Externalities in production economies and existence of competitive equilibria: A differentiable approach

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Abstract

We consider a general model of production economies with consumption and production externalities. Each household is characterized by an endowment of commodities and preferences described by a utility function. Each firm is owned by the households and it is characterized by technology described by a transformation function. That is, the choices of all agents (households and firms) affect individual preferences and production technologies. Describing equilibria in terms of first order conditions and market clearing conditions, and using a homotopy approach based on the seminal work by Smale (1974), under differentiability and boundary conditions, we prove the non-emptiness and compactness of the set of competitive equilibria with consumptions and prices strictly positive.

JEL classification: C62, D50, D62.

Key words: externalities, production economies, competitive equilibrium, homotopy approach.

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1 Introduction

We consider a general model of production economies with consumption and production externalities. In a differentiable framework, our purpose is to prove the non-emptiness and compactness of the set of competitive equilibria with consumptions and prices strictly positive.

Why do we care about the existence of equilibria from a differentiable viewpoint? The starting point of studying the set of regular economies is the non-empty and compact set of equilibria in a differentiable setting. The relevance of regular economies and issues related to the global approach of the equilibrium analysis can be found in Smale (1981), Mas-Colell (1985), Balasko (1988), Villanacci et al. (2002). So, because of the differentiable approach, this paper is a first step to study regular economies, and consequently, Pareto improving policies in production economies with externalities. Indeed, it is well-known that externalities might prevent competitive equilibrium allocations to be Pareto optimal. Thus, it is helpful to study Pareto improving policies in the presence of externalities. In different settings, see for instance Geanakoplos and Polemarchakis (1986, 2008), Citanna, Kajii and Villanacci (1998), Citanna, Polemarchakis and Tirelli (2006), Villanacci and Zenginobuz (2006, 2010).

Our model of externalities is based on the seminal works by Laffont and Laroque (1972), Laffont (1976, 1977, 1988), where the choices of all households and firms affect individual preferences and production technologies. For example, building of a mall in a residential area is an example of positive (or negative) externalities created by a firm on people living in that area. In counterpart, externalities may be created by consumers on firms. For instance, an over consumption of air-conditioner and consequently of electricity, might produce an electrical breakdown, breaking all production activities. Moreover, externalities may be created by firms on other firms. For instance an increasing of polluting production can damage the soil that an agricultural firm use, causing a decrease in production.

In this paper, we consider a private ownership economy with a finite number of commodities, households and firms. Each firm is characterized by a technology described by an inequality on a differentiable function called the transformation function. Each household is characterized by a consumption set which coincides with the strictly positive orthant of the commodity space, preferences and an initial endowment of commodities. Individual preferences are represented by a utility function. Firms are owned by households. Utility and transformation functions depend on the consumption of all households and on the production activity of all firms.

Facing a price, each household chooses a consumption bundle which solves his utility maximization problem under the budget constraint taking as given the choices of the others, i.e. given the level of externality created by the other households and firms. Facing a price, each firm chooses in his production set a production plan.
which solves his profit maximization problem taking as given the choices of the others, i.e. given the level of externality created by the other firms and households. The associated concept of competitive equilibrium is nothing else than an equilibrium à la Nash, the resulting allocation being feasible with the initial resources of agents. This notion includes as a particular case the classical equilibrium definition without externalities at all.

Our main result is Theorem 8 which states that for all strictly positive initial endowments, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact. Following the seminal work by Smale (1974), Villanacci and Zenginobuz (2005) and Bonnisseau and del Mercato (2010), we prove Theorem 8 using a homotopy approach and describing equilibria in terms of first order conditions and market clearing conditions. The homotopy idea is that any economy with externalities is connected by an arc to some economy without externalities at all. Along this arc, equilibria move in a continuous way without sliding off the boundary. The same idea is used by del Mercato (2006) where in a pure exchange economy only consumption externalities are considered.

We now compare our contribution with previous works. The existence results by Laffont and Laroque (1972), and Bonnisseau and Médecin (2001) are more general than ours since in these works individual consumption sets and firms technologies are represented by correspondences. Using fixed point arguments, Laffont and Laroque show the existence of free-disposal competitive equilibria where equilibrium consumptions are positive but not necessarily strictly positive. So, providing appropriate boundary conditions under which equilibrium consumptions are strictly positive we go further than these authors. In Bonnisseau and Médecin (2001), non-convexities in production are allowed. For that reason their existence result involve more sophisticated techniques than those we use.

We are interested in a model where one can perform comparative static analysis, and therefore Pareto improving policies from a differentiable viewpoint. So, at the cost of losing generality, we choose to use an inequality on differentiable functions, instead of more general correspondences, to describe firms technologies. Furthermore, in order to use standard first order conditions, fixing the externalities we require classical convexity assumptions to be satisfied. In this mild context, we provide an existence proof simpler than that of Bonnisseau and Médecin (2001).

One should notice that Villanacci and Zenginobuz (2005), del Mercato (2006), Mandel (2008) and Bonnisseau and del Mercato (2010) use the same homotopy idea we use. Villanacci and Zenginobuz (2005) focus on a specific kind of externalities, namely public goods. In order to prove existence and regularity, Mandel (2008) uses a Homotopy approach based on the Brouwer degree. Following Chapter 4 of Milnor (1965), our work is based on the theory of degree mod. 2. The degree theory Modulo 2 is simpler than the Brouwer degree that requires the concept of oriented manifold in order to deduce the existence result from regularity properties and from the Index Theorem. In Kung (2008), also public goods are taking into

\[ \text{The reader can find a survey of this approach in Villanacci et al. (2002).} \]
account. Differently to our paper, externalities affect only individual preferences and not the production technology. To get existence result, no specific assumptions on the transformation functions are necessary, but the author have to perturb all fundamentals of the economy. In del Mercato (2006), and Bonnisseau and del Mercato (2010), a general model of pure exchange economies with externalities on consumption sets and preferences is studied.

The paper is organized as follows. In Section 2, we present the model and the assumptions. In Section 3, we first focus on the equilibrium function which is built on first order conditions associated with households and and firms maximization problems, and we present our main result which states that for all initial endowments, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact (Theorem 8). Second, we construct the “test economy”, that is a Pareto optimal allocation of an appropriate production economy without externalities at all. Finally we describe the homotopy between the test economy and our economy with externalities. All the lemmas are proved in Section 4. Finally, in the Appendix, the reader can find the characterization of Pareto optimal allocation without externalities.

2 The model and the assumptions

There is a finite number $C$ of physical commodities labeled by the superscript $c \in C := \{1, \ldots, C\}$. The commodity space is $\mathbb{R}^C$. There are a finite number $J$ of firms labeled by the subscript $j \in J := \{1, \ldots, J\}$ and a finite number $H$ of households labeled by the subscript $h \in H := \{1, \ldots, H\}$. Each firm is owned by the households and it is characterized by a technology described by a transformation function. Each household is characterized by preferences described by a utility function, the shares on firms’ profit and an endowment of commodities. In this paper, we assume that all consumption sets coincide with the positive orthant of the commodity space. Utility and transformation functions may be affected by the consumption choices of all households and by the production activities of all firms. The notations are summarized below.

- $y_j := (y_j^1, \ldots, y_j^c, \ldots, y_j^C)$ is the production plan of firm $j$, as usual output components are positive and input components are negative, $y_{-j} := (y_k)_{k \neq j}$ denotes the production plan of firms other than $j$, $y := (y_j)_{j \in J}$.
- $x_h := (x_h^1, \ldots, x_h^{c_h}, \ldots, x_h^C)$ denotes household $h$’s consumption, $x_{-h} := (x_k)_{k \neq h}$ denotes the consumption of households other than $h$, $x := (x_h)_{h \in H}$.
- The technology of firm $j$ is described by an inequality on a function $t_j$ called the transformation function. The innovation of this paper comes from the dependency of the production set with respect to the production activities of the other firms and households consumption. That is, given $y_{-j}$ and $x$, the production set of the firm $j$ is described by the following set,

$$Y_j(y_{-j}, x) := \{ y_j \in \mathbb{R}^C : t_j(y_j, y_{-j}, x) \geq 0 \}$$
where the transformation function \( t_j \) is a function from \( \mathbb{R}^C \times \mathbb{R}^{C(J-1)} \times \mathbb{R}^{CH} \) to \( \mathbb{R} \). So, \( t_j \) describes the way firm \( j \)'s technology is affected by the actions of the other agents, \( t := (t_j)_{j \in J} \).

- Household \( h \) has preferences described by a utility function,

\[
  u_h : (x_h, x_{-h}, y) \in \mathbb{R}_+^C \times \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ} \longrightarrow u_h(x_h, x_{-h}, y) \in \mathbb{R}
\]

\( u_h(x_h, x_{-h}, y) \) is the utility level of household \( h \) associated with \( (x_h, x_{-h}, y) \). So, \( u_h \) describes the way household \( h \)'s preferences are affected by the actions of the other agents, \( u := (u_h)_{h \in H} \).

- \( s_{jh} \in [0, 1] \) is the share of firm \( j \) owned by household \( h \); \( s_h := (s_{jh})_{j \in J} \in [0, 1]^J \) denotes the vector of the shares owed by household \( h \); \( s := (s_h)_{h \in H} \in [0, 1]^{JH} \).

The set of shares is given by \( S := \{s \in [0, 1]^{JH} : \sum_{h \in H} s_{jh} = 1, \forall j \in J\} \).

- \( e_h := (e_{h1}, e_{h2}, ..., e_{hn}) \) denotes household \( h \)'s endowment, \( e := (e_h)_{h \in H} \).

- \( p' \) is the price of one unit of commodity \( c \); \( p := (p^1, p^2, ..., p^C) \in \mathbb{R}_+^C \).

- Given \( w = (w^1, w^2, ..., w^C) \in \mathbb{R}^C \), we denote \( w^j := (w^1, w^2, ..., w^{C-1}) \in \mathbb{R}^{C-1} \).

We make the following assumptions on the transformation functions.

**Assumption 1** Consider \( t = (t_j)_{j \in J} \). For all \( j \in J \),

1. The function \( t_j \) is a \( C^1 \) function.
2. For each \( (y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_+^{CH} \), \( t_j(0, y_{-j}, x) \geq 0 \).
3. For each \( (y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_+^{CH} \), the function \( t_j(\cdot, y_{-j}, x) \) is differentiably strictly decreasing, i.e. for every \( y'_j \in \mathbb{R}^C \), \( D_{y_j}t_j(y'_j, y_{-j}, x) < 0 \).
4. For each \( (y_{-j}, x) \in \mathbb{R}^{C(J-1)} \times \mathbb{R}_+^{CH} \), the function \( t_j(\cdot, y_{-j}, x) \) is \( C^2 \) and it is differentiably strictly quasi-concave, i.e. for every \( y'_j \in \mathbb{R}^C \), \( D^2_{y_j}t_j(y'_j, y_{-j}, x) \) is negative definite on \( \ker D_{y_j}t_j(y'_j, y_{-j}, x) \).

We remark that fixing the externalities, the assumptions on \( t_j \) are standard in “smooth” general equilibrium models. Indeed, from Point 1 of Assumption 1 the production set is closed and smooth, from Point 4 of Assumption 1 it is convex. Point 2 of Assumption 1 states that inactivity is possible. Point 3 of Assumption 1 represents the “free disposal” property.

Define the set \( Y_t \) of all production plans which are on the production sets whatever are the externalities, that is

\[
  Y_t := \{ y' \in \mathbb{R}^{CJ} \mid \exists (x, y) \in \mathbb{R}_+^{CH} \times \mathbb{R}^{CJ} : t_j(y'_j, y_{-j}, x) \geq 0, \forall j \in J \} \quad (1)
\]

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1. Let \( v \) and \( v' \) be two vectors in \( \mathbb{R}^n \), \( v \cdot v' \) denotes the *inner product* of \( v \) and \( v' \). Let \( A \) be a real matrix with \( m \) rows and \( n \) columns, and \( B \) be a real matrix with \( n \) rows and \( l \) columns, \( AB \) denotes the *matrix product* of \( A \) and \( B \). Without loss of generality, vectors are treated as row matrices and \( A \) denotes both the matrix and the following linear application \( A : v \in \mathbb{R}^n \rightarrow A(v) := Av^T \in \mathbb{R}^m \) where \( v^T \) denotes the transpose of \( v \) and \( \mathbb{R}^m := \{w^T : w \in \mathbb{R}^m\} \). When \( m = 1 \), \( A(v) \) coincides with the inner product \( A \cdot v \), treating \( A \) and \( v \) as vectors in \( \mathbb{R}^n \).
The following assumption can be interpreted as the asymptotic irreversibility and "no free lunch" assumptions at the aggregate level.\(^5\)

**Assumption 2** If \(y' \in CY_t\) and \(\sum_{j \in J} y'_j \geq 0\), then \(y'_j = 0\) for every \(j \in J\).\(^6\)

As usual, from Assumption 2 one gets the following lemma which ensures a fundamental property for the existence of an equilibrium, i.e. the compactness of the set of feasible allocations.

**Lemma 3** For every \(r \in \mathbb{R}^C_{++}\), the following set is bounded.

\[
A_{t,r} := \{(x', y') \in \mathbb{R}^C_{++} \times \mathbb{R}^C_{+} \mid \exists (x, y) \in \mathbb{R}^C_{++} \times \mathbb{R}^C_{+} : t_j(y'_j, y_{-j}, x) \geq 0, \forall j \in J \}
\]

and \(\sum_{h \in H} x'_h - \sum_{j \in J} y'_j \leq r\)

(2)

One should notice that Lemma 3 also guarantees the compactness of the set of feasible allocations for any fixed externalities. Furthermore, Lemma 3 implies that the set of feasible allocations is compact along the arc associated with the homotopy defined in Subsection 3.2.\(^7\)

We make the following assumptions on the utility functions.

**Assumption 4** Consider \(u = (u_h)_{h \in \mathcal{H}}\). For all \(h \in \mathcal{H}\),

1. The function \(u_h\) is continuous in its domain and it is \(C^1\) in the interior of its domain.
2. For each \((x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^C_{+}\), the function \(u_h(\cdot, x_{-h}, y)\) is differentiably strictly increasing, i.e. for every \(x'_h \in \mathbb{R}^C_{++}\), \(D_{x_h} u_h(x'_h, x_{-h}, y) > 0\).
3. For each \((x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^C_{+}\), the function \(u_h(\cdot, x_{-h}, y)\) is \(C^2\) and it is differentiably strictly quasi-concave, i.e. for every \(x'_h \in \mathbb{R}^C_{++}\), \(D^2_{x_h} u_h(x'_h, x_{-h}, y)\) is negative definite on \(\ker D_{x_h} u_h(x'_h, x_{-h}, y)\).
4. For each \((x_{-h}, y) \in \mathbb{R}^{C(H-1)}_{++} \times \mathbb{R}^C_{+}\) and for every \(u \in \text{Im} u_h(\cdot, x_{-h}, y)\),

\[
\text{cl}_{\mathbb{R}^C} \{ x'_h \in \mathbb{R}^C_{++} : u_h(x'_h, x_{-h}, y) \geq u \} \subseteq \mathbb{R}^C_{++}
\]

Fixing the externalities, the assumptions on \(u_h\) are standard in “smooth” general equilibrium models. In Points 1 and 4 of Assumption 4 we consider consumption \(x_{-h}\) in the closure of \(\mathbb{R}^{C(H-1)}_{++}\) just to look at the limit of a behavior (see Steps 2 in the proof of Lemma 11, Section 4).

\(^5\) Assumption 2 is in the same spirit as Assumption UB (Uniform Boundedness) in Bonnisseau and Médecin (2001) and Assumption P3 in Mandel (2008).

\(^6\) \(CY_t\) denotes the asymptotic cone of \(Y_t\).

\(^7\) For details, see Step 2.1 in the proof of Lemma 11, Section 4.
$\mathcal{T}$ denotes the set of $t$ satisfying Assumption 1 and Assumption 2, and $\mathcal{U}$ denotes the set of $u$ satisfying Assumption 4.

**Remark 5** From now on, $u \in \mathcal{U}$ and $s \in S$ are kept fixed and an economy is completely parameterized by transformation functions and initial endowments $(t, e)$ taken in the set $\mathcal{T} \times \mathbb{R}^{CH}_{++}$.

Without loss of generality, commodity $C$ is the “numeraire good”. So, given $p^\downarrow \in \mathbb{R}^{C-1}_{++}$ with innocuous abuse of notation, we denote $p := (p^\downarrow, 1) \in \mathbb{R}^{C}_{++}$.

**Definition 6** $(x^*, y^*, p^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \times \mathbb{R}^{C-1}_{++}$ is a competitive equilibrium for the economy $(t, e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++}$ if for all $j \in \mathcal{J}$, $y^*_j$ solves the following problem

\[
\max_{y_j \in \mathbb{R}^C} p^* \cdot y_j \\
\text{subject to } t_j(y_j, y^*_{-j}, x^*) \geq 0
\]

for all $h \in \mathcal{H}$, $x^*_h$ solves the following problem

\[
\max_{x_h \in \mathbb{R}^{C+}} u_h(x_h, x^*_{-h}, y^*) \\
\text{subject to } p^* \cdot x_h \leq p^* \cdot (e_h + \sum_{j \in \mathcal{J}} s_{jh} y^*_j)
\]

and $(x^*, y^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ}$ satisfies market clearing conditions, that is

\[
\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y^*_j
\]

**Remark 7** It is an easy matter to check that

- from Assumption 1, if $y^*_j$ is a solution to problem (3), then it is unique and it is completely characterized by Karush–Kuhn–Tucker conditions.
- From Point 2 of Assumption 1 and Assumption 4, there exists a unique solution $x^*_h$ to problem (4) and it is completely characterized by Karush–Kuhn–Tucker conditions.
- As usual, from Point 2 of Assumption 4, household $h$’s budget constraint holds with an equality at $x^*_h$. So, at equilibrium, due to the Walras law, the market clearing condition for commodity $C$ is “redundant”. Thus, one replaces condition (5) by $\sum_{h \in \mathcal{H}} x^*_h = \sum_{h \in \mathcal{H}} e_h + \sum_{j \in \mathcal{J}} y^*_j$.

### 3 Existence and compactness

In this section, using first order conditions associated with firms and households maximization problems, we first define the equilibrium function. Second, we prove that for all the economies $(t, e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++}$, the set of competitive equilibria with consumptions and prices strictly positive is non-empty and compact.
Let $\Xi := (\mathbb{R}^C_+ \times \mathbb{R}^J_+) \times (\mathbb{R}^C_+ \times \mathbb{R}^J_+) \times \mathbb{R}^{C-1}_+$ be the set of endogenous variables with generic element $\xi := (x, \lambda, y, \alpha, p^1) := ((x_h, \lambda_h)_{h \in \mathcal{H}}, (y_j, \alpha_j)_{j \in J}, p^1)$ where $\lambda_h$ denotes the Lagrange multiplier associated with household $h$’s budget constraint, and $\alpha_j$ denotes the Lagrange multiplier associated with firm $j$’s production constraint. For any given economy $(t, e) \in T \times \mathbb{R}^H_{++}$, using Karush–Kuhn–Tucker conditions, we can now describe competitive equilibria through the equilibrium function $F_{t,e} : \Xi \to \mathbb{R}^{\dim \Xi}$,

\[
F_{t,e}(\xi) := (F^{h,1}_{t,e}(\xi), F^{h,2}_{t,e}(\xi))_{h \in \mathcal{H}}(F^{j,1}_{t,e}(\xi), F^{j,2}_{t,e}(\xi))_{j \in J}, F^M_{t,e}(\xi))
\]

\[
F^{h,1}_{t,e}(\xi) := D_s u_h(x_h, x_{-h}, y) - \lambda_h p, \quad F^{h,2}_{t,e}(\xi) := -p \cdot (x_h - e_h - \sum_{j \in J} s_j y_j)
\]

\[
F^{j,1}_{t,e}(\xi) := p + \alpha_j D_{y_j}t_j(y_j, y_{-j}, x), \quad F^{j,2}_{t,e}(\xi) := t_j(y_j, y_{-j}, x)
\]

\[
F^M_{t,e}(\xi) := \sum_{h \in \mathcal{H}} x^h_h - \sum_{j \in J} y^j_j - \sum_{h \in \mathcal{H}} e^h_h
\]

From Remark 7, $(x^*, y^*, p^{\lambda^*})$ is an equilibrium for the economy $(t, e)$ if and only if there exists $(\lambda^*, \alpha^*) \in \mathbb{R}^H_{++} \times \mathbb{R}^J_{++}$ such that $F_{t,e}(x^*, \lambda^*, y^*, \alpha^*, p^{\lambda^*}) = 0$. We call $\xi^* = (x^*, \lambda^*, y^*, \alpha^*, p^{\lambda^*})$ simply an equilibrium.

**Theorem 8 (Existence and compactness)** For every economy $(t, e) \in T \times \mathbb{R}^H_{++}$, the equilibrium set $F_{t,e}^{-1}(0)$ is non-empty and compact.

In order to prove Theorem 8, following the seminal paper by Smale (1974) and more recent contributions by Villanacci and Zenginobuz (2005) and Bonnisseau and del Mercato (2010), we use homotopy arguments, that is the following theorem which is a consequence of the homotopy invariance of the topological degree mod 2.\(^8\)

**Theorem 9 (Homotopy Theorem)** Let $M$ and $N$ be $C^2$ boundaryless manifolds of the same dimension, $y \in N$ and $f, g : M \to N$ be such that $f$ and $g$ are $C^0$, $\# g^{-1}(y)$ is odd and $g$ is $C^1$ in an open neighborhood of $g^{-1}(y)$, $y$ is a regular value for $g$, there exists a continuous homotopy $L$ from $g$ to $f$ such that $L^{-1}(y)$ is compact. Then, $f^{-1}(y)$ is compact and $f^{-1}(y) \neq \emptyset$.\(^9\)

To apply Theorem 9, the equilibrium function $F_{t,e}$ plays the role of the function $f$. In order to construct the required homotopy and the function that plays the role of the function $g$, we proceed as follows. First, we construct the so called

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\(^8\) Following Chapter 4 of Milnor (1965), Theorem 9 is based on the degree mod 2. The theory of degree mod 2 is simpler than the one used in Mas-Colell (1985) which requires the concepts of oriented manifold and the associated topological degree – the Brouwer degree – in order to deduce the existence result from regularity properties of equilibria and from the Index Formula. The reader can find a survey of the theory of degree mod 2 in Villanacci et al. (2002).

\(^9\) As a consequence of these assumptions, the degree mod 2 of $g$ at $y$ is equal to 1. Since the homotopy $L$ is continuous and $L^{-1}(y)$ is compact, the homotopy invariance of the topological degree implies that the degree mod 2 of $f$ at $y$ is equal to 1, and so $f^{-1}(y) \neq \emptyset$. 

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“test economy”. The test economy is built using a Pareto optimal allocation of an appropriate production economy à la Arrow–Debreu without externalities at all. Second, using the test economy and the Second Welfare Theorem, we construct the equilibrium function \( G \) playing the role of the function \( g \). The test economy and the equilibrium function \( G \) are defined in Subsection 3.1. Finally, in Subsection 3.2, we provide the required homotopy \( H_{t,e} \) from \( G \) to \( F_{t,e} \) playing the role of the homotopy \( L \), and we verify that all the assumptions of Theorem 9 are satisfied. Namely, \( G^{-1}(0) \) is a singleton, 0 is a regular value of \( G \) and \( H_{t,e}^{-1}(0) \) is compact.

### 3.1 Test economy

In order to construct the test economy, first we fix the externalities and we consider a Pareto optimal allocation of a standard production economy à la Arrow–Debreu. Second, using the Second Welfare Theorem, we construct the equilibrium function \( G \) in such a way that, at the test economy, an equilibrium exists and it is unique.

Let \( \bar{x} \in \mathbb{R}^{CH}_{++} \) be an arbitrary list of consumptions and \( \bar{y} \in \mathbb{R}^{CJ} \) be an arbitrary list of productions. Fixing the externalities at \((\bar{x}, \bar{y})\), define \( \bar{u}_h(x_h) := u_h(x_h, \bar{x}_h, \bar{y}) \), \( \bar{t}_j(y_j) := t_j(y_j, \bar{y}_j, \bar{x}) \) and the corresponding production economy à la Arrow–Debreu, namely \( \bar{E} := ((\bar{u}_h)_{h \in \mathcal{H}}, (\bar{t}_j)_{j \in \mathcal{J}}, \sum_{h \in \mathcal{H}} e_h) \). Since there are no externalities at all, the notions of feasibility and Pareto optimality are standard in \( \bar{E} \). It is well known that, under Assumptions 1, 2 and 4, there exists a Pareto optimal allocation \((\tilde{x}, \tilde{y})\) in \( \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \) of the economy \( \bar{E} \), and there exist Lagrange multipliers \((\tilde{\theta}, \tilde{\gamma}, \tilde{\beta}) = ((\tilde{\theta}_h)_{h \neq 1}, \tilde{\gamma}, (\tilde{\beta}_j)_{j \in \mathcal{J}}) \in \mathbb{R}^{H-1}_{++} \times \mathbb{R}^C_{++} \times \mathbb{R}^{++}_+ \) such that \((\tilde{x}, \tilde{y}, \tilde{\gamma}, \tilde{\beta}) \) is the unique solution to the following system.\(^{10}\)

\[
\begin{align*}
&D_{x_1} \bar{u}_1(x_1) = \gamma, \quad \theta_h D_{x_h} \bar{u}_h(x_h) = \gamma, \quad \bar{u}_h(x_h) = \bar{u}_h(\tilde{x}_h), \quad \forall h \neq 1 \\
&\gamma + \beta_j D_{y_j} \bar{t}_j(y_j) = 0, \quad \bar{t}_j(y_j) = 0, \forall j \in \mathcal{J} \text{ and } \sum_{h \in \mathcal{H}} x_h - \sum_{j \in \mathcal{J}} y_j = \sum_{h \in \mathcal{H}} e_h
\end{align*}
\]

(7)

Also, it is well known that the Pareto optimal allocation \((\tilde{x}, \tilde{y})\) can be supported by some price system \( \tilde{p} \).\(^{11}\) From system (7), one easily deduces below the supporting price \( \tilde{p} \) and the equilibrium equations satisfied by \((\tilde{x}, \tilde{y})\) for appropriate Lagrange multipliers. More precisely, there exists \((\hat{\lambda}, \hat{\alpha}) \in \mathbb{R}^{H+}_{++} \times \mathbb{R}^J_{++} \) such that

\[
\begin{align*}
&\forall h \in \mathcal{H}, \quad D_{x_h} \bar{u}_h(\tilde{x}_h) = \hat{\lambda}_h \bar{p}, \quad \hat{\lambda} \cdot \tilde{x}_h = \tilde{p} \cdot (\hat{\epsilon} + \sum_{j \in \mathcal{J}} s_j \tilde{y}_j) \\
&\forall j \in \mathcal{J}, \quad \hat{\beta}_j D_{y_j} \bar{t}_j(\tilde{y}_j) = 0, \quad \bar{t}_j(\tilde{y}_j) = 0 \text{ and } \sum_{h \in \mathcal{H}} \tilde{x}_h - \sum_{j \in \mathcal{J}} \tilde{y}_j = \sum_{h \in \mathcal{H}} e_h
\end{align*}
\]

(8)

\(^{10}\)For a formal proof, see Appendix.

\(^{11}\)Using Debreu’s vocabulary, \((\tilde{x}, \tilde{y})\) is an equilibrium relative to some price system \( \tilde{p} \), see Section 6.4 of Debreu (1959).
where \( \lambda_1 := \gamma^C \), \( \lambda_h := \gamma^C \theta_h \) for every \( h \neq 1 \), \( \alpha_j := \frac{\beta_j}{\gamma^C} \), \( p^\lambda := \frac{\gamma^\lambda}{\gamma^C} \) and

\[
\tilde{e}_h := \tilde{x}_h - \sum_{j \in J} s_{jh} \bar{y}_j
\]

(9)

We call test economy the economy \( \tilde{E} := ((\pi_h, \tilde{e}_h, s_h)_{h \in H}, (\bar{f}_j)_{j \in J}) \) which is a standard private ownership economy à la Arrow–Debreu with no externalities at all. From Karush–Kuhn–Tucker conditions and (8), the following bundle

\[
\tilde{\xi} := (\tilde{x}, \tilde{\lambda}, \tilde{y}, \tilde{\alpha}, \tilde{p}^\lambda) \in \Xi
\]

(10)

is an equilibrium for the economy \( \tilde{E} \). Importantly, as will be shown in the proof of Lemma 10, Section 4, \( \tilde{\xi} \) is the unique equilibrium for the economy \( \tilde{E} \).

In a natural way, one deduces from (8) the function \( G : \Xi \to \mathbb{R}^{\dim \Xi} \) satisfying \( G(\tilde{\xi}) = 0 \) which is defined by

\[
G(\xi) := ((G^{h,1}(\xi) \cdot G^{h,2}(\xi))_{h \in H}, (G^{j,1}(\xi) \cdot G^{j,2}(\xi))_{j \in J}, G^M(\xi))
\]

\[
G^{h,1}(\xi) := D_{x_h} u_h(x_h, \pi_h - \lambda h p) - \lambda h p, \quad G^{h,2}(\xi) := -p \cdot (x_h - \tilde{e}_h - \sum_{j \in J} s_{jh} y_j)
\]

\[
G^{j,1}(\xi) := p + \alpha_j D_{y_j} t_j(y_j, \bar{g}_j - \pi), \quad G^{j,2}(\xi) := t_j(y_j, \bar{g}_j - \pi)
\]

\[
G^M(\xi) := \sum_{h \in H} x_h - \sum_{j \in J} y_j - \sum_{h \in H} e_h
\]

(11)

3.2 The homotopy and its properties

The basic idea is to homotopize the endowments and the externalities by an arc from the equilibrium conditions associated with the test economy to the ones associated with our economy. But, one finds the following difficulty. At equilibrium, the individual wealth is positive at the beginning as well as at the end of the arc. \(^{13}\) But, the individual wealth might not be positive along the homotopy arc, and consequently the individual budget constraint might be empty. We illustrate the reason below.

If one homotopizes the endowments by a segment, then for every \( \tau \in [0, 1] \) the

\[ \tilde{s}_{jh} := \frac{\tilde{p} \cdot \tilde{x}_h}{\tilde{p} \cdot \sum_{h \in H} \tilde{x}_h} \] and \( \tilde{e}_h := \tilde{s}_{jh} \sum_{h \in H} e_h \). But, in the latter case, the uniqueness of the equilibrium is not so obvious since the redistribution depends on the supporting price and on the Pareto optimal allocation.

\(^{12}\) One should notice that the endowments given by (9) are not necessarily positive. There are different redistributions that give rise to positive endowments. For example, \( \tilde{s}_{jh} := \frac{\tilde{p} \cdot \tilde{x}_h}{\tilde{p} \cdot \sum_{h \in H} \tilde{x}_h} \) and \( \tilde{e}_h := \tilde{s}_{jh} \sum_{h \in H} e_h \). But, in the latter case, the uniqueness of the equilibrium is not so obvious since the redistribution depends on the supporting price and on the Pareto optimal allocation.

\(^{13}\) Indeed, at the test economy \( \tilde{E} \), from (8) and the definition of \( \tilde{e}_h \) given by (9), the individual wealth is equal to \( \tilde{p} \cdot \tilde{x}_h \), which is positive. At the economy with externalities, the individual wealth is also positive by inactivity assumption and standard arguments from profit maximization.
individual wealth is given by \( p \cdot [\tau e_h + (1 - \tau)e_h] + p \cdot \sum_{j \in J} s_{jh}y_j \) which is equal to
\[
p \cdot [\tau e_h + (1 - \tau)e_h] + \sum_{j \in J} s_{jh}[y_j - (1 - \tau)\tilde{y}_j]
\]
from the definition of \( \tilde{e}_h \) given by (9). So, the individual wealth is positive if
\[
p \cdot y_j \geq p \cdot (1 - \tau)\tilde{y}_j, \quad \forall j \in J
\]
Using standard arguments from profit maximization, at equilibrium, this condition is satisfied if \((1 - \tau)\tilde{y}_j\) belongs to the production set of firm \( j \). On the other hand, if one homotopizes the externalities by a segment, then the production set becomes
\[
Y_j(\tau y_{-j} + (1 - \tau)\bar{y}_{-j}, \tau x + (1 - \tau)\bar{x})
\]
But, one does not know whether or not the production plan \((1 - \tau)\tilde{y}_j\) belongs to this production set unless it satisfies a strong convexity assumption, i.e. the production set is convex with respect to externalities.

To overcome the difficulty described above, we define the homotopy \( H_{t,e} \) in two times by two homotopies \( \Phi_{t,e} \) and \( \Gamma_{t,e} \). Namely,

- in the first homotopy \( \Phi_{t,e} \), we homotopize the endowments and the externalities **without** homotopizing the externalities in the production sets,
- in the second homotopy \( \Gamma_{t,e} \), we homotopize **only** the externalities in the production sets.\(^{14}\)

Formally, define the following convex combinations
\[
e_h(\tau) := \tau e_h + (1 - \tau)e_h, \quad x(\tau) := \tau x + (1 - \tau)x, \quad y(\tau) := \tau y + (1 - \tau)y
\]
the homotopies \( \Phi_{t,e}, \Gamma_{t,e} : \Xi \times [0, 1] \to \mathbb{R}^{\dim \Xi} \)
\[
\Phi_{t,e}(\xi, \tau) := ((\Phi_{t,e}^{h,1}(\xi, \tau), \Phi_{t,e}^{h,2}(\xi, \tau)))_{h \in H}, (\Phi_{t,e}^{j,1}(\xi, \tau), \Phi_{t,e}^{j,2}(\xi, \tau))_{j \in J}, \Phi_{t,e}^{M}(\xi, \tau)) \tag{12}
\]
where
\[
\Phi_{t,e}^{h,1}(\xi, \tau) = D_{x_h}u_h(x_h, x_{-h}(\tau), y(\tau)) - \lambda_h p; \quad \Phi_{t,e}^{h,2}(\xi, \tau) = -p \cdot [x_h - e_h(\tau) - \sum_{j \in J} s_{jh}y_j]; \quad \Phi_{t,e}^{j,1}(\xi, \tau) = p + \alpha_j D_{y_j}l_j(y_j, \bar{y}_{-j}, \bar{x}), \quad \Phi_{t,e}^{j,2}(\xi, \tau) = l_j(y_j, \bar{y}_{-j}, \bar{x}), \quad \Phi_{t,e}^{M}(\xi, \tau) = \sum_{h \in H} x_h^\downarrow - \sum_{j \in J} y_j^\downarrow - \sum_{h \in H} e_h^\downarrow,
\]
and
\[
\Gamma_{t,e}(\xi, \tau) := ((\Gamma_{t,e}^{h,1}(\xi, \tau), \Gamma_{t,e}^{h,2}(\xi, \tau)))_{h \in H}, (\Gamma_{t,e}^{j,1}(\xi, \tau), \Gamma_{t,e}^{j,2}(\xi, \tau))_{j \in J}, \Gamma_{t,e}^{M}(\xi, \tau)) \tag{13}
\]
\(^{14}\)We remark that if one assumes strong convexity on the production side, then one needs only one homotopy since endowments and externalities can be obviously homotoped at the same time.
where \(\Gamma^{h,1}_{i,e}(\xi, \tau) = D_{x_h}u_h(x_h, x_{-h}, y) - \lambda_h p, \Gamma^{h,2}_{i,e}(\xi, \tau) = -p \cdot [x_h - e_h - \sum_{j \in J} s_{jh} y_j],\)
\(\Gamma^{j,1}_{i,e}(\xi, \tau) = p + \alpha_j D_{y_j} t_j(y_j, y_{-j}(\tau), x(\tau)), \Gamma^{j,2}_{i,e}(\xi, \tau) = t_j(y_j, y_{-j}(\tau), x(\tau)), \Gamma^{M}_{i,e}(\xi, \tau) = \sum_{h \in H} x'_h - \sum_{j \in J} y'_j - \sum_{h \in H} e'_h.\)

Finally, define the homotopy \(H_{t,e} : \Xi \times [0, 1] \to \mathbb{R}^{\dim \Xi},\)
\[
H_{t,e}(\xi, \psi) := \begin{cases} 
\Phi_{t,e}(\xi, 2\psi) & \text{if } 0 \leq \psi \leq \frac{1}{2} \\
\Gamma_{t,e}(\xi, 2\psi - 1) & \text{if } \frac{1}{2} \leq \psi \leq 1 
\end{cases}
\]
(14)

The homotopy \(H_{t,e}\) is continuous since \(\Phi_{t,e}\) and \(\Gamma_{t,e}\) are composed by continuous functions. Importantly, \(H_{t,e}(\xi, \frac{1}{2})\) is well defined since \(\Phi_{t,e}(\xi, 1) = \Gamma_{t,e}(\xi, 0)\)

Furthermore, \(H_{t,e}(\xi, 0) = \Phi_{t,e}(\xi, 0) = G(\xi)\) and \(H_{t,e}(\xi, 1) = \Gamma_{t,e}(\xi, 1) = F_{t,e}(\xi)\).

Using the following two lemmas, all the assumptions of Theorem 9 are satisfied, and so Theorem 8 is completely proved.

**Lemma 10** \(G^{-1}(0) = \{\tilde{\xi}\}\) where \(\tilde{\xi}\) is given by (10), \(G\) is \(C^1\) and 0 is a regular value for \(G\).

**Lemma 11** For each \((t, e) \in T \times \mathbb{R}^{CH}_+, H^{-1}_{t,e}(0)\) is compact.

### 4 Proofs

In this section, we prove all the lemmas.

**Proof of Lemma 3.** Let \((x', y') \in A_{t,r}\. For every \(h \in \mathcal{H}, 0 \ll x'_h \ll \sum_{h \in \mathcal{H}} x'_h \leq \sum_{j \in J} y'_j + r\). Thus, to show that \(A_{t,r}\) is bounded it is enough to prove that the set \(Y_t \cap M_r\) is bounded where the set \(Y_t\) is defined by (1) and \(M_r := \{y' \in \mathbb{R}^{CJ} : \sum_{j \in J} y'_j + r \geq 0\}\). Since a subset of \(\mathbb{R}^n\) is bounded if and only if its asymptotic cone is reduced to zero, we show now that \(C(Y \cap M_r) = \{0\}\). One should notice that \(C(Y \cap M_r) \subseteq CY \cap CM_r.\)

Since the asymptotic cone of a set is immune to translation, we get \(CM_r = CM_0\), where \(M_0 := \{y' \in \mathbb{R}^{CJ} : \sum_{j \in J} y'_j \geq 0\}\). \(M_0\) is a closed cone with vertex 0, thus \(CM_0 = M_0\). So, in order to prove that \(CY \cap CM_r\) we have to show that \(CY \cap M_0 = \{0\}\) which directly follows from Assumption 2.

\[\text{Let } (B_i)_{i \in I} \subseteq \mathbb{R}^n\text{ be a family of subsets of } \mathbb{R}^n, C(\cap_{i \in I} B_i) \subseteq \cap_{i \in I} CB_i.\]
Therefore, \( \Delta = 0 \) where \( \Delta := \frac{\partial h}{\partial \tilde{x}} \) of system (7) we know that \( G \), So, the strict quasi-concavity of \( u \) follows from the definition of the components \( G^{h,1} \) and \( G^{j,1} \).

Proof of Lemma 10 First, we show that \( G^{-1}(0) = \{ \tilde{\xi} \} \) where \( \tilde{\xi} \) is given by (10) and \( G \) is defined by (11). From (8) and the definition of \( G \), one obviously deduces that \( \tilde{\xi} \in G^{-1}(0) \). Let \( \xi' \in \Xi \) be such that \( G(\xi') = 0 \), in order to prove that \( \xi' = \tilde{\xi} \), we show that \((x',y') = (\tilde{x},\tilde{y})\). Then, the uniqueness of multipliers and prices easily follows from the definition of the components \( G^{h,1} \) and \( G^{j,1} \).

Otherwise, suppose that \((x',y') \neq (\tilde{x},\tilde{y})\). Consider the convex combination \((x^*,y^*) = \frac{1}{2}(x',y') + \frac{1}{2}(\tilde{x},\tilde{y})\). First, we remark that \((x^*,y^*)\) is a feasible allocation. Indeed, since \( G^{j,2}(\xi') = G^{j,2}(\tilde{\xi}) = 0 \), by Point 4 of Assumption 1 one gets \( t_j(y^*_j,\bar{y}_{-j},\bar{p}) > 0 \) for every \( j \in J \). Since \((\tilde{x},\tilde{y})\) is a Pareto optimal allocation, from the last equation of system (7) we know that

\[
\sum_{h \in H} \tilde{x}_h - \sum_{j \in J} \tilde{y}_j = \sum_{h \in H} e_h
\]

So, using \( G^M(\xi') = 0 \) and summing over \( h \) the condition \( G^{h,2}(\xi') = 0 \), from the definition of \( \tilde{e}_h \) given by (9), we obtain \( \sum_{h \in H} x'_h - \sum_{j \in J} y'_j - \sum_{h \in H} e_h = 0 \) and consequently

\[
\sum_{h \in H} x'_h - \sum_{j \in J} y'_j = \sum_{h \in H} e_h.
\]

Second, we show that for all \( h \in H \)

\[
u_h(x'_h,\bar{x}_{-h},\bar{y}) \geq u_h(\tilde{x}_h,\bar{x}_{-h},\bar{y}) \tag{15}\]

So, the strict quasi-concavity of \( u_h(\cdot,\bar{x}_{-h},\bar{y}) \) (see Point 3 of Assumption 4) implies that for all \( h \in H \), \( u_h(x'_h,\bar{x}_{-h},\bar{y}) > \min\{u_h(x'_h,\bar{x}_{-h},\bar{y}),u_h(\tilde{x}_h,\bar{x}_{-h},\bar{y})\} = u_h(\tilde{x}_h,\bar{x}_{-h},\bar{y}) \) which contradicts the Pareto optimality of \((\tilde{x},\tilde{y})\).

Since \( G^{h,1}(\xi') = G^{h,2}(\xi') = 0 \), from Karush-Kuhn-Tucker sufficient conditions and the definition of \( \tilde{e}_h \) given by (9), \( x'_h \) solves the following maximization problem

\[
\max_{x_h \in \mathbb{R}^{d_h}_{+}} u_h(x_h,\bar{x}_{-h},\bar{y}) \tag{16}
\]

subject to \( p' \cdot x_h \leq p' \cdot \tilde{x}_h + \sum_{j \in J} \tilde{z}_{jh} p' \cdot (y'_j - \tilde{y}_j) \)

From \( G^{j,1}(\xi') = G^{j,2}(\xi') = 0 \) and Karush-Kuhn-Tucker sufficient conditions, \( y'_j \) solves the maximization problem \( \max \ p' \cdot y_j \) subject to \( t_j(y_j,\bar{y}_{-j},\bar{p}) \geq 0 \). From \( G^{j,2}(\tilde{\xi}) = 0 \), \( \tilde{y}_j \) belongs to constraint set of this problem, and so \( p' \cdot (y'_j - \tilde{y}_j) \geq 0 \).

Therefore, \( \tilde{x}_h \) belongs to the budget constraint of problem (16), and so (15) is completely proved.

Second, we remark that \( G \) is \( C^1 \) by Point 4 of Assumptions 1 and by Point 3 of Assumption 4. Finally, in order to show that 0 is a regular value for \( G \), one proves that \( \det D_{\xi} G(\xi) \neq 0 \). In this regard, we show that if \( D_{\xi} G(\xi)(\Delta) = 0 \), then \( \Delta = 0 \) where \( \Delta := (\Delta x_h,\Delta \lambda_h)_{h \in H},(\Delta y_j,\Delta \alpha_j)_{j \in J},\Delta p) \in \mathbb{R}^{\dim \Xi}. \) The system
\[ D \xi G(\xi)(\Delta) = 0 \] is given below.

\[
\begin{align*}
(h.1) & \quad \sum_{h \in \mathcal{H}} D^2_{x_h} u_h(\bar{x}_h, \bar{\tau}_h, \bar{\gamma})(\Delta x_h) - \Delta \lambda_h \bar{p}^T - \bar{\lambda}_h (\Delta p \setminus 0)^T = 0 \quad \forall \ h \in \mathcal{H} \\
(h.2) & \quad \bar{p} \cdot \left( \sum_{j \in J} s_{jh} \Delta y_j \right) - \bar{p} \cdot \Delta \bar{x}_h = 0 \quad \forall \ h \in \mathcal{H} \\
(j.1) & \quad \alpha_j D^2_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau})(\Delta y_j) + \Delta \alpha_j D_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) + (\Delta p \setminus 0)^T = 0 \quad \forall \ j \in J \\
(j.2) & \quad D_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) \cdot \Delta y_j = 0 \quad \forall \ j \in J \\
(M) & \quad \sum_{h \in \mathcal{H}} \Delta x_h \cdot \sum_{j \in J} \Delta y_j = 0
\end{align*}
\]

(17)

First, we show that if \( \Delta x_h = 0 \) for every \( h \in \mathcal{H} \), then \( \Delta = 0 \). In this case, considering commodity \( C \), from (h.1) in system (17) we get \( \Delta \lambda_h = 0 \) for all \( h \in \mathcal{H} \), and so \( \bar{\lambda}_h \Delta p \setminus 0 = 0 \) implies \( \Delta p \setminus 0 = 0 \) since \( \bar{\lambda}_h > 0 \). Since any differentiably strictly quasi-concave function with gradient different from zero has the bordered Hessian with determinant different from zero, \( t_j(\cdot, \bar{\gamma}_{j-}, \bar{\tau}) \) is strictly quasi-concave (see Point 4 of Assumption 1) and \( D_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) \neq 0 \) (Point 3 of Assumption 1), the following matrix has full rank.

\[
\begin{pmatrix}
\alpha_j D^2_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) & D_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) \\
D_{y_j} t_j(\bar{y}_j, \bar{\gamma}_{j-}, \bar{\tau}) & 0
\end{pmatrix}
\]

Thus, \( \Delta y_j, \Delta \alpha_j = (0, 0) \) by (j.1) and (j.2) in system (17), and one gets \( \Delta = 0 \).

Second, we prove that \( \Delta x_h = 0 \) for every \( h \in \mathcal{H} \). Suppose by contradiction that there is \( \bar{h} \in \mathcal{H} \) such that \( \Delta x_{\bar{h}} \neq 0 \). The proof goes through two claims. We first claim that

\[ \Delta p \setminus 0 \cdot (\sum_{h \in \mathcal{H}} \Delta x_h) < 0 \quad (18) \]

Multiplying both sides of \( G^{h,(1)}(\bar{\xi}) = 0 \) by \( s_{jh} \Delta y_j \), we get \( \bar{p} \cdot s_{jh} \Delta y_j = 0 \) by (j.2) in system (17). Summing over \( j \), for each \( h \in \mathcal{H} \) one gets \( \bar{p} \cdot (\sum_{j \in J} s_{jh} \Delta y_j) = 0 \) which implies

\[ \bar{p} \cdot \Delta x_h = 0 \quad (19) \]

by (h.2) in system (17). Thus, multiplying (h.1) in system (17) by \( \Delta x_h \) and summing over \( h \), one gets

\[ \sum_{h \in \mathcal{H}} \Delta x_h \frac{D^2_{x_h} u_h(\bar{x}_h, \bar{\tau}_h, \bar{\gamma})}{\bar{\lambda}_h} (\Delta x_h) = \Delta p \setminus 0 \cdot (\sum_{h \in \mathcal{H}} \Delta x_h) \quad (20) \]

Using (19) and multiplying \( G^{h,(1)}(\bar{\xi}) = 0 \) by \( \Delta x_h \), one also gets \( D_{x_h} u_h(\bar{x}_h, \bar{\tau}_h, \bar{\gamma}) \cdot \Delta x_h = 0 \). Therefore, Point 3 of Assumption 4 and (20) imply (18) which holds true with a strict inequality since \( \Delta x_{\bar{h}} \neq 0 \).
Second, we claim that \( \Delta p^\lambda \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^\lambda) \geq 0 \) which contradicts (18), and consequently \( \Delta x_h = 0 \) for all \( h \in \mathcal{H} \) which completes the proof of the lemma.

By (M) in system (17), one has

\[
\Delta p^\lambda \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^\lambda) = \Delta p^\lambda \cdot (\sum_{j \in \mathcal{J}} \Delta y_j^\lambda)
\]

By (j,2) in system (17), multiplying (j,1) in system (17) by \( \Delta y_j \) and summing over \( j \) one has

\[
\Delta p^\lambda \cdot (\sum_{j \in \mathcal{J}} \Delta y_j^\lambda) = -\sum_{j \in \mathcal{J}} \tilde{\alpha}_j \Delta y_j D_{y_j}^2 t_j(\tilde{y}_j, \tilde{y}_{-j}, \tilde{\tau})(\Delta y_j).
\]

Thus,

\[
\Delta p^\lambda \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^\lambda) = -\sum_{j \in \mathcal{J}} \tilde{\alpha}_j \Delta y_j D_{y_j}^2 t_j(\tilde{y}_j, \tilde{y}_{-j}, \tilde{\tau})(\Delta y_j)
\]

Therefore, \( \Delta p^\lambda \cdot (\sum_{h \in \mathcal{H}} \Delta x_h^\lambda) \geq 0 \) since \( t_j(\cdot, \tilde{y}_{-j}, \tilde{\tau}) \) is strictly quasi-concave (see Point 4 of Assumption 1).

**Proof of Lemma 11.** Let \( (t, e) \in \mathcal{T} \times \mathbb{R}^{CH}_{++} \). Observe that \( H_{t,e}^{-1}(0) = \Phi_{t,e}^{-1}(0) \cup \Gamma_{t,e}^{-1}(0) \). Since the union of a finite number of compact sets is compact, it is enough to show that \( \Phi_{t,e}^{-1}(0) \) and \( \Gamma_{t,e}^{-1}(0) \) are compact.

**Claim 1.** \( \Phi_{t,e}^{-1}(0) \) is compact.

We prove that, up to a subsequence, every sequence \( (\xi^\nu, \tau^\nu)_{\nu \in \mathbb{N}} \subseteq \Phi_{t,e}^{-1}(0) \) converges to an element of \( \Phi_{t,e}^{-1}(0) \), where \( \xi^\nu := (x^\nu, \lambda^\nu, y^\nu, \alpha^\nu, p^\nu, t^\nu)_{\nu \in \mathbb{N}} \). First observe that, since \( \{\tau^\nu : \nu \in \mathbb{N}\} \subseteq [0, 1] \), up to a subsequence, \( (\tau^\nu)_{\nu \in \mathbb{N}} \) converges to some \( \tau^* \in [0, 1] \). From Steps 1.1, 1.2, 1.3 and 1.4 below, we have that up to a subsequence, \( (\xi^\nu)_{\nu \in \mathbb{N}} \) converges to some \( \xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^*) \in \Xi \). Since the homotopy \( \Phi \) is continuous, taking the limit, we get the desired result, that is \( (\xi^*, \tau^*) \in \Phi_{t,e}^{-1}(0) \).

**Step 1.1.** Up to a subsequence, \( (x^\nu, y^\nu)_{\nu \in \mathbb{N}} \) converges to some \( (x^*, y^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \).

We first prove that the sequence \( (x^\nu, y^\nu)_{\nu \in \mathbb{N}} \) belongs to the set of feasible allocations for fixed externalities, which is included in the set \( A_{t,r} \) given by (2) of Lemma 3. Using a similar strategy as in Claim 1 of Lemma 10, by (9), \( \Phi^{h,2}_{t,e}(\xi^\nu, \tau^\nu) = 0 \) and \( \Phi^{M}_{t,e}(\xi^\nu, \tau^\nu) = 0 \) one easily gets

\[
\sum_{h \in \mathcal{H}} x_h^\nu - \sum_{j \in \mathcal{J}} y_j^\nu = \sum_{h \in \mathcal{H}} e_h, \quad \forall \nu \in \mathbb{N}
\]

So, \( (x^\nu, y^\nu)_{\nu \in \mathbb{N}} \subseteq A_{t,r} \) by \( \Phi^{h,2}_{t,e}(\xi^\nu, \tau^\nu) = 0 \). Consequently, the sequence \( (x^\nu, y^\nu)_{\nu \in \mathbb{N}} \) belongs to \( \text{cl} A_{t,r} \), which is compact by Lemma 3. Up to a subsequence, \( (x^\nu, y^\nu)_{\nu \in \mathbb{N}} \) converges to some \( (x^*, y^*) \in \text{cl} A_{t,r} \subseteq \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \), and thus \( (x^*, y^*) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{CJ} \).

**Step 1.2.** The consumption allocation \( x^* \) is strictly positive, i.e. \( x^* \gg 0 \). The proof is based on Point 4 of Assumption 4. By (9) and \( \Phi^{h,1}_{t,e}(\xi^\nu, \tau^\nu) = \Phi^{h,2}_{t,e}(\xi^\nu, \tau^\nu) = 0 \),
\( x_h^\nu \) solves the following problem for every \( \nu \in \mathbb{N} \):\(^{16}\)

\[
\begin{align*}
\max_{x_h \in \mathbb{R}^C_{++}} & \quad u_h(x_h, x_h^\nu(\tau^\nu), y^\nu(\tau^\nu)) \\
\text{subject to} & \quad p^\nu \cdot x_h \leq p^\nu \cdot [\tau^\nu e_h + (1 - \tau^\nu)\tilde{x}_h] + p^\nu \cdot \sum_{j \in J} s_{jh}(y_j^\nu - (1 - \tau^\nu)\tilde{y}_j)
\end{align*}
\]

(21)

We claim first that for every \( \nu \in \mathbb{N} \), the point

\[
\tau^\nu e_h + (1 - \tau^\nu)\tilde{x}_h
\]

(22)

belongs to the budget constraint of the problem above. By \( \Phi^{t,e}_1(\xi^\nu, \tau^\nu) = \Phi^{t,e}_2(\xi^\nu, \tau^\nu) = 0 \) and Karush–Kuhn–Tucker sufficient conditions, \( y_j^\nu \) solves the following problem for every \( \nu \in \mathbb{N} \):

\[
\begin{align*}
\max_{y_j \in \mathbb{R}^C} & \quad p^\nu \cdot y_j \\
\text{subject to} & \quad t_j(y_j, \bar{y}_{-j}, \bar{x}) \geq 0
\end{align*}
\]

(23)

Since inactivity is possible, \( t_j(0, \bar{y}_{-j}, \bar{x}) \geq 0 \) by Point 2 of Assumption 1. Since \((\tilde{x}, \tilde{y})\) is a Pareto optimal allocation, \( t_j(\tilde{y}_j, \bar{y}_{-j}, \bar{x}) = 0 \) by system (7). Since \( t_j(\cdot, \bar{y}_{-j}, \bar{x}) \) is strictly quasi-concave, we get

\[
t_j(\tau^\nu 0 + (1 - \tau^\nu)\tilde{y}_j, \bar{y}_{-j}, \bar{x}) = t_j((1 - \tau^\nu)\tilde{y}_j, \bar{y}_{-j}, \bar{x}) \geq 0
\]

So, the production plan \((1 - \tau^\nu)\tilde{y}_j\) belongs to the constraint set of problem (23), and thus \( p^\nu \cdot (y_j^\nu - (1 - \tau^\nu)\tilde{y}_j) \geq 0 \). Therefore,

\[
p^\nu \cdot \sum_{j \in J} s_{jh}(y_j^\nu - (1 - \tau^\nu)\tilde{y}_j) \geq 0
\]

which completes the proof of the claim.

We claim now that \( x_h^\nu \) belongs to the closure of some upper contour set. Obviously, for every \( \nu \in \mathbb{N} \)

\[
u \quad u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) \geq u_h(\tau^\nu e_h + (1 - \tau^\nu)\tilde{x}_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))
\]

By Point 2 of Assumption 4, for every \( \varepsilon > 0 \) we have that

\[
u \quad u_h(x_h^\nu + \varepsilon \mathbf{1}, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu)) > u_h(\tau^\nu e_h + (1 - \tau^\nu)\tilde{x}_h, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))
\]

where \( \mathbf{1} := (1, \ldots, 1) \in \mathbb{R}^C_{++} \). So, taking the limit for \( \nu \rightarrow +\infty \) and using the continuity of \( u_h \) given by Point 1 of Assumption 4, we get

\[
u \quad u_h(x_h^* + \varepsilon \mathbf{1}, x_{-h}^*(\tau^*), y^*(\tau^*)) \geq u_h(\tau^* e_h + (1 - \tau^*)\tilde{x}_h, x_{-h}^*(\tau^*), y^*(\tau^*)) := u
\]

That is, for every \( \varepsilon > 0 \) the point \((x_h^* + \varepsilon \mathbf{1})\) belongs to the following set

\[\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x_{-h}^*(\tau^*), y^*(\tau^*)) \geq u\}\]

\(^{16}\)Karush–Kuhn–Tucker conditions are sufficient to solve problem (21).
So, the point \( x_h^* \) belongs to the closure of set above which is included in \( \mathbb{R}^{C_+} \) by Point 4 of Assumption 4. Therefore, \( x_h^* \in \mathbb{R}^{CH}_{++} \). One should notice that, since \( \tau^* \in [0,1] \), \( x^*_h(\tau^*) \) is not necessarily strictly positive. For that reason, in Point 4 of Assumption 4 we consider \( x_{-h} \) in \( \mathbb{R}^{C(H-1)}_+ \).

**Step 1.3.** Up to a subsequence, \( (\lambda^\nu, p^\nu)_{\nu \in \mathbb{N}} \) converges to some \( (\lambda^*, p^*) \in \mathbb{H} \times \mathbb{R}^{C^{-1}}_+ \). By \( \Phi^{h,1}_{t,e}(\xi^\nu, \tau^\nu) = 0 \), considering commodity \( C \), for every \( \nu \in \mathbb{N} \) we get

\[
\lambda_h^\nu = D_{x^\nu_h} u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^\nu))
\]

Taking the limit and using the continuity of \( Du_h \) (see Point 1 of Assumption 4) we have

\[
\lambda_h^* = D_{x^*_h} u_h(x^*_h, x_{-h}^*(\tau^*), y^*(\tau^*))
\]

which is strictly positive since fixing the externalities the function \( u_h \) is strictly increasing (see Point 2 of Assumption 4).

By \( \Phi^{h,1}_{t,e}(\xi^\nu, \tau^\nu) = 0 \), for every commodity \( c \neq C \) and for all \( \nu \in \mathbb{N} \) we have

\[
p^\nu c = \frac{D_{x^\nu_h} u_h(x_h^\nu, x_{-h}^\nu(\tau^\nu), y^\nu(\tau^*))}{\lambda_h^\nu}, \quad \forall \nu \in \mathbb{N}
\]

Taking the limit and using Points 1 and 2 of Assumption 4, for all \( c \neq C \) we get

\[
p^* c = \frac{D_{x^*_h} u_h(x^*_h, x_{-h}^*(\tau^*), y^*(\tau^*))}{\lambda_h^*} > 0
\]

Therefore, \( p^* \) \( \in \mathbb{R}^{C^{-1}}_+ \).

**Step 1.4.** Up to a subsequence, \( (\alpha^\nu)_{\nu \in \mathbb{N}} \) converges to some \( \alpha^* \in \mathbb{F}_{++} \). By \( \Phi^{j,1}_{t,e}(\xi^\nu, \tau^\nu) = 0 \), considering commodity \( C \), we get

\[
\alpha_j^\nu = -\frac{1}{D_{y^\nu_j} t_j(y^\nu_j, \bar{y}_j, \bar{x})}, \quad \forall \nu \in \mathbb{N}
\]

Taking the limit for \( \nu \to +\infty \) and using the continuity of \( Dt_j \) and the “free disposal” property (see Points 1 and 3 of Assumption 1), the sequence \( (\alpha_j^\nu)_{\nu \in \mathbb{N}} \) converges to

\[
\alpha_j^* := -\frac{1}{D_{y_j^*} t_j(y_j^*, \bar{y}_j, \bar{x})} > 0
\]

**Claim 2.** \( \Gamma_{t,e}^{-1}(0) \) is compact.

Let \( (\xi^\nu, \tau^\nu)_{\nu \in \mathbb{N}} \) be a sequences in \( \Gamma_{t,e}^{-1}(0) \). As in Claim 1, \( (\tau^\nu)_{\nu \in \mathbb{N}} \) converges to \( \tau^* \in [0,1] \). From Seps 2.1, 2.2, 2.3 and 2.4 below, we have that, up to a subsequence, \( (\xi^\nu)_{\nu \in \mathbb{N}} \) converges to an element \( \xi^* := (x^*, \lambda^*, y^*, \alpha^*, p^\nu) \in \Xi \). Since \( \Gamma_{t,e} \) is a continuous function, taking limit one gets \( (\xi^*, \tau^*) \in \Gamma_{t,e}^{-1}(0) \).
**Step 2.1.** Up to a subsequence, \((x^\nu, y^\nu)_{\nu \in \mathbb{N}}\) converges to some \((x^*, y^*) \in \mathbb{R}^CH \times \mathbb{R}^{CJ}\). By \(\Gamma_{t,e}^{j,2}(\xi^\nu, \tau^\nu) = 0\), we have that for every \(\nu \in \mathbb{N}\) and for every \(j\)

\[
t_j(y^\nu_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) = 0
\]

Summing \(\Gamma_{t,e}^{h,2}(\xi^\nu, \tau^\nu) = 0\) over \(h\), by \(\Gamma_{t,e}^M(\xi^\nu, \tau^\nu) = 0\) we get

\[
\sum_{h \in H} x^\nu_h - \sum_{j \in J} y^\nu_j = \sum_{h \in H} e_h
\]

for all \(\nu \in \mathbb{N}\). Therefore, \((x^\nu, y^\nu)_{\nu \in \mathbb{N}}\) belongs to the set \(A_{t,r}\) given by (2). Consequently, the sequence \((x^\nu, y^\nu)_{\nu \in \mathbb{N}}\) converges to some \((x^*, y^*) \in \text{cl} A_{t,r} \subseteq \mathbb{R}^CH \times \mathbb{R}^{CJ}\), and thus \((x^*, y^*) \in \mathbb{R}^CH \times \mathbb{R}^{CJ}\).

**Step 2.2.** The consumption allocation \(x^*\) is strictly positive, i.e. \(x^* \gg 0\). The argument is similar to the one used in Step 1.2 of Claim 1. It suffices to replace

- (1) the problem (21) with the following problem

\[
\max_{x_h \in \mathbb{R}^C_{++}} u_h(x_h, x^*_{-h}, y^*)
\]

subject to \(p^\nu \cdot x_h \leq p^\nu \cdot e_h + p^\nu \cdot \sum_{j \in J} s_{jh} y^\nu_j\)

according to \(\Gamma_{t,e}^{h,1}(\xi^\nu, \tau^\nu) = \Gamma_{t,e}^{h,2}(\xi^\nu, \tau^\nu) = 0\),

- (2) the point given by (22) with \(e_h\),

- (3) the problem (23) with the following problem

\[
\max_{y_j \in \mathbb{R}^C_{++}} p^\nu \cdot y_j
\]

subject to \(t_j(y_j, y^\nu_{-j}(\tau^\nu), x^\nu(\tau^\nu)) \geq 0\) \quad (24)

according to \(\Gamma_{t,e}^{j,1}(\xi^\nu, \tau^\nu) = \Gamma_{t,e}^{j,2}(\xi^\nu, \tau^\nu) = 0\).

Next, as in Step 1.2 of Claim 1 one easily shows that \(x^*_h\) belongs to the closure of \(\{x_h \in \mathbb{R}^C_{++} : u_h(x_h, x^*_{-h}, y^*) \geq u := u_h(e_h, x^*_{-h}, y^*)\}\).

**Step 2.3.** Up to a subsequence, \((\lambda^\nu, p^\nu)_{\nu \in \mathbb{N}}\) converges to some \(\lambda^* \in \mathbb{R}^H_{++} \times \mathbb{R}^{CJ-1}_{++}\). Using Points 1 and 2 of Assumption 4, the proof is similar to the one of Step 1.3 in Claim 1.

**Step 2.4.** Up to a subsequence, \((\alpha^\nu)_{\nu \in \mathbb{N}}\) converges to some \(\alpha^* \in \mathbb{R}^J_{++}\). Using Points 1 and 3 of Assumption 1, the proof is similar to the one of Step 1.4 in Claim 1. ■

**Appendix**

**Characterization of Pareto optimality without externalities**

Let \(\mathcal{E} := ((\mathbb{R}^C_{++}, \mathbb{R}_{++}), (\overline{t}_j)_{j \in J}, r)\) be the economy defined in Section 3.1 where \(r := \sum_{h \in H} e_h\). Define the following sets
\[ A_{t,r} := \{(x', y') \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ} : \bar{I}_j(y'_j) \geq 0, \forall j \in J \text{ and } \sum_{h \in H} x'_h - \sum_{j \in J} y'_j \leq r \} \]

\[ U_r := \{(u_h)_{h \in H} \in \prod_{h \in H} \text{Im } \pi_h : \exists (x, y) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}, \bar{I}_j(y) \geq 0, \forall j \in J, \sum_{h \in H} x_h - \sum_{j \in J} y_j \leq r \text{ and } \pi_h(x_h) \geq u_h \forall h \in H \} \]

\[ \hat{U}_r := \{(u_h)_{h \neq 1} \in \prod_{h \neq 1} \text{Im } \pi_h : \exists u_1 \in \text{Im } \pi_1, (u_1, (u_h)_{h \neq 1}) \in U_r \} \]

By Point 2 of Assumption 1, the set \( A_{t,r} \) defined by (2) is non-empty. Thus, the sets \( U_r \) and \( \hat{U}_r \) are non-empty. Let \((u'_h)_{h \neq 1} \in \hat{U}_r \). Consider the following optimization problem

\[
\begin{align*}
\max_{(x,y) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}} & \quad \pi_1(x_1) \\
\text{subject to} & \quad \bar{I}_j(y) \geq 0 \text{ for every } j \in J \\
& \quad \pi_h(x_h) \geq u'_h \text{ for every } h \neq 1 \\
& \quad \sum_{h \in H} x_h - \sum_{j \in J} y_j \leq r
\end{align*}
\]  

(25)

**Proposition 12** There exists a unique solution \((\tilde{x}, \tilde{y})\) to problem (25).

**Proof.** In order to apply Weierstrass' Theorem, one replaces problem (25) with the following problem

\[
\begin{align*}
\max_{(x,y) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}} & \quad \pi_1(x_1) \\
\text{subject to} & \quad \bar{I}_j(y) \geq 0 \text{ for every } j \in J \\
& \quad \pi_h(x_h) \geq u'_h \text{ for every } h \neq 1 \\
& \quad \pi_1(x_1) \geq u'_1 \\
& \quad \sum_{h \in H} x_h - \sum_{j \in J} y_j \leq r
\end{align*}
\]  

(26)

where \( u'_1 \in \text{Im } \pi_1 \) is given by the definition of \( \hat{U}_r \). Since \((u'_h)_{h \in H} \in U_r \), there exists \((x', y') \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}\) such that \( \bar{I}_j(y'_j) \geq 0 \) for all \( j \in J \), \( \sum_{h \in H} x'_h - \sum_{j \in J} y'_j \leq r \) and \( \pi_h(x'_h) = u'_h, \forall h \in H \).

Denote \( K_1 \) the constraints set associated with problem (26). \( K_1 \) is non-empty since \((u'_h)_{h \in H} \in U_r \). We first claim that \( K_1 \) is a compact set included in \( \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ} \). We notice that \( K_1 = N \cap A_{t,r} \) where

\[
N := \{(x,y) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ} : \pi_h(x_h) \geq u'_h \forall h \in H \}
\]

So, from Lemma 3 we have that \( K_1 \) is bounded. Furthermore, \( K_1 \) is closed. Indeed, take a sequence \((x^\nu, y^\nu)_{\nu \in \mathbb{N}}\) in \( K_1 \) converging to some \((x,y)\). Since \((x^\nu, y^\nu)_{\nu \in \mathbb{N}} \subseteq N\), \((x,y)\) belongs to the set \( \text{cl}_{\mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ}} N \) which is included in \( \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ} \) by Point 4 of Assumption 4. So, \((x,y) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CJ} \). Since the function \( \pi_h \) are continuous (see Point 1 of Assumption 4), \((x,y) \in N \). Since the functions \( \bar{I}_j \) are continuous
(see Point 1 of Assumption 1), \((x, y) \in \bar{A}_{t,r}\) and so \((x, y) \in K_1\) which completes the proof of the claim.

By Weierstrass’ Theorem, there exists a solution \((\bar{x}, \bar{y})\) to problem (26). The solution to problem (26) is unique since the objective function is strictly quasi-concave (see Point 3 of Assumption 4) and the constraints set is convex (see Point 4 of Assumption 1 and Point 3 of Assumption 4).

We complete the proof showing that problems (25) and (26) are equivalent. Denote with \(K\) the constraints set associated with problem (25). Let \((\bar{x}, \bar{y})\) be a solution to problem (25), one gets
\[
\begin{align*}
\sum_{h \in H} x_h + \sum_{j \in J} y_j &= 0, \\
\bar{x}_h + \gamma + \beta_j \bar{y}_j &= 0, \forall j \in J
\end{align*}
\]
which completes the proof of the lemma.

**Proposition 13** Let \((\bar{x}, \bar{y})\) be the allocation given by Lemma 12. There exists \((\bar{\theta}, \bar{\gamma}, \bar{\beta}) := ((\bar{\theta}_h)_{h \neq 1}, \bar{\gamma}, (\bar{\beta}_j)_{j \in J}) \in \mathbb{R}^{H-1} \times \mathbb{R}^C_+ \times \mathbb{R}^J_+\) such that \((\bar{x}, \bar{y}, \bar{\theta}, \bar{\gamma}, \bar{\beta})\) is the unique solution to the following system.

\[
\begin{cases}
D_{x_1} \pi_1(x_1) - \gamma = 0 \\
\theta_h D_{x_h} \pi_h(x_h) - \gamma = 0, \forall h \neq 1 \\
\pi_h(x_h) - \pi_h(\bar{x}_h) = 0, \forall h \neq 1 \\
\gamma + \beta_j D_{y_j} \bar{t}_j(y_j) = 0, \forall j \in J \\
\bar{t}_j(y_j) = 0, \forall j \in J \\
-r - \sum_{h \in H} x_h + \sum_{j \in J} y_j = 0
\end{cases}
\]

(27)

**Proof.** The result follows showing that Karush-Kuhn-Tucker’ conditions are necessary conditions to solve problem (25). The Lagrangean function associated with problem (25) is given by

\[
\mathcal{L}(x, y, \theta, \gamma, \beta) = \pi_1(x_1) + \sum_{h \neq 1} \theta_h (\pi_h(x_h) - u'_h) + \sum_{j \in J} \beta_j \bar{t}_j(y_j) + \gamma (r - \sum_{h \in H} x_h + \sum_{j \in J} y_j)
\]

where \((\bar{\theta}, \bar{\gamma}, \bar{\beta}) := ((\bar{\theta}_h)_{h \neq 1}, \bar{\gamma}, (\bar{\beta}_j)_{j \in J}) \in \mathbb{R}^{H-1} \times \mathbb{R}^C_+ \times \mathbb{R}^J_+\) is the vector of the Lagrange multipliers associated with the constraints set of problem (25). So, the
Karush-Kuhn-Tucker conditions are given by

\begin{align*}
(1) \quad & D_{x_1} \bar{u}_1(x_1) - \gamma = 0 \\
(2) \quad & \theta_h D_{x_h} \bar{u}_h(x_h) - \gamma = 0, \; \forall \; h \neq 1 \\
(3) \quad & \gamma + \beta_j D_{y_j} \bar{t}_j(y_j) = 0, \forall \; j \in \mathcal{J} \\
(4) \quad & \min \{ \theta_h, \bar{u}_h(x_h) - u'_h \} = 0, \; \forall \; h \neq 1 \\
(5) \quad & \min \{ \beta_j, \bar{t}_j(y_j) \} = 0, \forall \; j \in \mathcal{J} \\
(6) \quad & \min \{ \gamma, r - \sum_{h \in \mathcal{H}} x_h + \sum_{j \in \mathcal{J}} y_j \} = 0
\end{align*}

It is enough to show that the Jacobian matrix associated with the constraints functions of problem (25) has full row rank. The Jacobian matrix is described below.

\[
\begin{bmatrix}
D_{x_2} \bar{u}_2(x_2) & \ldots & 0 & 0 & 0 & \ldots & 0 \\
: & \ldots & : & : & : & \ldots & : \\
0 & \ldots & D_{x_H} \bar{u}_H(x_H) & 0 & 0 & \ldots & 0 \\
-I_C & \ldots & -I_C & -I_C & I_C & \ldots & I_C \\
0 & \ldots & 0 & 0 & D_{y_1} \bar{t}_1(y_1) & \ldots & 0 \\
: & \ldots & : & : & : & \ldots & : \\
0 & \ldots & 0 & 0 & 0 & \ldots & D_{y_J} \bar{t}_J(y_J)
\end{bmatrix}
\]

The matrix above has full row rank since \( D_{x_h} \bar{u}_h(x_h) \gg 0 \) and \( D_{y_j} \bar{t}_j(y_j) \ll 0 \) (see Point 3 of Assumption 1 and Point 2 of Assumption 4) imply that the determinant
of the square sub-matrix $D$ defined below is different from zero.

\[
D := \begin{pmatrix}
    x_2^1 & \ldots & x_H^1 & x_1 & y_1^1 & \ldots & y_J^1 \\
    D_{x_2^1} \pi_2(x_2) & \ldots & 0 & 0 & 0 & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & D_{x_H^1} \pi_H(x_H) & 0 & 0 & \ldots & 0 \\
    -1 & \ldots & -1 & I_C & -1 & \ldots & -1 \\
    0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
    0 & \ldots & 0 & 0 & D_{y_1^1} \tilde{t}_1(y_1) & \ldots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & 0 & 0 & 0 & \ldots & D_{y_J^1} \tilde{t}_J(y_J)
\end{pmatrix}
\]

Therefore, the conditions given by (28) are necessary to solve problem (25). By Lemma 12 and equations (1), (2) and (3) in system (28), the Lagrange multipliers are unique. Furthermore, Point 3 of Assumption 1 and Point 2 of Assumption 4 imply that the Lagrange multipliers satisfying system (28) are strictly positive, and consequently, all the constraints in problem (25) are binding. So, in particular one gets

\[\pi_h(\tilde{x}_h) = u'_h \quad \forall \ h \neq 1\]

Therefore from system (28) one deduces system (27) and the lemma is completely proved. Using Proposition 13, one easily proves the following proposition.

**Proposition 14** If \((\tilde{x}, \tilde{y})\) is a solution of problem (25), then \((\tilde{x}, \tilde{y})\) solves the problem below

\[
\max_{(x,y) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{CJ}} \pi_1(x_1)
\]

subject to \(\tilde{t}_j(y_j) \geq 0\) for each \(j \in J\)

\[\pi_h(x_h) \geq \pi_h(\tilde{x}_h) \text{ for } h \neq 1\]

\[\sum_{h \in H} x_h - \sum_{j \in J} y_j \leq r\]  \hspace{1cm} (29)

**Proof.** It follows from system (27) in Lemma 13 and the fact that Karush-Kuhn-Tucker conditions are sufficient to solve problem (29). Indeed, by Points 2 and 3 of Assumption 4 the function \(\pi_1\) is quasi-concave with gradient different from zero, and by Point 4 of Assumption 1 and Point 3 of Assumption 4, the constraint functions associated with problem (29) are quasi-concave. □

We remind that \((\tilde{x}, \tilde{y}) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{CJ}\) is Pareto optimal allocation of the production economy \(\mathcal{E}\) if there is no other allocation \((\hat{x}, \hat{y}) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{CJ}\) such that

1. \(\tilde{t}_j(\tilde{y}_j) \geq 0\) for all \(j \in J\) and \(\sum_{h \in H} \tilde{x}_h \leq r + \sum_{j \in J} \tilde{y}_j\)
Proposition 15 \((\tilde{x}, \tilde{y})\) solves problem (29) if and only if it is a Pareto optimal allocation of \(\mathcal{Z}\).

Proof. By definition of Pareto optimal allocation, if \((\tilde{x}, \tilde{y})\) is a Pareto optimal allocation then \((\tilde{x}, \tilde{y})\) solves problem (29). Suppose now that \((\tilde{x}, \tilde{y})\) solves problem (29), we prove that \((\tilde{x}, \tilde{y})\) is a Pareto optimal allocation. By contradiction, suppose that there is an allocation \((\hat{x}, \hat{y})\) such that \(\sum_{j \in J} \bar{\pi}_j(\hat{y}_j) = 0\) for all \(j \in J\), \(\sum_{h \in H} \hat{x}_h \leq r + \sum_{j \in J} \bar{\pi}_h(\hat{x}_h) \geq \bar{\pi}_h(\tilde{x}_h)\) for all \(h \in H\) and \(\bar{\pi}_k(\hat{x}_k) > \bar{\pi}_k(\tilde{x}_k)\) for some \(k \in H\). If \(k = 1\), then we get a contradiction since \((\tilde{x}, \tilde{y})\) solves problem (29). If \(k \neq 1\), by the continuity of \(\bar{\pi}_k\) (see Point 1 of Assumption 4), there exists \(\varepsilon > 0\) such that \(\bar{\pi}_k(\hat{x}_k - \varepsilon \mathbf{1}_c) > \bar{\pi}_k(\tilde{x}_k)\) where \(\mathbf{1}_c \in \mathbb{R}^C_{++}\) has all the components equal to 0 except the component \(c\) which is equal to 1. Thus, the allocation \((x, y)\) defined below

\begin{align*}
  x_1 &:= \hat{x}_1 + \varepsilon \mathbf{1}_c \\
  x_k &:= \hat{x}_k - \varepsilon \mathbf{1}_c \\
  x_h &:= \hat{x}_h \quad \forall h \in H \setminus \{1, k\} \\
  y_j &:= \hat{y}_j \quad \forall j \in J
\end{align*}

satisfies the constraints of problem (29) and \(\bar{\pi}_1(x_1) > \bar{\pi}_1(\hat{x}_1)\) since \(\bar{\pi}_1\) is strictly increasing (see Point 2 of Assumption 4). So, once again we get a contradiction since \((\tilde{x}, \tilde{y})\) solves problem (29). ■

References


