FGT Poverty Measures and the Mortality Paradox: Theory and Evidence*

Mathieu Lefèbvre† Pierre Pestieau† Gregory Ponthiere‡

October 19, 2013

Abstract

Income-differentiated mortality, by reducing the share of poor persons in the population, leads to what can be called the "Mortality Paradox": the worse the survival conditions of the poor are, the lower the measured poverty is. We show that FGT measures (Foster Greer Thorbecke 1984) are, in general, not robust to variations in survival conditions. Then, following Kanbur and Mukherjee (2007), we propose to adjust FGT poverty measures by extending the income profiles of the prematurely dead, and we identify the condition under which so-adjusted FGT measures are robust to mortality changes. Finally, we show, on the basis of data on old-age poverty in 11 European economies (2007), that the effect of extending income profiles of the prematurely dead on poverty measurement varies significantly with: (1) the fictitious income assigned to the prematurely dead; (2) the degree of poverty aversion; (3) the shape of the (unadjusted) income distribution; (4) the strength of the income/mortality relationship.

Keywords: income-differentiated mortality, FGT poverty measures.

JEL classification code: I32.

---

*The authors would like to thank Marion Leturcq for her comments and suggestions on this paper.
†LAMETA, University Montpellier 1.
‡University of Liege, CORE, PSE and CEPR.
§Paris School of Economics and Ecole Normale Superieure, Paris, France. Address: ENS, 48 bd Jourdan, 75014 Paris, France. E-mail: gregory.ponthiere@ens.fr
1 Introduction

In the recent decades, a voluminous empirical literature has emphasized that mortality risks are negatively correlated with income.\(^1\) Lower incomes are statistically related with higher mortality risks. The relationship between income and life expectancy is increasing, but non-linear, and exhibits a stronger slope at low income levels (Backlund et al 1999).

Income-differentiated mortality raises serious problems for poverty measurement. Clearly, if low-income individuals tend to face higher mortality risks than high-income individuals, standard poverty measures capture not only the true poverty, but, also, the selection induced by income-differentiated mortality. Indeed, if poor persons tend to die, on average, earlier than non-poor persons, it follows that there exist some "missing poor" persons at the old age, to use an expression close to Sen’s (1998) "missing women".

The interferences or noise induced by income-differentiated mortality lead to what can be called the "Mortality Paradox": the worse the survival conditions of the poor are, the lower the measured poverty is. The Mortality Paradox is not caused by mortality \textit{per se}, but by the correlation between income and mortality risks. That correlation, by creating a selection bias, introduces some noise in the measurement of poverty.\(^2\)

At first glance, the selection bias induced by income-differentiated mortality seems to lead to an \textit{underestimation} of the poverty phenomenon. To illustrate this, take standard headcount poverty measures. If low-income individuals die earlier than non-poor individuals, those "missing poor" are not counted as poor. Assuming that income mobility is negligible, those poor individuals would have been counted as poor \textit{if} they had faced the same survival conditions as the non-poor. Therefore headcount measures underestimate the extent of poverty.

Note, however, that if there was a strong income mobility, that result would not hold. To illustrate this, take, for instance, the extreme example of a society where all poor become rich and all rich become poor. In that case, a worsening of the survival conditions faced by the young poor tends to reduce the proportion of rich at the old age, and, hence, tend to raise the measured poverty. Hence, in the case of extreme income mobility, income-differentiated mortality leads to an \textit{overestimation} - instead of an \textit{underestimation} - of the poverty phenomenon.

Furthermore, once we depart from headcount ratios and consider other poverty measures, it is not obvious that the Mortality Paradox still holds. Take, for instance, the class of poverty indicators known as the FGT measures (Foster Greer Thorbecke 1984). FGT measures are a parametric family of poverty measures where the parameter is an indicator of aversion to poverty. When that parameter equals 0, the poverty measure collapses to a simple headcount ratio,

---


\(^2\)Note that the difficulties raised by income-differentiated mortality concern all poverty measures. Indeed, multidimensional poverty indicators (even those taking life expectancy into account) compute poverty measures for a \textit{given} population, and, as such, are also subject to the noise induced by income-differentiated mortality.
but when that parameter is strictly positive, the poverty measure satisfies the Monotonicity Axiom (i.e. a reduction in the income of the poor must increase the poverty measure ceteris paribus). Moreover, if that parameter strictly exceeds 1, the poverty measure satisfies the Transfer Axiom (i.e. a pure transfer of income from a poor to someone richer must increase the poverty measure ceteris paribus). Distribution-sensitive poverty measures such as FGT measures may not necessarily decrease when the survival conditions of some poor are worsened. The measured poverty index may either go up or down, depending on the overall effect of that rise of mortality on the income distribution.

The goal of this paper is to examine how income-differentiated mortality affects FGT poverty measures, and, in particular, whether income-differentiated mortality leads FGT measures to over- or underestimate the extent of poverty. For that purpose, we develop a simple theoretical model with income mobility and income-differentiated mortality, and study the behavior of FGT poverty measures in that framework. We pay a particular attention to the following questions. Are FGT measures subject to the Mortality Paradox? If yes, are all subclasses of FGT measures equally subject to that selection bias?

In order to answer those questions, we will proceed in three stages. In a first stage, we examine whether FGT measures of old-age poverty are robust or not to changes in survival conditions, and we identify particular income mobility processes under which FGT measures of old-age poverty satisfy that robustness requirement. In a second stage, we propose, following the recent works by Kanbur and Mukherjee (2007) and Lefèbvre et al (2013), to construct adjusted FGT poverty measures by extending, through a fictitious income, the lifetime income profiles of the prematurely dead individuals, in such a way as to take those "missing poor" into account in the measurement of poverty. Then, we derive the formal condition under which adjusted FGT poverty measures are robust to mortality changes. Finally, the behavior of FGT measures is illustrated on the basis of old-age poverty data for 11 European countries (2007).

Anticipating on our results, we first show that standard FGT measures of old-age poverty are not, in general, robust to changes in survival conditions. We also show that the invariance of FGT measures to changes in survival conditions holds when we restrict ourselves to a particular family of income mobility processes under which members of all income groups face the same expected extent of poverty in case of survival to the old age. That assumption being quite restrictive, we then propose to adjust FGT measures by extending the lifetime income profiles of the prematurely dead, and we identify the condition under which such an extension makes FGT measures robust to variations in mortality risk. Finally, the empirical application to Europe reveals that the effect of extending income profiles of the prematurely dead on poverty measurement varies significantly with: (1) the fictitious income assigned to the prematurely dead; (2) the degree of poverty aversion; (3) the shape of the (unadjusted) income distribution; (4) the strength of the income/mortality relationship.

In the light of those results, the present paper complements the literature on the measurement of poverty under income-differentiated mortality in two main ways. Firstly, the present paper complements the theoretical papers by Kanbur
and Mukherjee (2007) and Lefèbvre et al (2013) by deriving, for the particular class of FGT poverty measures, conditions under which those measures are robust to changes in survival conditions, as well as conditions under which the extension of lifetime income profiles of the prematurely dead can make poverty measures robust to mortality changes. Secondly, we also provide, in the present paper, an empirical exploration of how the effects of extending the income profiles of the prematurely dead on poverty measurement vary across different adjustment techniques, across different degrees of poverty aversion, and across different countries. That empirical exploration, by showing how the treatment of the prematurely dead affects poverty measurement, suggests that the theoretical discussions in Kanbur and Mukherjee (2007) and Lefèbvre et al (2013) concern a general problem for poverty measurement, whose size varies with the degree of poverty aversion.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 studies the robustness of FGT measures to changes in survival conditions. Section 4 proposes to extend income profiles of the prematurely dead, in such a way as to make adjusted FGT measures robust to mortality changes. Section 5 uses data on old-age poverty in 11 European countries to compare, across countries and gender, the effect of extending the lifetime income profiles of the prematurely dead on old-age poverty measurement. Section 6 concludes.

2 The model

We consider a two-period economy, where a cohort, of size $N \in \mathbb{N}$, lives the young age (first period) for sure, whereas only some fraction of the population will enjoy the old age (second period). Within that economy, a higher income when being young leads to a higher probability of survival to the old age.\(^3\) Hence there exists a perfect rank correlation between, on the one hand, income levels at the young age, and, on the other hand, survival chances to the old age.

The economy takes the following form.\(^4\) There exists a finite number $K \in \mathbb{N}$ of possible income levels ($K > 1$). The set of income levels is: $Y = \{y_1, \ldots, y_K\}$. The number of young individuals with income $y_i \in Y$ is denoted by $n^{1}_{i}$.\(^5\) We denote by $\mathbf{n}$ the vector of size $K$, whose entries are $n^{1}_{k}$ for $k = 1, \ldots, K$. For the ease of presentation, income levels are indexed in an increasing order, so that:

$$y_1 < \ldots < y_K$$

(1)

Similarly, income-specific survival probabilities to the old age, denoted by $\pi_k$ for income groups $k = 1, \ldots, K$, are ranked in an increasing order:

$$\pi_1 < \ldots < \pi_K$$

(2)

\(^3\)That model is in line with the empirical evidence suggesting that income and longevity are positively correlated. See Duleep (1986), Denton and Paxson (1998), Jusot (2004) and Salm (2007).

\(^4\)This formal framework is similar to the one developed in Lefèbvre et al (2013).

\(^5\)We have: $\sum_{k=1}^{K} n^{1}_{k} = N$. 
We denote by $\pi$ the vector of size $K$ whose entries are the income-specific survival probabilities $\pi_k$. The number of surviving old individuals with income $y_i \in Y$ is denoted by $n^2_i$. We denote by $n^2$ the vector of size $K$, whose entries are $n^2_k$ for $k = 1, ..., K$.

The income mobility process depends both on the income-specific survival rates, and on the chances to shift to different income levels in case of survival. The number of surviving old individuals with, for instance, income $y_k$, denoted by $n^2_k$, is equal to the sum of all young individuals (with potentially any income level) who (1) survived to the old age and (2) turned out to move from their income level at the young age to the income group $k$ at the old age, that is:

$$n^2_k = \sum_{i=1}^{K} \pi_i n^1_i \lambda_{ik}$$  (3)

where $\lambda_{ik}$ is the probability that a young agent with income $y_i$ enjoys, in case of survival, an income $y_k$ at the old age. If there was no mortality ($\pi_i = 1 \forall i$) and no income mobility ($\lambda_{ii} = 1 \forall i$), we would have $n^2_k = n^1_k$. However, given the existence of mortality and income mobility, it is most likely that $n^2_k \neq n^1_k$.

In the following, we denote by $\Lambda$ the $K \times K$ matrix that describes the "pure" income mobility process, that is, the income mobility process conditionally on survival to the old age:

$$\Lambda \equiv \begin{pmatrix} \lambda_{11} & \lambda_{12} & \ldots & \lambda_{1K} \\ \lambda_{21} & \lambda_{22} & \ldots & \lambda_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{K1} & \lambda_{K2} & \ldots & \lambda_{KK} \end{pmatrix}$$  (4)

In the absence of income-differentiated mortality, the matrix $\Lambda$ would include all relevant information about the dynamics of income distribution over time. However, given the existence of income-differentiated mortality, that matrix does not provide a complete description of the income mobility process. Actually, the income mobility process can be described by means of another transition $K \times K$ matrix, denoted by $M$, which describes how the income distribution at the young age determines the income distribution at the old age:

$$n^2 = M n^1$$  (5)

where the $M$ matrix is defined as:

$$M \equiv \begin{pmatrix} \pi_1 \lambda_{11} & \pi_1 \lambda_{12} & \ldots & \pi_1 \lambda_{1K} \\ \pi_2 \lambda_{21} & \pi_2 \lambda_{22} & \ldots & \pi_2 \lambda_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_K \lambda_{K1} & \pi_K \lambda_{K2} & \ldots & \pi_K \lambda_{KK} \end{pmatrix}$$

The $M$ matrix fully describes the trajectories of individuals in our economy. The lifecycle trajectory depends on survival probabilities and on income transition probabilities, which are correlated in terms of rank. We can easily rewrite

\[ \sum_{k=1}^{K} n^2_k = \sum_{k=1}^{K} \pi_k n^1_k. \]
the matrix $M$ as the Hadamard product (i.e. the entrywise product) of the following two matrices:\footnote{The symbol $\odot$ refers to the Hadamard product.}

$$M = \Lambda \odot \Pi$$

where $\Pi \equiv \pi \times 1'_K$, $1_K$ being the identity vector of size $K$.

Given that $n^2 = M' n^1$, the income distribution at the old age $n^2$ is, for a given initial income distribution $n^1$, fully determined by the matrix $M = \Lambda \odot \Pi$. Therefore the income distribution at the old age can be described in two equivalent ways. Indeed, if one denotes the vector of possible income levels by $y$, the income distribution at the old age can be described either by pair:

$$\{y, n^2\}$$

or, equivalently, by the quartet:

$$\{y, n^1, \pi, \Lambda\}$$

Those two definitions of the income distribution at the old age - i.e. $\{y, n^2\}$ and $\{y, n^1, \pi, \Lambda\}$ - are equivalent, since $M = \Lambda \odot \Pi$. The only difference between the two definitions is that, while the definition $\{y, n^2\}$ is static (since it only shows the sizes of the different income groups at the old age), the definition $\{y, n^1, \pi, \Lambda\}$ is dynamic, since it tells us how the distribution of income at the old age $n^2$ was achieved while starting from the income distribution at the young age $n^1$. Given that dynamic nature, one can refer to $\{y, n^1, \pi, \Lambda\}$ either as the income distribution at the old age, or as the income mobility process.

That second, dynamic description of the income distribution at the old age is most relevant for the purpose of this study, and, hence, will be largely used throughout the rest of this paper. The reason lies in the fact that the definition of income distribution $\{y, n^1, \pi, \Lambda\}$ allows us to study how the income distribution at the old age depends on the vector $\pi$ of income-specific survival probabilities.

### 3 Robustness of FGT measures

Let us now consider how to measure poverty in the economy under study. As this is well-known, there exists various families of poverty measures, each of these satisfying some particular properties or axioms. We will focus here on a particular family of poverty measures, which are called FGT poverty measures (see Foster et al 1984). Moreover, given that we are concerned here with the impact of premature death on the measurement of poverty at the old age, we will concentrate exclusively on FGT measures of old-age poverty.

**Definition 1** Consider an economy with an income distribution at the old age $\{y, n^2\}$. Given a poverty threshold $y_P \in Y$ and an aversion to poverty $\alpha \geq 0$, the FGT poverty measure at the old age is the function:

$$P_\alpha (y, n^2) = \frac{1}{\sum_{j=1}^K n_j^2} \sum_{k=1}^{P-1} n_k^2 \left[ \frac{y_P - y_k}{y_P} \right]^{\alpha}$$
As stressed in Foster et al (1984), the parameter $\alpha$ can be interpreted as an indicator of aversion to poverty. When $\alpha = 0$, the poverty index is a headcount ratio, which, as such, is not reactive to income reductions of the poor. However, once $\alpha > 0$, income reductions of the poor increase, ceteris paribus, the measured poverty, in line with the Monotonicity Axiom. Moreover, when $\alpha > 1$, transfers of income from a poor to a richer person raise, ceteris paribus, the poverty measure, in line with the Transfer Axiom. In the special case where $\alpha = 1$, the FGT poverty measure is the average income gap (with respect to the poverty line), while it captures the average squared income gap when $\alpha = 2$.

While the above definition describes FGT old-age poverty measures in terms of the income distribution $\{y, n^2\}$, one can, alternatively, define poverty measures in terms of the income distribution $\{y, n^1, \pi, A\}$, as stated below.

**Remark 1** Given that $n_k^2 = \sum_{i=1}^{K} \pi_i n_1^1 \lambda_{ik}$, the FGT poverty measure at the old age can be written as:

$$\bar{P}_\alpha (y, n^1, \pi, A) = \frac{1}{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_1^1 \lambda_{ik}} \sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n_1^1 \lambda_{ik} \left[ \frac{yp - y_k}{yp} \right]^\alpha$$

By highlighting the role of survival conditions, that alternative definition of the old-age FGT poverty measure is most adequate for exploring the robustness of poverty measures to changes in survival conditions. Indeed, this allows us to examine how the measured poverty varies when there is a change in survival conditions, that is, a change in the vector $\pi$.

In order to study the sensitivity of FGT poverty measures to changes in survival conditions, we need first to define formally what we mean by a poverty measure that is robust to changes in mortality. The following property, entitled Robustness to Mortality Changes (RMC), captures some idea of robustness to changes in survival conditions.

**Definition 2** Consider an economy with an income distribution at the old age $\{y, n^1, \pi, A\}$. A poverty measure $\bar{P}_\alpha (y, n^1, \pi, A)$ satisfies Robustness to Mortality Changes (RMC) if and only if any change in the survival conditions leaves the measured poverty unchanged:

If $\pi$ and $\pi'$ are such that:

- $\pi_k \neq \pi'_k$ for some $k \in \{1, ..., K\}$
- $\pi_j = \pi'_j$ for other $j \neq k \in \{1, ..., K\}$

then $\bar{P}_\alpha (y, n^1, \pi, A) = \bar{P}_\alpha (y, n^1, \pi', A)$

The RMC property states that the poverty measure should be left unchanged by any variation in survival rates, whatever the income group considered. That requirement, if satisfied, makes poverty measures immunized against the noise due to income-differentiated mortality. Hence, one can regard the RMC property as a condition guaranteeing that the poverty measure escapes the Mortality Paradox, and thus does not exhibit lower values when the survival conditions of the poor are worsened.
Although the RMC property seems intuitive, it constitutes nonetheless a quite strong invariance requirement, since it requires the invariance of the old-age poverty measure to hold whatever the income mobility process \((y, n^1, \pi, A)\) is. Given the various forms that \((y, n^1, \pi, A)\) can take, the RMC condition imposes, in fact, a very strong requirement on poverty measures.

In the light of this, it does not come as a surprise that the FGT poverty measure does not satisfy RMC.

**Proposition 1** Consider an economy with an income distribution at the old age \(\{y, n^1, \pi, A\}\). The FGT measure \(P_0(y, n^1, \pi, A)\) does not satisfy RMC.

**Proof.** See the Appendix.

The reason why FGT poverty measures violate RMC lies in the strength of that robustness condition: RMC does not require \(P_0(y, n^1, \pi, A)\) to be invariant to some changes in survival probabilities, but to any of them. To put it differently, it could be the case that some variation in a survival probability leaves \(P_0(y, n^1, \pi, A)\) unchanged. However, requiring that any change in the survival conditions leaves the measured poverty unchanged is quite demanding.

The strength of the RMC requirement comes also from the absence of restrictions on the form of the income mobility process \(\{y, n^1, \pi, A\}\). Indeed, there could exist some subclasses of income mobility processes \(\{y, n^1, \pi, A\}\) such that the FGT measure \(P_0(y, n^1, \pi, A)\) satisfies RMC.

To illustrate this, let us first concentrate on a particular subclass of economies, for which the matrix \(A\) is such that all income groups face the same aggregate probabilities of falling into poverty in case of survival to the old age. The following proposition shows that, if one focuses on that particular class of income mobility processes, the FGT poverty measure satisfies RMC when the aversion to poverty is zero (i.e. when \(a = 0\)).

**Proposition 2** Consider a particular subclass of economies with an income distribution at the old age \(\{y, n^1, \pi, A\}\), where \(A\) is such that:

\[
\lambda_1 + \lambda_2 + \ldots + \lambda_{i-1} = \lambda_{j1} + \lambda_{j2} + \ldots + \lambda_{j_{p-1}}
\]

for all income levels \(y_i, y_j \in Y\). The FGT measure \(P_0(y, n^1, \pi, A)\) satisfies RMC.

**Proof.** See the Appendix.

Proposition 2 suggests that, if one focuses on a particular subclass of income mobility processes \(\{y, n^1, \pi, A\}\), FGT measures with zero poverty aversion - that is, headcount poverty measures - satisfy RMC. The specificity of the income mobility processes consists of similar probabilities of falling into poverty in case of survival to the old age whatever the initial income level is.

A particular corollary of Proposition 2 concerns economies with no income mobility, that is, for which the \(A\) matrix is a diagonal matrix. Indeed, if \(A\) is a diagonal matrix, the property mentioned in Proposition 2 cannot be satisfied. Hence RMC must be violated.
Let us illustrate further the result of Proposition 2 with the following economy, with two possible income levels $Y = \{y_1, y_2\}$. The poverty line is fixed to the level $y_2$, so that the population is partitioned between poor (i.e. with income $y_1$) and non-poor (i.e. with income $y_2$). The condition mentioned in Proposition 2 consists of assuming that the matrix $\Lambda$ takes the following form:

$$
\Lambda = \begin{pmatrix}
    x & 1 - x \\
    x & 1 - x
\end{pmatrix}
$$

It is clear that, whatever the survival conditions faced by the two income groups are (i.e. $\pi_1$ and $\pi_2$), and whatever the initial income distribution is, the proportion of poor persons at the old age is necessarily equal to $x$. Thus equal probabilities to fall into poverty in case of survival to the old age guarantee that the headcount poverty measure is robust to mortality changes.

To illustrate Proposition 2 further, let us consider the following matrix $\Lambda$, with $Y = \{y_1, y_2, y_3\}$:

$$
\Lambda = \begin{pmatrix}
    x & y & 1 - x - y \\
    x & z & 1 - x - z \\
    x & w & 1 - x - w
\end{pmatrix}
$$

When $y_P = y_2$, the proportion of poor persons at the old age is necessarily equal to $x$, whatever the survival rates and initial income distributions are. However, if $y_P = y_3$, this is no longer the case. Under that alternative poverty line, the RMC property requires necessarily $y = z = w$, so as to equalize the probabilities to fall into poverty across income groups.

In sum, Proposition 2 suggests that, if one focuses on economies where all individuals face the same probabilities of falling into poverty in case of survival to the old age, FGT poverty measures with zero aversion to poverty satisfy RMC. But does the same result hold for FGT poverty measures with positive aversion to poverty? Proposition 3 states the general condition under which FGT measures with various degrees of poverty aversion satisfy RMC.

**Proposition 3** Consider a particular subclass of economies with an income distribution at the old age $\{y, n^1, n^2, n^3, \pi^1, \pi^2, \pi^3, \Lambda\}$, where $y$ and $\Lambda$ are such that, for all income levels $y_i$, $y_j < y_P$:

$$
\lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_{i-1} z_{i-1} = \lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_{i-1} z_{i-1}
$$

where $z_i = \left(\frac{y_n - y_i}{y_P}ight)^{\alpha}$. The FGT measure $\tilde{P}_\alpha (y, n^1, n^2, n^3, \Lambda)$ satisfies RMC.

**Proof.** See the Appendix.

The intuition behind Proposition 3 goes as follows. When the members of all income groups face the same expected extent of poverty in case of survival to the old age, changes in survival conditions cannot affect the measured extent of poverty, since the measured poverty is then independent of survival conditions. It is straightforward to see that Proposition 3 generalizes Proposition 2.
to all FGT poverty measures, including those exhibiting a non-zero degree of aversion to poverty. Indeed, under headcount measures, we have \( z_i = 1 \) for all income groups, so that the condition of Proposition 3 vanishes to the condition of Proposition 2, that is, the equality of the probabilities to fall into poverty in case of survival to the old age.

To illustrate Proposition 3, let us turn back to the above example with \( Y = \{y_1, y_2, y_3\} \). Suppose that \( y_P = y_3 \), and that \( \Lambda \) and \( y \) are such that:

\[
\lambda_{11} z_1 + \lambda_{12} z_2 = \lambda_{21} z_1 + \lambda_{22} z_2 = \lambda_{31} z_1 + \lambda_{32} z_2 = Z
\]

in line with Proposition 3. The old-age poverty measure is:

\[
\tilde{P}_y (\mathbf{y}, \mathbf{n}, \pi, \Lambda) = \frac{n_1 \pi_1 Z + n_2 \pi_2 Z + n_3 \pi_3 Z}{n_1 \pi_1 + n_2 \pi_2 + n_3 \pi_3} = Z
\]

Hence the old-age poverty measure is equal to \( Z \) whatever the income-specific survival probabilities are, and is thus also invariant to any change in survival conditions.

Taken together, Propositions 2 and 3 show that there exist some subclasses of economies for which the FGT poverty measure satisfies the RMC requirement. However, the restrictions imposed on the income mobility process \( \{y, n^1, \pi, \Lambda\} \) so as to achieve RMC are extremely strong. Actually, one expects that, in the real world, members of different income groups are not characterized by the same probabilities of falling into poverty in case of survival to the old age, unlike what is required in Proposition 2. Indeed, poor persons are more likely to remain poor in case of survival to the old age, in comparison to non-poor young individuals. Moreover, it is also unlikely that members of different income groups are characterized by the same expected extent of poverty at the old age.

Hence, despite the particular cases underlined in Propositions 2 and 3, it is unlikely that FGT poverty measures satisfy the RMC requirement when applied to real world data. As a consequence, the FGT poverty measure is not, in general, robust to variations in survival conditions. As such, it is likely to be subject to the Mortality Paradox, in the sense that a worsening of the survival conditions of the poor may reduce the measured poverty.\(^8\)

In order to be more accurate about the circumstances in which a worsening of survival conditions makes the measured old-age poverty fall, the following proposition states the conditions under which a variation in the survival conditions of an income group \( m \) tends to increase or decrease the measured poverty.

**Proposition 4** Consider an economy with an income distribution at the old age \( \{\mathbf{y}, n^1, \pi, \Lambda\} \). Assume a variation in the survival rate, i.e. \( \pi'_m < \pi_m \). We

\(^8\)This is a likely case, even though the opposite result cannot be excluded: an increase in the mortality of the poor leading to an increase in the measured poverty.
have:

\[
\hat{P}_\alpha (y, n^1, \pi, \Lambda) \quad \text{iff} \quad \frac{\sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n^1_{ik} \lambda_{ik}}{\left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n^1_{ik} \lambda_{ik} \right)} 
\geq \frac{\sum_{k=1}^{P-1} n^1_m \lambda_{mk} \left( \frac{y_p - y_k}{y_p} \right)^\alpha}{\left( \sum_{k=1}^{K} n^1_m \lambda_{mk} \right)}
\]

**Proof.** See the Appendix. ■

Proposition 4 can be interpreted as follows. The left-hand side (LHS) of the condition in Proposition 4 consists of the average extent of poverty among all surviving individuals in the population, while the right-hand side (RHS) is the average extent of poverty for agents of type \( m \) conditionally on survival. Hence a worsening of the survival conditions of some income group may either reduce or increase the measured poverty, depending on whether the prematurely dead would have, in case of survival, suffered from an equal extent of poverty (RHS) in comparison with the average surviving population (LHS).

When the deterioration of survival conditions concerns a group with a higher probability to be poor at the old age than the average surviving population, as well as having a larger income gap with respect to the poverty line, the RHS exceeds the LHS in the condition of Proposition 4, and thus the worsening of survival conditions reduces the measured poverty. Alternatively, when gains in life expectancy only concern a high income group, with a very low probability to become poor in case of survival, those gains contribute also to reduce the measured poverty. Thus, there exist several ways in which changes in survival conditions can disconnect the measured poverty from the "true" poverty.

### 4 Adjusting FGT measures

As shown in the previous section, FGT old-age poverty measures are, in general, not robust to variations in survival conditions, and, hence, may be either increased or decreased when the survival chances of some income group vary. Such a lack of robustness is problematic because of two different reasons.

On the one hand, one may consider that poverty measures should, in general, satisfy RMC. Indeed, one can consider that income-differentiated mortality introduces interferences or noise in the measurement of old-age poverty, and that "good" poverty measures should be immunized against such interferences, and should count premature death as something "neutral" for poverty measurement. On the other hand, one may consider that poverty measures should not necessarily satisfy RMC, but should count premature death as a part of the poverty phenomenon to be measured, as argued by Sen (1998).

But whatever the precise motivation is, it is worth considering how one could adjust poverty measures, in such a way as to make these either robust to mortality changes, or, alternatively, to make these count premature death as a part of the poverty phenomenon to be measured.
For that purpose, we will follow here the approach proposed by Kanbur and Mukherjee (2007): the extension of the lifetime income profiles of the prematurely dead persons, in such a way as to count these when measuring the extent of the poverty phenomenon.\(^9\) The underlying intuition behind the adjustment proposed by Kanbur and Mukherjee (2007) consists of doing "as if" the prematurely dead individuals were still alive, in such a way as to make poverty measures robust to variations in survival conditions.

Such an extension of the lifetime income profiles of the prematurely dead leads to the following adjusted FGT poverty measures.

**Definition 3** Consider an economy with an income distribution at the old age \((y, n^1, \pi, \Lambda)\). Given a poverty threshold \(y_P \in Y\), and an aversion to poverty \(\alpha \geq 0\), the adjusted FGT poverty measure at the old age is the function:

\[
\hat{P}_{\alpha, \Sigma}(y, n^1, \pi, \Lambda) = \frac{\sum_{j=1}^{K} \pi_j n_j \left( \sum_{k=1}^{P-1} \lambda_{jk} \left[ \frac{y_P - y_k}{y_P} \right]^\alpha \right) + \sum_{j=1}^{K} (1 - \pi_j) n_j \left( \sum_{k=1}^{P-1} \sigma_{jk} \left[ \frac{y_P - y_k}{y_P} \right]^\alpha \right)}{\sum_{j=1}^{K} n_j}
\]

where \(\sigma_{ij}\) is the probability, for an individual with income \(y_i\) when being young, to have a fictitious income \(y_j\) assigned to him when he is dead.

The adjusted poverty measure \(\hat{P}_{\alpha, \Sigma}(y, n^1, \pi, \Lambda)\) can be interpreted as follows. The first term is standard: it counts the poor individuals among the old (surviving) population, and multiplies this by the transformed income gap. But the second term is less standard: it measures poverty among the individuals who did not survive, their fictitious incomes being assigned to them through the matrix \(\Sigma\).

The assignment of a fictitious income to the prematurely dead can take various forms, depending on: (1) whether the assignment of fictitious incomes concerns all individuals or only the initially poor; (2) whether fictitious incomes exceed or are below the poverty line \(y_P\). Those two features of the extension are captured by the matrix \(\Sigma\):

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \ldots & \ldots & \sigma_{1K} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\sigma_{K1} & \ldots & \ldots & \sigma_{KK}
\end{pmatrix}
\]

The matrix \(\Sigma\) determines how prematurely dead persons are taken into account in the measurement of poverty. In other words, each particular treatment of the prematurely dead persons corresponds to a particular form of the matrix \(\Sigma\).

As this is discussed in Lefèbvre *et al* (2013), various candidates are possible for the matrix \(\Sigma\). One can, for instance, use the identity matrix \(I\) as the

---

\(^9\)Note, however, an important difference with respect to Kanbur and Mukherjee (2007): the extension of lifetime income profiles concerns here all prematurely dead persons, and not only the prematurely dead poor ones.
matrix $\Sigma$. This amounts to assign to each prematurely dead person a fictitious income that is exactly equal to the income enjoyed when being alive. Such an adjustment amounts to treat a premature death as something "neutral". In case of very little income mobility, the matrix $\Lambda$ is close to the matrix $I$, so that assuming $\Sigma = I$ makes the adjusted FGT measures robust to changes in mortality.

Another possibility is to assign, to all prematurely dead persons, a fictitious income level that is inferior to the actual income enjoyed when being alive, in such a way as to reflect the fact that a premature death is a major cause of deprivation on its own. Under that alternative adjustment, poverty is not restricted to low incomes, but also includes premature death, in line with Sen (1998)'s emphasis on the necessity to count premature death as a component of poverty. For that purpose, one possibility, which is discussed in Lefèbvre et al (2013), consists of taking the "welfare-neutral" income $y_N$, which brings indifference, at the individual level, between, on the one hand, survival with that income, and, on the other hand, death:

$$U(u(y_i), u(y_N)) = U(u(y_i), \Omega)$$

(8)

where $U(u(y_i), u(y_N))$ is a separable lifetime welfare function, whereas $\Omega$ is the utility of being dead, usually normalized to zero. In that case, the matrix $\Sigma$ is the column matrix at the particular "welfare-neutral" income level. Clearly, in this case, the adjusted poverty measure is different from what would have prevailed in the absence of any income-differentiated mortality; it counts a premature death as something that is a source of poverty, and, hence, is not neutral.

The above discussion suggests that adjusted FGT poverty measures tend, by construction, to take into account not only the observable poverty (i.e. of existing persons), but, also, the poverty that would have prevailed in the absence of income-differentiated mortality. As such, adjusted FGT poverty measures seem to bring some beginning of solution to the Mortality Paradox. However, the adjustment can take various forms, and it is not straightforward to see which adjustment would make FGT measures robust to mortality changes.

Proposition 5 states the condition under which the adjusted FGT poverty measure satisfies RMC. That condition concerns the form of the matrix $\Sigma$.

**Proposition 5** Consider an economy with an income distribution at the old age $\{y, n, \pi, \Lambda\}$. The adjusted FGT measure $\hat{P}_{\alpha,\Sigma} (y, n, \pi, \Lambda)$ satisfies RMC if and only if the matrix $\Sigma$ is equal to the matrix $\Lambda$.

**Proof.** See the Appendix. ■

Proposition 5 states that adding the prematurely dead persons does not suffice to make old-age poverty measures robust to variations in survival conditions. Whether such a robustness will be achieved or not depends on the shape of the matrix $\Sigma$, that is, the income mobility matrix in terms of fictitious income. Actually, Proposition 5 states that adjusted FGT poverty measures are robust to variations in survival probabilities provided the matrix $\Sigma$, i.e. the income mobility matrix conditionally on premature death, is equal to the $\Lambda$ matrix, i.e.
the income mobility matrix conditionally on survival. That condition is quite intuitive: it is only when the precise way in which fictitious incomes are assigned to prematurely dead persons coincides with what would have been assigned to those persons in case of survival that the effect of differentiated mortality on old-age poverty measurement is neutralized.

At this stage, it should be stressed that, although fixing $\Sigma = \Lambda$ guarantees that the adjusted poverty measure satisfies Robustness to Mortality Changes, there is no obvious reason why adjusted poverty measures should necessarily satisfy RMC. Actually, there exists lots of reasons for adopting a matrix $\Sigma$ that is distinct from the matrix $\Lambda$. Indeed, RMC implies that mortality is neutral for poverty measurement. However, one may, like Sen (1998), believe that premature death is not neutral at all for poverty measurement, but is rather a component of the poverty phenomenon to be measured. Hence, in that case, fixing $\Sigma = \Lambda$ is by no means desirable, and one can instead assume that the $\Sigma$ matrix makes adjusted FGT measures rise when survival conditions deteriorate.

If the $\Sigma$ matrix differs from the $\Lambda$ matrix, the adjusted FGT measure is not robust to variations in survival conditions. As stated below, a change in survival conditions can either increase or decreased the extent of measured poverty.

**Proposition 6** Consider an economy with an income distribution at the old age $(y, n^1, \pi, \Lambda)$. Assume a variation in the survival rate, i.e. $\pi' < \pi_m$. We have:

$$\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, \Lambda) \geq \hat{P}_{\alpha, \Sigma} (y, n^1, \pi', \Lambda)$$

iff

$$\sum_{l=1}^{P-1} \lambda_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha} \geq \sum_{l=1}^{P-1} \sigma_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha}$$

**Proof.** See the Appendix. $\blacksquare$

Proposition 6 tells us how the adjusted FGT measure varies with changes in survival conditions when the matrix $\Sigma$ differs from the matrix $\Lambda$. If, for instance, one uses the identity matrix as the $\Sigma$ matrix, the above condition, under $y_m < y_P$, becomes:

$$\sum_{l=1}^{P-1} \lambda_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha} \geq \sum_{l=1}^{P-1} \sigma_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha}$$

Therefore, in the case of headcount ratio, it follows that $\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, \Lambda) < \hat{P}_{\alpha, \Sigma} (y, n^1, \pi', \Lambda)$. Note, however, that under $y_m \geq y_P$, the condition would be:

$$\sum_{l=1}^{P-1} \lambda_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha} \geq \sum_{l=1}^{P-1} \sigma_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha}$$

implying that $\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, \Lambda) > \hat{P}_{\alpha, \Sigma} (y, n^1, \pi', \Lambda)$ whatever $\alpha$ is. Hence whether the deterioration of survival conditions concerns initially poor or non-poor persons determines the direction of change when $\Sigma$ is the identity matrix.

If, alternatively, one used as a $\Sigma$ matrix equal to a column matrix at the particular "welfare-neutral" income level $y_N$ (see above), the condition of Proposition 6 would become:

$$\sum_{l=1}^{P-1} \lambda_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha} \geq \sum_{l=1}^{P-1} \sigma_{ml} \left[ \frac{y_P - y_l}{y_P} \right]^{\alpha}$$

Under a very low $y_N$, we thus have $\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, \Lambda) < \hat{P}_{\alpha, \Sigma} (y, n^1, \pi', \Lambda)$, so that the adjusted FGT measure value is increased by the rise in mortality. Note that this is true independently from the initial income level of the prematurely dead, unlike when $\Sigma$ is the identity matrix.
In the light of all this, it follows that the extension of lifetime income profiles of the prematurely dead persons can, in theory, lead to a more or less large rise in the measured poverty, to an extent that varies with several factors (the shape of matrix $\Sigma$, the degree of poverty aversion $\alpha$, the income distribution under study). The goal of the next section is to use data from 11 European countries to examine the impact of those determinants on the gap between unadjusted and adjusted FGT measures of poverty at the old age.

5 Evidence: old-age poverty in Europe

This section proposes to illustrate, on the basis of data on old-age poverty in 11 European countries, the above theoretical discussion, by exploring how the measured old-age poverty in Europe varies with the treatment of the prematurely dead within poverty measures. For that purpose, we compare standard FGT measures (with various degrees of poverty aversion $\alpha$) of old-age poverty with adjusted FGT measures (under various matrixes $\Sigma$) across different countries.

5.1 Data

The analysis is based on poverty data from the European household survey EU-SILC for the year 2007, and on the life expectancy by education level made available by Eurostat (2010).10 Due to the limited availability of comparable life expectancy statistics by educational level, 11 countries are included in the data set: Czech Republic, Denmark, Estonia, Finland, Hungary, Italy, Norway, Poland, Portugal, Slovenia and Sweden.

Given that the measurement interference induced by income-differentiated mortality is likely to be larger at higher ages, we will focus, throughout this section, on the measurement of old-age poverty, defined as poverty in the population aged 60 or more. The raw data on poverty in Europe in 2007 are presented in Table 1. Focusing first on simple headcount poverty measures ($P_0$), we see that the measured poverty at age 60+ varies strongly across countries. For instance, whereas only 5.5 % of the population aged 60 and more is below the poverty threshold in the Czech Republic, fractions as high as one fourth of the population aged 60 and more are below the poverty line in countries such as Estonia and Portugal. Note also that, although women exhibit, in all countries under study, higher headcount poverty measures than men, the distribution of poverty across genders varies significantly across countries. In some countries, such as Hungary or Poland, the proportion of persons in poverty at the old age is approximately the same for men and women. On the contrary, in countries like the Czech Republic or Norway, the poverty gap between women and men is much larger: the prevalence of poverty at the old age is, in those countries, about three times larger among women than among men.

If one now considers the extent of poverty, measured through the average income gap ($P_1$) or squared income gap ($P_2$), further observations can be made.

A first observation is that the ranking of countries in terms of poverty varies significantly according to the FGT measure used. For instance, Norway exhibits a higher proportion of poor persons in the population in comparison to Hungary (14.3 % against 10.7 %), but the extent of poverty, as measured by the average income gap, is lower in Norway than in Hungary (2.2 % against 2.5 %). Furthermore, the size and sign of the poverty gender gap also varies with the FGT measure used. For instance, poverty among men is lower than among women in Poland when using headcount ratios, but the extent of poverty is larger among men than among women when focusing on the squared income gap.

<table>
<thead>
<tr>
<th>Countries</th>
<th>$P_0$ (%)</th>
<th>$P_1$ (%)</th>
<th>$P_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>total</td>
<td>males</td>
<td>females</td>
</tr>
<tr>
<td>Czech Rep.</td>
<td>5.5</td>
<td>2.6</td>
<td>7.6</td>
</tr>
<tr>
<td>Denmark</td>
<td>13.0</td>
<td>11.2</td>
<td>14.5</td>
</tr>
<tr>
<td>Estonia</td>
<td>24.5</td>
<td>15.8</td>
<td>29.2</td>
</tr>
<tr>
<td>Finland</td>
<td>19.1</td>
<td>14.9</td>
<td>22.2</td>
</tr>
<tr>
<td>Hungary</td>
<td>10.7</td>
<td>8.9</td>
<td>11.8</td>
</tr>
<tr>
<td>Italy</td>
<td>20.8</td>
<td>17.6</td>
<td>23.3</td>
</tr>
<tr>
<td>Norway</td>
<td>14.3</td>
<td>7.1</td>
<td>20.0</td>
</tr>
<tr>
<td>Poland</td>
<td>9.1</td>
<td>8.3</td>
<td>9.6</td>
</tr>
<tr>
<td>Portugal</td>
<td>24.9</td>
<td>23.1</td>
<td>26.3</td>
</tr>
<tr>
<td>Slovenia</td>
<td>18.1</td>
<td>11.7</td>
<td>22.5</td>
</tr>
<tr>
<td>Sweden</td>
<td>10.2</td>
<td>6.7</td>
<td>13.1</td>
</tr>
</tbody>
</table>

Table 1: FGT poverty measures at age 60+, for year 2007.

Table 1 provides a contrasted picture of old-age poverty in Europe: old-age poverty levels vary across countries and gender, and are also sensitive to the FGT measure that is used. Note, however, that the picture provided by Table 1 may actually hide even larger discrepancies across European economies. Actually, differential income-specific survival conditions across countries may, by leading to a more or less large number of "missing poor" - and to a more or less large "hidden poverty" - distort the picture provided by Table 1.

In order to identify the impact of income-differentiated mortality on poverty measurement, we need data on survival conditions by income levels. There is, to our knowledge, no lifetable by income for European countries. However, Eurostat produces comparable information on mortality by education. To illustrate the differentials in life expectancy between and within countries, Tables 2 and 3 show, respectively, the life expectancy at age 60 by gender and by education.

11 $P_1$ measures can be interpreted as follows: individuals whose income is below the poverty line in e.g. Estonia have, on average, an income that is equal to 100 - 4.1 = 95.9 % of the poverty line. Regarding $P_2$ measures, these can be interpreted as follows. Persons whose income is below the poverty line in e.g. Estonia have, on average, an income whose relative gap with respect to the poverty line raised to the power 2 is equal to 1.3 %.

12 For each country, the poverty threshold is fixed at 60 % of the median income.

level (primary, secondary and tertiary).\textsuperscript{14}

<table>
<thead>
<tr>
<th>Countries</th>
<th>Life expectancy at 60</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>total</td>
</tr>
<tr>
<td>Czech Rep</td>
<td>20.8</td>
</tr>
<tr>
<td>Denmark</td>
<td>21.9</td>
</tr>
<tr>
<td>Estonia</td>
<td>19.7</td>
</tr>
<tr>
<td>Finland</td>
<td>23.3</td>
</tr>
<tr>
<td>Hungary</td>
<td>19.4</td>
</tr>
<tr>
<td>Italy</td>
<td>24.3</td>
</tr>
<tr>
<td>Norway</td>
<td>23.4</td>
</tr>
<tr>
<td>Poland</td>
<td>20.8</td>
</tr>
<tr>
<td>Portugal</td>
<td>23.1</td>
</tr>
<tr>
<td>Slovenia</td>
<td>22.2</td>
</tr>
<tr>
<td>Sweden</td>
<td>23.6</td>
</tr>
</tbody>
</table>

Table 2: Life expectancy at age 60 by gender 2007.

<table>
<thead>
<tr>
<th>Countries</th>
<th>Life expectancy at 60</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>primary</td>
</tr>
<tr>
<td>Czech Rep</td>
<td>20.68</td>
</tr>
<tr>
<td>Denmark</td>
<td>21.25</td>
</tr>
<tr>
<td>Estonia</td>
<td>16.53</td>
</tr>
<tr>
<td>Finland</td>
<td>22.91</td>
</tr>
<tr>
<td>Hungary</td>
<td>17.91</td>
</tr>
<tr>
<td>Italy</td>
<td>23.87</td>
</tr>
<tr>
<td>Norway</td>
<td>22.57</td>
</tr>
<tr>
<td>Poland</td>
<td>20.28</td>
</tr>
<tr>
<td>Portugal</td>
<td>23.06</td>
</tr>
<tr>
<td>Slovenia</td>
<td>21.86</td>
</tr>
<tr>
<td>Sweden</td>
<td>23.06</td>
</tr>
</tbody>
</table>

Table 3: Life expectancy at age 60 by education level, 2007.

Table 2 shows the existence of significant inequalities in longevity across Europe. The lowest life expectancy at age 60 is measured in Hungary (19.4 years), while the largest one is measured in Italy (24.3 years). Table 2 also highlights that the gender gap between women and men varies across countries, from 3 years in Denmark to 6.6 years in Estonia. However, as shown in Table 3, aggregate life expectancy statistics hide also large inequalities within countries, depending on the education level. The education gap in terms of life expectancy is very small in some countries, such as Sweden, where the life expectancy at age 60 for individuals with tertiary education is only 1.5 year larger than the one for individuals with primary education only. On the contrary, the education gap is much larger in Estonia, where it is equal to about 6.2 years.

\textsuperscript{14}Note that the data for Poland and Portugal are for year 2008 and 2009 respectively.
The varying life expectancy gap across countries suggests that the interferences in poverty measurement caused by income-differentiated mortality are likely to be varying across countries. In order to have a confirmation of that conjecture, we need first to use the education-specific lifetables provided by Eurostat in order to extrapolate lifetables by income levels. For that purpose, we use a weighted ordinary least square regression, in line with Bossuyt et al (2004) and Van Oyen et al (2005). Taking into account the high correlation that exists between education and income, we can extrapolate mortality by income class on the basis of the mortality by education, by relating the distributions of individuals on both dimensions. For that purpose, we first transform the absolute educational status into a relative educational status. Indeed among cohorts, the size of educational groups has changed. Young people studied more than older ones. For a given cohort, we represent each category of education by its size in the population. We then order these categories from the lowest level to the highest on a scale from 0 to 100%. That is each category of income represents a percentage of the total population of the cohort. This scale gives us a distribution of the cohort population according to education. We assume that the reference of an education category is determined by its relative position, defined as the mid-point of the proportion of the category represented on the ordered scale of 100% (Pamuk, 1985, 1988). We then regress the life expectancy by education on the reference mid-point of the education category by weighting for the prevalence of the category, i.e. the relative size of the educational level. The slope of the regression line represents the difference in mortality between the bottom and the top of the education hierarchy. Once estimated, the coefficients can be used to compute lifetables according to income. This is done by assuming that the social hierarchy given by the income is similar to the one given by education. We can thus apply the coefficient of one education category to the corresponding categories of income.

Figures 1 and 2 show, for the 11 European countries under study, the estimated life expectancy at age 60 by income class, for males and females respectively. For each country, life expectancy at age 60 is increasing with the income class considered. However, the longevity differential related to income inequality varies strongly across countries. The life expectancy differential is large in Estonia and in the Czech Republic. On the contrary, it is much lower in Sweden and Denmark. Moreover, the life expectancy gap tends to be larger for males (Figure 1) than for females (Figure 2). The significant variation in the size of the life expectancy gap in terms of income levels across countries and across gender implies that the selection bias in poverty measurement resulting from income-differentiated mortality is likely to vary across countries and gender.

\footnote{For example, if the first category is given by those with at most a primary degree and represents 10% of the cohort, the mid-point reference will be 5%. If the second category represents, let’s say those with a secondary degree, 20% of the population, the bounds of the category in the distribution are 10 and 30% and the mid-point is 20%.}
Figure 1: Life expectancy at age 60 by income class in Europe, males.

Figure 2: Life expectancy at age 60 by income class in Europe, females.
5.2 The adjustment technique

The adjustment of FGT poverty measures is made in two steps. First, we need to compute the number of "missing" persons for each country and each gender. Second, we need to assign a fictitious income to those "missing persons".

Regarding the first task, the number of "missing" individuals in each income class is computed, for each country and each gender, by calculating the hypothetical number of individuals of that class who would have survived if they had benefited from the survival conditions of the highest income class, for that country and that gender. Assuming a stable demography, that number of "missing" individuals in an income class can be obtained by multiplying the number of surviving individuals in that class by a coefficient equal to the ratio of income-specific life expectancies of the highest income class to the actual income class. As an illustration of that adjustment, Table 4 shows, for each country and each gender, the life expectancy statistics at age 60 for the bottom income class and for the top income class, as well as the corresponding adjustment coefficient.

<table>
<thead>
<tr>
<th>Countries</th>
<th>males</th>
<th>females</th>
<th>males</th>
<th>females</th>
<th>males</th>
<th>females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Czech Rep.</td>
<td>14.6</td>
<td>21.6</td>
<td>22.9</td>
<td>24.6</td>
<td>1.57</td>
<td>1.14</td>
</tr>
<tr>
<td>Denmark</td>
<td>19.3</td>
<td>22.5</td>
<td>22.0</td>
<td>25.1</td>
<td>1.14</td>
<td>1.12</td>
</tr>
<tr>
<td>Estonia</td>
<td>11.7</td>
<td>20.4</td>
<td>21.6</td>
<td>25.1</td>
<td>1.85</td>
<td>1.23</td>
</tr>
<tr>
<td>Finland</td>
<td>19.6</td>
<td>25.1</td>
<td>22.4</td>
<td>26.3</td>
<td>1.14</td>
<td>1.05</td>
</tr>
<tr>
<td>Hungary</td>
<td>14.1</td>
<td>20.8</td>
<td>21.8</td>
<td>23.7</td>
<td>1.54</td>
<td>1.14</td>
</tr>
<tr>
<td>Italy</td>
<td>20.0</td>
<td>25.2</td>
<td>25.1</td>
<td>28.0</td>
<td>1.25</td>
<td>1.11</td>
</tr>
<tr>
<td>Norway</td>
<td>20.0</td>
<td>24.2</td>
<td>23.1</td>
<td>26.4</td>
<td>1.16</td>
<td>1.09</td>
</tr>
<tr>
<td>Poland</td>
<td>15.8</td>
<td>22.4</td>
<td>20.4</td>
<td>24.3</td>
<td>1.29</td>
<td>1.09</td>
</tr>
<tr>
<td>Portugal</td>
<td>20.0</td>
<td>24.6</td>
<td>22.1</td>
<td>25.8</td>
<td>1.11</td>
<td>1.05</td>
</tr>
<tr>
<td>Slovenia</td>
<td>17.1</td>
<td>23.9</td>
<td>21.6</td>
<td>25.3</td>
<td>1.26</td>
<td>1.06</td>
</tr>
<tr>
<td>Sweden</td>
<td>21.4</td>
<td>24.7</td>
<td>23.7</td>
<td>26.7</td>
<td>1.11</td>
<td>1.08</td>
</tr>
</tbody>
</table>

Table 4: Life expectancy at age 60 for bottom and top income classes and the associated adjustment coefficient, 2007.

Adjustment factors for the lowest income class are larger for males than for females, in line with the higher gaps in terms of life expectancy by income class. There is also a variation in adjustment factors across countries: these are large in Estonia and Czech Republic, but much smaller for Finland and Portugal.

Regarding the second task, which consists of assigning a fictitious income to all those "missing" persons, we will adopt two alternative approaches, which consists of two distinct matrices $\Sigma$. The first approach consists of assigning, to each missing person, a fictitious income equal to the income previously enjoyed. That approach consists of assuming that $\Sigma$ is an identity matrix. In that case, a premature death is, in some sense, treated as neutral for poverty measurement. Another, alternative approach, consists of counting a premature death as a source of deprivation, which leads to assign, as a fictitious income, the income equivalent of death, that is, $y_N$. 

20
That welfare-neutral income, which makes an agent indifferent between, on the one hand, further life with that income, and, on the other hand, death, can be calibrated by following the work by Becker et al (2005). Taking income as a proxy for consumption, and assuming that individuals have time-additive preferences with a temporal utility function of the form \( u(y) = \frac{y^{1-1/\gamma}}{1-1/\gamma} + \alpha \), it is possible to derive the welfare-neutral income \( y_N \). \( y_N \) makes the utility associated to a life-period equal to the utility of being dead:

\[
\frac{y_N^{1-1/\gamma}}{1-1/\gamma} + \alpha = 0
\]

Following Becker et al (2005), we take \( \gamma = 1.25 \). Regarding the calibration of \( \alpha \), we also follow Becker et al (2005), who use the estimation of \( \epsilon = \frac{u(y)u_y}{u(y)} = 0.346 \) from Murphy and Topel (2003) to extrapolate the level of \( \alpha \). The value of \( \epsilon \) is estimated from compensating differentials for occupational mortality risks; it captures how individuals make trade-off between more income and more risk. Then, for each country, we calculate the level of \( \alpha \) on the basis of the average income, while assuming \( \gamma = 1.25 \) and \( \epsilon = 0.346 \). Then, in a last stage, we compute, for each country, and on the basis of the parameters \( \alpha \) and \( \gamma \) (the former being country-specific), the level of the welfare-neutral income \( y_N \). Table 5 shows the values of the welfare-neutral income \( y_N \) for each country.\(^{16}\)

\[
\begin{array}{|c|c|}
\hline
\text{Countries} & \text{Welfare-neutral fictitious income (euros 2007)} \\
\hline
\text{Czech Rep} & 77 \\
\text{Denmark} & 449 \\
\text{Estonia} & 54.5 \\
\text{Finland} & 347 \\
\text{Hungary} & 67 \\
\text{Italy} & 265 \\
\text{Norway} & 626 \\
\text{Poland} & 53 \\
\text{Portugal} & 136 \\
\text{Slovenia} & 149 \\
\text{Sweden} & 339 \\
\hline
\end{array}
\]


The welfare-neutral income \( y_N \) is extremely low, which is not surprising. Moreover, it varies strongly across countries, because of differences in standards of living (i.e. the level of the average income in the country), which lead to different levels of the intercept \( \alpha \). Those differences may seem, at first glance, surprising. However, similar inequalities would be obtained under alternative calibration techniques using country-specific income and risk-taking attitudes.\(^{17}\)

\(^{16}\)Those figures are expressed in yearly terms.
\(^{17}\)See, for instance, the meta-analysis made by Miller (2000) showing large differentials in the value of a statistical life across countries, depending on the income level.
5.3 Results

Table 6 shows the adjusted FGT measures for poverty at age 60 and more obtained while assigning to each missing individual a fictitious income equal to the past income enjoyed. In comparison with the unadjusted FGT poverty measures (Table 1), adjusted FGT measures are significantly higher. Those higher levels reflect the inclusion, within all income classes, of the prematurely dead persons. Given that low income classes are also characterized by worse survival conditions - and thus require the addition of a larger number of missing persons -, low income classes include, proportionally, higher numbers of added people than high income classes. Note, however, that, in theory, there was no obvious reason why FGT measures would necessarily be increased by the adjustment: this depends, in fine, on whether the prematurely dead would have, in case of survival, suffered from a more severe poverty than the average surviving population. Table 6, when compared to Table 1, shows that adjusted poverty measures are unambiguously higher than unadjusted poverty measures.

<table>
<thead>
<tr>
<th>Countries</th>
<th>$P_0$ (%) total</th>
<th>$P_0$ (%) males</th>
<th>$P_0$ (%) females</th>
<th>$P_1$ (%) total</th>
<th>$P_1$ (%) males</th>
<th>$P_1$ (%) females</th>
<th>$P_2$ (%) total</th>
<th>$P_2$ (%) males</th>
<th>$P_2$ (%) females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Czech Rep.</td>
<td>5.7</td>
<td>3.1</td>
<td>8.0</td>
<td>0.7</td>
<td>0.4</td>
<td>0.9</td>
<td>0.2</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>Denmark</td>
<td>13.5</td>
<td>11.6</td>
<td>15.0</td>
<td>1.7</td>
<td>1.2</td>
<td>2.1</td>
<td>0.5</td>
<td>0.3</td>
<td>0.6</td>
</tr>
<tr>
<td>Estonia</td>
<td>26.2</td>
<td>19.3</td>
<td>31.0</td>
<td>4.5</td>
<td>4.0</td>
<td>4.9</td>
<td>1.5</td>
<td>1.5</td>
<td>1.4</td>
</tr>
<tr>
<td>Finland</td>
<td>19.5</td>
<td>15.7</td>
<td>22.5</td>
<td>2.8</td>
<td>2.2</td>
<td>3.3</td>
<td>0.7</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>Hungary</td>
<td>11.7</td>
<td>10.8</td>
<td>12.4</td>
<td>2.8</td>
<td>3.0</td>
<td>2.6</td>
<td>1.3</td>
<td>1.6</td>
<td>1.2</td>
</tr>
<tr>
<td>Italy</td>
<td>21.9</td>
<td>19.2</td>
<td>24.1</td>
<td>5.4</td>
<td>4.6</td>
<td>6.0</td>
<td>2.4</td>
<td>2.1</td>
<td>2.7</td>
</tr>
<tr>
<td>Norway</td>
<td>14.7</td>
<td>7.6</td>
<td>20.5</td>
<td>2.3</td>
<td>1.5</td>
<td>3.0</td>
<td>0.7</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>Poland</td>
<td>9.6</td>
<td>9.3</td>
<td>9.9</td>
<td>1.9</td>
<td>2.0</td>
<td>1.9</td>
<td>0.7</td>
<td>0.8</td>
<td>0.6</td>
</tr>
<tr>
<td>Portugal</td>
<td>25.5</td>
<td>23.9</td>
<td>26.7</td>
<td>6.6</td>
<td>6.1</td>
<td>7.0</td>
<td>2.8</td>
<td>2.6</td>
<td>2.9</td>
</tr>
<tr>
<td>Slovenia</td>
<td>18.6</td>
<td>12.9</td>
<td>22.9</td>
<td>4.3</td>
<td>3.2</td>
<td>5.1</td>
<td>1.6</td>
<td>1.5</td>
<td>1.9</td>
</tr>
<tr>
<td>Sweden</td>
<td>10.4</td>
<td>7.0</td>
<td>13.4</td>
<td>1.8</td>
<td>1.1</td>
<td>2.4</td>
<td>0.6</td>
<td>0.4</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 6: Adjusted FGT poverty measures at age 60+, for year 2007 (fictitious income = past income).

The size of the adjustment varies significantly across countries. The adjustment is very small in Sweden (+ 0.2 % for the headcount ratio), in Czech republic (+ 0.2 %), in Norway (+0.4 %) and in Finland (+ 0.4 %), but is much larger in countries such as Estonia (+ 1.7 % for the headcount ratio), Italy (+ 1.1 %) and Hungary (+1.0 %). Another observation concerns the gender poverty gap. Table 6 suggests that, once poverty measures are adjusted, the gap between poverty prevalences among men and women is significantly reduced. For

---

18 Throughout this section, the poverty line is assumed to keep the same level as before the adjustment. That assumption is in line with the framework developed in Sections 2-4. Note that this assumption constitutes an obvious simplification, since the addition of prematurely dead persons to the population may potentially affect the level of the poverty line, and, hence, poverty measures. That effect is discussed in Lefèbvre et al (2013).

19 For each country, the poverty threshold is fixed at 60 % of the median income.
instance, whereas the gender gap in Estonia was equal to 29.2% - 15.8% = 13.4% in unadjusted terms (headcount), it is reduced to 31% - 19.3% = 11.7% once poverty measures are adjusted. Hence the inclusion of the "missing persons" does not only affect the overall poverty prevalence, but also lowers the gender poverty gap, even though women remain, on average, more subject to poverty.

In order to quantify the effect of extending lifetime income profiles of the prematurely dead on poverty measurement at the old age, we will use the following index, which measures the differential between, on the one hand, the poverty phenomenon as measured under standard FGT measures, and, on the other hand, the poverty phenomenon as measured under adjusted FGT measures:

\[ G_{\alpha\Sigma} = 1 - \frac{\bar{P}_\alpha (y, n^1, \pi, \Lambda)}{\bar{P}_{\alpha, \Sigma} (y, n^1, \pi, \Lambda)} \] (10)

Throughout the rest of this section, we refer to \(G_{\alpha\Sigma}\) as to the "gap index".

Figure 3 shows the levels of the gap index under headcount measures (\(\alpha = 0\)), income gap measures (\(\alpha = 1\)) and squared income gap measures (\(\alpha = 2\)), for the total population. The size of the gap index varies strongly across countries. Whereas it remains below 5% in Denmark, Finland, Norway, Portugal, Slovenia and Sweden, the gap index reaches much higher levels in Estonia, Hungary and Poland. Hence the effect of extending lifetime income profiles of the prematurely dead persons on old-age poverty measurement varies strongly across countries.

Another important lesson from Figure 3 concerns the variation of the gap index across FGT measures of poverty for a given country. Figure 3 shows that, for the countries under study, the gap index tends to be higher for squared income gap measures (\(\alpha = 2\)) than for average income gap measures (\(\alpha = 1\)) and for headcount measures (\(\alpha = 0\)). Hence, the extension of income profiles of the prematurely dead leads to larger adjustments when the degree of poverty aversion is larger. Note that the extent to which the gap index increases with \(\alpha\)
varies across countries. Those variations reflect the differentials between income distributions across countries.

Let us now contrast those results with what is obtained under alternative fictitious incomes. For that purpose, Table 7 shows the adjusted FGT poverty measures when the fictitious income used for the extension of income profiles of the prematurely dead persons consists of the welfare-neutral income $y_N$. Note that the income gap or the squared income gap is expected to be more sensitive to the level of fictitious incomes than the headcount. The reason is that adopting the number 0 for all the poor made to survive or their past income that is below the poverty line has, by construction, the same impact on the headcount, but not on the income gap.

In the light of Table 7, several observations can be made. Firstly, the adjusted FGT poverty measures take, under that alternative fictitious income, much larger levels than when fictitious incomes are equalized to past incomes. That result comes from the low levels of the welfare-neutral income $y_N$ (see Table 5). Hence, when one considers premature death as a source of poverty and deprivation, and include it in poverty measures under the form of the income equivalent to death, the measured old-age poverty becomes much larger.

A second important point to be stressed concerns the strong differentials across countries. The adjustment using the welfare-neutral income as a fictitious income increases the old-age poverty rate (headcount) by 3.3% in Denmark and by 4.1% in Sweden (in comparison to the unadjusted poverty rate), but by 14% for Estonia, and by 13.5% in Czech Republic, and by 10.5% in Hungary. Those large adjustments reflect the stronger differentials in life expectancy across income classes in those countries (Figures 1 and 2).

<table>
<thead>
<tr>
<th>Countries</th>
<th>$P_1$ (%)</th>
<th>$P_1$ (%)</th>
<th>$P_2$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>total</td>
<td>males</td>
<td>females</td>
</tr>
<tr>
<td>Czech Rep.</td>
<td>19.0</td>
<td>24.3</td>
<td>14.4</td>
</tr>
<tr>
<td>Denmark</td>
<td>16.3</td>
<td>14.9</td>
<td>17.5</td>
</tr>
<tr>
<td>Estonia</td>
<td>38.5</td>
<td>40.1</td>
<td>37.4</td>
</tr>
<tr>
<td>Finland</td>
<td>23.0</td>
<td>21.2</td>
<td>24.4</td>
</tr>
<tr>
<td>Hungary</td>
<td>21.2</td>
<td>25.8</td>
<td>17.9</td>
</tr>
<tr>
<td>Italy</td>
<td>27.1</td>
<td>26.6</td>
<td>27.5</td>
</tr>
<tr>
<td>Norway</td>
<td>19.4</td>
<td>14.0</td>
<td>23.9</td>
</tr>
<tr>
<td>Poland</td>
<td>15.0</td>
<td>17.7</td>
<td>13.2</td>
</tr>
<tr>
<td>Portugal</td>
<td>27.8</td>
<td>27.2</td>
<td>28.3</td>
</tr>
<tr>
<td>Slovenia</td>
<td>23.6</td>
<td>21.8</td>
<td>25.1</td>
</tr>
<tr>
<td>Sweden</td>
<td>14.3</td>
<td>11.4</td>
<td>16.8</td>
</tr>
</tbody>
</table>

Table 7: Adjusted FGT poverty measures at age 60+, for year 2007 (fictitious income = welfare-neutral income).^{20}

A third observation concerns the size of the gender gap. Once the welfare-neutral income is used to extend the income profiles of the prematurely dead, for each country, the poverty threshold is fixed at 60% of the median income.
the form of the poverty gender gap is strongly altered in some countries. In Czech Republic, Estonia, Hungary and Poland, the headcount poverty measure is larger among men than among women, whereas the opposite was prevailing in unadjusted poverty measures. The reversal of the poverty gender gap observed in those countries is due to the fact that income-related differentials in survival conditions are much larger among men than among women in those countries. Therefore, once the "missing" persons are added and are assigned $y_N$ as a fictitious income, the poverty gender gap is reversed. Note that, in other countries, females remain, after the adjustment, more subject to poverty than men.

Finally, let us compute the gap index under that alternative adjustment of FGT poverty measures (Figure 4). The comparison of Figures 3 and 4 suggests that the effect of extending income profiles of the prematurely dead on old-age poverty measurement is larger when using $y_N$ as a fictitious income in comparison to the adjustment based on past incomes. This reflects that the adjustment of old-age poverty measures is larger the adjusted poverty measure takes into account not only the "missing poor" (as on Figure 3), but, also, the "missing poverty" (premature death being counted as a source of poverty).

![Figure 4: Gap index for FGT measures, total population (fictitious income = welfare-neutral income)](image)

Note also that, on Figure 4, the size of the gap index varies strongly across FGT poverty measures, and is much larger for squared poverty gap measures ($\alpha = 2$) than for average poverty gaps ($\alpha = 1$) and headcount ratios ($\alpha = 0$). The size of the rise of the gap index when $\alpha$ is increased is substantial, especially for countries such as Denmark, Finland, Norway, and Sweden. The intuition behind this lies in the fact that the intensity of poverty in unadjusted terms is very low in those countries. Hence, given that unadjusted average poverty gap and squared poverty gap measures are low, the inclusion, within the income distribution, of the prematurely dead persons with very low incomes (equal to $y_N$) generates a quite strong rise in the intensity of poverty, in comparison to a low intensity in unadjusted terms. That rise is reinforced by the fact that
poverty lines are larger in those countries. Those larger poverty threshold lead to a higher intensity of poverty when the "missing persons" are added with a fictitious income equal to the welfare-neutral income (which is much lower than the poverty line).

In the light of Figures 3 and 4, it appears that the effects of extending income profiles of the prematurely dead on old-age poverty measurement vary strongly across countries and across classes of FGT measures. Eastern economies are characterized by larger income-related differentials in survival conditions. Therefore, the adjustment strongly raises headcount poverty measures for those countries. On the contrary, Nordic economies suffer from lower income-related differentials in survival conditions, so that the number of "missing" persons is lower. This explains why Nordic economies exhibit lower gap indexes when $\alpha = 0$. However, for Nordic countries, the adjustment has a bigger impact on distribution-sensitive poverty indicators ($\alpha > 0$), since these were very low in unadjusted terms, and since the poverty line is larger in Nordic economies. Hence, once we take the intensity of poverty into account, Nordic countries exhibit larger adjustments that are close to the ones of Eastern economies.

In sum, the effect of extending lifetime income profiles of the prematurely dead on old-age poverty measurement varies depending on: (1) the fictitious incomes assigned to the prematurely dead persons; (2) the postulated degree of poverty aversion $\alpha$; (3) the shape of the (unadjusted) income distribution; (4) the strength of the income/mortality relationship. The determinant (1) plays a crucial role: when the fictitious income assigned to the prematurely dead persons is equal to the welfare-neutral income, adjusted poverty measures are much larger than unadjusted poverty measures. But even for a given adjustment technique, there exist significant variations in the adjustment depending on the degree of poverty aversion $\alpha$ (2): distribution-sensitive measures lead to larger adjustments than headcount measures. Factors (3) and (4) are well illustrated by international comparisons. International differentials in the size of adjustment - in particular the opposition between Nordic and Eastern Europe - mirror both international differentials in income-related survival conditions and in the income distribution (including the level of the poverty line).

### 6 Concluding remarks

By mechanically reducing the proportion of poor persons in the population, income-differentiated mortality introduces some noise in the measurement of poverty. This leads to the Mortality Paradox: a deterioration of the survival conditions faced by the poor can generate a decline in the measured poverty. That reduction is puzzling, and is a mere consequence of the absence of the "missing poor" in the population on which poverty is measured, and of the neglect of premature death as a major aspect of poverty (Sen 1998).

This paper examined whether this puzzle for poverty measurement affects FGT poverty measures. Are FGT measures subject to the Mortality Paradox? If yes, are all subclasses of FGT measures equally subject to the Mortality
Paradox, whatever the degree of poverty aversion is?

To answer those questions, we developed a model of income mobility with risky lifetime to study how robust FGT measures are to variations in survival conditions. We showed that FGT old-age poverty measures do not, in general, satisfy the Robustness to Mortality Changes condition (RMC). Actually, it is only under some particular income mobility process - where all individuals face the same expected extent of poverty in case of survival to the old age - that FGT measures satisfy RMC.

Under general conditions, FGT measures do not satisfy RMC. This motivated us to propose an adjustment of FGT measures, by extending the lifetime income profiles of the prematurely dead, in line with Kanbur and Mukherjee (2007). Then, we identified conditions under which so-adjusted FGT measures satisfy RMC. Actually, adjusted FGT measures satisfy RMC when the assignment of fictitious income to the prematurely dead is similar to the pure income mobility process conditional on survival.

Finally, we showed, on the basis of data on old-age poverty in 11 European economies (2007), that the measured extent of old-age poverty varies strongly, depending on the particular treatment of the prematurely dead, that is, depending on how the "missing poor" are taken into account (or not) in the poverty measure. The effect of extending the lifetime income profiles varies with the fictitious incomes assigned to the prematurely dead, and, also, with the degree of aversion to poverty within FGT measures. The adjustment is lower for headcount measures than for measures taking the intensity of poverty into account. The size of the adjustment varies also across countries, depending on the income distribution, and on the severity of overmortality due to low income. Whereas Eastern European countries exhibit larger adjustments than Nordic European countries under headcount measures, both Eastern and Nordic countries exhibit large adjustments when the intensity of poverty is also taken into account.

All in all, our study illustrates that the interferences caused by income-differentiated mortality constitute a general problem for poverty measurement. Economies with large (unadjusted) poverty rates and strong overmortality for the poor are concerned by the Mortality Paradox. But more surprisingly, richer economies with little income-differentiated mortality are also subject to it. The reason is that, in their case, taking the "missing poor" and the "hidden poverty" into account creates a much bigger contrast with the standards of the surviving populations. Hence, even in rich economies, how one treats the prematurely dead affects the measured poverty.

7 References


Appendix

8.1 Proof of Proposition 1

The FGT poverty measure at the old age is equal to:

\[ \bar{P}_\alpha (y, n^1, \pi, \Lambda) = \frac{1}{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n^1_i \lambda_{ik}} \sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n^1_i \lambda_{ik} \left[ \frac{y_P - y_k}{y_P} \right] \alpha \]

Let us now compute the impact of a change in a survival rate \( \pi_m \), with \( m \leq K \).
Differentiating $\tilde{P}_\alpha (y, \mathbf{n}^1, \pi, \mathbf{A})$ with respect to $\pi_m$ yields:

$$\frac{\partial \tilde{P}_\alpha (y, \mathbf{n}^1, \pi, \mathbf{A})}{\partial \pi_m} = \sum_{k=1}^{P-1} n_m^1 \lambda_{mk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{i=1}^{K} \sum_{k=1}^{K} \pi_i n_i^1 \lambda_{ik} \right) \left( \sum_{i=1}^{K} \sum_{k=1}^{K} \pi_i n_i^1 \lambda_{ik} \right)^2 - \sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{k=1}^{K} n_m^1 \lambda_{mk} \right) \left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right)^2$$

The first term of the RHS is positive, while the second term is negative.

RMC requires: $\frac{\partial \tilde{P}_\alpha (y, \mathbf{n}^1, \pi, \mathbf{A})}{\partial \pi_m} = 0$. This is equivalent to:

$$\sum_{k=1}^{P-1} n_m^1 \lambda_{mk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right) = \sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{k=1}^{K} n_m^1 \lambda_{mk} \right)$$

That condition can be simplified to:

$$\frac{\sum_{k=1}^{P-1} n_m^1 \lambda_{mk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right)}{\left( \sum_{k=1}^{K} n_m^1 \lambda_{mk} \right)} = \frac{\sum_{k=1}^{P-1} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha}{\left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right)}$$

The LHS is the average extent of poverty for agents of type $m$ conditionally on survival, while the RHS is the average extent of poverty among all surviving individuals.

It is easy to find examples where that equality is not verified. To see this, take the simple case where $\alpha = 0$ and where $Y = \{y_1, y_2\}$. Assume also that the matrix $\mathbf{A}$ is equal to $\begin{pmatrix} 0.9 & 0.1 \\ 0.6 & 0.4 \end{pmatrix}$, that $\pi_1 = 0.1$ and $\pi_2 = 0.9$ and that $n_1^1 = n_2^1 = 1$. In that case, and considering a change in $\pi_1$, the above condition is:

$$\frac{n_1^1 \lambda_{11}}{n_1^1} = \frac{\pi_1 n_1^1 \lambda_{11} + \pi_2 n_2^1 \lambda_{21}}{\pi_1 n_1^1 \lambda_{11} + \pi_2 n_2^1 \lambda_{21} + \pi_1 n_1^2 \lambda_{12} + \pi_2 n_2^1 \lambda_{22}}$$

Given our numerical values that condition is not verified, since:

$$0.9 \neq 0.63$$

That simple counterexample suffices to show that $\tilde{P}_\alpha (y, \mathbf{n}^1, \pi, \mathbf{A})$ does not satisfy RMC.

### 8.2 Proof of Proposition 2

Assume that $\alpha = 0$. RMC requires that the ratio $\frac{n_1^1 + n_2^1 + \ldots + n_k^1}{n_1^1 + n_2^1 + \ldots + n_k^1}$ remains unchanged. The condition for constancy of the poverty ratio when $\pi_1$ varies is
now:

\[
\left( \pi_1 n_1^1 (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1}) + \pi_2 n_2^1 (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1}) \right) \\
\left( + \pi_K n_K^1 (\lambda_{K1} + \lambda_{K2} + \ldots + \lambda_{Kp-1}) \right) \\
\frac{\left( + \pi_K n_K^1 \right)}{\left( \pi_1 n_1^1 + \pi_2 n_2^1 + \ldots + \pi_K n_K^1 \right)}
\]

\[
\left( \pi_1 n_1^1 (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1}) + \pi_2 n_2^1 (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1}) \right) \\
\left( + \pi_K n_K^1 \right) \\
\frac{\left( + \pi_K n_K^1 \right)}{\left( \pi_1 n_1^1 + \pi_2 n_2^1 + \ldots + \pi_K n_K^1 \right)}
\]

Using the notation \( x_j = \pi_j n_j^1 \), that condition can be written as:

\[
x_2 (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1} - (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1})) \\
+ \ldots + x_K (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1} - (\lambda_{K1} + \lambda_{K2} + \ldots + \lambda_{Kp-1})) \\
= 0
\]

Consider now a variation in \( \pi_2 \). The condition is now:

\[
\left( \pi_1 n_1^1 (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1}) + \pi_2 n_2^1 (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1}) \right) \\
\left( + \pi_K n_K^1 \right) \\
\frac{\left( + \pi_K n_K^1 \right)}{\left( \pi_1 n_1^1 + \pi_2 n_2^1 + \ldots + \pi_K n_K^1 \right)}
\]

The condition can be simplified to:

\[
x_1 (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1} - (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1})) \\
+ \ldots + x_K (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1} - (\lambda_{K1} + \lambda_{K2} + \ldots + \lambda_{Kp-1})) \\
= 0
\]

Consider now a variation in \( \pi_K \). The condition is now:

\[
\left( \pi_1 n_1^1 (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1}) + \pi_2 n_2^1 (\lambda_{21} + \lambda_{22} + \ldots + \lambda_{2p-1}) \right) \\
\left( + \pi_K n_K^1 \right) \\
\frac{\left( + \pi_K n_K^1 \right)}{\left( \pi_1 n_1^1 + \pi_2 n_2^1 + \ldots + \pi_K n_K^1 \right)}
\]

The condition can be simplified to:

\[
x_1 (\lambda_{K1} + \lambda_{K2} + \ldots + \lambda_{Kp-1} - (\lambda_{11} + \lambda_{12} + \ldots + \lambda_{1p-1})) + \ldots \\
+ x_{K-1} (\lambda_{K1} + \lambda_{K2} + \ldots + \lambda_{Kp-1} - (\lambda_{K-11} + \lambda_{K-12} + \ldots + \lambda_{K-1p-1})) \\
= 0
\]
Taking all conditions together, we have:

\[
\begin{align*}
&x_2 (\lambda_{11} + \ldots + \lambda_{1p-1} - (\lambda_{21} + \ldots + \lambda_{2p-1})) \\
&+ x_K (\lambda_{11} + \ldots + \lambda_{1p-1} - (\lambda_{K1} + \ldots + \lambda_{KP-1})) = 0 \\
&x_1 (\lambda_{21} + \ldots + \lambda_{2p-1} - (\lambda_{11} + \ldots + \lambda_{1p-1})) \\
&+ x_K (\lambda_{21} + \ldots + \lambda_{2p-1} - (\lambda_{K1} + \ldots + \lambda_{KP-1})) = 0 \\
&\vdots \\
&x_1 (\lambda_{K1} + \ldots + \lambda_{KP-1} - (\lambda_{11} + \ldots + \lambda_{1p-1})) \\
&+ x_K (\lambda_{K1} + \ldots + \lambda_{KP-1} - (\lambda_{K1} + \ldots + \lambda_{KP-1})) = 0
\end{align*}
\]

This is true independently of the levels of \(x_1, x_2, \ldots, x_K\) if and only if:

\[
\lambda_{11} + \ldots + \lambda_{1p-1} = \lambda_{21} + \ldots + \lambda_{2p-1} = \ldots = \lambda_{K1} + \ldots + \lambda_{KP-1}
\]

This means that the probability of falling into poverty must be independent from the initial income level. To be certain, one can, without loss of generality, prove the last stage by considering a 3x3 case, where \(y_2 = y_p\). The conditions are:

\[
\begin{align*}
x_2 ((\lambda_{21} - \lambda_{11}) + (\lambda_{22} - \lambda_{12})) + x_3 ((\lambda_{31} - \lambda_{11}) + (\lambda_{32} - \lambda_{12})) &= 0 \\
x_1 ((\lambda_{21} - \lambda_{11}) + (\lambda_{22} - \lambda_{12})) + x_3 ((\lambda_{31} - \lambda_{11}) + (\lambda_{32} - \lambda_{12})) &= 0 \\
x_1 ((\lambda_{31} - \lambda_{11}) + (\lambda_{32} - \lambda_{12})) + x_2 ((\lambda_{31} - \lambda_{21}) + (\lambda_{32} - \lambda_{22})) &= 0
\end{align*}
\]

We can prove by reductio ad absurdum that we need \(\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda_{31} + \lambda_{32}\). Suppose that \(\lambda_{21} + \lambda_{22} > \lambda_{11} + \lambda_{12}\). Then by (1) we need also \(\lambda_{31} + \lambda_{32} < \lambda_{11} + \lambda_{12}\). Hence by transitivity \(\lambda_{31} + \lambda_{32} < \lambda_{21} + \lambda_{22}\). Moreover, if \(\lambda_{21} + \lambda_{22} > \lambda_{11} + \lambda_{12}\), then by (2) \(\lambda_{31} + \lambda_{32} < \lambda_{31} + \lambda_{32}\). But this is in contradiction with \(\lambda_{31} + \lambda_{32} < \lambda_{21} + \lambda_{22}\). The same argument holds for the case where \(\lambda_{21} + \lambda_{22} < \lambda_{11} + \lambda_{12}\). If true; this requires by (1) \(\lambda_{31} + \lambda_{32} > \lambda_{11} + \lambda_{12}\). But then by transitivity, we have \(\lambda_{21} + \lambda_{22} < \lambda_{31} + \lambda_{32}\). Moreover, by (2), we have \(\lambda_{21} + \lambda_{22} < \lambda_{31} + \lambda_{32}\). Again a contradiction is reached. The same kind of proof could be provided for the general \(K \times K\) case.

### 8.3 Proof of Proposition 3

Let us now relax the assumption \(\alpha = 0\). Obviously, the above results remain valid when the poverty line is fixed to \(y_2\), since in that case all poor persons have the same income level, and thus the same extent of poverty. Hence, in those cases, the condition for RMC is invariant to the extent of poverty, and thus invariant to \(\alpha\).

RMC requires that the ratio \(\frac{n_1^2 \left[ \frac{y_k - y_{k-1}}{y_k} \right]^\alpha + n_2^2 \left[ \frac{y_k - y_{k-1}}{y_k} \right]^\alpha + \ldots + n_{p-1}^2 \left[ \frac{y_p - y_{p-1}}{y_p} \right]^\alpha}{n_1 + n_2 + \ldots + n_K}\) remains unchanged.
The condition for constancy of the poverty ratio when \( \pi_1 \) varies is now:

\[
\left( \pi_1 n_1^1 + \pi_2 n_2^2 \sum_{K} n_K K \right) \left( \lambda_1 \left[ \frac{y_p - y_1}{y_p} \right]^\alpha + \lambda_2 \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{1P-1} \left[ \frac{y_p - y_{1P-1}}{y_p} \right]^\alpha \right) + \frac{\pi_1 n_1^1 \lambda_{11} \left[ \frac{y_p - y_1}{y_p} \right]^\alpha + \lambda_2 \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{1P-1} \left[ \frac{y_p - y_{1P-1}}{y_p} \right]^\alpha}{\pi_1 n_1^1 + \pi_2 n_2^2 + \ldots + \pi_K n_K K} 
\]

Using the notation \( x_j = \pi_j n_j^1 \), and denoting \( \left[ \frac{y_p - y_1}{y_p} \right]^\alpha \) by \( z_i \), that condition can be rewritten as:

\[
x_2 (\lambda_{11} z_1 + \lambda_{12} z_2 + \ldots + \lambda_{1P-1} z_{P-1}) + z_2 (\lambda_{11} z_1 + \lambda_{12} z_2 + \ldots + \lambda_{1P-1} z_{P-1}) = 0
\]

Consider now a variation in \( \pi_2 \). The condition is now:

\[
\left( \pi_1 n_1^1 + \pi_2 n_2^2 \sum_{K} n_K K \right) \left( \lambda_1 \left[ \frac{y_p - y_1}{y_p} \right]^\alpha + \lambda_2 \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{1P-1} \left[ \frac{y_p - y_{1P-1}}{y_p} \right]^\alpha \right) + \frac{\pi_1 n_1^1 \lambda_{11} \left[ \frac{y_p - y_1}{y_p} \right]^\alpha + \lambda_2 \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{1P-1} \left[ \frac{y_p - y_{1P-1}}{y_p} \right]^\alpha}{\pi_1 n_1^1 + \pi_2 n_2^2 + \ldots + \pi_K n_K K} 
\]

Using the notation \( x_j = \pi_j n_j^1 \), and denoting \( \left[ \frac{y_p - y_1}{y_p} \right]^\alpha \) by \( z_i \), that condition can be rewritten as:

\[
x_1 (\lambda_{11} z_1 + \lambda_{12} z_2 + \ldots + \lambda_{1P-1} z_{P-1}) + z_2 (\lambda_{11} z_1 + \lambda_{12} z_2 + \ldots + \lambda_{1P-1} z_{P-1}) = 0
\]

32
Consider now a variation in $\pi_K$. The condition is now:

$$
\begin{align*}
\left( \pi_1 n_1^1 \left( \lambda_{11} \left[ \frac{y_p - y_i}{y_p} \right]^\alpha + \lambda_{12} \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{1p-1} \left[ \frac{y_p - y_{p-1}}{y_p} \right]^\alpha \right) \right. \\
+ \pi_2 n_2^2 \left( \lambda_{21} \left[ \frac{y_p - y_i}{y_p} \right]^\alpha + \lambda_{22} \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{2p-1} \left[ \frac{y_p - y_{p-1}}{y_p} \right]^\alpha \right) \\
+ \ldots + \pi_K n_K^K \left( \lambda_{K1} \left[ \frac{y_p - y_i}{y_p} \right]^\alpha + \lambda_{K2} \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{Kp-1} \left[ \frac{y_p - y_{p-1}}{y_p} \right]^\alpha \right) \\
\left. + \pi_{K'} n_{K'}^K \left( \lambda_{K'1} \left[ \frac{y_p - y_i}{y_p} \right]^\alpha + \lambda_{K'2} \left[ \frac{y_p - y_2}{y_p} \right]^\alpha + \ldots + \lambda_{Kp-1} \left[ \frac{y_p - y_{p-1}}{y_p} \right]^\alpha \right) \right) \\
&= \frac{1}{(\pi_1 n_1^1 + \pi_2 n_2^2 + \ldots + \pi_K n_K^K)}
\end{align*}
$$

Using the notation $x_j \equiv \pi_j n_j^1$, and denoting $\left[ \frac{y_p - y_i}{y_p} \right]^\alpha$ by $z_i$, that condition can be simplified to:

$$
\begin{align*}
x_1 \left( \lambda_{K1} z_1 + \lambda_{K2} z_2 + \ldots + \lambda_{Kp-1} z_{p-1} - (\lambda_{11} z_1 + \lambda_{12} z_2 + \ldots + \lambda_{1p-1} z_{p-1}) \right) \\
+ x_2 \left( \lambda_{K1} z_1 + \lambda_{K2} z_2 + \ldots + \lambda_{Kp-1} z_{p-1} - (\lambda_{21} z_1 + \lambda_{22} z_2 + \ldots + \lambda_{2p-1} z_{p-1}) \right) + \ldots \\
= 0
\end{align*}
$$

Taking all conditions together, and denoting $\left[ \frac{y_p - y_i}{y_p} \right]^\alpha$ by $z_i$, we have:

$$
\begin{align*}
\begin{bmatrix}
  x_2 \left( (\lambda_{21} - \lambda_{11}) z_1 + \ldots + (\lambda_{2k-1} - \lambda_{1k-1}) z_{k-1} \right) \\
  \vdots \\
  x_1 \left( (\lambda_{K1} - \lambda_{11}) z_1 + \ldots + (\lambda_{Kk-1} - \lambda_{1k-1}) z_{k-1} \right) \\
  \vdots \\
  x_1 \left( (\lambda_{K1} - \lambda_{21}) z_1 + \ldots + (\lambda_{Kk-1} - \lambda_{2k-1}) z_{k-1} \right) \\
  + x_2 \left( (\lambda_{K1} - \lambda_{21}) z_1 + \ldots + (\lambda_{Kk-1} - \lambda_{2k-1}) z_{k-1} \right) + \ldots
\end{bmatrix}
&= 0
\end{align*}
$$

Given that $z_i > 0 \forall i$, those conditions can only be verified for any values of $x_1, x_2, \ldots, x_K$ when, for all $i$:

$$
\lambda_{i1} z_1 + \lambda_{i2} z_2 + \ldots + \lambda_{ik-1} z_{k-1} = \lambda_{j1} z_1 + \lambda_{j2} z_2 + \ldots + \lambda_{jk-1} z_{k-1}
$$

That condition states that the expected extent of poverty must be equal whatever the initial income level is.

To prove that claim, let us turn to the example of the 3x3 case with $y_p = y_2$.

In that case, the conditions are:

$$
\begin{align*}
x_2 \left( (\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 \right) + x_3 \left( (\lambda_{31} - \lambda_{11}) z_1 + (\lambda_{32} - \lambda_{12}) z_2 \right) &= 0 \\
x_1 \left( (\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 \right) + x_2 \left( (\lambda_{21} - \lambda_{31}) z_1 + (\lambda_{22} - \lambda_{32}) z_2 \right) &= 0 \\
x_1 \left( (\lambda_{31} - \lambda_{11}) z_1 + (\lambda_{32} - \lambda_{12}) z_2 \right) + x_2 \left( (\lambda_{31} - \lambda_{21}) z_1 + (\lambda_{32} - \lambda_{22}) z_2 \right) &= 0
\end{align*}
$$
Let us show that we need: $\lambda_{21} z_1 + \lambda_{22} z_2 = \lambda_{11} z_1 + \lambda_{12} z_2 = \lambda_{31} z_1 + \lambda_{32} z_2$. To show that, let us add the first two conditions. We obtain:

$$(x_2 + x_1) (\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 + x_3 (\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 = 0$$

That condition can be rewritten as:

$$(x_2 + x_1 + x_3) (\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 = 0$$

Given that $x_2 + x_1 + x_3 > 0$, that equality is achieved only if $(\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 = 0$. This is equivalent to $\lambda_{21} z_1 + \lambda_{22} z_2 = \lambda_{11} z_1 + \lambda_{12} z_2$. But if $(\lambda_{21} - \lambda_{11}) z_1 + (\lambda_{22} - \lambda_{12}) z_2 = 0$, it must also be the case that $(\lambda_{31} - \lambda_{11}) z_1 + (\lambda_{32} - \lambda_{12}) z_2$, which is equivalent to $\lambda_{21} z_1 + \lambda_{22} z_2 = \lambda_{11} z_1 + \lambda_{12} z_2$. Hence by transitivity of equality, we have $\lambda_{21} z_1 + \lambda_{22} z_2 = \lambda_{11} z_1 + \lambda_{12} z_2 = \lambda_{31} z_1 + \lambda_{32} z_2$. The same rationale apply to the $K \times K$ case.

### 8.4 Proof of Proposition 4

The FGT poverty measure at the old age is equal to:

$$\hat{P}_\alpha (y, n^1, \pi, A) = \frac{1}{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik}} \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha$$

Let us now compute the impact of a change in a survival rate $\pi_m$, with $m \leq K$.

Differentiating $\hat{P}_\alpha (y, n^1, \pi, A)$ with respect to $\pi_m$ yields:

$$\frac{\partial \hat{P}_\alpha (y, n^1, \pi, A)}{\partial \pi_m} = \frac{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{mk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right) \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik}^2}{\left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{mk} \right) \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik}^2}$$

We thus have $\frac{\partial \hat{P}_\alpha (y, n^1, \pi, A)}{\partial \pi_m} \geq 0$ when

$$\frac{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{mk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha}{\left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{mk} \right)} \leq \frac{\sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha}{\left( \sum_{k=1}^{K} \sum_{i=1}^{K} \pi_i n_i^1 \lambda_{ik} \right)}.$$

### 8.5 Proof of Proposition 5

The adjusted FGT poverty measure at the old age is equal to:

$$\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A) = \frac{1}{\sum_{j=1}^{K} n_j^1} \left[ \sum_{j=1}^{K} \pi_j n_j^1 \left( \sum_{k=1}^{P-1} \lambda_{jk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \right) + \sum_{j=1}^{K} (1 - \pi_j) n_j^1 \left( \sum_{k=1}^{P-1} \sigma_{jk} \left[ \frac{y_p - y_k}{y_p} \right]^\alpha \right) \right]$$

Let us now compute the impact of a change in a survival rate $\pi_m$, with $m \leq K$. 34
Differentiating $\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)$ with respect to $\pi_m$ yields:

$$
\frac{\partial \hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)}{\partial \pi_m} = \frac{1}{\sum_{j=1}^{K} n_j} \left[ n \left( \sum_{k=1}^{P-1} \lambda_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) - n_1 \left( \sum_{k=1}^{P-1} \sigma_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) \right]
$$

We have $\frac{\partial \hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)}{\partial \pi_m} = 0$ if and only if:

$$
\sum_{k=1}^{P-1} \lambda_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha = \sum_{k=1}^{P-1} \sigma_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha
$$

This equality is satisfied if and only if $\lambda_{mk} = \sigma_{mk}$ for all $k$. That is, for a change in a survival rate $\pi_m$, we need that the $m$-rows of the $\Lambda$ and $\Sigma$ matrix are identical. Hence, if we want $\frac{\partial \hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)}{\partial \pi_m} = 0$ for any $\pi_m$, it follows that all rows must be identical in matrix $\Lambda$ and $\Sigma$, which implies the equality of those matrices.

### 8.6 Proof of Proposition 6

The adjusted FGT poverty measure at the old age is equal to:

$$
\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A) = \frac{1}{\sum_{j=1}^{K} n_j} \left[ \sum_{j=1}^{K} \pi_j n_j \left( \sum_{k=1}^{P-1} \lambda_{jk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) + \sum_{j=1}^{K} \left( 1 - \pi_j \right) n_j \left( \sum_{k=1}^{P-1} \sigma_{jk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) \right]
$$

Differentiating $\hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)$ with respect to $\pi_m$ yields:

$$
\frac{\partial \hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)}{\partial \pi_m} = \frac{1}{\sum_{j=1}^{K} n_j} \left[ \sum_{j=1}^{K} \pi_j n_j \left( \sum_{k=1}^{P-1} \lambda_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) - n_1 \left( \sum_{k=1}^{P-1} \sigma_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \right) \right]
$$

We thus have $\frac{\partial \hat{P}_{\alpha, \Sigma} (y, n^1, \pi, A)}{\partial \pi_m} \geq 0$ when $\sum_{k=1}^{P-1} \lambda_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha \geq \sum_{k=1}^{P-1} \sigma_{mk} \left[ \frac{y^p - y_k}{y^p} \right]^\alpha$. 

35