Ambiguity and Coordination in a Global Game Model of Financial Crises

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Abstract

By using a non-Bayesian approach with ambiguity averse players, we highlight a new channel through which uncertainty can affect crises in global game models of financial crises. We show that ambiguity reduces the amount of coordination perceived by each player. In the model we consider, which is a two-player global game where creditors finance some project, this effect contributes to increasing the probability of a financial crisis, and therefore provides an additional argument in favor of transparency.

JEL classification: C72, D81, D82, G01.

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1 Introduction

Global games have often been used in order to analyse financial crises. In a "global game", the game is not common knowledge and, instead, each player receives an imperfect signal on an underlying parameter which characterizes the game. The rationale of using global games for the analysis of financial crises is that, by introducing in this way uncertainty on the underlying parameter which represents the value of "fundamentals", one can get rid of the multiplicity of equilibria which would otherwise arise in models of financial crises if the value of fundamentals were common knowledge1.

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1See Carlson and van Damme (1993), which have introduced global games. See also Morris and Shin (2003) for a survey on global games. The global game approach has been
In models of financial crises, multiple equilibria arise because of complementarities between the decisions of markets participants. Thus, the possibility of a bank run depends on the amount of depositors who withdraw their money deposited at the bank; and, in the same way, when a group of creditors finance some investment project, a financial crisis, due to an early liquidation by creditors, occurs only when an insufficient amount of creditors roll over their loans. Also, if we consider a currency crisis, such a crisis requires that a sufficient amount of speculators attack the currency.

This implies that a market participant will be more willing to take the corresponding action if (s)he believes that a larger number of market participants choose the same action, i.e. if (s)he perceives that there is more coordination between his/her action and the actions of other players. As it has been shown in the literature, this may lead to multiple equilibria, at least for some intermediate range of values of fundamentals. Thus, if each depositor expects that other depositors keep their money deposited at the bank, then a bank run will not occur. But such a bank run will occur if, on the contrary, each depositor expects other depositors to withdraw their money. Similarly, a failure of the investment project due to an early liquidation by creditors will occur if each creditor expects that other creditors do not roll over their loans, but there will be no failure if each creditor expects that other creditors roll over their loans. Concerning currency crises, a currency crisis will occur if everyone expects other participants to attack the currency, but the crisis will not occur if, on the contrary, each one expects that other participants do not attack the currency. In each case, the corresponding game has an equilibrium with a financial crisis, and another one without a financial crisis\(^2\).

In a global game, instead of assuming that the value of fundamentals is common knowledge, it is assumed that each player (market participant) only receives an imperfect signal on the value of fundamentals. It can then be shown that the multiplicity of equilibria disappears: there is a unique equilibrium. Then, comparative statics can easily be done, and one issue which has been explored is whether more uncertainty increases the probability of a financial crisis or not. The answer to such a question has policy implications. For, if more uncertainty can trigger financial crises, then it would be desirable for policymakers to be transparent about the information they have; and they should also follow policies aiming at reducing uncertainty. According to the existing literature, the comparative statics exercises done

\(^2\) See Diamond and Dybvig (1983) and Obstfeld (1996) for such analyses.
with these models lead to results which are not very clear-cut. Depending on the model and on the parameters of the model, greater uncertainty may either increase or decrease the probability of a crisis. In the present paper, we want to highlight a new channel, which is absent in the existing literature, through which uncertainty can affect crises in global game models of financial crises. This channel goes through the amount of coordination perceived by each player when (s)he chooses the action which requires coordination. Because of the complementarities involved, the more coordination each player perceives, the greater is his/her willingness to choose this action. For example, consider the case of creditors who have to decide whether to roll over their loans for some investment project. If a player perceives that, when (s)he rolls over his/her loan, more players coordinate, i.e. more players also roll over their loans, then this player has a greater incentive to roll over his/her loan. As a consequence, this decreases the probability of a financial crisis.

However, in the existing literature, uncertainty, in some sense, does not affect the amount of coordination perceived by each player in a global game. This means that, in the literature, when one consider the effect of uncertainty on financial crises in global games, there is no channel going through this perceived coordination. More precisely, the literature has underlined the following property. Assume that there is a non informative prior on fundamentals and consider the situation where a player receives a signal equal to the threshold value which makes himself/herself indifferent between the two actions (s)he can choose. Then, at the equilibrium, any other player always has a probability 1/2 of taking the action which requires coordination. This means that, in this case, each player always perceives a degree of coordination equal to 1/2. The amount of uncertainty has no effect on this perceived degree of coordination, which is always equal to 1/2.

In our analysis, we will show that this property is actually very dependent on the approach to uncertainty which is taken. The literature on global games generally uses the standard Bayesian approach of expected utility maximization. However, such an approach has been challenged. Knight (1921) had already made the distinction between a situation of "risk" where there is a known probability distribution, and a situation of "(Knightian) uncertainty" where no known probability distribution exists. Ellsberg (1961), through some experiments, has shown that decision makers exhibit some "aversion to

\footnote{\textsuperscript{3}See for example Metz (2003).}
\footnote{\textsuperscript{4}See Morris and Shin (2003)).}
\footnote{\textsuperscript{5}As it is the case in these global game models of crises, we consider a binary action game where only two actions are possible: depending on the model of crises, they consist in withdrawing deposits or not; rolling over the loan or not; attacking the currency or not.}
ambiguity", and further experiments confirmed this finding. Decision makers usually prefer a situation where there is a known probability distribution to a situation where this distribution is unknown. This is why, in the past two or three decades, some new non-Bayesian approaches to uncertainty, where such an aversion to ambiguity could be taken into account, have been developed.

We will therefore use such a non-Bayesian approach to uncertainty and introduce ambiguity (or "Knightian uncertainty"), with ambiguity averse players, in a global game model of financial crises. We will show that the amount of ambiguity will affect the perceived degree of coordination. When players receive a signal equal to the threshold value (and have an uninformative prior), this perceived degree of coordination will no longer be equal to 1/2, as in the standard analysis of the literature on global games. We will actually show that, in the presence of ambiguity, it is in general less than 1/2, and that it is a decreasing function of ambiguity. If ambiguity is large enough it could even become equal to zero. This effect of ambiguity on perceived coordination gives a new channel through which uncertainty can affect financial crises.

We will consider a two-player global game between two creditors who finance some investment project. At an interim stage, each player has to decide whether to roll over his/her loan or not. As it will be shown, more ambiguity tends to decrease the degree of coordination of players which is perceived by each player when s(he) considers rolling over his/her loan. This will make each player less willing to roll over the loan, and therefore makes a financial crisis more likely. From a policy point of view, our result is in favor of transparency, because more transparency from the public authorities may reduce the amount of ambiguity that each player has to face.

In the literature, there are actually some analyses which introduce ambiguity in global games (Kawagoe and Ui (2010) and Ui (2009)). However, in these analyses, ambiguity has no effect on the perceived degree of coordination between players, which actually remains equal to 1/2, as in the literature which uses the standard expected utility approach. Therefore, the channel that we underline, and which goes through the effect of ambiguity on the perceived degree of coordination, does not exist in these analyses. As we will show, the reason is that they make an implicit assumption which precludes

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6 Two classical references are Gilboa and Schmeidler (1989) and Schmeidler (1989). In these non-Bayesian frameworks, the presence of some aversion to ambiguity leads the decision maker to give more weight to the bad outcomes implied by each decision.

Such a non-Bayesian approach has been applied to a large variety of issues in economics. It has been shown that this approach could help explain some stylized facts, and could allow us to better understand some results which otherwise looked quite paradoxical (see for example Mukerji and Tallon (2004)).
such a channel. They a priori assume that the probability distributions of the signals received by all the players are identical. Such an assumption seems actually restrictive, and in the present paper we will relax this assumption. We will therefore allow the probability distributions of the signals to be different for different players. Then, it will be shown that ambiguity has an adverse effect on the belief that each player holds on the amount of coordination.

The model is presented in Section 2. Section 3 derives the equilibrium of the game. Section 4 considers the effect of more ambiguity. Section 5 concludes.

2 Model

As in the global game model of Morris and Shin (2004), a group of creditors are financing some investment project through some collateralized debt. At some interim stage, each creditor can either roll over the loan until the maturity of the project, or seize the collateral, which has some given known value $\lambda$. The value of the investment project at maturity will depend on the state of fundamentals, represented by some real variable $\theta$. The value of fundamentals $\theta$ is not known, and each creditor receives an imperfect signal on $\theta$. The value of the project at maturity will also be negatively affected if an insufficient amount of creditors do not roll over the loan. This creates some complementarities between the decisions of creditors to roll over their loans or not. The model considers the corresponding game between the creditors. As in Kawagoe and Ui (2010), we will consider a two-player version of the game, and we will introduce some ambiguity on $\theta$ with ambiguity averse players.

The actions and payoffs of this two-player game are the following. Each player has two possible actions: to roll over the loan (R), or not to roll over the loan (N). By choosing N, the player receives the value of the collateral $\lambda$, where $\lambda$ satisfies the inequality $0 < \lambda < 1$. By choosing to roll over the loan (R), a player gets a payoff equal to 1 when the project succeeds, and a payoff equal to 0 when the project fails. It is assumed that the project always succeeds when the fundamentals are good enough, i.e. when we have $\theta > 1$; and that the project always fails when the fundamentals are bad enough, i.e. when we have $\theta < 0$. For intermediate values of the fundamentals, i.e. in the case $0 \leq \theta \leq 1$, it is assumed that the investment project succeeds only in the case where both players roll over the loan. The payoff matrix in the case $0 \leq \theta \leq 1$ (which is the case where the coordination of players matters) is therefore:
If the value of $\theta$ were common knowledge, then, in the case $\theta > 1$, R would be a dominant strategy; while in the case $\theta < 0$, N would be a dominant strategy. In the case $0 \leq \theta \leq 1$, both (N,N) and (R,R) would be equilibria: (N,N) is a financial crisis equilibrium, where no player rolls over the loan and where, consequently, the project fails; and (R,R) is an equilibrium without a financial crisis, where each player rolls over the loan, and where, as a consequence, the project succeeds.

In our analysis, however, as in the global game literature, it is assumed that $\theta$ is not common knowledge. Each player receives a signal $x_i$ given by

$$x_i = \theta + \xi_i$$

where $\xi_i$ is a random variable.

The literature on global games usually assumes that $\xi_i$ follows some given known probability distribution. Then, it can be shown that, even in the case $0 \leq \theta \leq 1$, there is a unique equilibrium. This occurs even when the uncertainty becomes small, i.e. when the variance of $\xi_i$ goes to zero.

As in Kawagoe and Ui (2010), the assumption that the probability distribution of $\xi_i$ is known will be removed and it will be assumed that there is (Knightian) uncertainty, or "ambiguity", on some parameter underlying the distribution of $\xi_i$. As indicated in the Introduction, Kawagoe and Ui (2010) assume that the probability distributions of the signals are the same for all the players, but, in our analysis, we will relax this assumption. Such an assumption seems restrictive. Each player may not necessarily get the same kind of information. Each player may get information from different sources or may obtain the opinions of different experts, or may ponder them in a different way. This could change not only the signal received but also the underlying probability distribution of the signal. In our analysis, we will therefore assume that the signal received by each player follows a probability distribution which may not be the same for all the players. As we will see, this will allow ambiguity to have an effect on the perceived coordination of the players.

We will assume that the mean of the probability distribution of $\xi_i$ is uncertain. Thus player $i$ receives a signal with a possible "bias" $\mu_i$, which is assumed to be uncertain\(^7\). We will write

$$\xi_i = \mu_i + \varepsilon_i$$

\(^7\)In Kawagoe and Ui (2010), the mean is assumed to be equal to zero (as it is usually the case in the global game literature) and there is ambiguity on the variance of the probability distribution.
where $\varepsilon_i$ is a zero-mean random variable which has a known probability distribution, and where the mean $\mu_i$ of $\xi_i$ depends on $i$, and therefore is not necessarily the same for the two players. The random variables $\varepsilon_1$ and $\varepsilon_2$ are identically distributed, and the variables $\theta$, $\varepsilon_1$ and $\varepsilon_2$ are mutually independent. From (1) and (2) we have

$$x_i = \theta + \mu_i + \varepsilon_i$$

There is ambiguity on the value of the mean (or "bias") $\mu_i$. We will assume that each $\mu_i$, $i = 1, 2$, belongs to the interval $[\mu, \mu']$ and that each player has a corresponding maxmin criterion of expected utility\(^8\). We allow the distributions of the signals received by the players to be different, by allowing $\mu_1 \neq \mu_2$.

As in Kawagoe and Ui (2010), we assume that $\theta$ is uniformly distributed on $[-\delta, 1 + \delta]$; and that $\varepsilon_i$ is uniformly distributed on $[-\gamma, \gamma]$. We will assume\(^9\) that we have $0 < \gamma \leq \frac{\delta}{2}$, and, to simplify the presentation\(^10\), it will also be assumed that we have $\gamma \leq \frac{1}{2}$.

\(^{7}\) distribution (i.e. on the length $\gamma$ of the support of the uniform distribution of $\varepsilon_i$). To take the mean rather than the variance as the ambiguous parameter simplifies the analysis. Cheli and Della Posta (2007) have argued that it might be justified to introduce biased signals in global game models.

\(^{8}\)This is a standard criterion in the literature on decision under uncertainty with ambiguity aversion (see Gilboa and Schmeidler (1989)). This was also the criterion used by Ui (2009) and Kawagoe and Ui (2010).

The minimum will be taken with respect to $\mu_i$ belonging to $[\mu, \mu']$, but we could as well take the minimum with respect to all probability distributions having support included in $[\mu, \mu']$. The worst case would be the same because it would lead to a distribution concentrated at either $\mu$ or $\mu'$.

Note that the range $[\mu, \mu']$, which enters the maxmin criterion, may be considered as reflecting both the ambiguity of the available information and the aversion to ambiguity of the players (see Gajdos et al. (2004)).

\(^{9}\)In Kawagoe and Ui (2010), the less restrictive assumption $0 < \gamma \leq \delta$ is made. This was necessary to encompass the specific parameter values they took in their experiments. Then, when they developed their theoretical analysis, they had to restrict the possible values of $\lambda$. Thus, parameter $\lambda$ had to belong to the interval $[\frac{3}{8}, \frac{5}{8}]$. Here, by making the assumption $0 < \gamma \leq \frac{\delta}{2}$, our analysis can be made without further restraining $\lambda$. Any value of $\lambda$ satisfying $0 < \lambda < 1$ is allowed.

\(^{10}\)Note that this is compatible with the case, often considered, of a small uncertainty. The case $\frac{1}{2} < \gamma \leq \frac{\delta}{2}$ could be taken into account without any difficulties, and would lead to the same kind of basic qualitative results. However, adding this case would have made the analysis and the presentation of the results more cumbersome.
3 Equilibrium

As it is usually done in such global games, we will look for equilibria in "switching strategies". When player $i$ follows the switching strategy with switching point $k$, s(he) chooses $R$ if $x_i > k$ and chooses $N$ if $x_i \leq k$.

From the payoff structure previously given, the expected payoff of a player who chooses $N$ (i.e. who does not roll over the loan) is equal to $\lambda$, while the expected payoff of a player who chooses $R$ (i.e. who rolls over the loan) is equal to the probability that the investment project succeeds in that case.

Let $\pi_{\mu_i,\mu_j}(k, x_i)$ denote the probability, conditional on having received the signal $x_i$, that player $i$ holds on the event that the project succeeds when the other player $j$ follows the switching strategy with switching point $k$. This probability is therefore given by

$$\pi_{\mu_i,\mu_j}(k, x_i) = \Pr_{\mu_i,\mu_j}[(0 \leq \theta \leq 1 \text{ and } x_j > k) \text{ or } (\theta > 1) \mid x_i] \quad (4)$$

Note that this probability depends on both parameters $\mu_i$ and $\mu_j$ because it depends on the probability distributions of both random variables $\varepsilon_i$ and $\varepsilon_j$.

As underlined before, each player has some aversion to ambiguity and uses a maxmin criterion. This means that player $i$ chooses action $R$ if and only if we have\footnote{As in Morris and Shin (2004), we assume that player $i$ chooses $N$ if (s)he is indifferent between $N$ and $R$. This assumption is actually without any consequences on the analysis and the results.} $\min_{\mu_i \in [\mu, \overline{\mu}], \mu_j \in [\overline{\mu}, \mu]} \pi_{\mu_i,\mu_j}(k, x_i) > \lambda$.

3.1 Worst case for each player

We first have the following result:

Proposition 1 We have $\min_{\mu_i \in [\mu, \overline{\mu}], \mu_j \in [\overline{\mu}, \mu]} \pi_{\mu_i,\mu_j}(k, x_i) = \pi_{\mu,\mu}(k, x_i)$. The worst case for player $i$ is therefore obtained for $\mu_i = \overline{\mu}$ and $\mu_j = \mu$.

Proof: From (3), we have $\theta = x_i - \mu_i - \varepsilon_i$. A larger bias $\mu_i$ shifts the probability distribution that player $i$ holds on $\theta$, conditional on having received a signal $x_i$, towards lower values of $\theta$. This reduces (or at most leaves unchanged) the probability of success of the project through two channels. First, this may decrease the probability of being in the case $\theta > 1$ where the project is always successful, or this may increase the probability of being in the case $\theta < 0$ where the project is never successful. Second, as, from (3), we have $x_j = \theta + \mu_j + \varepsilon_j$, this reduces the probability that the other player
rolls over the loan, because this decreases the probability of having $x_j > k$. As a consequence of these two effects, player $i$ considers that the worst case is obtained when the bias $\mu_i$ of her message takes its maximum value $\mu$.

Now, consider the effect of the bias $\mu_j$ of the signal received by the other player. This bias changes the probability distribution (conditional on $x_i$) that player $i$ holds on the signal $x_j$ received by the other player. From (3), we have $x_j = \theta + \mu_j + \varepsilon_j$, and therefore a smaller bias $\mu_j$ shifts this probability distribution toward lower values of $x_j$. This implies that a smaller bias $\mu_j$ decreases (or at most leaves unchanged if this probability was already equal to zero) the probability of having $x_j > k$, and therefore decreases the probability that player $j$ rolls over the loan. This tends to reduce the success of the investment project. Consequently, the worst case would occur when this bias would be the lowest ($\mu_j = \mu$). QED

Each player has only to consider what happens in the worst case. As a consequence, Proposition 1 implies that each player behaves as if (s)he believes that the bias of his/her own signal is the maximum bias $\mu$ and that the bias of the signal of the other player is the minimum bias $\underline{\mu}$.

### 3.2 Equilibrium switching point

Proposition 1 implies that player $i$ (strictly) prefers action R (and therefore chooses R) if and only if we have $\pi_{\underline{\mu},\mu}(k,x_i) > \lambda$. Let $b(k)$ be the value of $x_i$ which makes player $i$ indifferent between R and N, and which therefore satisfies the equality $\pi_{\underline{\mu},\mu}(k,b(k)) = \lambda$. It can be seen\(^\text{12}\) that $\pi_{\underline{\mu},\mu}(k,x_i)$ is a non-decreasing function of $x_i$, and, in the neighborhood of $k$, a (strictly) increasing function of $x_i$. Consequently, if we have $x_i > b(k)$, then we have $\pi_{\underline{\mu},\mu}(k,x_i) > \lambda$, and therefore player $i$ chooses R; and if we have $x_i \leq b(k)$ then we have $\pi_{\underline{\mu},\mu}(k,x_i) \leq \lambda$ and therefore player $i$ chooses N. This implies that the switching strategy with the switching point $b(k)$ is the best response of player $i$ to the switching strategy with switching point $k$ of player $j$.

As a consequence, when both players use switching strategies with switching point $k^*$, where $k^*$ is a solution of the equation $b(k) = k$, we have an

\(^\text{12}\)When we consider $\pi_{\underline{\mu},\mu}(k,x_i)$, we could make the same kind of analysis as is done for $\pi_{\underline{\mu},\mu}(k,k)$ in the Appendix. The difference is that point E in the Figures would have coordinates $(x_i - \overline{\mu},0)$ instead of $(k - \overline{\mu},0)$. An increase in $x_i$ would then shift toward the right the support of the uniform distribution which represents the conditional distribution of the joint variable $(\theta,\varepsilon_j)$ (represented by the square ABCD in the Figures). This would increase $\pi_{\underline{\mu},\mu}(k,x_i)$, or possibly leave it unchanged in some cases, but it can be seen that in the case where we have $0 < \pi_{\underline{\mu},\mu}(k,x_i) < 1$ (which is satisfied in the neighborhood of $k$ for $x_i$ because we have $\pi_{\underline{\mu},\mu}(k,k) = \lambda$ with $0 < \lambda < 1$), this would strictly increase $\pi_{\underline{\mu},\mu}(k,x_i)$.
equilibrium of the game. This equation can be written \( \pi_{\mu,\mu}(k, k) = \lambda \). The following proposition shows that (in general) the solution of this equation is unique, and it explicitly gives its value\(^{13}\).

**Proposition 2** Let us consider the ambiguity parameter \( \eta \) defined by

\[
\eta \equiv \frac{1}{2} (\mu - \mu)
\]  
(5)

1. In the case \( \eta < \gamma \), if we have \( \lambda \neq \frac{1}{2} \left( 1 - \frac{\eta}{\gamma} \right)^2 \), there is a unique equilibrium in switching strategies, where the equilibrium switching point \( k^* \) is

\[
k^* = \mu + g(\eta, \gamma, \lambda)
\]  
(6)

where the function \( g(\eta, \gamma, \lambda) \) is given by:

(i) if we have \( 0 < \lambda < \frac{1}{2} \left( 1 - \frac{\eta}{\gamma} \right)^2 \), then we have

\[
g(\eta, \gamma, \lambda) = \gamma \left( 1 - 2 \sqrt{(1 - \frac{\eta}{\gamma})^2 - 2\lambda} \right) - 2\eta
\]  
(7)

(ii) if we have \( \frac{1}{2} \left( 1 - \frac{\eta}{\gamma} \right)^2 < \lambda < 1 - \frac{\eta}{\gamma} \), then we have

\[
g(\eta, \gamma, \lambda) = 1 - \gamma + 2\sqrt{\eta^2 + \gamma^2 \left( 2\lambda - \left( 1 - \frac{\eta}{\gamma} \right)^2 \right)} - 2\eta
\]  
(8)

(iii) if we have \( 1 - \frac{\eta}{\gamma} \leq \lambda < 1 \), then we have

\[
g(\eta, \gamma, \lambda) = 1 - \gamma + 2\lambda \gamma
\]  
(9)

(and therefore does not depend on \( \eta \) in this case)

In the case \( \eta < \gamma \), if we have \( \lambda = \frac{1}{2} \left( 1 - \frac{\eta}{\gamma} \right)^2 \), then any switching point \( k^* \) such that \( \gamma - 2\eta \leq k^* - \mu \leq 1 - \gamma \) gives an equilibrium in switching strategies.

2. In the case \( \eta \geq \gamma \), then, for all \( \lambda \) (where \( 0 < \lambda < 1 \)), there is a unique equilibrium in switching strategies, where the equilibrium switching point \( k^* \) is given by (6) and (9) (and therefore \( k^* - \mu \) does not depend on \( \eta \) in this case).

The proof is given in the Appendix.  

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\(^{13}\)Following the same kind of argument as in, for example, Morris and Shin (2003), it could be shown that the switching strategy with switching point \( k^* \), where \( k^* \) is a solution of the equation \( b(k) = k \), survives the iterated deletion of interim-dominated strategies. And, when there is a unique solution to this equation, the corresponding switching strategy is the unique strategy which survives the iterated deletion of interim-dominated strategies.
4 Effect of ambiguity on financial crises

4.1 Two channels

Ambiguity has been defined by the couple of values \((\mu, \pi)\), where \(\mu \leq \pi\). We will say that \((\mu', \pi')\) is more ambiguous than \((\mu, \pi)\) if we have \([\mu, \pi] \supset [\mu', \pi']\), which can be written: \(\mu' \leq \mu\) and \(\pi' \geq \pi\), with at least one strict inequality. Note that this implies \(\eta' > \eta\), where parameter \(\eta\) is defined by (5).

Let \(\mu_1\) and \(\mu_2\) be the true given biases of the signals received by player 1 and player 2, respectively. Then, we will compare \((\mu, \pi)\) to \((\mu', \pi')\), where \((\mu', \pi')\) is more ambiguous than \((\mu, \pi)\), and where we have \(\mu \leq \mu_1 \leq \pi, \mu \leq \mu_2 \leq \pi\), \(\mu' \leq \mu_1 \leq \pi\) and \(\mu' \leq \mu_2 \leq \pi\).

In the model, a financial crisis occurs when the investment project fails because of a problem of early liquidation by creditors. This happens in the case \(0 \leq \theta \leq 1\) when at least one of the two creditors does not roll over the loan. Therefore, more ambiguity increases the probability of a crisis if and only if we have \(k^\eta > k^*\).

From equation (6) of Proposition 214, we see that ambiguity affects the equilibrium switching point \(k^*\), and therefore the probability of a crisis, through two channels. The first goes through the highest bias \(\pi\). The reason is that, in the worst case, we have \(\mu_i = \pi\), and therefore player \(i\) acts as if (s)he believed that the bias of his/her signal was at its maximum value \(\pi\). As we have \(\pi' \geq \pi\), this makes \(k^\eta \geq k^*\). Thus, as more ambiguity generally raises the highest possible bias, each player interprets less favorably any message (s)he receives: (s)he infers lower values for the fundamentals \(\theta\), the conditional distribution of the fundamentals \(\theta\) being shifted toward lower values. This kind of channel was present in the literature which introduced ambiguity in global games15.

The second channel goes through the function \(g(\eta, \gamma, \lambda)\), which, in cases 1-(i) and 1-(ii) of Proposition 2, depends on the ambiguity parameter \(\eta\) defined

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14This equation is valid in the general case where there is a unique equilibrium. In the special case given by \(\eta < \gamma\) and \(\lambda = \frac{1}{2} \left(1 - \frac{3}{4}\right)^2\), where, from Proposition 2, we have a continuum of equilibria for \(k\) in the interval \([\pi + \gamma - 2\eta, \pi + 1 - \gamma]\), we should add an upward jump from \(k^*\) to \(\pi + 1 - \gamma\) in the case of more ambiguity, or a downward jump from \(k^*\) to \(\pi + \gamma - 2\eta\) in the case of less ambiguity. These additional jumps would then reinforce the effect found in the general case where there is a unique equilibrium \(k^*\).

15In Kawagoe and Ui (2010), it is not the mean but the variance, or equivalently the length \(\gamma\) of the support of the uniform distribution of \(\epsilon_i\) which is ambiguous: we have \(\gamma \in [\frac{1}{3}, \frac{2}{3}]\). Then this channel occurs through the maximum \(\gamma\) (or the minimum \(\gamma\), depending on the value of \(\lambda\)), rather than through the maximum mean \(\pi\).
by (5). In these cases, from (7) or (8), we have $\frac{\partial g(\eta, \gamma, \lambda)}{\partial \eta} > 0$, which makes $k^{*} > k^{\ast}$. Thus, through this second channel, more ambiguity also increases the probability of a crisis. This kind of channel was not present in the literature on the effect of uncertainty in global games. It is actually the new channel we want to underline in our analysis, and we will discuss it in more details in the next subsection.

The two channels go into the same direction. As a consequence, we find that more ambiguity increases the probability of a financial crisis. This makes more ambiguity harmful and, therefore, gives an argument for more transparency\textsuperscript{16}.

4.2 The role of the perceived coordination of players

The second channel is entirely due to the fact that we have allowed $\mu_i \neq \mu_j$, because it comes from the property that, in the worst case, $\mu_i$ and $\mu_j$ actually differ and take the opposite extreme values (from Proposition 1 we have $\mu_i = \overline{\mu}$ and $\mu_j = \underline{\mu}$ in the worst case). If, as it is done in the literature, we had assumed that the distributions of the signals of the two players were the same, and therefore that we had $\mu_i = \mu_j$, this channel would have disappeared and only the channel going through $\overline{\mu}$ would have remained\textsuperscript{17}.

Thus, each player, in the worst case, substracts to his/her own signal the maximum bias $\overline{\mu}$ but considers that the other player receives a signal with the minimum bias $\underline{\mu}$. As a consequence of such a gap $\overline{\mu} - \underline{\mu}$ (which is equal

\textsuperscript{16}This result contrasts with what would be the effect of more uncertainty under a Bayesian approach in the model. Thus, take the Bayesian case where we have $\eta = 0$, and consider the effect of more uncertainty, in the sense of an increase in the variance $\sigma^2_i$ of the noise $\varepsilon_i$. This is actually equivalent to an increase in parameter $\gamma$ of the uniform distribution followed by each $\varepsilon_i$.

In the Bayesian case $\eta = 0$, Proposition 2 would give : $g(0, \gamma, \lambda) = \gamma \left( 1 - 2\sqrt{1 - 2\lambda} \right)$ in the case $\lambda < \frac{1}{2}$; and $g(0, \gamma, \lambda) = 1 + \gamma \left( 2\sqrt{2\lambda - 1} - 1 \right)$ in the case $\lambda > \frac{1}{2}$ (Note that these expressions also appears in Kawagoe and Ui (2010) p.6). As a consequence, an increase in $\gamma$ increases the probability of a crisis in the case $\frac{1}{2} < \lambda < \frac{1}{2}$ or $\frac{1}{2} < \lambda < 1$, but decreases the probability of a crisis in the case $0 < \lambda < \frac{1}{2}$ or $\frac{1}{2} < \lambda < \frac{3}{2}$ (This corresponds to the result underlined by Kawagoe and Ui (2010) that the effect of the "quality" of information depends on parameter $\lambda$).

Thus, as it is often the case in global game models of financial crises under a Bayesian approach (see the Introduction), the sign of the effect of uncertainty (i.e. of "risk") on the probability of a crisis would depend on the value of some parameter of the model. In contrast, under the present non-Bayesian approach, more ambiguity always increases the probability of a crisis.

\textsuperscript{17}If we had imposed $\mu_i = \mu_j$, we can easily see that the worst case would have been obtained for $\mu_i = \mu_j = \overline{\mu}$. Therefore, ambiguity would have had an effect (only) through $\overline{\mu}$, as in the first channel.
to $2\eta$), in the worst case each player believes that the other player receives signals which are on average less favorable (i.e. lower) than the signal (s)he receives. This lowers the probability that the other player rolls over his/her loan. More ambiguity, which increases this gap, has therefore the effect of decreasing this probability. In other words, more ambiguity decreases the amount of coordination that each player expects when (s)he rolls over his/her loan. This makes each player less inclined to roll over his/her loan and, consequently, increases the probability of a financial crisis.

As we have indicated in the Introduction, the existing literature had underlined the following property: when player $i$ receives a signal equal to the equilibrium switching point ($x_i = k^*$), and if the prior on $\theta$ is non informative, then the other player has probability $\frac{1}{2}$ of rolling over the loan. In other words, when, in this case, a player rolls over the loan, (s)he perceives a degree of coordination of $\frac{1}{2}$. In our analysis, this property does not hold anymore because, through the effect just described, ambiguity decreases the probability that the other player rolls over the loan.

This can be seen formally in the following way. The probability that player $j$ rolls over the loan is given by the probability of having $x_j > k^*$. According to (3), this is equal to the probability of having $\varepsilon_j > k - \mu_j - \theta$. On the other hand, from (3), the equality $x_i = k^*$ can be written $\theta = k^* - \mu_i - \varepsilon_i$. When the prior on $\theta$ is non informative\textsuperscript{18} this equality gives the distribution of $\theta$ conditional on $x_i = k^*$. Then, replacing $\theta$ by this expression in the previous inequality, the probability (conditional on $x_i = k^*$) of having $x_j > k^*$ is equal to the probability of having $\varepsilon_j - \varepsilon_i > \mu_i - \mu_j$. If we had a priori assumed, as in Ui (2009) and Kawagoe and Ui (2010), that the probability distributions of the signals are the same for the two players, which would imply $\mu_i = \mu_j$, this probability would be equal to $\frac{1}{2}$. But, in our analysis, player $i$ evaluates this probability by taking the worst case for $\mu_i$ and $\mu_j$. Therefore player $i$ has instead to consider the probability of having $\varepsilon_j - \varepsilon_i > \mu_j - \mu_i$, which can be written $\varepsilon_j - \varepsilon_i > 2\eta$. This probability is equal to $\frac{1}{2}$ in the usual Bayesian case $\eta = 0$ of no ambiguity, but it is less than $\frac{1}{2}$ when we have $\eta > 0$, i.e. when there is ambiguity; and this probability is a decreasing function of the ambiguity parameter $\eta$. In this case, when a player rolls over the loan, the perceived degree of coordination is less than $\frac{1}{2}$; and more ambiguity decreases this perceived coordination.

Note that the effect of more ambiguity going through the perceived coordination of players does not play a role in cases 1-(iii) and 2 of Proposition 2, where $g(\eta, \gamma, \lambda)$ is given by (9) and does not depend on $\eta$. The reasons are the following. First, consider case 2 of Proposition 2. This case arises when

\textsuperscript{18}As shown in the Appendix this is satisfied when we have $\mu_i = \overline{\mu}$. 

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the level of ambiguity $\eta$ is already large enough to make the probability of having $\varepsilon_j - \varepsilon_i > 2\eta$ equal to zero. This happens in the case $\eta \geq \gamma$, because the values of $\varepsilon_j - \varepsilon_i$ satisfying the inequality $\varepsilon_j - \varepsilon_i > 2\eta$ would then be outside the support of $\varepsilon_j - \varepsilon_i$. In that case any marginal increase in the value of $\eta$ would be without any effect on this probability, which would remain equal to zero. Second, when the inequality $x_j > k$ can be satisfied only for values of $\theta$ greater than 1, then the coordination of players does not matter because the investment project is always successful. When we have $\eta < \gamma$, this corresponds to case 1-(iii) of Proposition 2.

5 Conclusion

We have considered a standard two-player global game of coordination between creditors who have to decide whether to roll over their loans or not, in order to finance some investment project. We have introduced some ambiguity, players being assumed to have some aversion to ambiguity. We have shown that more ambiguity tends to reduce the perceived coordination of players. More ambiguity leads each player to act under the belief that, when he rolls over his/her loan, there is less coordination of the players. This reduces the incentive to roll over the loan, and therefore increases the probability of a financial crisis due to early liquidation by creditors.

This effect of ambiguity, going through the belief that each creditor has on the degree of coordination, actually reinforces a channel which was present in the analyses of the literature which introduced ambiguity in global games. But, this effect, going through the perceived coordination, was absent in the analyses on global games which can be found in the literature. This occurred either because this literature used a Bayesian expected utility approach to uncertainty, or because, when it used a non-Bayesian approach with some aversion to ambiguity, it did so under the implicit restrictive assumption that the probability distributions of the signals received by the players were the same. By removing this last assumption, we have been able to show that ambiguity reduces the perceived coordination of players in rolling over their loans. This highlights a new channel through which more ambiguity

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19 This can easily be seen in the following way. From (3) and $\mu_j = \mu$, the inequality $x_j > k^*$ can be written $\theta > k^* - \mu - \varepsilon_j$. Using (6) and (9), this inequality becomes $\theta > 1 + 2\eta + (2\lambda - 1)\gamma - \varepsilon_j$. Consequently, this inequality will be satisfied only for values of $\theta$ greater than 1 if and only if we have $2\eta + (2\lambda - 1)\gamma - \varepsilon_j \geq 0$. This is verified for all $\varepsilon_j$ if and only if it is verified for $\varepsilon_j = \gamma$. As we have $2\eta + (2\lambda - 1)\gamma - \varepsilon_j = 0$ for $\varepsilon_j = \gamma$ and $\lambda = 1 - \frac{\eta}{\gamma}$, then the inequality is satisfied if and only if we have $\lambda \geq 1 - \frac{\eta}{\gamma}$, which gives the inequalities for $\lambda$ of case 1-(iii) of Proposition 2.
increases the probability of a financial crisis.

From a policy point of view, the results we obtain are in favor of more transparency because more ambiguity increases the probability of a crisis. The central point of our analysis that we have emphasized is that, by reducing ambiguity, more transparency would increase the perceived coordination between creditors, which would help to prevent a financial crisis. In their communication to the private sector, the public authorities should therefore try to make statements which may not be interpreted in an ambiguous way, and should provide information which would reduce the amount of ambiguity on the underlying fundamentals.

We have nonetheless to be careful in interpreting these results because they may depend on the specific global game model used. Actually, the main point of our analysis is that ambiguity could change the belief that each player holds on the degree of coordination between his/her own action and the actions of the other players. We have shown that ambiguity tend to decrease this perceived coordination of players. In the model considered, which is a model of coordination between creditors who roll over their loans, the coordination of players has a beneficial role, because it helps to prevent a crisis; and this is why the negative effect of ambiguity on perceived coordination can trigger a crisis and is therefore harmful.

However, the same kind of effect on perceived coordination could on the contrary make ambiguity beneficial if the coordination of players has a harmful role and can trigger a crisis. This would be the case in a global game model of currency attacks. For, in such a model, a player (speculator) is more willing to attack the currency if (s)he believes that there is more coordination with the other players when attacking the currency. If more ambiguity reduced the degree of perceived coordination of players (speculators) who attack the currency, this would decrease the incentive of each player to attack the currency. Consequently, more ambiguity would, on the contrary, decrease the probability of a crisis (i.e. of a currency attack). This would make ambiguity beneficial and, consequently, transparency harmful.\footnote{Although the effect on the perceived coordination of players that we underline in the present paper was not present in the literature, the existing literature (Kawagoe and Ui (2010)) and Ui (2009)) has shown that more ambiguity has also opposite effects on the probability of crises in these two different models. As in our analysis, more ambiguity increases the probability of a crisis in a model of creditors having to choose whether to roll over their loans (or in a bank run model), but decreases the probability of a crisis in a model of currency attacks.

As explained in Ui (2009), the reason is that, in the model of creditors having to roll over their loans or not, the action which has a payoff which depends on the value of fundamentals is the action which helps to prevent a crisis ("to roll over the loan"); while, in the currency attack model, the action which has a payoff which depends on the value
it may therefore be worthwhile, both from a theoretical and an empirical point of view, to further investigate the implications of such opposite and symmetric effects that ambiguity may have on different types of crises.

Appendix

Proof of Proposition 2

As indicated in Section 3, the equilibrium condition is \( \pi_{\mu,\mu}(k, k) = \lambda \). From (3) and (4), we have \( \pi_{\mu,\mu}(k, k) = \Pr_{\mu_i=\mu_j=k}(0 \leq \theta \leq 1 \text{ and } \theta + \varepsilon_j > k - \mu) \) or \( (\theta > 1) \mid x_i = k \). Consider the distribution of \( \theta \) conditional on having received the signal \( x_i = k \), when we have \( \mu_i = \mu \). From (3), the equality \( x_i = k \) can be written \( \theta = k - \mu - \varepsilon_i \). Consider a value of \( k \) which belongs to the interval \( [\mu - \frac{\delta}{2}, \mu + 1 + \frac{\delta}{2}] \). As we have assumed \( \gamma \leq \frac{\delta}{2} \), this equality implies \( -\delta \leq \theta \leq 1 + \delta \). Consequently, for \( k \in [\mu - \frac{\delta}{2}, \mu + 1 + \frac{\delta}{2}] \) all the possible values of \( \theta \) which are compatible with having received the signal \( x_i \), belong to the support of the uniform prior on \( \theta \). This implies that the conditional distribution of \( \theta \) is a uniform distribution with mean \( k - \mu \) and support \( [k - \mu - \gamma, k - \mu + \gamma] \). As \( \varepsilon_j \) is independent of \( \varepsilon_i \), this implies that the conditional distribution of the joint variable \((\theta, \varepsilon_j)\) is a uniform distribution having support \( [k - \mu - \gamma, k - \mu + \gamma] \times [-\gamma, \gamma] \).

Then, we could calculate \( \pi_{\mu,\mu}(k, k) \) through some integration calculus, but it will be simpler, or at least more illuminating, to do it graphically. In the plane of coordinates \( \theta \) and \( \varepsilon_j \), the support of the uniform conditional distribution of the joint variable \((\theta, \varepsilon_j)\) can be represented by a square ABCD centered on point E of coordinates \((k - \mu, 0)\) on the horizontal axis. This is done in Figures 1, 2 and 3. As we have assumed \( \gamma \leq \frac{\delta}{2} \), there are only three possible cases which are represented by these three figures. In Figure 1, the square ABCD is entirely in the area where we have \( 0 \leq \theta \leq 1 \). Figure 2 corresponds to the case where the square ABCD intercepts the vertical axis, in which case we can have \( \theta < 0 \) or \( 0 \leq \theta \leq 1 \) (but never \( \theta > 1 \)). In figure 3, where the square ABCD intercepts the vertical line where \( \theta = 1 \), we can have \( \theta > 1 \) or \( 0 \leq \theta \leq 1 \) (but never \( \theta < 0 \)).

of fundamentals is the action which helps to trigger a crisis (“to attack the currency”). As more ambiguity decreases the incentive to choose the action which has a payoff which depends on the level of fundamentals, this explains the opposite results that ambiguity has in these two models, which are found in Ui (2009) and Kawagoe and Ui (2010).

In our analysis, we have emphasized that an other characteristic of the action can also play a role: it is whether the action has a payoff which depends on the coordination of players or not. As we have indicated, the implied channel going through the perceived degree of coordination reinforces the previous channel underlined in the literature, and should also lead to opposite effects when we go from a model of debt with creditors to a model of currency attacks by speculators.
The length of each side of the square ABCD is equal to $2\gamma$. We have $AB = CD = AC = BD = 2\gamma$. In these figures we have represented the straight line $(Z)$ of slope $-1$ of equation $\theta + \varepsilon_j = k - \mu$. We have also drawn the straight line parallel to $(Z)$ going through point $E$, which is a diagonal of the square ABCD. This diagonal would be the same as $(Z)$ in the special case $\mu = \underline{\mu}$ where there would be no ambiguity.

The straight line $(Z)$ intercepts the horizontal axis at point $F$, which has coordinates $(x_i - \mu, 0)$ and is therefore at the right of point $E$. We have $EF = \underline{\mu} - \mu = 2\eta$. Therefore, $(Z)$ intersects the square ABCD in two points $G$ and $H$ if and only if we have $\eta < \gamma$.

In Figure 1, we always have $0 \leq \theta \leq 1$, which occurs when we have $\gamma \leq k - \underline{\mu} \leq 1 - \gamma$. (This case is possible because we have assumed $\gamma \leq \frac{\pi}{2}$).

Then, we simply have $\pi_{\underline{\mu}}(k, k) = \text{Pr}_{\mu_i=\mu, \mu_j=\underline{\mu}}(\theta + \varepsilon_j > k - \underline{\mu} \mid x_i = k)$. In the case $\eta < \gamma$, this probability is given by $\frac{\mathcal{A}(GBH)}{\mathcal{A}(ABCD)}$, where $\mathcal{A}(\mathcal{F})$ represents the area of figure $\mathcal{F}$. We have $\mathcal{A}(ABCD) = AB^2 = 4\gamma^2$ and $\mathcal{A}(GBH) = \frac{1}{2}\gamma GB^2$. As we have $GB = AB - AG = AB - EF = 2(\gamma - \eta)$, we get $\mathcal{A}(GBH) = 2(\gamma - \eta)^2$. This gives $\pi_{\underline{\mu}}(k, k) = \frac{1}{\gamma^2} \left(1 - \frac{\eta}{\gamma}\right)^2$. In the case $\eta \geq \gamma$, where the straight line $(Z)$ does not intersect the square ABCD, we have $\pi_{\underline{\mu}}(k, k) = \text{Pr}_{\mu_i=\mu, \mu_j=\underline{\mu}}(\theta + \varepsilon_j > k - \underline{\mu} \mid x_i = k) = 0$.

The case represented in Figure 2 occurs when we have $k - \underline{\mu} < \gamma$. In this case, the area where we have $\theta < 0$ should be excluded in order to calculate $\pi_{\underline{\mu}}(k, k)$. If we have $\eta < \gamma$, then, from Figure 2, this gives $\pi_{\underline{\mu}}(k, k) = \frac{\mathcal{A}(GBH) - \mathcal{A}(GPR)}{\mathcal{A}(ABCD)}$ in the case where (as in Figure 2) $G$ is at the left of $P$, i.e. when we have $AG < AP$. We have $AG = EF = 2\eta$, and $AP = SO = SE + EO = \gamma - (k - \underline{\mu})$ (because, in Figure 2, point $E$ being on the left of point $O$, we have $k - \underline{\mu} < \gamma$ and therefore $EO = -(k - \underline{\mu})$). Thus, the inequality $AG < AP$ gives the condition $k - \underline{\mu} < \gamma - 2\eta$. As we have $\mathcal{A}(GPR) = \frac{1}{2}\gamma GP^2$ and $GP = AP - AG = \gamma - (k - \underline{\mu}) - 2\eta$, this gives $\pi_{\underline{\mu}}(k, k) = \frac{1}{\gamma^2} \left(1 - \frac{\eta}{\gamma}\right)^2 - \frac{1}{8\gamma^2} \left[\gamma - (k - \underline{\mu}) - 2\eta\right]^2$. If $G$ is on the right of $P$, which occurs when we have $k - \mu \geq \gamma - 2\eta$, then there is no relevant area where we have $\theta < 0$, and, as in the first case, we have $\pi_{\underline{\mu}}(k, k) = \frac{1}{\gamma^2} \left(1 - \frac{\eta}{\gamma}\right)^2$. Finally, as in the first case, when we have $\eta \geq \gamma$, we always have $\pi_{\underline{\mu}}(k, k) = 0$.

In the third case, depicted in Figure 3, the square ABCD intercepts the vertical line $\theta = 1$. This occurs when we have $k - \underline{\mu} > 1 - \gamma$. Then all the area where we have $\theta > 1$ should be included. If we have $\eta < \gamma$, this means that, in Figure 3, to the area of the triangle GBH, we have to add the area of LNDH. We therefore have $\pi_{\underline{\mu}}(k, k) = \frac{\mathcal{A}(GBH) + \mathcal{A}(LHND)}{4\gamma^2}$. We can write $\mathcal{A}(LHND) = \mathcal{A}(L IH) + \mathcal{A}(IH\bar{N}D)$. We have $\mathcal{A}(L IH) = \frac{1}{2}IH^2$.
and $A(IHND) = IH.HD$. We have $HD = EF = 2\eta$ and $IH = OT - 1 = OE + ET - 1 = k - \pi + \gamma - 1$. Let us define $\nu \equiv k - \pi + \gamma - 1$. We get $A(\text{LIH}) = \frac{1}{2}\nu^2$ and $A(IHND) = 2\eta\nu$. This implies $\pi_{\mu,\mu}(k,k) = \frac{1}{2} \left( \frac{k - \mu + 1}{\gamma} \right)^2 + \frac{1}{8\gamma^2} (\nu^2 + 4\eta\nu)$. However this calculus is valid only when $G$ is on the left of $M$, as in Figure 3. This occurs when we have $AG < AM$. As we have $AG = EF = 2\eta$ and $AM = AB - IH = 2\gamma - \nu$, this inequality can be written $2\eta < 2\gamma - \nu$, or equivalently $\nu < 2(\gamma - \eta)$, which can also be written $k - \pi < 1 - \gamma + 2(\gamma - \eta)$. When this inequality is not verified, i.e. when we have $k - \pi \geq 1 - \gamma + 2(\gamma - \eta)$, then $G$ is on the right of $M$, and we simply have $\pi_{\mu,\mu}(k,k) = A(MBND) = \frac{1}{2}\nu$. Finally, when we have $\eta \geq \gamma$, then we also have $\pi_{\mu,\mu}(k,k) = 0$.

These results can be summarized in the following Proposition:

**Proposition 3**

(i) In the case $k - \pi < \gamma - 2\eta$, when we have $\eta < \gamma$, we get

$$\pi_{\mu,\mu}(k,k) = \frac{1}{2} \left( \frac{k - \mu + 1}{\gamma} \right)^2 - \frac{1}{8\gamma^2} \left[ \gamma - (k - \pi \pi) - 2\eta \right]^2 \quad (10)$$

and, when we have $\eta \geq \gamma$, we get $\pi_{\mu,\mu}(k,k) = 0$

(ii) In the case $\gamma - 2\eta \leq k - \pi \leq 1 - \gamma$, when we have $\eta < \gamma$, we get

$$\pi_{\mu,\mu}(k,k) = \frac{1}{2} \left( \frac{k - \mu + 1}{\gamma} \right)^2 \quad (11)$$

and, when we have $\eta \geq \gamma$, we get $\pi_{\mu,\mu}(k,k) = 0$

(iii) In the case $\eta < \gamma$ and $1 - \gamma \leq k - \pi < 1 - \gamma + 2(\gamma - \eta)$ we get

$$\pi_{\mu,\mu}(k,k) = \frac{1}{2} \left( \frac{k - \mu + 1}{\gamma} \right)^2 + \frac{1}{8\gamma^2} (\nu^2 + 4\eta\nu) \quad (12)$$

where we have defined

$$\nu \equiv k - \pi + \gamma - 1 \quad (13)$$

(iv) In the case $k - \pi \geq 1 - \gamma + 2(\gamma - \eta)$ if we have $\eta < \gamma$, and in the case $k - \pi > 1 - \gamma$ if we have $\eta \geq \gamma$, we get

$$\pi_{\mu,\mu}(k,k) = \frac{1}{2}\frac{\nu}{\gamma} \quad (14)$$

The equilibrium condition is $\pi_{\mu,\mu}(k,k) = \lambda$. First, consider the case $\eta < \gamma$. Then, if we look for a solution satisfying the inequality $k - \pi < \gamma - 2\eta$ of
case (i) of Proposition 3. In the case \( \eta < d \), using (10), \( \pi_{\pi,\mu}(k, k) = \lambda \) gives a second order equation in the variable \((k - \pi)\). It can easily be shown that this equation has solutions if and only if we have \( 0 < \lambda < \frac{1}{2} \left(1 - \frac{\eta}{\gamma}\right)^2 \); and, when this last inequality is satisfied, there is a unique solution satisfying the inequality \( k - \pi < \gamma - 2\eta \), and this is the solution given by (6) and (7) of Proposition 2.

In the same way, we can look for a solution satisfying \( 1 - \gamma < k - \pi < 1 - \gamma + 2(\gamma - \eta) \) of case (iii) of Proposition 3. Using (13), this inequality can be written \( 0 < \nu < 2(\gamma - \eta) \). Using (12), \( \pi_{\pi,\mu}(k, k) = \lambda \) gives a second order equation in \( \nu \) which is \( \nu^2 + 4\eta\nu - 4\gamma^2(2\lambda - \left(1 - \frac{\eta}{\gamma}\right)^2) = 0 \). It can easily be shown that this equation has a solution satisfying the inequalities \( 0 < \nu < 2(\gamma - \eta) \) if and only if we have \( \frac{1}{2} \left(1 - \frac{\eta}{\gamma}\right)^2 \leq \lambda < 1 - \frac{\eta}{\gamma} \), and this solution is then unique and is given by (6) and (8) of Proposition 2.

When we look for a solution satisfying inequality \( k - \pi \geq 1 - \gamma + 2(\gamma - \eta) \) of case (iv) of Proposition 3, using (14), we get \( \nu = 2\lambda\gamma \); using (13), this gives the solution for \( k \) given by (6) and (9) of Proposition 2. This solution satisfies the corresponding inequality \( k - \pi \geq 1 + \gamma - 2\eta \), if and only if we have \( \lambda \geq 1 - \frac{\eta}{\gamma} \).

The case (ii) of Proposition 3, where we look for a solution satisfying \( \gamma - 2\eta \leq k - \pi \leq 1 - \gamma \) is peculiar because \( \pi_{\pi,\mu}(k, k) \) is always equal to \( \frac{1}{2} \left(1 - \frac{\eta}{\gamma}\right)^2 \) in the case \( \eta < \gamma \) (and to zero if \( \eta \geq \gamma \)). Consequently, if we have \( \lambda = \frac{1}{2} \left(1 - \frac{\eta}{\gamma}\right)^2 \) and \( \eta < \gamma \), then the equilibrium condition \( \pi_{\pi,\mu}(k, k) \) can be satisfied for any value of \( k - \pi \) such that we have \( \gamma - 2\eta \leq k - \pi \leq 1 - \gamma \). Then, any switching point satisfying this inequality gives an equilibrium.

Now consider the case \( \eta \geq \gamma \). Proposition 3 implies that we have \( \pi_{\pi,\mu}(k, k) = 0 \) if we have \( k - \pi \leq 1 - \gamma \), and \( \pi_{\pi,\mu}(k, k) = \frac{1}{2}\gamma \) if we have \( k - \pi > 1 - \gamma \). As we have \( \lambda > 0 \), the equilibrium condition \( \pi_{\pi,\mu}(k, k) = \lambda \) in the case \( \eta \geq \gamma \) requires \( k - \pi > 1 - \gamma \). Then, this solution is given by \( \nu = 2\lambda\gamma \), which yields the solution for \( k \) given by case 2, and therefore by equations (6) and (9) of Proposition 2. Note that for all possible values of \( \lambda \) (which satisfy \( 0 < \lambda < 1 \)) this solution verifies the required inequality \( k - \pi > 1 - \gamma \).

Finally, at the beginning of the Appendix, in order to calculate the probability distribution of \( \theta \), conditional on having \( x_i = k \), we have assumed that \( k \) belongs to the interval \([\pi - \frac{\delta}{2}, \pi + 1 + \frac{\delta}{2}]\). We therefore have to verify that the equilibrium values of \( k \) we found satisfy this requirement. This can be seen in the following way. In the extreme case \( k = \pi - \frac{\delta}{2} \), we would be in the case of Figure 2 where \( E \) would have coordinates \((-\frac{\delta}{2}, 0)\). As we have
assumed $\gamma \leq \frac{\delta}{2}$, this implies that the square ABCD would be necessarily entirely contained in the region $\theta \leq 0$ on the left of the vertical axis. We would thus have $\pi_{\mu,\mu}(k, k) = 0$ in this case. In the same way, in the other extreme case $k = \ubar{m} + 1 + \frac{\delta}{2}$, we would be in the case of Figure 3 where E would have coordinates $(1 + \frac{\delta}{2}, 0)$. As we have assumed $\gamma \leq \frac{\delta}{2}$, this implies that the square ABCD would be necessarily entirely contained in the region $\theta \geq 1$ on the right of the vertical axis $\theta = 1$. This would give $\pi_{\mu,\mu}(k, k) = 1$ in this case. But, in Figures 1, 2 and 3, an increase in $k$ has the effect of shifting the square ABCD and the straight line $(Z)$ to the right by the same amount. Thus, a greater $k$ decreases the probability of having $\theta < 0$ or increases the probability of having $\theta > 1$ (or possibly could leave them unchanged). This implies that $\pi_{\mu,\mu}(k, k)$ is a non decreasing function of $k$ (this property could also easily be seen from Proposition 3). As a consequence, because we have $0 < \lambda < 1$, the solution of the equation $\pi_{\mu,\mu}(k, k) = \lambda$ necessarily belongs to the interval $[\ubar{m} - \frac{\delta}{2}, \ubar{m} + 1 + \frac{\delta}{2}]$. QED

References


