can arise in the (π_t, k_t) space. Such cycles are both *economic* cycles (i.e. in terms of capital, output and consumption) and *demographic* cycles (i.e. in terms of life expectancy and population size). The existence of long-run cycles is subject to some specific conditions on the structural parameters of the economy.

Proposition 4 *There exists no long-run cycle around steady-state equilibria* $(\bar{\pi}, 0)$, (π_1, k_1), (π_3, k_3), (π_4, k_4), (π_5, k_5), (π_7, k_7) *and* $(1, k^*)$.

There exist long-run cycles around the steady-states (π_2, k_2) *and* (π_6, k_6) *if and only if the following conditions are satisfied:*

(i) $\left[\frac{\alpha\left[1+\eta(c^*-c)^2\right]^2-2\eta(c^*-c)\left(\frac{\beta c}{1+\beta\pi}\right)}{\left[1+\eta(c^*-c)^2\right]^2}\right]^2 + 8\frac{\alpha\eta(c^*-c)\left(\frac{\beta c}{1+\beta\pi}\right)\left(1+\frac{1}{\beta\pi}\right)}{\left[1+\eta(c^*-c)^2\right]^2} < 0$

(ii) $\sqrt[2]{\frac{-2\alpha\eta(c^*-c)\left(\frac{\beta c}{1+\beta\pi}\right)\left(1+\frac{1}{\beta\pi}\right)}{\left[1+\eta(c^*-c)^2\right]^2}} = 1$

where π *and* c *take their equilibrium values.*

Proof. See the Appendix. ■

It is easy to see why cycles cannot arise around underconsumption equilibria like $(\bar{\pi}, 0)$, (π_1, k_1), (π_3, k_3), (π_4, k_4) or (π_5, k_5). Indeed, in those cases, the condition (i) is necessarily violated, as $c < c^*$. It is thus only at overconsumption equilibria, like (π_2, k_2) and (π_6, k_6), that condition (i) can be satisfied, and if condition (ii) is also true, then a cycle exists around the steady-state.

Note, here again, the crucial role played by the parameter η. If η is close to zero, cycles cannot occur, as conditions (i) and (ii) are necessarily violated. Moreover, for too high levels of η, it is condition (ii) that would not be satisfied: the economy would just diverge in the long-run, without exhibiting a cycle. Thus, the existence of long-run cycles requires a particular set of conditions, including a sensitivity of life expectancy to consumption behaviour that is neither too small, neither too large.

While Proposition 4 informs us about the general conditions under which long-run cycles exist in the (π_t, k_t) space, it is difficult to know *a priori* whether conditions (i) and (ii) are strong or weak, and whether these are compatible with standard values for the parameters of the economy. The task of the next section is to discuss this by means of numerical simulations.

4 Numerical illustrations

This Section illustrates numerically the dynamics of production and longevity in the economy under study. For that purpose, we will concentrate here on

[17] The intuition behind this lies in the mere fact that, if β is low, the agent's reactions to a change in their expected time horizon are necessarily of smaller size, which favours stability.

the cases of advanced economies, i.e. on economies with a high productivity. Hence, the equilibria under study belong here to the cases (2) and (5) of Proposition 1 (i.e. steady-states (π_2, k_2) and (π_6, k_6)). We will rely on the following benchmark values for the structural parameters of the economy.[18]

Parameters	values
A	30
α	0.300
β	0.400
η	0.010
c^*	30

We shall also take, as initial conditions, $k_0 = 0.1$ and $\pi_0 = 0.05$ (for periods of 30 years, this coincides with an initial life expectancy of about 61.5 years). It is easy to check that, under those parameters values, we have $\left[c^* \frac{1+\beta}{A(1-\alpha)}\right]^{\frac{1}{\alpha}} < \left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}}$, so that, given $1/\alpha$ odd, we are in case (2).

In the light of the discussions in Section 3, one can expect that the dynamics of production and longevity depends on the parameter η, which determines the reaction of longevity when consumption departs from its healthy level. Those intuitions are confirmed by Figures 6a-6d, which show the dynamics of the economy in the (π_t, k_t) space. Under low values of η, the convergence to the long-run equilibrium is monotonic, except when the economy is very close to the long-run equilibrium (see Figures 6a and 6b). A converging spiral holds under $\eta = 0.020$ (Figure 6c). However, under $\eta = 0.030$ (Figure 6d), there is a non-converging cycle, as the economy satisfies the conditions of Proposition 4.

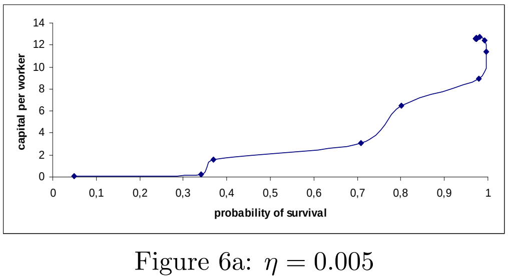

Figure 6a: $\eta = 0.005$

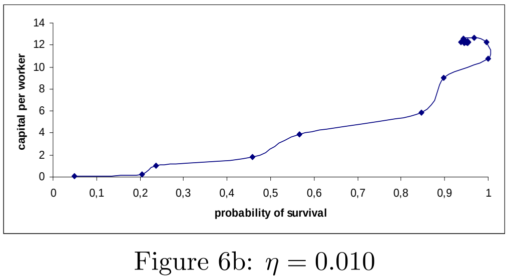

Figure 6b: $\eta = 0.010$

[18]Note that the time preference parameter β, which is fixed to 0.40, is slightly larger than the usual value of 0.30. Actually, 0.30 is generally used, as this coincides with a quarterly discount factor of 0.99. Here we rely on a higher value for β, as there is already some "natural" discounting through the survival probability.

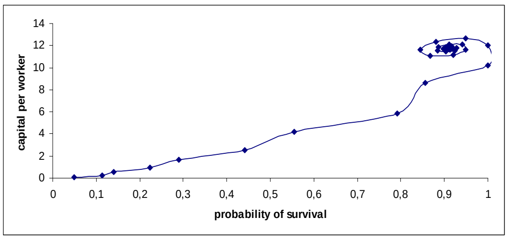

Figure 6c: $\eta = 0.020$

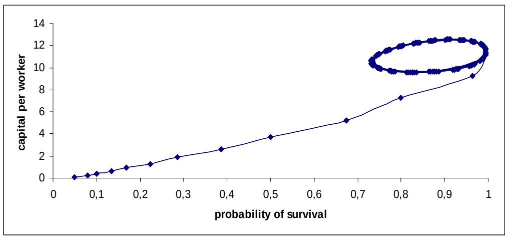

Figure 6d: $\eta = 0.030$

An interesting feature of Figures 6c and 6d is that, if we look at the first 10 periods of time, the dynamics is quasi *identical* whatever η equals 0.020 or 0.030. Given the substantial length of each period (equal to about 30 years), it follows from this that empirical evidence covering something like three centuries of data on output and longevity would not help us to distinguish between $\eta = 0.020$ and $\eta = 0.030$, even though the long-run dynamics induced by those two parametrizations are very different. This constitutes a quite negative result, as one can thus hardly, on the basis of existing data sets, discriminate between different levels of η, and, hence, detect the possible existence of cycles.

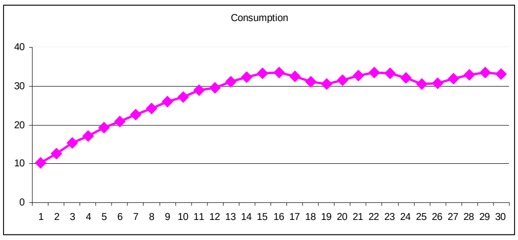

Figure 7a: the dynamics of consumption

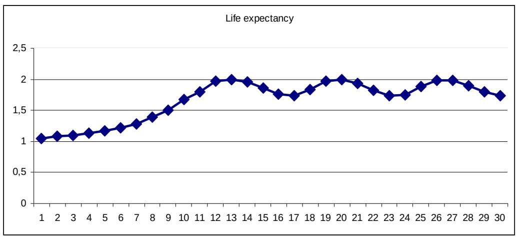

Figure 7b: the dynamics of life expectancy

In order to understand the mechanism behind long-run cycles, Figures 7a and 7b show the time series of consumption and life expectancy under η equal to 0.030. As long as consumption is below the healthy consumption level c^* (equal to 30), the capital accumulation process makes consumption grow towards c^*, implying a growth of life expectancy. Thanks to the horizon effect, a high life expectancy keeps on reinforcing capital accumulation, which raises consumption beyond c^*. This overconsumption leads to a fall of life expectancy. For a low level of η, departures from c^* cause only small changes in life expectancy, which will have minor effects on capital accumulation, and, thus, on future consumption and life expectancy. Hence, for low levels of η, there will be a convergence of c_t and π_t towards the long-run equilibrium. However, for larger values of η, the fall of life expectancy induced by an excessive consumption is significant. Due to the horizon effect, that fall of life expectancy reduces capital accumulation, and, *in fine*, consumption, which falls down towards its healthy level. This raises life expectancy again, which increases savings and capital accumulation, and so forth. A large η, by implying strong upwards and downwards reactions of life expectancy to consumption and *vice versa*, is thus the cause of the instability, which takes here the form of a cycle.[19]

Let us now examine the robustness of those results to the calibration of the other parameters. As shown on Figures 8a-8b, a slight change in the level of healthy consumption c^* has large effects on the long-run dynamics of longevity and capital. On Figure 8a, we can see that, under $\eta = 0.030$, cycles disappear once c^* is raised from 30 to 31. On the contrary, Figure 8b shows that cycles become larger once healthy consumption is slightly reduced.

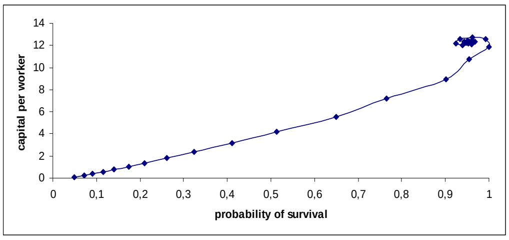

Figure 8a: $\eta = 0.030, c^* = 31$

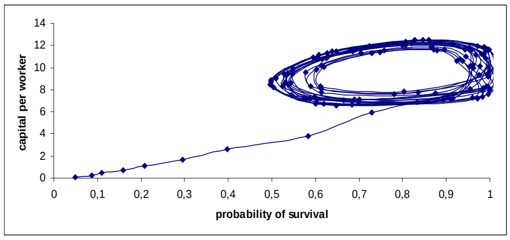

Figure 8b: $\eta = 0.030, c^* = 29$

Beyond η and c^*, other parameters enter the stability condition of Proposition 3, and can be expected to influence the overall dynamics of the economy. Take, for instance, the time preference parameter β. The higher it is, the stronger is the horizon effect *ceteris paribus*, and so the stronger the feedback from capital accumulation to survival conditions is. Note, however, that numerical simulations under the benchmark values of the other parameters do not point to a significant sensitivity of the long-run dynamics to the level of β.[20]

On the contrary, the dynamics of output and longevity is strongly sensitive to the parameter α, i.e. the elasticity of output with respect to capital. This influence is obvious in the light of the stability condition of Proposition 3: the higher α is, the less plausible local stability is. As shown on Figure 9a, once α is raised to 0.4, a level of η as low as 0.008 suffices to bring a non-converging cycle, whereas, under $\alpha = 0.5$, non-converging cycles appear for $\eta = 0.004$ (Figure 9b).

[19] Note that larger levels of η can lead to diverging spirals around the equilibrium.

[20] But this does not mean that this parameter is benign, as it influences naturally the position of the steady-state equilibrium levels of output and life expectancy. Because of space constraints, those simulations are not included here.

Hence, more capital-intensive economies are also likely to exhibit an unstable stationary equilibrium.

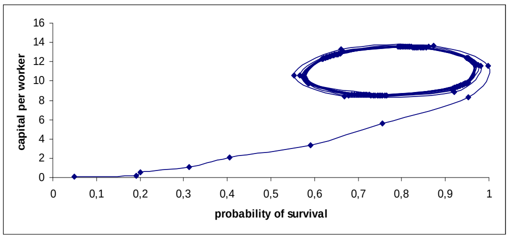

Figure 9a: $\alpha = 0.4, \eta = 0.008$

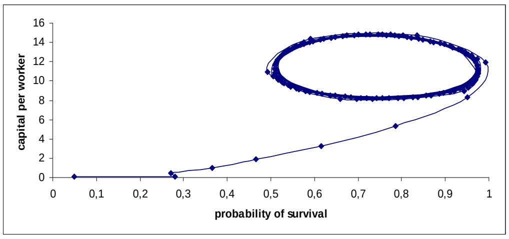

Figure 9b: $\alpha = 0.5, \eta = 0.004$

Finally, note that, although the productivity parameter A does not explicitly enter the stability condition in Proposition 2, it tends, however, to have a significant influence on the dynamics of output and longevity. As shown on Figure 6b, there is, under η equal to its benchmark value of 0.01, a non-monotonic convergence of the economy to the steady-state equilibrium under $A = 30$. Figures 10a and 10b show that changes in A affect the dynamics significantly: the convergence becomes monotonic in π_t and k_t under $A = 25$ (Figure 10a), and there exists a non-converging cycle under $A = 35$ (Figure 10b).

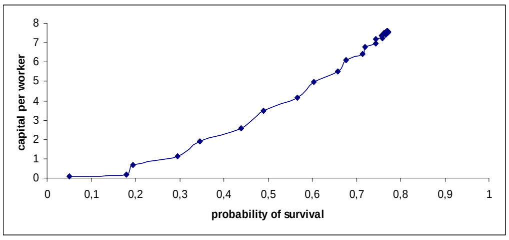

Figure 10a: $A = 25$

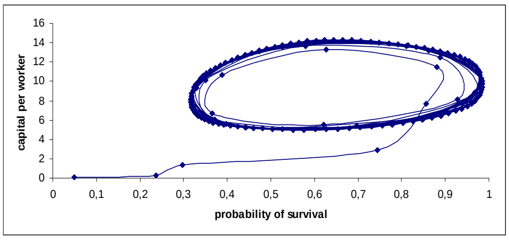

Figure 10b: $A = 35$

Hence, more productive economies are more likely to exhibit a cyclical dynamics than less productive economies. Indeed, in more productive economies, the impact of capital accumulation in terms of output and consumption is larger. As a consequence, the reactions of longevity to capital accumulation are also larger, reinforcing the likelihood of cycles.

That result is worth being stressed, especially if one thinks that a major limitation of the model developed in Section 2 lies in the absence of technological progress. Figure 10b suggests that if some exogenous technological progress was assumed instead (i.e. a variable $A_t \equiv A(1+g)^t$ in the production function), the high levels of A_t reached after some periods of time would lead the economy to fluctuations in output and life expectancy. Hence the introduction of some exogenous technological progress would reinforce the likelihood of long-run cycles. However, *endogenous* technological progress may or may not do so, depending on the precise modelling of the determinants of technological change.

5 Conclusions

The large theoretical literature on the relation between economic growth and longevity gains assumes usually that survival is increasing monotonically with the level of (physical or human) capital, either directly or indirectly (e.g. through health spending). Nevertheless, that postulate has a counterintuitive corollary: survival must, in general, be correlated positively with consumption, whatever the consumption level is. That corollary is not compatible with the epidemiological literature showing that excess consumption leads to a larger mortality.

This paper developed a two-period OLG model where the probability of survival to the second period is non-monotonic in consumption, and is increasing in consumption only as long as consumption lies below a healthy consumption level. The study of the existence, uniqueness and stability of steady-state equilibria revealed that the dynamics of output and longevity varies strongly with the structural parameters of the economy, which can lead to either an overconsumption or an underconsumption equilibrium. The stability of the equilibrium is not guaranteed: cycles may exist around long-run equilibria, in the sense that periods of economic growth and longevity improvement would be followed by periods of economic contraction and lower life expectancy, and so forth.

Long-run economic and demographic cycles exist only around overconsumption equilibria, and under particular conditions on the structural parameters of the economy. Cycles are more plausible in economies with a high sensitivity of life expectancy to consumption behaviour, and where, because of exogenous reasons, the healthy consumption level is lower. Cycles are also more likely the more productive and capital-intensive the economy is. Those features are all shared by advanced economies, so that the present findings do not allow us to exclude *a priori* the possibility of future economic and demographic cycles due to the nefast effect of overconsumption on survival conditions.

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7 Appendix

7.1 Proposition 1

The existence of a steady-state equilibrium can be explored as the issue of the intersection of the kk locus and the $\pi\pi$ locus in the two-dimensional space (π_t, k_t). For that purpose, let us first derive the basic properties of those two loci.

We know that $G(0) = 0$ and that, under $k_t = 0$, π_t is maintained constant at a level $\bar{\pi}$.

The first-order derivative of $G(.)$ is

$$G'(\pi_t) = \frac{1}{1-\alpha}\left[A(1-\alpha)\beta\right]^{\frac{1}{1-\alpha}}\pi_t^{\frac{1}{1-\alpha}-1}\left[\frac{1}{1+\beta\pi_t}\right]^{\frac{2-\alpha}{1-\alpha}} > 0$$

Regarding $H_1(\pi_t)$, we have

$$\begin{aligned}\lim_{\pi_t\to 0} H_1(\pi_t) &= \left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{0}\right)^{1/2}\right)\frac{1}{A(1-\alpha)}\right]^{\frac{1}{\alpha}} \\ &= -\infty \text{ if } 1/\alpha \text{ is odd} \\ &= +\infty \text{ if } 1/\alpha \text{ is even}\end{aligned}$$

Moreover

$$\begin{aligned}H_1'(\pi_t) &= \frac{1}{\alpha}\left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1} \\ &\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{2}\frac{\frac{1}{\pi_t}-\beta}{\pi^{1/2}\left(1-\pi\right)^{1/2}A(1-\alpha)}\right) + c^*\frac{\beta}{A(1-\alpha)}\right]\end{aligned}$$

If $1/\alpha$ is odd, $(1/\alpha)-1$ is even, so that the first factor is always positive, whatever π_t is. Given that the second factor is always positive (as $\beta \leq 1 \leq \frac{1}{\pi_t}$), we have thus $H_1'(\pi_t) > 0$. However, if $1/\alpha$ is even, $(1/\alpha)-1$ is odd, so that the first factor is negative for low levels of $\pi_t < \bar{\pi} = \frac{1}{1+\eta c^{*2}}$, but positive for $\pi_t > \bar{\pi}$. Given that the second factor is always positive, we have $H_1'(\pi_t) < 0$ for $\pi_t < \bar{\pi}$ and $H_1'(\pi_t) > 0$ for $\pi_t > \bar{\pi}$. At $\pi_t = \bar{\pi}$, we have $H_1'(\pi_t) = 0$.

Note also that:

$$\begin{aligned}\lim_{\pi_t\to 0} H_1'(\pi_t) &= \frac{1}{\alpha}\left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{0}\right)^{1/2}\right)\frac{1}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1} \\ &\left[\frac{1}{2}\left(\frac{1}{0}\right)^{-1/2}\left(\frac{1}{\eta}\right)^{1/2}\frac{1}{0}\left(\frac{1}{A(1-\alpha)}\right) + \left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{0}\right)^{1/2}\right)\frac{\beta}{A(1-\alpha)}\right] \\ &= +\infty \text{ if } 1/\alpha \text{ is odd} \\ &= -\infty \text{ if } 1/\alpha \text{ is even}\end{aligned}$$

and

$$\begin{aligned}\lim_{\pi_t\to 1} H_1'(\pi_t) &= \frac{1}{\alpha}\left[c^*\frac{(1+\beta)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1}\left[\frac{1}{2}\left(\frac{0}{1}\right)^{-1/2}\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1+\beta}{A(1-\alpha)}\right) + c^*\frac{\beta}{A(1-\alpha)}\right] \\ &= +\infty\end{aligned}$$

Regarding $H_2(\pi)$, we have

$$\begin{aligned}\lim_{\pi_t\to 0} H_2(\pi_t) &= \left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{0}\right)^{1/2}\right)\frac{1}{A(1-\alpha)}\right]^{\frac{1}{\alpha}} \\ &= +\infty\end{aligned}$$

Moreover,

$$H_2'(\pi_t) = \frac{1}{\alpha}\left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1}$$
$$\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{-\frac{1}{2}\pi_t^{1/2}(1+\beta\pi_t)+\beta(1-\pi_t)}{\pi_t^2(1-\pi_t)^{1/2}A(1-\alpha)}\right)+c^*\frac{\beta}{A(1-\alpha)}\right]$$

The first factor is always positive, whatever π_t is. But the second factor can be positive or negative, depending on the level of π_t.Actually, if $\pi_t < \bar{\pi} = \frac{1}{1+\eta c^{*2}}$, we always have $H_2'(\pi_t) < 0$, but this may not be the case for $\pi_t > \bar{\pi}$.

Note also that:

$$\lim_{\pi_t \to 0} H_2'(\pi_t) = \frac{1}{\alpha}\left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-0}{0}\right)^{1/2}\right)\frac{1}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1}$$
$$\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{-\frac{1}{2}0_t^{1/2}+\beta}{0A(1-\alpha)}\right)+c^*\frac{\beta}{A(1-\alpha)}\right]$$
$$= +\infty$$

and

$$\lim_{\pi_t \to 1} H_2'(\pi_t) = \frac{1}{\alpha}\left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}(0)^{1/2}\right)\frac{(1+\beta)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1}$$
$$\left[\frac{1}{2}(0)^{-1/2}\left(\frac{1}{\eta}\right)^{1/2}\frac{-1}{1}\left(\frac{1+\beta}{A(1-\alpha)}\right)+\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{0}{1}\right)^{1/2}\right)\frac{\beta}{A(1-\alpha)}\right]$$
$$= -\infty$$

Let us now consider the existence problem on a case-by-case basis.

First of all, given that a zero capital level maintains itself over time for any level of π_t, and that a zero level of capital makes the probability of survival constant at a minimum level $\bar{\pi}$, it follows that $(\bar{\pi}, 0)$ is a stationary equilibrium, at which underconsumption prevails.

In order to treat the existence of other steady-states, let us distinguish between the different cases.

Regarding case (1), where $\frac{1}{\alpha}$ is odd and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} < \Omega$, there must exist at least one steady-state with $\pi > \bar{\pi}$ and $k > 0$. Indeed, the $\pi\pi$ locus lies, at $\pi = 1$, above the kk locus: $H_1(1) = H_2(1) > G(1)$. However, given that $H_1(0) < 0 = G(0)$, it follows, by continuity of $H_1(.)$ and $G(.)$, that $H_1(.)$ must intersect $G(.)$ at least once. That underconsumption equilibrium is denoted by (π_1, k_1) in Proposition 1.

Regarding case (2), where $\frac{1}{\alpha}$ is odd and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} > \Omega$, there must also exist at least one steady-state with $\pi > 0$ and $k > 0$. Indeed, the $\pi\pi$ locus lies, at $\pi = 1$, below the kk locus: $H_1(1) = H_2(1) < G(1)$. However, given that $G(.)$ tends to zero as π_t tends to 0, whereas $H_2(.)$ tends to infinity as π_t tends to 0, $G(\pi_t)$ and $H_2(\pi_t)$ must intersect somewhere. That overconsumption equilibrium is denoted by (π_2, k_2) in Proposition 1. Note that, as it is an overconsumption equilibrium, we know that $k_2 > k^*$, but not whether $\pi_2 \gtrless \bar{\pi}$.

Regarding case (3), where $\frac{1}{\alpha}$ is odd and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} = \Omega$, the $\pi\pi$ locus lies, at $\pi_t = 1$, at exactly the same level as the kk locus: $H_1(1) = H_2(1) = G(1)$, so that $(1, k^*)$ is an equilibrium. There cannot be any other healthy consumption equilibrium, because a unique level of k can yield $\pi = 1$.

Regarding case (4), where $\frac{1}{\alpha}$ is even and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} < \Omega$, there must exist at least two steady-states with $\pi > 0$ and $k > 0$. Indeed, $H_1(1) = H_2(1) > G(1)$. Furthermore, $H_1(.)$ tends to 0 when π_t tends to $\bar{\pi}$, whereas $G(.)$ tends to zero as π_t tends to 0, so that $G(\pi_t)$ and $H_1(\pi_t)$ must necessarily intersect somewhere, at an equilibrium denoted by (π_4, k_4), at which $\pi_4 > \bar{\pi}$ and $k_4 > 0$. That equilibrium is an underconsumption equilibrium. Moreover, under $\frac{1}{\alpha}$ even, $H_1(.)$ tends also to $+\infty$ when π_t tends to zero. Because of this, it must also be the case that $G(\pi_t)$ and $H_1(\pi_t)$ intersect at an equilibrium with $\pi < \bar{\pi}$ and $k > 0$. That equilibrium, denoted (π_3, k_3), is also an underconsumption equilibrium, and must lie below (π_4, k_4), as it must be at $\pi_3 < \bar{\pi}$, and as $G(.)$ increasing in π_t. Thus we must have $\pi_4 > \pi_3$ and $k_4 > k_3$.

Regarding case (5), where $\frac{1}{\alpha}$ is even and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} > \Omega$, there must exist at least two steady-states with $\pi > 0$ and $k > 0$. Indeed, $H_1(1) = H_2(1) < G(1)$. Furthermore, we know that $H_2(.)$ tends to $+\infty$ when π_t tends to zero, whereas $G(.)$ tends to zero as π_t tends to 0, so that $G(\pi_t)$ and $H_2(\pi_t)$ must necessarily intersect somewhere, at (π_6, k_6), which is an overconsumption equilibrium, at which $k_6 > k^*$, and $\pi_6 \lesseqgtr \bar{\pi}$. Moreover, under $\frac{1}{\alpha}$ even, $H_1(.)$ tends to $+\infty$ when π_t tends to zero. Because of this, it must be the case that $G(\pi_t)$ and $H_1(\pi_t)$ intersect somewhere else, at another equilibrium with $\pi > 0$ and $k > 0$. That equilibrium, denoted (π_5, k_5), is an underconsumption equilibrium, and lies below (π_6, k_6), as $H_1(.) < H_2(.)$ for any $\pi_t \neq 1$ and $G(.)$ increasing in π_t. Thus we must have $\pi_6 > \pi_5$ and $k_6 > k_5$.

Regarding case (6), where $\frac{1}{\alpha}$ is even and where $\left[\frac{\beta A(1-\alpha)}{1+\beta}\right]^{\frac{1}{1-\alpha}} = \Omega$, we have $H_1(1) = H_2(1) = G(1)$, so that $(1, k^*)$ is an equilibrium. Moreover, under $\frac{1}{\alpha}$ even, $H_1(.)$ tends to $+\infty$ when π_t tends to zero. Because of this, it must also be the case that $G(\pi_t)$ and $H_1(\pi_t)$ intersect somewhere else, at an equilibrium with $\pi < \bar{\pi}$ and $k < k^*$. That equilibrium, denoted (π_7, k_7), is an underconsumption equilibrium, and must lie below $(1, k^*)$.

Finally, it should be stressed that the above statements concern the necessary existence of some minimal number of stationary equilibria. However, the ambiguous signs of the second-order derivatives of both $G(\pi_t)$, $H_1(\pi_t)$ and $H_2(\pi_t)$ do not allow us to draw conclusions regarding the possible existence of a larger number of steady-state equilibria. Indeed, the second-order derivative of the kk locus is

$$G''(\pi_t) = \frac{1}{1-\alpha}\left[A(1-\alpha)\beta\right]^{\frac{1}{1-\alpha}}\left[\frac{1}{1+\beta\pi_t}\right]^{\frac{2-\alpha}{1-\alpha}}\pi_t^{\frac{1}{1-\alpha}-2}\left[\frac{\alpha - 2\beta\pi_t(1-\alpha)}{(1-\alpha)1+\beta\pi_t}\right]$$

Thus, the kk locus is convex when $\alpha > 2\beta\pi_t(1-\alpha)$ and concave for $\alpha < 2\beta\pi_t(1-\alpha)$. Put it differently, we have $G''(.) > 0$ when $\pi_t < \tilde{\pi} \equiv \frac{\alpha}{2\beta(1-\alpha)}$, and $G''(.) < 0$ when $\pi_t > \tilde{\pi} \equiv \frac{\alpha}{2\beta(1-\alpha)}$.

Moreover, after simplifications, the second-order derivative of the low branch of the $\pi\pi$ locus is

$$\begin{aligned} H_1''(\pi_t) &= \frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)\left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-2} \\ &\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1}{2}\frac{\frac{1}{\pi_t}-\beta}{\pi^{1/2}(1-\pi)^{1/2}A(1-\alpha)}\right)+c^*\frac{\beta}{A(1-\alpha)}\right]^2 \\ &+\frac{1}{\alpha}\left[\left(c^* - \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1} \\ &\left[\left(\frac{1}{\eta}\right)^{1/2}\frac{1}{2}\left(\frac{-(1-\pi_t)^{1/2}-\frac{1}{2}\left[\frac{\left(\frac{1}{\pi_t}-\beta\right)}{\pi_t^{1/2}(1-\pi_t)^{1/2}}\right]}{\left[\pi_t^{1/2}(1-\pi_t)^{1/2}\right]^2 A(1-\alpha)}\right)\right] \end{aligned}$$

If $1/\alpha$ is odd, so is $(1/\alpha)-2$, so that the first term is positive if $\pi_t < \bar{\pi}$ and negative if $\pi_t > \bar{\pi}$. However, the second term is always negative. Hence, for $\pi_t > \bar{\pi}$, we always have $H_1''(\pi_t) < 0$, that is, the convexity of the $\pi\pi$ locus in the positive orthan of the (π_t, k_t) space. If $1/\alpha$ is even, so is $(1/\alpha)-2$, so that the first term is always positive. However, the second term is negative if $\pi_t < \bar{\pi}$ and positive if $\pi_t > \bar{\pi}$. Hence, for $\pi_t > \bar{\pi}$, we always have $H_1''(\pi_t) > 0$, that is, the convexity of the $\pi\pi$ locus in its increasing part.

Finally, the second-order derivative of the high branch of the $\pi\pi$ locus is

$$\begin{aligned} H_2''(\pi_t) &= \frac{1}{\alpha}\left(\frac{1}{\alpha}-1\right)\left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-2} \\ &\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{-\frac{1}{2}\pi_t^{1/2}(1+\beta\pi_t)+\beta(1-\pi_t)}{\pi_t^2(1-\pi_t)^{1/2}A(1-\alpha)}\right)+c^*\frac{\beta}{A(1-\alpha)}\right]^2 \\ &+\frac{1}{\alpha}\left[\left(c^* + \left(\frac{1}{\eta}\right)^{1/2}\left(\frac{1-\pi_t}{\pi_t}\right)^{1/2}\right)\frac{(1+\beta\pi_t)}{A(1-\alpha)}\right]^{\frac{1}{\alpha}-1} \\ &\left[\left(\frac{1}{\eta}\right)^{1/2}\left(\frac{-\frac{1}{4}\pi_t^{-1/2}(1+\beta\pi_t)-\beta\frac{1}{2}\pi_t^{1/2}-\beta+\frac{\frac{1}{2}(1+\beta\pi_t)(4-5\pi_t)}{2\pi_t^{1/2}(1-\pi_t)}-\frac{\beta(4-5\pi_t)}{2\pi_t}}{\pi_t^2(1-\pi_t)^{1/2}A(1-\alpha)}\right)\right] \end{aligned}$$

The first term is unambiguously positive, as the second factor is at the power two. However, the sign of the second term is ambiguous, as the second factor may be either positive or negative.

It follows from the ambiguous signs of the second-order derivatives that the uniqueness of intersections of those loci in different areas of the (π_t, k_t) space cannot be taken for granted.

7.2 Proposition 2

Let us now study formally the stability of the long-run equilibria identified in Section 3. First of all, let us notice that the present dynamic system is non linear. Indeed, the transition functions for capital per worker and the survival probability are:

$$\begin{aligned} k_{t+1} &= \frac{\beta\pi_t}{1+\beta\pi_t} A(1-\alpha)k_t^{\alpha} \\ \pi_{t+1} &= \frac{1}{1+\eta\left(c^* - \left(\frac{1}{1+\beta\pi_t}A(1-\alpha)k_t^{\alpha}\right)\right)^2} \end{aligned}$$

The non linearity of the system of dynamic equations implies that the conventional analysis of the Jacobian matrix (composed of the first-order derivatives of dynamic equations with respect to state variables) can only inform us on the stability of equilibria provided these are hyperbolic.

Actually, if a fixed point is hyperbolic, the Hartman-Grobman theorem states that the stability of the linearized system (or its non stability) implies the local stability of the non-linear system (or its non stability) (see Medio and Lines, 2001). However, if a fixed point is not hyperbolic, then the analysis of the linearized system does not allow us to draw conclusions on the local stability of the non linear system.

As stated by Medio and Lines (2001), fixed points in discrete-time systems are hyperbolic if none of the eigenvalues of the Jacobian matrix, evaluated at the equilibrium, is equal to 1 in modulo.

Thus, to discuss the hyperbolicity of the equilibria characterized in Section 3, let us first compute the Jacobian matrix and study its properties. The Jacobian matrix is

$$J \equiv \begin{pmatrix} \frac{\partial k_{t+1}}{\partial k_t} & \frac{\partial k_{t+1}}{\partial \pi_t} \\ \frac{\partial \pi_{t+1}}{\partial k_t} & \frac{\partial \pi_{t+1}}{\partial \pi_t} \end{pmatrix}$$

where the entries are estimated at the equilibrium (π, k).

The entries of the Jacobian matrix are:

$$\begin{aligned} \frac{\partial k_{t+1}}{\partial k_t} &= \alpha < 1 \\ \frac{\partial k_{t+1}}{\partial \pi_t} &= \frac{\beta c}{1+\beta\pi} > 0 \\ \frac{\partial \pi_{t+1}}{\partial k_t} &= \frac{2\alpha\eta\left(c^* - c\right)}{\beta\pi\left[1+\eta\left(c^* - c\right)^2\right]^2} \\ \frac{\partial \pi_{t+1}}{\partial \pi_t} &= \frac{-2\eta\left(c^* - c\right)\left(\frac{\beta c}{1+\beta\pi}\right)}{\left[1+\eta\left(c^* - c\right)^2\right]^2} \end{aligned}$$

where c denotes the equilibrium consumption, i.e. $\left(\frac{1}{1+\beta\pi}\right)^{\frac{1}{1-\alpha}}\left[A(1-\alpha)\right]^{\frac{1}{1-\alpha}}\left[\beta\pi\right]^{\frac{\alpha}{1-\alpha}}$.

Hence the determinant of the Jacobian matrix is

$$\det(J) = -2\alpha\eta\left(c^* - c\right)\frac{\left(\frac{\beta c}{1+\beta\pi}\right)\left(1+\frac{1}{\beta\pi}\right)}{\left[1+\eta\left(c^* - c\right)^2\right]^2}$$

The trace of the Jacobian matrix is

$$tr(J) = \alpha - \frac{2\eta\left(c^{*}-c\right)\left(\frac{\beta c}{1+\beta\pi}\right)}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}$$

As stated in Medio and Lines (2001, p. 52), the conditions that are necessary and sufficient having two eigenvalues of the Jacobian matrix smaller than 1 in modulo can be written as

$$\begin{aligned} 1 + tr(J) - \det(J) &> 0 \\ 1 - tr(J) + \det(J) &> 0 \\ 1 - \det(J) &> 0 \end{aligned}$$

Substituting for the trace and the determinant of the Jacobian matrix yields the conditions of Proposition 2. Note that those conditions are necessary and sufficient for the hyperbolicity and the stability of the equilibrium.

Note that, as far as the equilibrium $(\bar{\pi}, 0)$ are concerned, it is easy to see that those conditions are always satisfied. Indeed, under a zero equilibrium capital level, $tr(J)$ equals α and $\det(J)$ equals 0.

7.3 Proposition 3

Having stated the general conditions for stability, let us look, on a case-by-case basis, at the various possible scenarios for the dynamics. For that purpose, let us notice, following Medio and Lines (2001), that the characteristic equation is

$$\lambda^{2} - tr(J)\lambda + \det(J) = 0$$

Thus the eigenvalues can be written as

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left(tr(J) \pm \sqrt[2]{\left[tr(J)\right]^{2} - 4\det\left(J\right)}\right) \\ &= \frac{1}{2}\left(tr(J) \pm \sqrt[2]{\Delta}\right) \end{aligned}$$

The term Δ is equal to

$$\begin{aligned} \Delta &\equiv \left[\frac{\alpha\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2} - 2\eta\left(c^{*}-c\right)\left(c\frac{\beta}{1+\beta\pi}\right)}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}}\right]^{2} + 8\frac{\alpha\eta\left(c^{*}-c\right)\left(\frac{\beta c}{1+\beta\pi}\right)\left(1+\frac{1}{\beta\pi}\right)}{\left[1+\eta\left(c^{*}-c\right)^{2}\right]^{2}} \\ &> 0 \text{ if } c^{*} \geq c \\ &\lesseqgtr 0 \text{ if } c^{*} < c \end{aligned}$$

Let us now distinguish between the different cases.

Case (1) Under case (1), we know that the capital per worker at the equilibrium is lower than the one maximizing π. Hence, it must be the case that we are in an equilibrium of underconsumption: $c < c^{*}$. Thus we have $\frac{\partial \pi_{t+1}}{\partial k_t} > 0$ and $\frac{\partial \pi_{t+1}}{\partial \pi_t} < 0$. Therefore, $\det(J)$ is negative, indicating that the eigenvalues of the Jacobian matrix have opposite signs. The trace has an ambiguous sign, as the second term is negative. Moreover, we have $\Delta > 0$. Hence three cases can arise:

- $|\lambda_1| < 1, |\lambda_2| < 1$, we have a stable equilibrium. Given that $\det(J) < 0$, we have improper oscillations, but there is a non-monotonic convergence.
- $|\lambda_1| > 1, |\lambda_2| > 1$, we have an unstable equilibrium, with improper oscillations. There is no convergence.
- $|\lambda_1| > 1, |\lambda_2| < 1$, the equilibrium is a saddle point.

Hence, the stability of the equilibrium (π_1, k_1) depends on the one of those three cases under which we fall.

Case (2) Under case (2), we have $c > c^*$. Hence, we have $\frac{\partial \pi_{t+1}}{\partial k_t} < 0$ and $\frac{\partial \pi_{t+1}}{\partial \pi_t} > 0$. Thus, in that case, $\det(J)$ is positive, and the trace is positive. Hence the two eigenvalues are positive. However, we do not know the sign of Δ.

If $\Delta > 0$, three cases can arise:

- $|\lambda_1| < 1, |\lambda_2| < 1$, we have a stable equilibrium. Given that $\det(J) > 0$, we have a monotonic convergence.
- $|\lambda_1| > 1, |\lambda_2| > 1$, we have an unstable equilibrium, with a monotonic divergence.
- $|\lambda_1| > 1, |\lambda_2| < 1$, the equilibrium is a saddle point.

If $\Delta < 0$, the two eigenvalues are a complex conjugate pair $\lambda_1, \lambda_2 = \sigma \pm \theta i$, and the solutions are sequences of points situated on spirals. Three cases can arise:

- If $\sqrt[2]{\det(J)} < 1$, the solutions converge to equilibrium point, which is a stable equilibrium.
- If $\sqrt[2]{\det(J)} > 1$, the solutions diverge and the equilibrium point is unstable.
- If $\sqrt[2]{\det(J)} = 1$, the eigenvalues lie exactly on the unit circle. Hence the equilibrium point is unstable, as we have a cycle around it.

If $\Delta = 0$, there is a repeated real eigenvalue $\lambda = Tr(J)/2$. Hence the eigenvalues are

$$\frac{\alpha - \frac{2\eta(c^*-c)\left(c\frac{\beta}{1+\beta\pi}\right)}{\left[1+\eta(c^*-c)^2\right]^2}}{2}$$

Given that we have here $(c^* - c) < 0$, this expression is always positive. If this is smaller than 1, we have a convergence towards the steady-state. If it is equal or larger than 1, we have a divergence and the equilibrium is unstable.

Note that if we impose the condition of Proposition 3:

$$\Lambda \equiv \alpha - \frac{2\eta\left(c^* - c\right)\left(c\frac{\beta}{1+\beta\pi}\right)}{\left[1 + \eta\left(c^* - c\right)^2\right]^2} < 1$$

it is easy to see that the three conditions of Proposition 2 become:

$$\begin{aligned} 1+\Lambda+\alpha\left[\Lambda-\alpha\right]\left(1+\frac{1}{\beta\pi}\right) &> 0 \\ 1-\Lambda+\alpha\left[\Lambda-\alpha\right]\left(1+\frac{1}{\beta\pi}\right) &> 0 \\ 1-\alpha\left[\Lambda-\alpha\right]\left(1+\frac{1}{\beta\pi}\right) &> 0 \end{aligned}$$

Under case (2), we have $\Lambda > 0$. Hence, if $\Lambda < 1$, the three conditions above are necessarily satisfied. Hence the above condition suffices to guarantee the local stability of the equilibrium (π_2, k_2). However, the precise form of the convergence - monotonic or not - depends on whether $\Delta \geq 0$ or $\Delta < 0$. In the former case, we will observe a monotonic convergence, whereas in the latter, there will be a spiral converging towards the equilibrium.

Case (3) Under case (3), we have $c = c^*$, so that $\Delta > 0$. We also have $\det(J) = 0$ and $tr(J) = \alpha$, from which it follows that the two eigenvalues must be 0 and α. As a consequence, we fall under the case where $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so that we have a stable equilibrium. We also have monotonic convergence towards that equilibrium. Thus $(1, k^*)$ is locally stable.

Case (4) Under case (4), we have, at each equilibrium, $c < c^*$. Thus $\det(J)$ is negative, indicating that the eigenvalues of J have opposite signs. The trace has an ambiguous sign. One can apply the same procedure as in case (1) to determine the stability of equilibria (π_3, k_3) and (π_4, k_4).

Case (5) Under case (5), the low equilibrium (π_5, k_5) is such that $c < c^*$. Hence $\det(J)$ is, at that equilibrium, negative, so that the eigen values are of opposite signs, and the trace is of an ambiguous sign. On the contrary, at the high equilibrium (π_6, k_6), we have $c > c^*$. Hence, in that case, $\det(J)$ is positive, and the trace is positive. Hence the two eigen values are positive.

The analysis of local stability is the same as under case (2). Given that the two eigenvalues of the Jacobian matrix are positive, the condition of Proposition 3 guarantees the hyperbolicity of the equilibrium and its local stability. Indeed, under that condition, the two eigen values, which are positive, are strictly lower than 1 (as the trace, i.e. their sum, is lower than 1) and so hyperbolicity prevails. Moreover, the conditions of Proposition 2 are also satisfied.

Case (6) As in case (3), we have $c = c^*$, so that $\Delta > 0$. We also have $\det(J) = 0$ and $tr(J) = \alpha$, from which it follows that the two eigenvalues must be 0 and α. As a consequence, we fall under the case where $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so that we have a stable equilibrium. We also have monotonic convergence towards that equilibrium. Thus $(1, k^*)$ is locally stable.

At (π_7, k_7), we have $c < c^*$. Thus $\det(J)$ is negative, indicating that the eigen values of the Jacobian matrix have opposite signs. The trace has an ambiguous sign. One can apply the same procedure as in case (1) to determine the stability of the equilibrium (π_7, k_7).

7.4 Proposition 4

Note first that, as far as the equilibria $(\bar{\pi}, 0)$ are concerned, the stability conditions are always satisfied, so that there can be no cycle in those cases.

Actually, in order to have long-run cycles, we need, as shown above, $\Delta < 0$, that is, that the eigenvalues of the Jacobian matrix are complex.

Under cases (1) and (4), where the long-run equilibrium is an underconsumption equilibrium, we always have $\Delta > 0$, so that cycles cannot occur around equilibria (π_1, k_1), (π_3, k_3) and (π_4, k_4).

Under cases (2) and (5), we can have $\Delta < 0$. The first condition of Proposition 4 states the condition for $\Delta < 0$. Note that this condition excludes the possibility of cycles around the low equilibrium of case (5), that is, around (π_5, k_5), because that equilibrium is an underconsumption equilibrium, at which we necessarily have $\Delta > 0$. But the condition $\Delta < 0$ may be true at (π_2, k_2) and (π_6, k_6), which are overconsumption equilibria.

The second condition of Proposition 4 is equivalent to $\sqrt[2]{\det(J)} = 1$, as required for the existence of a cycle. Here again, that condition may be satisfied at equilibria (π_2, k_2) and (π_6, k_6), depending on the structural parameters of the economy. Hence taken together, the 2 conditions of Proposition 4 guarantee the existence of long-run cycles around (π_2, k_2) and (π_6, k_6).

Under cases (3) and (6), we have $\Delta > 0$, so that cycles cannot occur around equilibria $(1, k^*)$ and (π_7, k_7).