Fear of miscoordination and the robustness of cooperation in dynamic global games with exit

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Abstract

This paper develops a framework to assess the impact of miscoordination fear on agents' ability to sustain dynamic cooperation. Building on theoretical insights from Carlsson and van Damme (1993), it explores the effect of small amounts of private information on a class of dynamic cooperation games with exit. It is shown that lack of common knowledge creates a fear of miscoordination which pushes players away from the full-information Pareto frontier. Unlike in one-shot two-by-two games, the global games information structure does not yield equilibrium uniqueness, however, by making it harder to coordinate, it does reduce the range of equilibria and gives bite to the notion of local dominance solvability. Finally, the paper provides a simple criterion for the robustness of cooperation to miscoordination fear, and shows it can yield predictions that are qualitatively different from those obtained by focusing on Pareto efficient equilibria under full information.

keywords: cooperation, strategic risk, miscoordination risk, global games, dynamic games, exit games, rationalizability, local strong rationalizability, local dominance solvability. jel classification codes: C72, C73

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1 Introduction

The folk theorem for repeated games teaches us that even though short run incentives may lead to suboptimal outcomes, continued interaction can allow players to sustain efficient cooperation when promises of future benefits are large enough. One notable property of repeated games is that they admit a large set of equilibria. As a consequence, much of the applied work using dynamic cooperation games focuses on Pareto optimal equilibria for the purpose of deriving comparative statics. One may however worry that using Pareto efficiency as a selection criterion overestimates the players' ability to coordinate. Indeed, there exists a substantial experimental literature on coordination failure in one-shot coordination games[1] indicating that empirically, Pareto efficiency is not a fully satisfying selection criterion and that risk-dominance, in the sense of Harsanyi and Selten (1988), is often a better predictor of experimental outcomes. The work of Carlsson and van Damme (1993) sheds theoretical light on these empirical findings by showing that the Pareto efficiency criterion relies heavily on common knowledge and that for a natural family of small departures from full information, the risk-dominant action will be the unique rationalizable outcome.

This paper uses the information structure of Carlsson and van Damme (1993) to model miscoordination risk in a class of games with exit that replicates much of the intuition underlying repeated games, while being simple enough to study the effects of small amounts of private information. The exit games considered are two-player games with infinite horizon and positive discount rate, in which players decide each period whether they want to stay or exit. Under the global games information structure, in each period t, players' payoffs are affected by an i.i.d. state of the world w_t, on which players make noisy observations.

The paper's main result is a characterization of rationalizable strategies as players' signals become arbitrarily precise. Although the likelihood of miscoordination becomes vanishingly small as signals get more precise, the ghost of miscoordination is enough to push players away from the Pareto efficient frontier. The set of surviving equilibria – which are inter-

[1] See for instance Cooper, DeJong, Forsythe, and Ross (1990) or Battalio, Samuelson, and Van Huyck (2001).

preted as those equilibria that are robust to miscoordination risk – depends both on the magnitude of miscoordination losses and on the distribution of states of the world w_t. Unlike the case of one-shot coordination games studied by Carlsson and van Damme (1993) and Frankel, Morris and Pauzner (2003), the global games information structure does not yield unique selection in infinite horizon games. However, the dominance solvability of static global games does carry over in the weaker form of local dominance solvability. As players' signals get arbitrarily precise it is possible to characterize local dominance solvability explicitly for a focal class of equilibria. This allows us to identify equilibria that are robust to strategic uncertainty in addition to being robust to miscoordination fear. Finally, the paper provides a simple criterion for cooperation to be robust in games with approximately constant payoffs, and shows how taking into account the impact of miscoordination fear on cooperation can yield predictions that are qualitatively different from those obtained by focusing on full-information Pareto-efficient equilibria. This is illustrated in an applied model which investigates the question of how wealth affects people's ability to cooperate.

From a methodological perspective, the paper shows how the Abreu, Pearce, and Stacchetti (1990) approach to dynamic games can be used to study the impact of a global games information structure in a broader set of circumstances than one-shot coordination games. The approach has two steps: the first step is to recognize that one-shot action profiles in a perfect Bayesian equilibrium must be Nash equilibria of an augmented one-shot game incorporating continuation values; the second step is to apply global games selection results that hold uniformly over the family of possible augmented games, and derive a fixed point equation for possible continuation values. This approach can accommodate the introduction of an observable Markovian state variable and auto-correlated states of the world.

This paper contributes to the literature on the effect of private information in infinite horizon cooperation games. Since Green and Porter (1984), Abreu, Pearce, and Stacchetti (1986), and Radner, Myerson, and Maskin (1986), much of this literature[2] has focused on the issue of imperfect monitoring of other players' actions and on the amount of inefficient punishment that must occur on an equilibrium path. In this paper however, actions are

[2]See for instance Fudenberg, Levine, and Maskin (1994), Compte (1998), or Kandori (2003)

observable. It is the players' assessment of the state of the world that is private information. Interestingly, this form of private information prevents the players from attaining the full-information Pareto frontier even as the players' assessments become arbitrarily precise.

This paper also fits in the growing literature on dynamic global games. Much of this literature however avoids intertemporal incentives. Levin (2001) studies a global game with overlapping generations. Chamley (1999), Morris and Shin (1999), and Angeletos, Hellwig and Pavan (2006) consider various models of dynamic regime change, but assume a discount rate equal to zero, and focus on the endogenous information dynamics that result from agents observing others' actions and new signals of the state of the world. In this sense, these models are models of dynamic herds rather than models of repeated interaction. Closer to the topic of this paper is Giannitsarou and Toxvaerd (2003), which extends results from Frankel, Morris, and Pauzner (2003) and discusses equilibrium uniqueness in a family of finite, dynamic, supermodular global games. From the perspective of the present paper, which is concerned with infinite horizon games, their uniqueness result is akin to equilibrium uniqueness in a finitely repeated dominance solvable game. Finally, in two papers that do not rely on private noisy signals as the source of miscoordination, but carry a very similar intuition, Burdzy, Frankel, and Pauzner (2001), and Frankel and Pauzner (2000) obtain full selection for a model in which players' actions have inertia and fundamentals follow a random walk. However, their unique selection result hinges strongly on the random walk assumption and does not rule out multiplicity in settings where fundamentals follow different processes.

The paper is organized as follows. Section 2 presents the setup. Section 3 is the core of the paper and proves selection and local dominance solvability results. It illustrates how tools developed for one-shot global games can be applied to study perfect Bayesian equilibria in dynamic games. Section 4 applies the results of Section 3 and makes the case that the model of miscoordination fear proposed in this paper is practical and can yield qualitatively new comparative statics. Section 5 concludes. Proofs are contained in Appendix A, unless mentioned otherwise. The results of Section 3 are extended to non-stationary games in Appendix B.

2 Stationary exit games

2.1 The setup

Consider an infinite-horizon game with discrete time $t \in \{1, \dots, +\infty\}$ and two players $i \in \{1, 2\}$ with discount rate β. The two players act simultaneously and can take two actions: $\mathcal{A} = \{Stay, Exit\}$. Payoffs are indexed by a state of the world $w_t \in \mathbb{R}$, which is independently drawn each period. Given the state of the world w_t, player i expects flow payoffs,

	S	E
S	$g^i(w_t)$	$W^i_{12}(w_t)$
E	$W^i_{21}(w_t)$	$W^i_{22}(w_t)$

where i is the row player. States of the world $\{w_t\}_{t\in\{1,\dots,\infty\}}$ form an i.i.d. sequence of real numbers drawn from a distribution with density f, c.d.f. F and convex support $I \subseteq \mathbb{R}$. All payoffs, $g^i, W^i_{12}, W^i_{21}, W^i_{22}$ are continuous in w_t.

At time t, the state of the world w_t is unknown, but each player gets a signal $x_{i,t}$ of the form

$$x_{i,t} = w_t + \sigma\varepsilon_{i,t}$$

where $\{\varepsilon_{i,t}\}_{i\in\{1,2\},\, t\geq 1}$ is an i.i.d. sequence of independent random variables taking values in the interval $[-1, 1]$. For simplicity w_t is ex-post observable[3].

Whenever there is an exit, the game ends and players get a continuation value equal to zero. This is without loss of generality since termination payoffs can be included in the flow-payoffs upon exit W^i_{12}, W^i_{21} and W^i_{22}. For all $\sigma \geq 0$, let Γ_σ denote this dynamic game with imperfect information. Note that Γ_0 corresponds to the game with full information. The paper is concerned with equilibria of Γ_σ with σ strictly positive but arbitrarily small.

[3]Note that the analysis that follows would hold if players' final payoffs were shifted by some idiosyncratic noise $\eta_{i,t}$ independent of all other random variables and with zero expectation.

2.2 Example: a partnership game

As a benchmark, consider the following – extremely simple – partnership game. Flow payoffs are symmetric and given by,

	S	E
S	w_t	$w_t - C + \beta V_E$
E	$b + V_E$	V_E

where payoffs are given for the row player only, and $C > b \geq 0$. Parameter w_t is the expected return from putting effort in the partnership at time t; C represents the diminished value of being in the partnership when the other player walks out; parameter b (which can be set to 0) represents a potential benefit from cheating on a cooperating partner; and V_E is the present value of the players' constant outside option. States of the world w_t are drawn from a distribution with density f and support $\mathbb{R}$. We assume that $\mathbf{E}|w_t| < \infty$ and $V_E > 0$.

As a benchmark, let us study subgame perfect equilibria under full information. Whenever $w_t \leq (1-\beta)V_E + C$, playing (E, E) is a possible equilibrium outcome. Similarly, there exists a lowest value $\underline{w}$ of w_t for which (S, S) can be an equilibrium play. This cooperation threshold is associated with the greatest equilibrium continuation value $\overline{V}$. The following equations characterize $\overline{V}$ and $\underline{w}$:

$$\underline{w} + \beta\overline{V} = b + V_E \tag{1}$$

$$\overline{V} = \mathbf{E}\left[(w_t + \beta\overline{V})\mathbf{1}_{w_t > \underline{w}}\right] + F(\underline{w})V_E. \tag{2}$$

Whenever w_t belongs to $[\underline{w}, (1-\beta)V_E + C]$, any symmetric pair of actions is an equilibrium play and any pair of actions is rationalizable. In fact, within these bounds, any symmetric pair of actions can be an outcome of a Markovian equilibrium, and hence, any action is rationalizable by a Markovian strategy. When w_t is greater than $(1-\beta)V_E + C$, staying is the dominant action. When w_t is smaller than $\underline{w}$, exit is the dominant action.

Under full information, the criterion of Pareto efficiency would imply that players coordinate on using $\underline{w}$ as their threshold for cooperation, independently of C, which does not enter equations (1) and (2). Is this prediction robust when players' assessments of w_t are private? If not, what equilibria are robust to such a departure from common knowledge? How do these robust equilibria move with respect to C? Section 3 develops tools to answer such questions for a variety of games.

2.3 Assumptions

To exploit existing results on one-shot global games, we make a few assumptions which essentially ensure that the assumptions of Carlsson and van Damme (1993) hold for the family of one-shot stage games augmented with the players' potential continuation values. While it is possible to find weaker conditions under which the results of Section 3 will hold, the assumptions given here have the advantage that they can be checked in a straightforward way from primitives.

Assumption 1 (boundedness) *Let m_i and M_i respectively denote the min-max and maximum values of player i in the full information game Γ_0. Both m_i and M_i are finite.*

This assumption is typically unrestrictive but is still important given that in many natural examples, w_t will have unbounded support. The min-max value m_i will appear again in Assumptions 4 and 5, while M_i will be used in Assumption 2.

In the partnership example of Section 2.2, we have $m_i = m_{-i} = \mathbf{E}\max\{V_E, w_t - C + \beta V_E\}$ and $M_i = M_{-i} = M$ where M satisfies $M = \mathbf{E}\max\{w_t + \beta M, b + V_E\}$.

Assumption 2 (dominance) *There exist $\underline{w}$ and $\overline{w}$ such that for all $i \in \{1, 2\}$,*

$$g^i(\underline{w}) + \beta M_i - W^i_{21}(\underline{w}) < 0 \quad \textit{and} \quad W^i_{12}(\underline{w}) - W^i_{22}(\underline{w}) < 0 \quad \textit{(Exit dominant)}$$
$$\textit{and} \quad W^i_{12}(\overline{w}) - W^i_{22}(\overline{w}) > 0 \quad \textit{and} \quad g^i(\overline{w}) + \beta m_i - W^i_{21}(\overline{w}) > 0 \quad \textit{(Staying dominant).}$$

Assumption 3 (increasing differences in the state of the world) *For all $i \in \{1, 2\}$, $g^i(w_t) - W^i_{21}(w_t)$ and $W^i_{12}(w_t) - W^i_{22}(w_t)$ are strictly increasing over $w_t \in [\underline{w}, \overline{w}]$, with a slope* *greater than some real number $r > 0$.*

Note that the assumption that $W^i_{12} - W^i_{22}$ is strictly increasing in the state of the world may rule out examples in which staying yields a constant zero payoff when the other player exits.

Definition 1 *For any functions $V_i, V_{-i} : \mathbb{R} \to \mathbb{R}$, let $G(V_i, V_{-i}, w_t)$ denote the full information one-shot game*

	S	E
S	$g^i(w_t) + \beta V_i(w_t)$	$W^i_{12}(w_t)$
E	$W^i_{21}(w_t)$	$W^i_{22}(w_t)$

where i is the row player. Let $\Psi_\sigma(V_i, V_{-i})$ denote the corresponding one-shot global game in which players observe signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 4 (equilibrium symmetry) *For all states of the world w_t, $G(m_i, m_{-i}, w_t)$ has a pure strategy Nash equilibrium and all pure equilibria belong to $\{(S,S),(E,E)\}$.*

Recall that m_i is player i's min-max value in the game with full information Γ_0. If Assumption 4 is satisfied, then for any function $\mathbf{V} = (V_i, V_{-i})$ taking values in $[m_i, +\infty) \times [m_{-i}, +\infty)$, the game $G(\mathbf{V}, w_t)$ also has a pure strategy equilibrium, and its pure equilibria also belong to $\{(S,S),(E,E)\}$. Indeed, whether (E,E) is an equilibrium or not does not depend on the value of (V_i, V_{-i}), and if (S,S) is an equilibrium when $\mathbf{V} = (m_i, m_{-i})$, then it is also an equilibrium when the continuation values of player i and $-i$ are respectively greater than m_i and m_{-i}.

Note that when Assumptions 2 and 3 hold, Assumption 4 is equivalent to the condition that for all $i \in \{1,2\}$, at the state w_i such that $W^i_{12}(w_i) - W^i_{22}(w_i) = 0$, we have $g^i(w_i) + \beta m_i - W^i_{21}(w_i) > 0$ and $g^{-i}(w_i) + \beta m_{-i} - W^{-i}_{21}(w_i) > 0$. Assumption 4 holds for the partnership game since $C > b$.

Together, Assumptions 3 and 4 insure that at any state of the world w and for any pair of individually rational continuation values $\mathbf{V}$, either (S,S) or (E,E) is the risk-dominant equilibrium of $G(\mathbf{V}, w)$, and that there is a unique risk-dominant threshold $x^{RD}(\mathbf{V})$ —

(S, S) being risk-dominant above this threshold and (E, E) being risk-dominant below. In conjunction with Assumption 2 this is in fact the unidimensional version of Carlsson and van Damme's assumption that states of the world should be connected to dominance regions by a path that is entirely contained in the risk-dominance region of one of the equilibria.

Definition 2 *For any function $V : \mathbb{R} \to \mathbb{R}$, and $w \in \mathbb{R}$, define $A_i(V, w)$ and $B_i(w)$ by,*

$$A_i(V, w) = g^i(w) + \beta V(w) - W^i_{12}(w) \quad \textit{and} \quad B_i(w) = W^i_{21}(w) - W^i_{22}(w).$$

Assumption 5 (staying is good) *For all players $i \in \{1, 2\}$ and all states of the world $w \in [\underline{w}, \overline{w}]$, $A_i(m_i, w) \geq 0$ and $B_i(w) \geq 0$.*

Recall that $[\underline{w}, \overline{w}]$ corresponds to states of the world where there need not be a dominant action. Assumption 5 is restrictive but not unreasonable: it means that under full information, at a state $w \in [\underline{w}, \overline{w}]$ with no clearly dominant action, player i is weakly better off whenever player $-i$ stays, independently of her own action.

The partnership game of Section 2.2 satisfies this assumption since for all $w \in \mathbb{R}$, $A_i(m_i, w) = C + \beta(m_i - V_E) > 0$ and $B_i(w) = b \geq 0$.

2.4 Solution concepts

Because of exit, at any decision point, a history $h_{i,t}$ is characterized by a sequence of past signals and past outcomes: $h_{i,t} \equiv \{x_{i,1}, \dots, x_{i,t} \ ; \ w_{i,1}, \dots, w_{i,t-1}\}$. Let $\mathcal{H}$ denote the set of all such sequences. A pure strategy is a mapping $s : \mathcal{H} \mapsto \{S, E\}$. Denote by Ω the set of pure strategies. For any set of strategies $S \subset \Omega$, let $\Delta(S)$ denote the set of probability distributions over S that have a countable support. The two main solution concepts we will be using are *perfect Bayesian equilibrium* and *sequential rationalizability.* To define these concepts formally, it is convenient to denote by $h^0_{i,t} \equiv \{x_{i,1}, \dots, x_{i,t-1} \ ; \ w_{i,1}, \dots, w_{i,t-1}\}$ the histories before players receive period t's signal but after actions of period $t-1$ have been taken. A strategy s_{-i} of player $-i$, conditional on the history $h^0_{-i,t}$ having been observed, will be denoted $s_{-i|h^0_{-i,t}}$. A conditional strategy $s_{-i|h^0_{-i,t}}$ and player i's conditional belief $\mu_{|h^0_{i,t}}$ over

$h^0_{-i,t}$ induce a mixed strategy denoted by $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$. Player i's sequential best-response correspondence, denoted by $BR_{i,\sigma}$, is defined as follows.

Definition 3 (sequential best-response) *$\forall\, s_{-i} \in \Omega$, $s_i \in BR_{i,\sigma}(s_{-i})$ if and only if:*

(i) At any history $h^0_{i,t}$ that is attainable given s_{-i} and s_i, the conditional strategy $s_{i|h^0_{i,t}}$ is a best-reply of player i to the mixed strategy $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$, where conditional beliefs $\mu_{|h^0_{i,t}}$ over $h^0_{-i,t}$ are obtained by Bayesian updating;

(ii) At any history $h^0_{i,t}$ that is not attainable given s_{-i} and s_i, $s_{i|h^0_{i,t}}$ is a best-reply of player i to a mixed strategy $(s_{-i|h^0_{-i,t}}, \mu_{|h^0_{i,t}})$ for some *(any) conditional beliefs $\mu_{|h^0_{i,t}}$ over $h^0_{-i,t}$.*

With this definition of sequential best-response, a strategy s_i of player i is associated with a perfect Bayesian equilibrium of Γ_σ if and only if, $s_i \in BR_{i,\sigma} \circ BR_{-i,\sigma}(s_i)$. Sequential rationalizability is defined as follows.

Definition 4 (sequential rationalizability) *A strategy s_i belongs to the set of sequentially rationalizable strategies R_i of player i if and only if*

$$s_i \in \bigcap_{n \in \mathbb{N}} (BR^\Delta_{i,\sigma} \circ BR^\Delta_{-i,\sigma})^n(\Omega)$$

where $BR^\Delta_{i,\sigma} \equiv BR_{i,\sigma} \circ \Delta$.

Given strategies s_i, s_{-i} and beliefs upon unattainable histories, let $V_i(h_{i,t})$ denote the value player i expects from playing the game at history $h_{i,t}$. Pairs of value functions will be denoted $\mathbf{V} \equiv (V_i, V_{-i})$.

3 Selection and local dominance solvability

The first class of results presented in this section aims at characterizing the extent to which lack of common knowledge and the fear of miscoordination prevent players from achieving

Pareto efficient levels of cooperation. It is shown that payoffs upon miscoordination influence equilibrium selection, although in equilibrium, as σ goes to 0, miscoordination happens with a vanishing probability. Theorem 2 characterizes the limit set of equilibrium values explicitly. Section 4.1 applies these results to the partnership game introduced in Section 2.2 and derives simple comparative statics that do not hold when focusing on Pareto efficient equilibria under full information.

The second class of results given in this section explores how the dominance solvability result of Carlsson and van Damme (1993) extends to dynamic games. Since dynamic global games can admit multiple equilibria, it would seem that these results do not carry over. However, Section 3.4 shows that the global games structure gives bite to the notion of local dominance solvability, extensively discussed in Guesnerie (2002). Theorem 3 shows that for the class of exit games defined in Section 2, local dominance solvability, which is a high dimensional property of sets of strategies, is asymptotically characterized by the stability of the fixed points of an easily computable, increasing mapping from $\mathbb{R}$ to $\mathbb{R}$. This will allow us to discuss the issue of robustness of equilibria to strategic uncertainty.

3.1 General methodology

A useful methodological insight of this paper is to recognize that tools from equilibrium selection in one-shot global games can be exploited to study the impact of a global games information structure in dynamic games. Using the dynamic programming approach to dynamic games developed in Abreu, Pearce, and Stacchetti (1990), actions prescribed by a perfect Bayesian equilibrium of a dynamic game must be outcomes of a Nash equilibrium in the Bayesian one-shot game that incorporates the players' continuation values. The idea is to apply global games selection results to a family of such augmented games in order to characterize the equilibrium continuation values of the dynamic game.

The main difficulty is that the selection results of Carlsson and van Damme (1993) only hold pointwise – they take the payoff structure as given – while selection results will need to hold uniformly[4] to apply dynamic programming techniques. For this reason, the present paper draws on results from Chassang (2006) which show that selection does happen uniformly over equicontinuous[5] families of two-by-two games satisfying the assumptions of Carlsson and van Damme (1993). Mostly, we will use the following implication of Theorems 2, 3 and 4 of Chassang (2006).

Lemma 1 (uniform selection) *For any compact subset $\mathcal{V} \subset \mathbb{R}^2$, consider the family of one-shot global games $\Psi_\sigma(\mathbf{V})$ indexed by $\mathbf{V} \in \mathcal{V}$. If for all $\mathbf{V} \in \mathcal{V}$ the full information one-shot game $G(\mathbf{V}, w)$ has pure equilibria which are all symmetric and admits dominance regions with respect to w, then under Assumptions 2 and 3*

(i) There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, all one-shot global games $\Psi_\sigma(\mathbf{V})$, indexed by values $\mathbf{V} \in \mathcal{V}$, have a unique rationalizable equilibrium;

(ii) This equilibrium takes a threshold form[6] with thresholds denoted by $\mathbf{x}^*_\sigma(\mathbf{V}) \in \mathbb{R}^2$. The mapping $\mathbf{x}^*_\sigma(\cdot)$ is continuous over $\mathcal{V}$;

(iii) As σ goes to 0, each component of $\mathbf{x}^*_\sigma(\mathbf{V})$ converges uniformly over $V \in \mathcal{V}$ to the risk-dominance threshold of $\Psi_0(\mathbf{V})$, denoted by $x^{RD}(\mathbf{V})$.

The analysis will proceed as follows. Section 3.2 shows that for an appropriate order over strategies, the game Γ_σ exhibits a restricted form of monotone best response which suffices to show that the set of sequentially rationalizable strategies is bounded by extreme Markovian equilibria. Section 3.3 characterizes the continuation values of Markovian equilibria by iteratively applying selection results on one-shot global games to families of augmented stage games. Section 3.4 shows how the dominance solvability of one-shot global games can be used to characterize the local dominance solvability of equilibria of the dynamic game Γ_σ.

[4]The reason for this will become clear in Section 3.3.

[5]More precisely, families of games whose associated family of payoff functions is equicontinuous.

[6]A strategy s_i in game $\Psi_\sigma(\mathbf{V})$ takes a threshold form if and only if there exists $x \in \mathbb{R}$ such that almost surely, $s_i(x_i) = S$ if and only if $x_i \geq x$.

3.2 Monotone best response and rationalizability

This section exploits assumptions of Section 2.3 and the exit game structure to prove simplifying structural properties on Γ_σ. In particular, it shows that for σ small enough, the set of rationalizable strategies of Γ_σ is bounded by extreme Markovian equilibria.

Definition 5 (Markovian strategies) *A strategy s_i is said to be Markovian if $s_i(h_{i,t})$ depends only on player i's current signal, $x_{i,t}$.*

A Markovian strategy s_i is said to take a threshold form if there exists a constant value x such that for almost all $x_{i,t} \geq x$, s_i prescribes player i to stay, and for almost all $x_{i,t} < x$, s_i prescribes player i to exit. The threshold of a threshold form strategy s will be denoted x_s and a strategy of threshold x will be denoted s_x.

Definition 6 *We define a partial order $\preceq$ on pure strategies by*

$$s' \preceq s \iff \{a.s.\, \forall h \in \mathcal{H},\ s'(h) = Stay \Rightarrow s(h) = Stay\}\,.$$

In other words, a strategy s is greater than s' with respect to $\preceq$ if and only if players stay more under strategy s. Consider a strategy s_{-i} of player $-i$ and a history $h_{i,t}$ observed by player i. From the perspective of player i, the one period action profile of player $-i$ is a mapping from player $-i$'s current signal to lotteries over $\{stay, exit\}$, which we denote by $a_{-i|h_{i,t}} : \mathbb{R} \to \Delta\{stay, exit\}$. The order $\preceq$ on dynamic strategies extends to one-shot action profiles as follows:

$$a' \preceq a \iff \{a.s.\, \forall x \in \mathbb{R},\ Prob[a'(x) = Stay] \leq Prob[a(x) = Stay]\}\,.$$

Note that if s_{-i} is Markovian, then $a_{-i|h_{i,t}}$ is effectively a mapping from $\mathbb{R}$ to $\{stay, exit\}$. For any mapping V_i that maps player i's current signal, $x_{i,t} \in \mathbb{R}$, to a continuation value $V_i(x_{i,t})$, and any mapping $a_{-i} : \mathbb{R} \to \Delta\{stay, exit\}$, one can define $BR_{i,\sigma}(a_{-i}, V_i)$, as the one period best response correspondence of player i when she expects a continuation value V_i and player $-i$ uses an action profile a_{-i}. Given a continuation value function V_i, the expected payoffs upon staying and exit – respectively denoted by $\Pi^i_S(V_i)$ and Π^i_E – are

$$\Pi^i_S(V_i) = \mathbf{E}\Big[W^i_{12}(w) + \{g^i(w) + \beta V_i(h_{i,t}, w) - W^i_{12}(w)\}\mathbf{1}_{s_{-i}=S}|h_{i,t}, s_{-i}\Big] \tag{3}$$

$$\Pi^i_E = \mathbf{E}\Big[W^i_{22}(w) + \{W^i_{21}(w) - W^i_{22}(w)\}\mathbf{1}_{s_{-i}=S}|h_{i,t}, s_{-i}\Big]. \tag{4}$$

Lemma 2 *For any one-shot action profile a_{-i} and value function V_i, the one-shot best-response correspondence $BR_{i,\sigma}(a_{-i}, V_i)$ admits a lowest and a highest element with respect to $\preceq$. These highest and lowest elements are respectively denoted $BR^H_{i,\sigma}(a_{-i}, V_i)$ and $BR^L_{i,\sigma}(a_{-i}, V_i)$.*

Proof: An action profile a_i belongs to the set of one-shot best-replies $BR_{i,\sigma}(a_{-i}, V_i)$ if and only if a_i prescribes S when $\Pi^i_S(V_i) > \Pi^i_E$ and prescribes E when $\Pi^i_S(V_i) < \Pi^i_E$. Because ties are possible $BR_{i,\sigma}(a_{-i}, V_i)$ need not be a singleton. However, by breaking the ties consistently in favor of either S or E, one can construct strategies a^H_i and a^L_i that are respectively the greatest and smallest elements of $BR_{i,\sigma}(a_{-i}, V_i)$ with respect to $\preceq$. ■

Lemma 3 *There exists $\overline{\sigma} > 0$ and $\nu > 0$ such that for all constant functions V_i taking value in $[m_i - \nu, M_i + \nu]$, and all $\sigma \in (0, \overline{\sigma})$, $BR^H_{i,\sigma}(a_{-i}, V_i)$ and $BR^L_{i,\sigma}(a_{-i}, V_i)$ are increasing in a_{-i} with respect to $\preceq$.*

The proof of this result exploits the fact that Assumption 4 implies a family of single-crossing conditions already identified in Milgrom and Shannon (1994). Note that the results of Athey (2002) do not apply directly since the conditions on distributions they require are only satisfied at the limit where σ is equal to 0.

Lemma 4 *Consider continuation value functions V and V' such that for all $h_{i,t} \in \mathcal{H}$, $V(h_{i,t}) \leq V'(h_{i,t})$. Then, for any a_{-i},*

$$BR^H_{i,\sigma}(a_{-i}, V) \preceq BR^H_{i,\sigma}(a_{-i}, V') \quad \textit{and} \quad BR^L_{i,\sigma}(a_{-i}, V) \preceq BR^L_{i,\sigma}(a_{-i}, V').$$

Proof: The result is proven for the greatest one-shot best-reply $BR^H_{i,\sigma}$. Player i chooses S over E whenever $\Pi^i_S(V_i) \geq \Pi^i_E$. As equation (3) shows, $\Pi^i_S(V_i)$ is increasing in V_i, while Π^i_E

does not depend on V_i. This yields that $BR^H_{i,\sigma}(a_{-i}, V) \preceq BR^H_{i,\sigma}(a_{-i}, V')$. The same proof applies for the lowest one-shot best-reply. ■

Lemma 5 *Whenever s_{-i} is a Markovian strategy, $BR_{i,\sigma}(s_{-i})$ admits a lowest and a highest element with respect to $\preceq$. These strategies are Markovian and are respectively denoted $BR^L_{i,\sigma}(s_{-i})$ and $BR^H_{i,\sigma}(s_{-i})$.*

Proof: Let V be the value player i obtains from best replying to s_{-i}. Since s_{-i} is Markovian, at any history $h^0_{-i,t}$ the conditional strategy $s_{-i|h^0_{-i,t}}$ is identical to s_{-i}, and the value player i expects conditional on $h^0_{i,t}$ is always V. Hence, $s_i \in BR_{i,\sigma}(s_{-i})$ if and only if action profiles prescribed by s_i at a history $h^0_{i,t}$ belong to $BR_{i,\sigma}(s_{-i}, V)$, where s_{-i} is identified with its one-shot action profile. Since $BR_{i,\sigma}(s_{-i}, V)$ admits highest and lowest elements a^H_i and a^L_i, the Markovian strategies s^H_i and s^L_i respectively associated with the one-shot profiles a^H_i and a^L_i are the greatest and a smallest elements of $BR_{i,\sigma}(s_{-i})$ with respect to $\preceq$. ■

We now show that game Γ_σ exhibits monotone best response as long as there is a Markovian strategy on one side of the inequality.

Proposition 1 (restricted monotone best response) *There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, whenever s_{-i} is a Markovian strategy, then, for **all** strategies s'_{-i},*

$$s'_{-i} \preceq s_{-i} \Rightarrow \left\{\forall s'' \in BR_{i,\sigma}(s'_{-i}),\ s'' \preceq BR^H_{i,\sigma}(s_{-i})\right\}$$
$$\textit{and} \quad s_{-i} \preceq s'_{-i} \Rightarrow \left\{\forall s'' \in BR_{i,\sigma}(s'_{-i}),\ BR^L_{i,\sigma}(s_{-i}) \preceq s''\right\}.$$

Proof: Let us show the first implication. Consider s_{-i} a Markovian strategy and s'_{-i} such that $s'_{-i} \preceq s_{-i}$. Define V_i and V'_i the continuation value functions respectively associated to player i's best response to s_{-i} and s'_{-i}. Note that since s_{-i} is Markovian, V_i is a constant function. Assumption 5, that "staying is good", implies that at all histories $h_{i,t}$, $V'_i(h_{i,t}) \leq V_i(h_{i,t})$. From Lemma 4, we have that

$$BR^H_{i,\sigma}(a'_{-i}, V'_i(h_{i,t})) \preceq BR^H_{i,\sigma}(a'_{-i}, V_i(h_{i,t})). \tag{5}$$

Since $V_i(h_{i,t})$ is constant we want to apply Lemma 3. For this, let us show that $a'_{-i|h_{i,t}} \preceq a_{-i|h_{i,t}}$. This follows directly from s_{-i} being Markovian, and the fact that $s'_{-i} \preceq s_{-i}$. Indeed, whenever $Prob\{a'_{-i|h_{i,t}} = stay\} > 0$, we must have $Prob\{a_{-i|h_{i,t}} = stay\} = 1$. Applying Lemma 3 yields that

$$BR^H_{i,\sigma}(a'_{-i}, V_i(h_{i,t})) \preceq BR^H_{i,\sigma}(a_{-i}, V_i(h_{i,t})). \tag{6}$$

Combining equations (5) and (6) we obtain that indeed, for all $s'' \in BR_{i,\sigma}(s'_{-i})$, $s'' \preceq BR^H_{i,\sigma}(s_{-i})$. An identical proof holds for the other inequality. ■

Proposition 1 will allow us to prove the existence of extreme threshold-form equilibria. For this we will use the following lemma which shows that for σ small enough, the best response to a threshold-form strategy is unique and takes a threshold form.

Lemma 6 *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$ and any $x \in \mathbb{R}$, there exists $x' \in \mathbb{R}$ such that $BR_{i,\sigma}(s_x) = \{s_{x'}\}$, i.e. the best response to a threshold form Markovian strategy is a unique threshold form Markovian strategy. Moreover, x' is continuous in x.*

Theorem 1 (extreme strategies) *There exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, sequentially rationalizable strategies of Γ_σ are bounded by a highest and lowest Markovian Nash equilibria, respectively denoted by $\mathbf{s}^H_\sigma = (s^H_{i,\sigma}, s^H_{-i,\sigma})$ and $\mathbf{s}^L_\sigma = (s^L_{i,\sigma}, s^L_{-i,\sigma})$.*

Those equilibria take threshold forms : for all $i \in \{1, 2\}$ and $j \in \{H, L\}$, there exists $x^j_{i,\sigma}$ such that $s^j_{i,\sigma}$ prescribes player i to stay if and only if $x_{i,t} \geq x^j_{i,\sigma}$.

Indeed, although Γ_σ is not supermodular, Proposition 1 is sufficient for the construction of Milgrom and Roberts (1990) to hold. The first step is to note that the strategies corresponding to staying always, and exiting always are threshold form Markovian strategies that bound the set of possible strategies. The idea is then to apply the best response mappings iteratively to these extreme strategies. A formal proof is given in Appendix A.

Let us denote by $\mathbf{x}^H_\sigma$ and $\mathbf{x}^L_\sigma$ the pairs of thresholds respectively associated with the highest and lowest equilibria with respect to $\preceq$. Note that $\mathbf{s}^L_\sigma \preceq \mathbf{s}^H_\sigma$, but $\mathbf{x}^L_\sigma \geq \mathbf{x}^H_\sigma$. Let $\mathbf{V}^H_\sigma$ and $\mathbf{V}^L_\sigma$ be the value pairs respectively associated with $\mathbf{s}^H_\sigma$ and $\mathbf{s}^L_\sigma$.

Lemma 7 $\mathbf{s}_\sigma^H$ *and* $\mathbf{s}_\sigma^L$ *are respectively associated with the highest and lowest possible pairs of rationalizable value functions,* $\mathbf{V}_\sigma^H$ *and* $\mathbf{V}_\sigma^L$*. More precisely, if* s_{-i} *is a rationalizable strategy, the value function* $V_{i,\sigma}$ *associated with player i's best reply to* s_{-i} *is such that at all histories* $h_{i,t}$*,* $V_{i,\sigma}^L \leq V_{i,\sigma}(h_{i,t}) \leq V_{i,\sigma}^H$*.*

Proposition 1 and Theorem 1 are the main benefits of using an exit game structure. They also provide a first justification for why we are specifically interested in Markovian equilibria: they provide tight bounds for rationalizable behavior. This focus will be further justified in Section 3.4.

3.3 Dynamic selection

We can now state the first selection result of the paper. It shows that continuation values associated with Markovian equilibria of Γ_σ must be fixed points of a mapping $\phi_\sigma(\cdot)$ that converges uniformly to an easily computable mapping Φ from $\mathbb{R}^2$ to $\mathbb{R}^2$. This provides explicit bounds for the set of rationalizable value functions and shows that the set of Markovian equilibria – which is a continuum under full information – typically shrinks to a finite number of elements under a global games information structure.

Theorem 2 *Under Assumptions 1, 2, 3, 4 and 5 there exists* $\overline{\sigma} > 0$ *such that for all* $\sigma \in (0, \overline{\sigma})$*, there exists a continuous mapping* $\phi_\sigma(\cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$*, mapping value pairs to value pairs such that,*

(i) $\mathbf{V}_\sigma^L$ *and* $\mathbf{V}_\sigma^H$ *are the lowest and highest fixed points of* $\phi_\sigma(\cdot)$*;*

(ii) A vector of continuation values is supported by a Markovian equilibrium if and only if it is a fixed point of $\phi_\sigma(\cdot)$*;*

(iii) As σ *goes to* 0*, the family of functions* $\phi_\sigma(\cdot)$ *converges uniformly over any compact set of* $\mathbb{R}^2$ *to an increasing mapping* $\Phi : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *defined by*

$$(7)\quad \Phi(V_i, V_{-i}) = \left(\begin{array}{c} \mathbf{E}_w \left[(g^i + \beta V_i)\, \mathbf{1}_{w > x^{RD}(V_i, V_{-i})} + W_{22}^i(w) \mathbf{1}_{w < x^{RD}(V_i, V_{-i})} \right] \\ \mathbf{E}_w \left[(g^{-i} + \beta V_{-i})\, \mathbf{1}_{w > x^{RD}(V_i, V_{-i})} + W_{22}^{-i}(w) \mathbf{1}_{w < x^{RD}(V_i, V_{-i})} \right] \end{array} \right)$$

where $x^{RD}(V_i, V_{-i})$ is the risk-dominant threshold of the one-shot game $\Psi_0(V_i, V_{-i})$.

Proof: For any fixed σ, any Markovian equilibrium of Γ_σ is associated with a vector of constant continuation values $\mathbf{V}_\sigma = (V_{i,\sigma}, V_{-i,\sigma})$. By continuity of the min-max values, for any $\nu > 0$, there exists $\overline{\sigma} > 0$, such that for all $\sigma \in (0, \overline{\sigma})$, $V_{i,\sigma} \in [m_i - \nu, M_i]$. Stationarity implies that equilibrium actions at any time t must form a Nash equilibrium of the one-shot game

	S	E
S	$g^i(w_t) + \beta V_{i,\sigma}$	$W^i_{12}(w_t)$
E	$W^i_{21}(w_t)$	$W^i_{22}(w_t)$

where i is the row player and players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$. All such one-shot games $\Psi_\sigma(\mathbf{V})$, indexed by $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$ and $\sigma > 0$ have a global game structure à la Carlsson and van Damme (1993).

Assumption 4 implies that there exists $\nu > 0$ such that for all $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$ and all $w \in I$, the one-shot game $G(\mathbf{V}, w)$ admits pure equilibria and they are all symmetric. Hence, Lemma 1 (uniform selection) implies that the following are true

1. There exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$ and $V \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$, the game $\Psi_\sigma(\mathbf{V})$ has a unique pair of rationalizable strategy. These strategies take a threshold-form and the associated pair of thresholds is denoted by $\mathbf{x}^*_\sigma(\mathbf{V})$;

2. The pair of thresholds $\mathbf{x}^*_\sigma(\mathbf{V})$ is continuous in $\mathbf{V}$;

3. As σ goes to 0, $\mathbf{x}^*_\sigma(\mathbf{V})$ converges to the risk dominant threshold $x^{RD}(\mathbf{V})$ uniformly over $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$.

The first result, joint selection, implies that there is a unique expected vector of values from playing game $\Psi_\sigma(\mathbf{V})$, which we denote $\phi_\sigma(\mathbf{V})$. The other two results imply that $\phi_\sigma(\mathbf{V})$ is continuous in $\mathbf{V}$, and that as σ goes to 0, $\phi_\sigma(\mathbf{V})$ converges uniformly over $\mathbf{V} \in \times_{i\in\{1,2\}}[m_i - \nu, M_i]$ to the vector of values $\Phi(\mathbf{V})$ players expect from using the risk-dominant strategy under full information.

Stationarity implies that the value vector $\mathbf{V}$ of any Markovian equilibrium of Γ_σ must satisfy the fixed point equation $\mathbf{V} = \phi_\sigma(\mathbf{V})$. Reciprocally, any vector of values $\mathbf{V}$ satisfying $\mathbf{V} = \phi_\sigma(\mathbf{V})$ is supported by the Markovian equilibrium in which players play the unique equilibrium of game $\Psi_\sigma(\mathbf{V})$ each period. This gives us (ii).

Furthermore, we know that the equilibrium strategies of game $\Psi_\sigma(\mathbf{V})$ converge to the risk-dominant strategy as σ goes to 0. This allows us to compute explicitly the limit function Φ. Because the risk-dominance threshold is decreasing in the continuation value, and using Assumption 5, it follows that Φ is increasing in $\mathbf{V}$. This proves (iii).

Finally, (i) is a straightforward implication of (ii). Values associated with Markovian equilibria of Γ_σ are the fixed points of $\phi_\sigma(\cdot)$. Hence the highest and lowest values associated with Markovian equilibria are also the highest and lowest fixed points of $\phi_\sigma(\cdot)$. ■

Theorem 2 states that extreme equilibria of games Γ_σ are characterized by the extreme fixed points of an operator $\phi_\sigma(\cdot)$ that converges uniformly to an explicit operator Φ as σ goes to 0. To show that the mapping Φ gives us a precise description of Markovian equilibria of Γ_σ however, we must show that the uniform convergence of the mapping $\phi_\sigma(\cdot)$ implies the convergence of its fixed points. This corresponds to the upper- and lower-hemicontinuity of fixed points of ϕ_σ.

The first important property we consider is upper-hemicontinuity. The next lemma states that fixed points of $\phi_\sigma(\cdot)$ converge to a subset of fixed points of Φ as σ goes to 0. In that sense, considering fixed points of Φ is sufficient: we do not need to worry about other equilibria.

Lemma 8 (upper-hemicontinuity) *The set of fixed points of $\phi_\sigma(\cdot)$ is upper-hemicontinuous at $\sigma = 0$. That is, for any sequence of positive numbers $\{\sigma_n\}_{n\in\mathbb{N}}$ converging to 0, whenever $\{\mathbf{V}_n\}_{n\in\mathbb{N}} = \{(V_{i,\sigma_n}, V_{-i,\sigma_n})\}_{n\in\mathbb{N}}$ is a converging sequence of fixed points of $\phi_{\sigma_n}(\cdot)$, the sequence $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ converges to a fixed point $\mathbf{V}$ of Φ.*

Theorem 2 and Lemma 8 imply that whenever Φ has a unique fixed point, the set of rationalizable strategies of game Γ_σ converges to a single pair of strategies as σ goes to 0. Section 4.2 will exploit that property to define a robustness criterion for cooperation in games with approximately constant payoffs.

As another illustration, Lemma 9 shows that conditional on continuation values belonging to some bounded set, whenever the states of the world have sufficient variance, then, equilibrium is unique. Let $||\cdot||_1$ denote the norm on $\mathbb{R}^2$ defined by $||\mathbf{V}||_1 = |V_i| + |V_{-i}|$.

Lemma 9 (uniqueness) *Let K be a bounded interval of $\mathbb{R}$. Under the maintained constraint that individually rational values V_i belong to K, there exists a constant $\eta > 0$, that depends only on payoff functions, such that whenever the distribution of states of the world f satisfies $\max_{[\underline{w},\overline{w}]} f < \eta$, then Φ is a contraction mapping with rate $\delta < 1$ with respect to the norm on vectors $||\cdot||_1$. That is, for all V_i taking values in K, $||\Phi(\mathbf{V}) - \Phi(\mathbf{V}')||_1 \leq \delta||\mathbf{V} - \mathbf{V}'||_1$.*

The question is now what happens when Φ has multiple fixed points (see Section 4.1 for examples)? Does the game Γ_σ have multiple equilibria? This is not a trivial question. If all fixed points of Φ are indeed associated to equilibria of Γ_σ for σ small, this shows that while a global games information structure may yield uniqueness in static settings, this does not hold anymore when players have an infinite horizon. This question is closely related to the problem of lower-hemicontinuity: when is it that a fixed point of Φ is associated with a sequence of fixed points of $\phi_\sigma(\cdot)$ as σ goes to 0? This is the point of Proposition 2.

So far we have been characterizing Markovian equilibria by their continuation values. For the remainder of this section, it becomes convenient to characterize Markovian equilibria by their cooperation threshold. This is authorized by the following lemma.

Lemma 10 (threshold-form Markovian equilibria) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, all Markovian equilibria of Γ_σ take a threshold form.*

Furthermore, if $(x_{i,\sigma}, x_{-i,\sigma})$ is a pair of equilibrium thresholds, then $|x_{i,\sigma} - x_{-i,\sigma}| \leq 2\sigma$.

Proof: Consider a Markovian equilibrium of Γ_σ denoted by (s_i, s_{-i}). This Markovian equilibrium is associated to a pair of values (V_i, V_{-i}). The one-shot action profile (a_i, a_{-i}) associated with (s_i, s_{-i}) has to be a Nash equilibrium of the global game $\Psi_\sigma(V_i, V_{-i})$. Lemma 1 implies that there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, and all $\mathbf{V} \in [m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $\Psi_\sigma(\mathbf{V})$ has a unique Nash equilibrium. Furthermore, this unique equilibrium takes a threshold form. This proves the first part of the lemma: for σ small enough, all Markovian equilibria of Γ_σ take a threshold form. The second part of the lemma is a direct application of Lemma 4 of Chassang (2006). ■

Note that the second part of this lemma shows that as σ goes to 0, Markovian equilibria of Γ_σ are asymptotically symmetric, so that the likelihood of actual miscoordination vanishes. This illustrates that the ghost of miscoordination, rather than miscoordination itself is enough to drive players away from efficient behavior.

Definition 7 *For all $\sigma \geq 0$, let $BRV_{i,\sigma}(x)$ denote the value that player i gets in game Γ_σ from best replying to a player $-i$ using a threshold form strategy s_x.*

For σ small enough for Lemma 10 to hold, denote by $\mathbf{x}^*_\sigma(\mathbf{V})$ the unique rationalizable pair of strategies of game $\Psi_\sigma(\mathbf{V})$. Note that $\mathbf{x}^*_\sigma(\mathbf{V})$ belongs to $\mathbb{R}^2$, while the risk-dominant threshold $x^{RD}(\mathbf{V})$ of game $\Psi_0(\mathbf{V})$ belongs to $\mathbb{R}$.

For any pair of thresholds $\mathbf{x} \in \mathbb{R}^2$, define $\xi_\sigma(\mathbf{x}) \equiv \mathbf{x}^*_\sigma(BRV_{i,\sigma}(x_{-i}), BRV_{-i,\sigma}(x_i))$ and $\xi(\mathbf{x}) \equiv x^{RD}(BRV_{i,0}(x_{-i}), BRV_{-i,0}(x_i))$. When $x \in \mathbb{R}$, $\xi(x)$ will be used to denote $\xi(x, x)$.

Lemma 11 (properties of ξ_σ) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, ξ_σ is a well defined, continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. Furthermore, the following properties hold:*

(i) A pair of strategies (s_i, s_{-i}) is a Markovian equilibrium of Γ_σ if and only if it takes a threshold form and the associated pair of thresholds, $\mathbf{x} = (x_i, x_{-i})$, satisfies $\mathbf{x} = \xi_\sigma(\mathbf{x})$;

(ii) As σ goes to 0, $\xi_\sigma(\mathbf{x})$ converges uniformly over $\mathbf{x} \in \mathbb{R}^2$ to the symmetric pair $(\xi(\mathbf{x}), \xi(\mathbf{x}))$;

(iii) The mapping $\xi : \{\mathbb{R} \to \mathbb{R},\ x \mapsto \xi(x)\}$ is weakly increasing.

Note that point (i) of Lemma 11 implies that there is a bijection between fixed points of $\xi_\sigma(\cdot)$ and fixed points of $\phi_\sigma(\cdot)$.

Definition 8 (non-singular fixed points) *A fixed point x of ξ is non-singular if and only* *if there exists* $\epsilon > 0$ *such that either*

$$\forall y \ \in [x-\epsilon, x),\ \xi(y) < y \quad \textit{and} \quad \forall y \ \in (x, x+\epsilon],\ \xi(y) > y$$
$$\textit{or} \quad \forall y \ \in [x-\epsilon, x),\ \xi(y) > y \quad \textit{and} \quad \forall y \ \in (x, x+\epsilon],\ \xi(y) < y.$$

In other terms x *is non-singular whenever* ξ *cuts strictly through the* 45^o *line at* x*.*

Proposition 2 (lower hemicontinuity) *Whenever* x *is a non-singular fixed point of* ξ*, then, for any sequence of positive numbers* $\{\sigma_n\}_{n\in\mathbb{N}}$ *converging to 0, there exists a sequence of fixed points of* ξ_{σ_n}*,* $\{\mathbf{x}_n\}_{n\in\mathbb{N}} = \{(x_{i,\sigma_n}, x_{-i,\sigma_n})\}_{n\in\mathbb{N}}$*, converging to* (x, x) *as* n *goes to infinity.*

This shows that all non-singular fixed point of ξ are the limit of threshold-form equilibria of the game Γ_σ as σ goes to 0. This shows that all equilibria (s_x, s_x) of Γ_0, with x a non-singular fixed point of ξ, are robust to miscoordination risk. Theorem 3 will enrich this result by showing that robustness to miscoordination risk implies robustness to strategic risk only if x is a *stable* non-singular fixed point of ξ. The next lemma shows that for an appropriate distance on payoff structures, fixed points of ξ are generically non-singular.

Definition 9 (topology on C^1 payoff structures) *A* C^1 *payoff structure* π *is a* 9*-tuple of* C^1 *functions* $\pi = \times_{i\in\{1,2\}}(g^i, W^i_{12}, W^i_{21}, W^i_{22}) \times F$*, that satisfies the assumptions of Section 2.3. Let* Π^1 *denote the set of* C^1 *payoff structures. The distance* $||\cdot||_{\Pi^1}$ *over payoff structures is defined as,*

$$||\pi - \tilde{\pi}||_{\Pi^1} = \sum_{l\in\{1,...,9\}} ||\pi_l - \tilde{\pi}_l||_\infty + \left|\left|\frac{\partial \pi_l}{\partial w} - \frac{\partial \tilde{\pi}_l}{\partial w}\right|\right|_\infty$$

where $||\cdot||_\infty$ *denotes the supremum norm.*

Lemma 12 (generic non-singularity) *There exists a subset* P *of* Π^1 *that is open and dense in* Π^1 *with respect to* $||\cdot||_{\Pi^1}$ *and such that whenever* $\pi \in P$*, the fixed points of* ξ *are all non-singular.*

Proposition 2 and Lemma 12 imply that typically, all fixed points of Φ are indeed associated with Markovian equilibria of Γ_σ as σ goes to 0. This shows that a global games information structure need not always yield full-fledged selection[7]. In that sense coordination in dynamic games is qualitatively different from coordination in one-shot games.

Although dominance solvability is clearly an attractive feature of one-shot global games, the possibility of multiplicity should not be considered a negative result in this context. As the example of Section 4.4 shows, trigger equilibria in a fully repeated global game are also equilibria of a related exit game in which payoffs upon exit are those obtained from reverting to the one-shot Nash equilibrium. In that setting, one can show that the one-shot Nash equilibrium is always an equilibrium of this exit game. If dynamic global games with exit were always dominance solvable, this would imply that the one-shot Nash is the only equilibrium in trigger strategies that is robust to private noisy assessments of the state of the world. From that perspective, the fact that a global games information structure does not always imply dominance solvability is reassuring.

Furthermore, Section 3.4 shows that the dominance solvability of one shot global games does survive in dynamic exit games, albeit in a weaker form. While equilibria may not be globally uniquely rationalizable, it is shown that the global games information structure can make them locally uniquely rationalizable.

3.4 Local dominance solvability, stability, and strategic uncertainty.

Local dominance solvability, discussed at length by Guesnerie (2002) in a macroeconomic context, can be viewed as an intermediary notion between Nash equilibrium and dominance solvability. For any two-player game, consider a set of strategies $\mathcal{Z}$ of player i and a strategy $s \in \mathcal{Z}$. The game is said to be locally dominance solvable at s with respect to $\mathcal{Z}$ whenever the sequence[8] $\{(BR_i^\Delta \circ BR_{-i}^\Delta)^n(\mathcal{Z})\}_{n\in\mathbb{N}}$ converges to $\{s\}$ as n goes to infinity. In this case, we say that s is locally strongly rationalizable with respect to $\mathcal{Z}$. Equivalently, the game is said to be locally dominance solvable at s with respect to $\mathcal{Z}$ if and only if s is the only

[7]See Section 4.1 for examples.

[8]Recall that $BR_i^\Delta = BR_i \circ \Delta$.

rationalizable outcome when it is common knowledge among players that player i uses a strategy that belongs to $\mathcal{Z}$. From this perspective, s is a strict Nash equilibrium if and only if it is locally strongly rationalizable with respect to itself, and s is the unique rationalizable strategy of player i if and only if it is locally strongly rationalizable with respect to the set of all possible strategies.

The purpose of local dominance solvability is to be a middle ground between Nash equilibrium, which may not be demanding enough, and dominance solvability, which may be too demanding. The approach of Guesnerie (2002) is to introduce a topology on strategies and then define a strategy s as locally strongly rationalizable – without reference to any set – whenever there exists a neighborhood $\mathcal{N}$ of s such that s is locally strongly rationalizable with respect to $\mathcal{N}$. This effectively defines a stability criterion with respect to iterated best response. The object of this section is to characterize both the stability of Markovian equilibria of Γ_σ, and the size of their basin of attraction. We must first define a topology on strategies.

Definition 10 (balls in the "noise" topology) *Consider two histories h_t and h'_t. These are vectors of real numbers of length $2t-1$. Hence, we can define the distance $d(h_t, h'_t) = ||h_t - h'_t||_\infty$ and the Lebesgue measure λ over histories of length t. For any strategy s and any $\delta > 0$, the ball $\mathcal{B}_\delta(s)$ of center s and radius δ is defined as*

$$\mathcal{B}_\delta(s) \equiv \{s' \mid a.s.\ \forall h_t \in \mathcal{H}, \quad \lambda(\{h'_t \mid d(h'_t, h_t) < \delta \ \ and \ \ s(h'_t) = s'(h_t)\}) > 0\}.$$

A neighborhood $\mathcal{N}$ of a strategy s is a set that contains a ball of center s and radius $\delta > 0$.

Note that the choice of topology is not innocuous. Depending on the topology, the same equilibrium may be locally strongly rationalizable or not. In the topology defined above, a ball $\mathcal{B}_\delta(s_i)$ corresponds to the set of strategies an uninformed observer might deem possible when observing perfectly the moves of player i but observing a version of player i's signal that is garbled by a noise term of maximum amplitude δ. Alternatively, one can view a ball $\mathcal{B}_\delta(s_i)$ as the set of strategies deemed possible by a player getting a description of s_i that potentially misclassifies histories that differ by less than δ. In this sense, this topology is appropriate to discuss strategic uncertainty[9]. A strategy s will be locally strongly rationalizable with respect to balls in this topology whenever it is robust to small amounts of doubt regarding the players' common understanding of s.

Under full information, exit games admit no locally strongly rationalizable strategies because given any equilibrium, it is always possible to find another equilibrium that is arbitrarily close. The rest of this section shows that as σ goes to 0, the dominance solvability result of Carlsson and van Damme (1993) for one-shot two-by-two games translates into local dominance solvability for exit games. Furthermore, it is shown that as σ goes to 0, the local strong rationalizability of Markovian equilibria of Γ_σ – which is a stability property in the space of strategies – is asymptotically characterized by the stability of the increasing mapping $\xi : \mathbb{R} \to \mathbb{R}$ introduced in Definition 7. Let us first define asymptotic local dominance solvability formally.

Definition 11 (Asymptotic Local Dominance Solvability) *Consider a pair of strategies (s_i, s_{-i}). We say that the family of games $\{\Gamma_\sigma\}_{\sigma>0}$ is asymptotically locally dominance solvable (ALDS) at (s_i, s_{-i}) if there exist neighborhoods of s_i and s_{-i}, denoted $\mathcal{N}_i$ and $\mathcal{N}_{-i}$ such that $\forall i \in \{1, 2\}$,*

$$\lim_{\sigma\to 0}\lim_{n\to\infty}(BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma})^n(\mathcal{N}_i) = \{s_i\} \tag{8}$$

$$\text{and} \quad \lim_{\sigma\to 0}\lim_{n\to\infty} BR^{\Delta}_{-i,\sigma} \circ (BR^{\Delta}_{i,\sigma} \circ BR^{\Delta}_{-i,\sigma})^n(\mathcal{N}_i) = \{s_{-i}\} \tag{9}$$

The basin of attraction of (s_i, s_{-i}) is the greatest neighborhood $\mathcal{N}_i \times \mathcal{N}_{-i}$ of (s_i, s_{-i}) such that equations (8) and (9) hold.

The central result of this section is that the asymptotic local dominance solvability and the basin of attraction of Markovian threshold form equilibria are largely characterized by the stability and basins of attraction of fixed points of the mapping $\xi : \mathbb{R} \mapsto \mathbb{R}$.

[9] In fact the size of the balls $\mathcal{B}_\delta(s)$ for $\delta > 0$ is a good measure of the strategic uncertainty inherent to a strategy s. For instance, if $s = s_x$ then $B_\delta(s) = [s_{x+\delta}, s_{x-\delta}]$, while if s is defined by $s(w) = Stay$ if and only if $int[w/\delta]$ is an even number, then, $B_\delta(s)$ is the set of all possible strategies.

Proposition 3 is the key step to characterize local dominance solvability. It shows that whenever x is a stable fixed point of ξ, then for σ small enough, the first step of iterated best-response shrinks neighborhoods of s_x. Using the partial monotone best-response result of Proposition 1 this will allow us to prove asymptotic local dominance solvability.

Proposition 3 *Consider a stable fixed point x of ξ and y in the basin of attraction of x. If $y < x$, then there exists $x' \leq y$ and $\overline{\sigma} > 0$ such that x' belongs to the basin of attraction of x and, for all $\sigma \in (0, \overline{\sigma})$ and $i \in \{1, 2\}$, we have*[10] $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) \preceq s_{x'}$.

Similarly, if $y > x$, there exists $x'' \geq y$ and $\overline{\sigma}$ such that x'' belongs to the basin of attraction of x and for all $\sigma \in (0, \overline{\sigma})$ and $i \in \{1, 2\}$, $s_{x''} \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x''})$.

Proposition 3 is a key step to understand the impact of a global games information structure on local dominance solvability, which is why a proof in the case of symmetric games is given here. It is instructive of how dominance solvability results for one-shot global games can be exploited in dynamic games. The proof in the case of asymmetric games is more delicate but follows the same intuition. It is given in Appendix A.

Proof (symmetric games): This proof applies to the case where players have the same payoff functions. This implies that $BR_{i,\sigma} = BR_{-i,\sigma} = BR_\sigma$. Let us show the first part of the lemma.

Pick $\overline{\sigma}$ small enough such that Proposition 1 applies. Then, for all $\sigma \in (0, \overline{\sigma})$, it is sufficient to prove that there exists $x' \leq y$ such that $BR_\sigma(s_{x'}) \preceq s_{x'}$. Let $BR_\sigma(a, V)$ denote the one-shot best response of a player expecting a continuation value V and facing a one-shot action profile a. Pick any[11] $x' \leq y$ that belongs to the basin of attraction of x. It must be that $\xi(x') > x'$. Using the fact that for Markovian strategies, one-shot action profiles are equivalent to full-fledged strategies, we can write, $BR_\sigma(s_{x'}) = BR_\sigma\left(s_{x'}, BRV_\sigma(x')\right)$.

The idea is to use this formulation to apply dominance solvability results from one-shot global games. From Lemma 1, we know that for $\overline{\sigma}$ small enough, all games $\Psi_\sigma(\mathbf{V})$, with $\sigma \in (0, \overline{\sigma})$ and $\mathbf{V} \in [m_i - \nu, M_i] \times [m_{-i} - \nu, M_{-i}]$, are dominance solvable. This and Lemma

[10] Recall that if a and b are thresholds such that $a > b$ then the corresponding strategies satisfy $s_a \preceq s_b$.

[11] In the case of asymmetric payoffs the choice of an appropriate x' becomes relevant, which is why the proposition allows for this extra degree of freedom.

3 implies that when n goes to infinity, the sequence $\{(BR_\sigma(\cdot, BRV_\sigma(x')))^n(s_{x'})\}_{n\in\mathbb{N}}$ converges monotonously to the unique rationalizable equilibrium of the one-shot global game[12] $\Psi_\sigma(BRV_\sigma(x'))$. This equilibrium is associated to the threshold $x^*_\sigma(BRV_\sigma(x')) = \xi_\sigma(x')$. We know that ξ_σ converges uniformly to ξ and that $\xi(x') > x'$. This implies that for σ small enough, $\xi_\sigma(x') > x'$. This implies that the sequence $\{(BR_\sigma(\cdot, BRV_\sigma(x')))^n(s_{x'})\}_{n\in\mathbb{N}}$ is decreasing with respect to $\preceq$. Hence, we must have $BR_\sigma(s_{x'}) \preceq s_{x'}$. This proves the first part of the lemma. The second part results from an entirely symmetric reasoning. ■

We can now prove the main result of this section. It states that asymptotically, basins of attraction of Markovian strategies are largely characterized by the basins of attraction of ξ.

Theorem 3 (Asymptotic Local Dominance Solvability) *Consider any symmetric pair of threshold form strategies (s_x, s_x). Whenever x is a stable fixed point of ξ, then the family $\{\Gamma_\sigma\}_{\sigma>0}$ is ALDS at (s_x, s_x).*

More strongly, if an interval $[y, z]$ is included in the basin of attraction of x with respect to ξ, and $x \in (y, z)$, then, $[s_z, s_y]^2$ is included in the basin of attraction of (s_x, s_x) with respect to asymptotic local dominance.

Proof: The second part of the theorem implies the first one. We prove the second part directly. Using Proposition 3, we know there exist $\overline{\sigma}$, $x_- \le y$ and $x_+ \ge z$, with $[x_-, x_+]$ included in the basin of attraction of x, such that for all $\sigma \in (0, \overline{\sigma})$, and $i \in \{1, 2\}$,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x^-}) \preceq s_{x^-} \quad \text{and} \quad s_{x^+} \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x^+}).$$

These inequalities and Proposition 1 imply by iteration that for all $n \in \mathbb{N}$,

$$\begin{aligned}\left(BR^\Delta_{i,\sigma} \circ BR^\Delta_{-i,\sigma}\right)^n([s_{x^+}, s_{x^-}]) \quad &\subset \quad [(BR_{i,\sigma} \circ BR_{-i,\sigma})^n(s_{x^+}), (BR_{i,\sigma} \circ BR_{-i,\sigma})^n(s_{x^-})] \\ &\subset \quad [(BR_{i,\sigma} \circ BR_{-i,\sigma})^{n-1}(s_{x^+}), (BR_{i,\sigma} \circ BR_{-i,\sigma})^{n-1}(s_{x^-})] \\ &\subset \quad \cdots \subset [s_{x^+}, s_{x^-}]\end{aligned}$$

[12]Given that we are considering symmetric games, the arguments of many previously defined functions become redundant. Such redundant arguments are dropped in all relevant cases.

Consider the decreasing sequence $\{(BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^-})\}_{n \in \mathbb{N}}$. As n goes to ∞, it must converge to a threshold form strategy with threshold $x^-_{i,\sigma} \in [x_-, x_+]$. Moreover $(s_{x^-_{i,\sigma}}, BR_{-i,\sigma}(s_{x^-_{i,\sigma}}))$ must be a Markovian threshold form equilibrium of Γ_σ. Lemma 8 implies that as σ goes to 0, any converging subsequence of $\{(x^-_{i,\sigma}, x^-_{-i,\sigma})\}_{\sigma>0}$ must converge to a symmetric pair $(\tilde{x}, \tilde{x})$ such that $\tilde{x}$ is a fixed point of ξ and $\tilde{x} \in [x_-, x_+]$. The only fixed point of ξ in $[x_-, x_+]$ is x. This implies that as σ goes to 0, $x^-_{i,\sigma}$ must converge to x. Similarly, as n goes to ∞, the sequence $(BR_{i,\sigma} \circ BR_{-i,\sigma})^n (s_{x^+})$ converges to a threshold strategy with a threshold $x^+_{i,\sigma}$ that converges to x as σ goes to 0. This concludes the proof. ■

The value of this result lies in the fact that the stability of strategies with respect to a complex iterated best response mapping is characterized by the stability of fixed points of a simple[13] mapping ξ from $\mathbb{R}$ to $\mathbb{R}$.

It is also interesting to note that the closure of basins of attraction of an increasing mapping is a partition of $\mathbb{R}$. In other words, any value $x \in \mathbb{R}$ is either a fixed point of ξ, or belongs to the basin of attraction of a fixed point of ξ. This implies that if a Markovian equilibrium is associated to a threshold x that is an unstable fixed point of ξ, then s_x is asymptotically unstable with respect to iterated best response. As σ goes to 0, arbitrarily small amounts of pessimism or optimism will push players' behavior away from s_x. Hence, a Markovian equilibrium associated to a fixed point x of ξ will be robust to strategic uncertainty if and only if x is a stable fixed point of ξ. The basin of attraction of x with respect to ξ measures the amount of strategic uncertainty that can be introduced before the players' behavior is perturbed away.

Finally, this result restricts possible non-Markovian strategies: asymptotically, there can be no non-Markovian equilibrium that is strictly contained within two consecutive Markovian equilibria with respect to $\preceq$.

[13]Computations can be further simplified by considering the mapping $\zeta : \mathbb{R} \mapsto \mathbb{R}$ defined by, $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)} \mathbf{E}\left[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}\right]$. Computing ζ is simpler than computing ξ and both functions coincide around their respective fixed points. See Lemma 17 in Appendix A for more details.

4 Applications

This section makes the case that the class of exit games introduced in Section 2 provides a practical framework to model miscoordination risk, and yields predictions that are qualitatively different from those obtained by focusing on Pareto-efficient equilibria under full information. Section 4.1 revisits the partnership game of Section 2.2 and shows how under private information, miscoordination fear can drive players to immediate exit even though the likelihood of miscoordination is vanishing. Section 4.2 shows how results from Section 3 can be used to define a simple criterion for the robustness of equilibria to miscoordination fear in exit games with approximately constant payoffs. As an example, Section 4.3 explores how wealth affects agents' ability to cooperate and shows that taking into account miscoordination fear yields predictions that are both intuitive and qualitatively distinct from those obtained under full information. Finally, Section 4.4 uses the example of repeated Cournot competition to show how exit games can be used to study the properties of trigger equilibria in repeated games with noisily observed states.

4.1 Miscoordination fear in the partnership game

Consider the partnership game introduced in Section 2.2. Flow payoffs are symmetric and given by

	S	E
S	w_t	$w_t - C + \beta V_E$
E	$b + V_E$	V_E

where payoffs are given for the row player only and $C > b \geq 0$. Under full information there exists a Pareto dominant equilibrium, of value $\overline{V}$, in which players stay whenever the state w_t is greater than a minimum threshold $\underline{w}$ defined by

$$\begin{aligned} \underline{w} + \beta\overline{V} &= b + V_E \\ \overline{V} &= \mathbf{E}\left[(w_t + \beta\overline{V})\mathbf{1}_{w_t > \underline{w}}\right] + F(\underline{w})V_E. \end{aligned}$$

If we use Pareto efficiency as a selection criterion in this full-information game, then the magnitude of C has no impact on players' behavior. Let us show this is not the case anymore when players privately assess the state of the world.

From Theorem 2, we know that the extreme equilibria of Γ_σ are asymptotically characterized by the fixed points of the mapping Φ. Hence, we are interested in the comparative statics of extreme fixed points of Φ with respect to C. Because the game is symmetric, fixed points of Φ will be symmetric and Φ can be restricted to a mapping from $\mathbb{R}$ to $\mathbb{R}$. The risk dominant threshold of the augmented game $\Psi_0(V)$ is given by the equation

$$x^{RD}(V) + \beta V - b - V_E = V_E - x^{RD}(V) + C - \beta V_E$$

so that $x^{RD}(V) = (1-\beta)V_E + \frac{b+C}{2} + \beta\frac{V_E - V}{2}$. The mapping Φ is defined by,

$$\text{(10)} \qquad \forall\, V \in [V_E, +\infty), \quad \Phi(V) = V_E + \int_{w\in\mathbb{R}} (w + \beta V - V_E)\mathbf{1}_{w > x^{RD}(V)} f(w)\mathrm{d}w.$$

Figure 1 summarizes simulations of Φ in which f is a Gaussian distribution of parameters (μ, η^2). In the cases represented in Figure 1, $V_E = 5$, $\beta = 0.7$, $C = 3$, $b = 1$ and $\mu = 3$. As Figure 1(a) shows in the case of $\eta = 1$, private assessments of the state of the world can dramatically reduce the set of rationalizable strategies. The range of equilibrium values shrinks from the interval $[5.3, 9.9]$ under full-information to the singleton $\{7.4\}$ once players' fear of miscoordination is taken into account. Interestingly, as the standard-error η diminishes, the set of equilibria that are robust to miscoordination risk changes a lot even though extreme equilibrium values under full-information vary very little. Figure 1(b) corresponds to the case $\eta = 0.2$. Under full information, the set of equilibrium values is $[5.1, 9.8]$ and does not differ from the case $\eta = 1$ by much. However, unlike the case $\eta = 1$, the game with private information now exhibits multiple asymptotic equilibria: one middle equilibrium that is unstable with respect to iterated best-reply, and two extreme equilibria associated with values 5.2 and 9.5, that are stable with respect to iterated best-reply. Note that the two extreme equilibrium values under miscoordination risk are actually very close to the extreme equilibrium values under full information[14].

One can also derive comparative statics with respect to C directly from expression (10). Indeed, we have, $\frac{\partial\Phi(V)}{\partial C} = -f(x^{RD}(V))\left(x^{RD}(V) + \beta V - V_E\right)$. Since $x^{RD}(V)$ is the risk-dominant threshold of $\Psi_0(V)$, it must be that for $w = x^{RD}(V)$, staying is a strict Nash equilibrium of the game $G(V, w)$. Hence, we obtain that $(x^{RD}(V) + \beta V - V_E) > 0$. This shows that $\frac{\partial\Phi(V)}{\partial C} < 0$. Since Φ is an increasing mapping, downward shifts of Φ also shift its extreme fixed points downwards. Hence, we conclude that the extreme fixed points of Φ are strictly decreasing in C. Under a global games information structure, worsening the payoffs upon miscoordination diminishes the players' ability to cooperate, even though the probability of actual miscoordination is vanishingly small.

In fact, as C goes to $+\infty$, $x(V)$ goes uniformly to $-\infty$ over any compact. This implies that over any compact, $\Phi(V)$ converges uniformly to the constant V_E. Since we know that independently of C, fixed points of Φ must belong to $[V_E, \overline{V}]$, this implies that as $\lim_{C\to\infty}\lim_{\sigma\to 0} V_\sigma^H(C) = V_E$, and immediate exit is asymptotically the only rationalizable strategy that is robust to miscoordination fear. Given that having C go to $+\infty$ does not affect the Pareto efficient equilibrium of the full information game, this shows in a stark way how modeling miscoordination fear can generate new predictions. Section 4.3 illustrates this point in a richer economic context by using the robustness criterion developed in Section 4.2.

4.2 Robustness of cooperation in games with constant payoffs

This section considers the limit where the distribution f of states of the world w becomes arbitrarily concentrated around a particular state w_0. Interestingly, at the limit, the sensitivity of cooperation to miscoordination risk only depends on the payoffs at w_0. This allows us to define a simple explicit criterion for the robustness of cooperation in games with constant payoffs. The example of Section 4.3 will exploit that criterion to investigate the effect of wealth on agents' ability to cooperate.

[14]This could potentially suggest that as the underlying distribution f becomes concentrated around a particular state of the world w_0 (here $w_0 = \mu$), miscoordination risk has no impact on the sustainability of cooperation. Section 4.2 shows that this is not the case.

Definition 12 (global games extension) *Consider a vector of payoff functions*

$$\gamma = (g^i, W_{12}^i, W_{21}^i, W_{22}^i) \times (g^{-i}, W_{12}^{-i}, W_{21}^{-i}, W_{22}^{-i})$$

and a sequence $\{f_n\}_{n \in \mathbb{N}}$ of density functions with convex support, converging weakly to a Dirac mass at w_0 when n goes to infinity. The sequence of game structures $\pi_n = (\gamma, f_n)$ is said to be a global games extension of the full information game with constant payoffs $\gamma(w_0)$ whenever, for all $n \in \mathbb{N}$, the payoff structure π_n satisfies Assumptions 1, 2, 3, 4 and 5.

Note that a game with constant payoffs $\gamma(w_0)$ can admit multiple global game extensions, which can have different payoff functions and different densities.

Assumptions 2 and 3 only make sense in a global game context, Assumptions 1, 4, and 5 however naturally extend to games with constant payoffs. Indeed Assumption 1 is trivially satisfied, and Assumptions 4, 5 are required to hold only when the state of the world is w_0.

Lemma 13 *Any exit game with certain flow-payoffs $\gamma(w_0)$ satisfying Assumptions 4 and 5 admits a global game extension.*

Consider a global game extension $\{\pi_n\}_{n \in \mathbb{N}}$ of some game with constant payoffs $\gamma(w_0)$, and $\{\Phi_n\}_{n \in \mathbb{N}}$ the associated value mappings. Let $\mathbf{V}_n^H$ and $\mathbf{V}_n^L$ denote the highest and lowest fixed points of Φ_n. Denote by $\mathbf{V}^H$ the vector of values obtained by players if they stayed every period in game $\gamma(w_0)$ and $\mathbf{V}^L = (W_{22}^i(w_0), W_{22}^{-i}(w_0))$ the values they would obtain upon immediate exit.

The case of greatest interest - considered in the remainder of this section - is the one in which staying is a Nash equilibrium whenever players expect continuation values $\mathbf{V}^H$, but exit is the only Nash equilibrium when players expect continuation values $\mathbf{V}^L$. As before, we denote by $G(\mathbf{V}, w)$ the full information augmented game

	S	E
S	$g^i(w_t) + \beta V_i$	$W_{12}^i(w_t)$
E	$W_{21}^i(w_t)$	$W_{22}^i(w_t)$

where i is the row player.

The next proposition shows that in games with approximately constant payoffs, the robustness of cooperation to miscoordination fear is entirely characterized by the payoffs at w_0.

Proposition 4 (robustness to miscoordination fear) *Whenever staying is the risk-dominant equilibrium of game $G(\mathbf{V}^H, w_0)$, as n goes to infinity, $\mathbf{V}_n^L$ converges to $\mathbf{V}^L$ and $\mathbf{V}_n^H$ converges to $\mathbf{V}^H$.*

Whenever exit is the risk-dominant equilibrium of game $G(\mathbf{V}^H, w_0)$, as n goes to infinity, $\mathbf{V}_n^L$ converges to $\mathbf{V}^L$ and $\mathbf{V}_n^H$ converges to $\mathbf{V}^L$.

Proof: Denote by $\mathbf{V}_n(x)$ the vector of values players would obtain by best replying to s_x under full information for the payoff structure π_n.

Let us first prove that $\mathbf{V}_n^L$ always converges to $\mathbf{V}^L$ as n goes to ∞. Since $\mathbf{V}^L$ is the value of immediate exit, it is clear that $\liminf \mathbf{V}_n^L \geq \mathbf{V}^L$. Since staying is not an equilibrium action when players expect continuation values $\mathbf{V}^L$, it must be that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^L) > w_0 + \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $||\mathbf{V} - \mathbf{V}^L||_\infty < \delta$, we have $x^{RD}(\mathbf{V}) > w_0 + \tau/2$. Convergence of f_n to a Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $||\mathbf{V}_n(w_0 + \tau/2) - \mathbf{V}^L||_\infty < \delta$. This implies that for all $n \geq N$, $\xi_n(w_0 + \tau/2) \geq w_0 + \tau/2$. Hence ξ_n must have a fixed point above $w_0 + \tau/2$. Since f_n converges to a Dirac mass at w_0 the value pair associated with such an equilibrium converges to $\mathbf{V}^L$ as n goes to infinity. Hence, $\lim \mathbf{V}_n^L = \mathbf{V}^L$.

Assume that staying is risk-dominant in $G(\mathbf{V}^H, w_0)$. This means that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^H) < w_0 - \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $||\mathbf{V} - \mathbf{V}^H||_\infty < \delta$, we have $x^{RD}(\mathbf{V}) < w_0 - \tau/2$. Convergence of f_n to a Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $||\mathbf{V}_n(w_0 - \tau/2) - \mathbf{V}^H||_\infty < \delta$. This implies that for all $n \geq N$, $\xi_n(w_0 - \tau/2) \leq w_0 - \tau/2$. Hence ξ_n must have a fixed point below $w_0 - \tau/2$. This and the convergence of f_n to a Dirac mass at w_0 implies that as n goes to infinity, $\mathbf{V}_n^H$ converges to $\mathbf{V}^H$.

Assume now that exit is risk-dominant in $G(\mathbf{V}^H, w_0)$. This means that there exists $\tau > 0$ such that $x^{RD}(\mathbf{V}^H) > w_0 + \tau$. By continuity of x^{RD} this implies that there exists $\delta > 0$ such that for all $\mathbf{V}$ satisfying $\mathbf{V} < \mathbf{V}^H + \delta$, we have $x^{RD}(\mathbf{V}) > w_0 + \tau/2$. Convergence of f_n to a

Dirac mass at w_0 implies that there exists N such that for all $n \geq N$, $\mathbf{V}_n(+\infty) < \mathbf{V}^H + \delta$. This implies that for all $n \geq N$, and all $x \in \mathbb{R}$, $\xi_n(x) > w_0 + \tau/2$. Hence all fixed points of ξ_n are above $w_0 + \tau/2$. This and the convergence of f_n to a Dirac mass at w_0 implies that as n goes to infinity, $\mathbf{V}_n^H$ converges to $\mathbf{V}^L$. ■

Because this result does not depend on the particular global games extension of the game with constant payoffs $\gamma(w_0)$, Proposition 4 can be used to define a simple robustness criterion for cooperation in exit games with constant payoffs. According to this criterion, whenever the continuation value associated with full cooperation is high enough for staying to be risk-dominant in the augmented one-shot game, then full cooperation is robust to the fear of miscoordination. However if exit is the risk-dominant equilibrium in the augmented game, then immediate exit is the only robust equilibrium of the game with constant payoffs $\gamma(w_0)$. Section 4.3 provides an illustration of how the robustness criterion of Proposition 4 can yield predictions that are qualitatively different from those obtained when focusing on Pareto-efficient equilibria of the full-information game.

4.3 Wealth, miscoordination fear, and cooperation

This section investigates whether wealth facilitates cooperation or not. It is shown that taking into account the players' fear of miscoordination generates qualitatively new insights about the forces that affect players' ability to cooperate. In the exit game considered here, two symmetric players can cooperate on a project which increases their regular income I by an amount Π. Each player can either cooperate (Stay) or defect (Exit). When both players stay, the life of the project is extended by one period, otherwise the project dies next period and the players get their baseline stream of income. More precisely, we consider the symmetric exit game with the following, constant, flow payoffs

	S	E
S	$u(I + \Pi)$	$u(I - L) + \frac{\beta}{1-\beta}u(I)$
E	$u(I + G) + \frac{\beta}{1-\beta}u(I)$	$u(I) + \frac{\beta}{1-\beta}u(I)$

where payoffs are given for the row player, $G > \Pi > 0$, $L > 0$, $I \geq L$, and u is a concave twice differentiable utility function defined over $(0, +\infty)$.

In this game, the value of full cooperation is $V^H = \frac{1}{1-\beta}u(I+\Pi)$ while the value of immediate exit is $V^L = \frac{1}{1-\beta}u(I)$. Under full information, full cooperation will be sustainable if and only if

$$\frac{\beta}{1-\beta}\left[u(I+\Pi) - u(I)\right] \geq u(I+G) - u(I+\Pi) \tag{11}$$

which is equivalent to $g(I) \equiv \frac{u(I+G)-u(I+\Pi)}{u(I+\Pi)-u(I)} \leq \frac{\beta}{1-\beta}$.

Proposition 5 (wealth makes cooperation harder under full information) *Whenever u exhibits (strictly) decreasing absolute risk aversion ($r \equiv -\frac{u''}{u'}$ decreasing), then g is (strictly) increasing in I.*

Decreasing absolute risk aversion is a standard property of utility functions. For instance it is satisfied for the class of CRRA functions $u(x) = \rho(x^\rho - 1)$, with $\rho \in (-\infty, 1)$. Hence Proposition 5 implies that for natural utility functions, focusing on the Pareto efficient outcome of the game with full information yields the prediction that wealth makes it harder to cooperate. While this prediction is not entirely counter-intuitive – it simply states that the rich just cannot be bothered to cooperate – the fact that it holds for all feasible levels of wealth is rather surprising. This result, however, does not hold anymore once we consider the impact of miscoordination fear.

The game defined above satisfies the conditions of Lemma 13, hence, it admits a global game extension, and we can use the robustness criterion of Proposition 4. Cooperation is robust to miscoordination fear if and only if staying is the risk-dominant action in the augmented symmetric one-shot game

	S	E
S	$u(I+\Pi) + \beta V^H$	$u(I-L) + \frac{\beta}{1-\beta}u(I)$
E	$u(w+G) + \frac{\beta}{1-\beta}u(I)$	$u(I) + \frac{\beta}{1-\beta}u(I)$.

Because the game is symmetric, staying will be risk-dominant if and only if

$$\frac{\beta}{1-\beta}\underbrace{[u(I+\Pi)-u(I)]}_{\text{value of coop.}} \geq \underbrace{u(I+G)-u(I+\Pi)}_{\text{dev. tempt.}} + \underbrace{u(I)-u(I-L)}_{\text{miscoord. loss}} \tag{12}$$

which is equivalent to $h(I) \equiv g(I) + \frac{u(I-D)-u(I-L)}{u(I+\Pi)-u(I)} \leq \frac{\beta}{1-\beta}$.

Condition (12) has an intuitive interpretation: cooperation is robust to miscoordination risk if and only if the value of continued cooperation is greater than the sum of the deviation temptation and the miscoordination loss. For the same reason that $g(I)$ is increasing in I when u exhibits decreasing absolute risk aversion (DARA), the second term of h is decreasing in I when u is DARA, and hence, the monotonicity of h is unclear. The forces of deviation temptation and miscoordination fear push in opposite directions. The following proposition shows that when the loss L upon miscoordination is large enough, the prediction of Proposition 5 is entirely overturned: wealth facilitates cooperation at every income level.

Proposition 6 (wealth facilitates cooperation under miscoordination fear) *Whenever $L \geq G$ and the coefficient of absolute risk aversion $r \equiv -\frac{u''}{u'}$ is decreasing and (strictly) convex over $(0, \infty)$, then h is (strictly) decreasing for $I \in (L, +\infty)$.*

For DARA utility functions, it is quite natural for r to be convex. It simply states that the players' risk tolerance is increasing, but at a diminishing rate. This property is satisfied, for instance, for all CRRA functions. Proposition 6 implies that when strategic risk is significant enough, the impact of wealth on miscoordination fear always dominates the impact of wealth on the deviation temptation. Moreover, even when r is not convex, or $L < G$, the following lemma shows that at least for the very poor, wealth facilitates cooperation once the players' fear of miscoordination is taken into account.

Lemma 14 *Consider any concave function u such that $\lim_{x \to 0} u'(x) = +\infty$, then there exists $I^* > L$ such that h is strictly decreasing over the range (L, I^*).*

Because the miscoordination loss looms very large for the poor, they are particularly wary of miscoordination risk and choose not to purse projects that require them to rely on a partner.

This example shows how taking into account the robustness of cooperation to miscoordination fear can yield new comparative statics that are qualitatively different from those obtained by focusing on Pareto efficiency in the full-information game.

4.4 Trigger strategies in a game of repeated Cournot competition

This section uses the example of repeated Cournot competition to illustrate the point that selection results from Section 3 can be used to study trigger strategy equilibria in two-by-two repeated games with noisy assessments. In particular, the class of perfect Bayesian equilibria supported by trigger strategies can be mapped into the class of subgame perfect equilibria of the exit game in which players get the repeated Nash continuation value upon exit.

In each period, two firms $i \in \{1, 2\}$ can produce a quantity of good $Q_i \in \{Q, (1+\rho)Q\}$. The additional cost of producing ρQ units is $C > 0$. The unit price of the good is $P_t = \frac{D_t}{Q_1+Q_2}$, where D_t represents the strength of demand[15]. Parameters ρ, C, and Q are common knowledge, positive, and fixed in time. The intensity of demand, $\{D_t\}_{t\in\mathbb{N}}$, is an i.i.d. sequence of positive numbers drawn from some distribution f_D with c.d.f. F_D and support $[0, +\infty)$. Each firm gets a signal of current demand strength, $x_{i,t} = D_t + \sigma\varepsilon_{i,t}$. Each player's production decision is ex-post observable. Firms are risk neutral.

We say that a firm cooperates when its production is Q and defects when its production is $(1+\rho)Q$. Under full information, one-shot payoffs (for the row player) are given by

	Coop.	*Defect*
Coop.	$\frac{1}{2}D_t$	$\frac{1}{2+\rho}D_t$
Defect	$\frac{1+\rho}{2+\rho}D_t - C$	$\frac{1}{2}D_t - C.$

Clearly, for any D_t, cooperation is the efficient outcome of this one-shot game. Define $D^{NE} = 2\frac{2+\rho}{\rho}C$. Whenever $D_t > D^{NE}$, then defection is a dominant strategy. Inversely, whenever $D_t < D^{NE}$, then cooperation is a dominant strategy. Hence, this one shot game is dominance solvable. Denote V^{NE} the value of playing this one-shot Nash equilibrium

[15]Note that this particular functional form facilitates the analysis by making the total value of sales constant.

repeatedly under full information. Because this game exhibits increasing differences with respect to the state of the world $w_t \equiv -D_t$, it satisfies the assumptions of Carlsson and van Damme (1993). Hence, for σ small enough, this game is also dominance solvable under a global games information structure. Denote by V_σ^{NE} the value of repeatedly playing the one-shot Nash equilibrium under a global games information structure. Note that as σ goes to 0, V_σ^{NE} converges to V^{NE}.

The question is whether repeated interaction allows firms to sustain greater cooperation under trigger equilibria. Under trigger strategies players revert to repeatedly playing the one shot Nash equilibrium following any defection. In between periods, players also have the option to return to the repeated one-shot equilibrium. This insures that players always expect a continuation value weakly greater than V_σ^{NE}. Any trigger strategy equilibrium must be an equilibrium of the exit game Γ_σ with flow payoffs

	S	E
S	$\frac{1}{2}D_t$	$\frac{1}{2+\rho}D_t + \beta V_\sigma^{NE}$
E	$\frac{1+\rho}{2+\rho}D_t - C + \beta V_\sigma^{NE}$	$\frac{1}{2}D_t - C + \beta V_\sigma^{NE}$.

Because in this game payoffs upon exit are indexed by σ, we are not exactly in the framework of Section 2.3, the game, however, satisfies the more general assumptions of Appendix C: since players have the option to exit in-between periods, rational players expect values V greater than V_σ^{NE} and the equilibrium symmetry assumption holds since the one-shot game with common knowledge has only symmetric equilibria; increasing differences in the state of the world holds with respect to $-D_t$; dominance holds since staying is dominant for D_t close enough to 0, and exit is dominant for any D_t high enough; finally, the assumption that "staying is good" holds because

$$A_i(D_t, V_\sigma^{NE}) = \frac{\rho}{2(2+\rho)} D_t = B_i(\gamma_t) \geq 0.$$

As σ goes to 0, rationalizable strategies of game Γ_σ are bounded by extreme Markovian equilibria[16] whose continuation values converge to the highest and lowest fixed points of

$$\Phi(V) \;=\; \frac{1}{2}\mathbf{E}[D_t] - C + \beta V^{NE} + (C + \beta V - \beta V^{NE}) F_D(x^{RD}(V))$$

Where $x^{RD}(V) = \frac{2+\rho}{\rho}(2C + \beta V - \beta V^{NE})$. It is interesting to note that $\Phi(V^{NE}) = V^{NE}$, hence asymptotically, the one-shot Nash equilibrium is always an equilibrium of Γ_σ. Furthermore $\xi(D^{NE}) = D^{NE}$, independently of the particular distribution of states of the world.

We are particularly interested in the case where there can be multiple equilibria, and in their asymptotic stability properties. The asymptotic local strong rationalizability of Markovian equilibria of Γ_σ is characterized by the stability properties of fixed points of the mapping ξ. Because we are considering symmetric games, the stability of fixed points of ξ is equivalent to the stability of fixed points of Φ.

Lemma 15 *Define $g = \log f_D$. Then whenever $\partial^2 g/\partial D^2 \leq 0$ and $\partial^3 g/\partial D^3 \leq 0$, the mapping Φ is S-shaped.*

Notably, this lemma covers the case of exponential distributions and truncated Gaussian distributions. For such distributions, Φ will admit at most three fixed points.

Figure 2 presents various simulation in the case where f_D follows a truncated Gaussian of parameters (μ, η^2). In all cases, $C = 2, \rho = 1, \eta = 1$, and $\beta = 0.7$. The different cases correspond to different values of μ. Low values of μ put greater weight on states D_t that make cooperation easier to sustain while high values of μ make cooperation typically harder to sustain. For all simulations $D^{NE} = 12$ is an equilibrium threshold. Note that fixed points of ξ below D^{NE} do not correspond to equilibria, since they are associated with continuation values less than V^{NE} which players can opt out of. For $\mu = 8$, Figure 2(a) shows there is a unique rationalizable strategy with threshold D^{NE}. In this case, players already cooperate most of the time in the one-shot Nash equilibrium. Hence incremental

[16] Note that in this example, because the game exhibits increasing differences in $-D_t$ rather than D_t, for a given cooperation threshold x, players cooperate when D_t is less than x rather than greater than x. Hence the greatest equilibrium with respect to $\preceq$ is the one which is associated to the greatest cooperation threshold rather than the smallest cooperation threshold.

amounts of cooperation bring only very little gain in utility and cannot be self-sustained. Figure 2(b) corresponds to the case where $\mu = 12$. It is particularly interesting because only two of the fixed points of ξ correspond to Markovian equilibria. From Theorem 3, we know that the lowest one – corresponding to the one-shot Nash equilibrium – is unstable with respect to iterated best response, while the highest one is stable. This highest equilibrium is the only stable equilibrium of the game. Hence this higher equilibrium can be viewed as the natural outcome: any small amount of optimism will lead players to coordinate on the high cooperation equilibrium. Finally, Figure 2(c) corresponds to the case where $\mu = 15$ which puts greater weight on states of the world that make it difficult to sustain cooperation. There are three equilibria, D^{NE} being the lowest. In this case both extreme equilibria are stable.

5 Conclusion

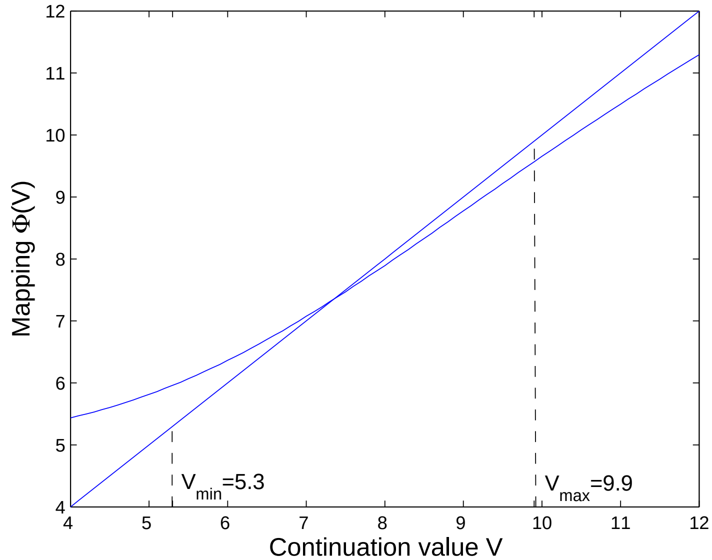
This paper provides a framework to model miscoordination fear in dynamic games. In particular it analyzes the robustness of cooperation to small amounts of observational noise in a class of dynamic games with exit. In equilibrium, this departure from common knowledge generates a fear of miscoordination that pushes players away from the full information Pareto efficient frontier, even though actual miscoordination happens with a vanishing probability. Payoffs upon miscoordination, which play no role when considering the Pareto efficient frontier under full information, determine the extent of the efficiency loss. The greater the loss upon miscoordination, the further will players be pushed away from the full information Pareto frontier.

The first step of the analysis is to show that for cooperation games with exit, the set of rationalizable strategies is bounded by extreme Markovian equilibria. The second step uses the dynamic programming approach to subgame perfection of Abreu, Pearce, and Stacchetti (1990) to recursively apply selection results in one-shot global games. As players' signals become increasingly correlated, this yields a fixed point equation for continuation values associated with Markovian equilibria. Whenever this mapping has a unique fixed point, the set of rationalizable strategies of the game with perturbed information converges to a singleton as signals become arbitrarily precise. However, unlike in one-shot two-by-two games, infinite horizon exit games can still admit multiple equilibria under the global game information structure.

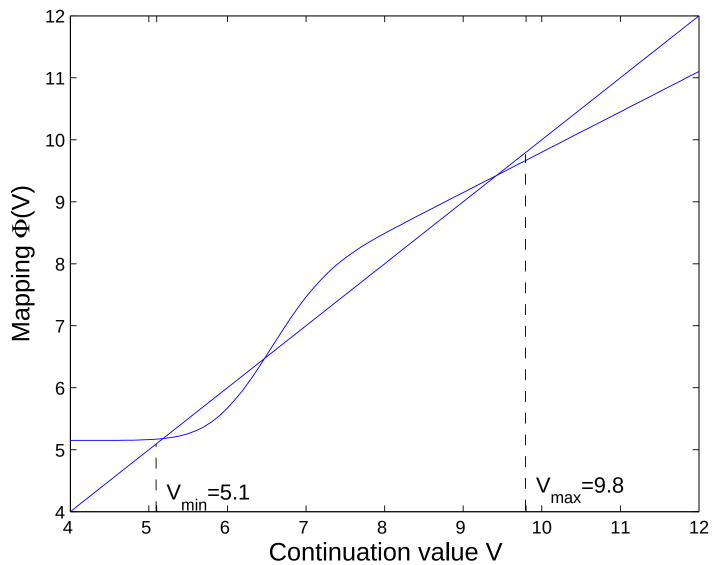
The dominance solvability of one-shot global games carries over in the weaker form of local dominance solvability, which can be interpreted both as a stability property and as a form of robustness to strategic uncertainty. As noise vanishes, the local dominance solvability and basins of attraction of Markovian equilibria are characterized by the stability of fixed points of an explicitly computable increasing mapping from $\mathbb{R}$ to $\mathbb{R}$. The greater the basin of attraction of an equilibrium s, the more robust it is to strategic uncertainty.

Finally, by considering various examples, the paper makes the case that this framework is simple and flexible enough to be used for applied purposes, and that it provides new insights about cooperation that could not be obtained by focusing on Pareto efficiency under full information. In particular, the model can be used to define a robustness criterion for cooperation in exit games with constant payoffs: whenever staying is the risk-dominant strategy of the one-shot game augmented with the players' continuation values, cooperation is robust to any global game extension; whenever defection is the risk-dominant strategy of the one-shot augmented game, then for any global game extension, the set of rationalizable strategies shrinks to immediate exit. This criterion can be readily used in applied games and provides insights on the determinants of cooperation that are qualitatively different from those obtained under full-information.

Figures

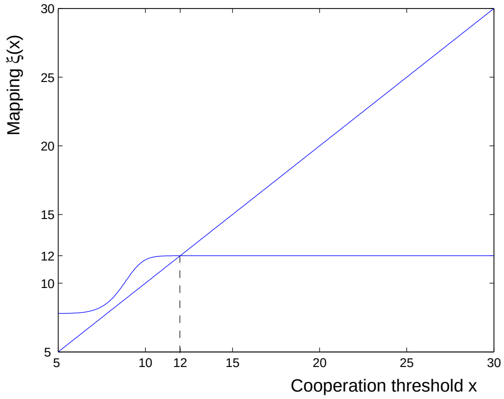


(a) Unique fixed point: $\eta = 1$.

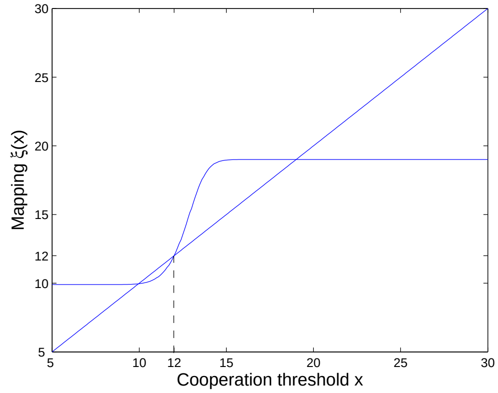


(b) Multiple fixed points: $\eta = 0.2$.

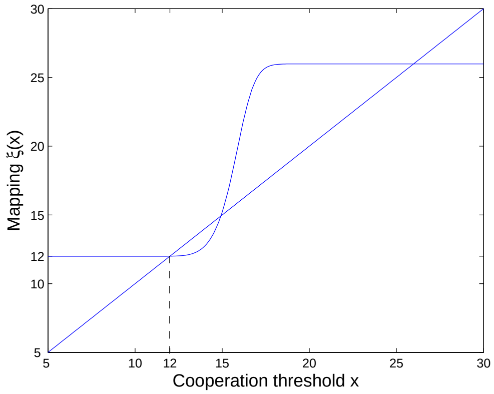
Figure 1: Equilibria of the partnership game depending on η. $V_E = 5$, $\beta = 0.7$, $\mu = 3$, $C = 3$ and $b = 1$



(a) Unique stable equilibrium: $\mu = 8$.



(b) One unstable and one stable equilibria: $\mu = 12$.



(c) One unstable and two stable equilibria: $\mu = 15$.

Figure 2: Trigger equilibria in a game of repeated Cournot competition depending on μ. $C = 2$, $\beta = 0.7$, $\rho = 1$, $\eta = 1$. Fixed points of ξ below $x = 12$ do not correspond to equilibria.

Appendix A: Proofs

Proof of Lemma 1: This is a direct application of Theorems 2, 3 and 4 of Chassang (2006). ■

Proof of Lemma 3: This is a direct application of Theorem 1 of Chassang (2006). ■

Proof of Lemma 6: Consider $s' \in BR_{i,\sigma}(s_x)$, and denote V the value player i expects from best-responding. The one-shot action profile a' induced by s' must belong to $BR_{i,\sigma}(s_x, V)$. Lemma 3 of Chassang (2006) implies that there exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$, there is a unique such one-shot best-reply. It takes a threshold form $a_{x'}$ and the threshold x' is continuous in both x and V. This concludes the proof. ■

Proof of Theorem 1: This is a corollary of Proposition 1. The methodology of Milgrom and Roberts (1990) and Vives (1990) applies almost directly. Let R_i denote the set of rationalizable strategies of player i , U the set of all possible strategies and, $\mathbf{S}$ and $\mathbf{E}$ the strategies corresponding to "always staying" and "always exiting". Define $BR = BR_i \circ BR_{-i}$. R_i is the largest set of strategies such that $R_i \subset BR(R_i)$ and $R_{-i} = BR_{-i}(R_i)$.

Noting that $U = [\mathbf{E}, \mathbf{S}]$, since $\mathbf{S}$ and $\mathbf{E}$ are Markovian, Proposition 1 implies that $BR(U) \subset [BR(\mathbf{E}), BR(\mathbf{S})]$. Since the best response to a Markovian strategy is Markovian, we know that $BR(\mathbf{E})$ and $BR(\mathbf{S})$ are Markovian. This implies we can iterate forward. For all $k \in \mathbb{N}$, we obtain that $R_i \subset [BR^k(\mathbf{E}), BR^k(\mathbf{S})]$. Because $\{BR^k(\mathbf{E})\}_{k\in\mathbb{N}}$ and $\{BR^k(\mathbf{S})\}_{k\in\mathbb{N}}$ are monotone sequences of Markovian strategies, they are equivalent to monotone sequences of indicator functions specifying for all state of the world $w \in \mathbb{R}$ whether the player should stay or exit. As k goes to infinity, these sequences converge in probability to limits $BR^\infty(\mathbf{E})$ and $BR^\infty(\mathbf{S})$. We get that, $R_i \subset [BR^\infty(\mathbf{E}), BR^\infty(\mathbf{S})]$. Denote by s_i^L and s_i^H these extreme strategies (omitting the σ subscript for simplicity). By continuity of the best response mapping with respect to convergence in probability, we have $s_i^H = BR(s_i^H)$ and $s_i^L = BR(s_i^L)$, so that the pairs of strategies $(s_i^H, BR_{-i}(s_i^H))$ and $(s_i^L, BR_{-i}(s_i^L))$ are Nash equilibria in addition to being rationalizable.

It is easy to check that whenever s is a threshold form Markovian strategy, then Assumption 3 implies that $BR_i(s)$ is also Markovian and takes a threshold form. Since $\mathbf{E}$ and $\mathbf{S}$ take threshold forms, then by induction, extreme strategies also take a threshold form. ■

Proof of Lemma 7: Consider the highest equilibrium s_σ^H. For any rationalizable strategy s_{-i}, $s_{-i} \preceq s_{-i}^H$. Assumption 5, implies that player i gets a higher value from best-replying against $s_{-i,\sigma}^H$ than s_{-i}. Thus $V_i \leq V_{i,\sigma}^H$ in the functional sense. Identical reasoning yields the other inequality. ■

Proof of Lemma 8: Since $\mathbf{V}_n$ converges to $\mathbf{V}$ and Φ is continuous, for all $\tau > 0$, there exists N_1 such that for all $n \geq N_1$

$$||\Phi(\mathbf{V}) - \mathbf{V}||_\infty \leq ||\Phi(\mathbf{V}_n) - \mathbf{V}_n||_\infty + \tau/2.$$

Since $\phi_{\sigma_n}(\cdot)$ converges uniformly to Φ and $\mathbf{V}_n$ is a fixed point of ϕ_{σ_n}, there exists N_2 such that for all $n \geq N_2$, $||\Phi(\mathbf{V}_n) - \mathbf{V}_n||_\infty \leq \tau/2$. This yields that $||\Phi(\mathbf{V}) - \mathbf{V}|| \leq \tau$ for all $\tau > 0$. Hence, $\mathbf{V}$ must be a fixed point of Φ. ■

Proof of Lemma 9: Indeed, it results from expression (7) that

$$||\Phi(V) - \Phi(V')||_1 \leq \beta||V - V'||_1 + ||f||_\infty \sum_{i\in\{1,2\}} ||g^i_{11} + \beta V_i - W^i_{22}||_\infty \left|\left| \frac{\partial x^{RD}}{\partial V_i} + \frac{\partial x^{RD}}{\partial V_{-i}} \right|\right|_\infty ||V - V'||_1$$

Since $\sum_{i\in\{1,2\}} ||g^i_{11} + \beta V_i - W^i_{22}||_\infty || \frac{\partial x^{RD}}{\partial V_i} + \frac{\partial x^{RD}}{\partial V_{-i}} ||_\infty$ is finite, for any $\delta > \beta$, there exists $||f||_\infty$ small enough such that $||\Phi(V) - \Phi(V')||_1 \leq \delta||V - V'||_1$. ■

Proof of Lemma 11: $BRV_{i,\sigma}(x)$ is continuous in x since it is the maximum of a bounded function continuous in x. In conjunction with Theorems 2 and 4 of Chassang (2006), this yields the first part of the lemma.

Lemma 10 implies that for σ small enough all Markovian equilibria must take a threshold form. Such an equilibrium is associated with values (V_i, V_{-i}) and thresholds $\mathbf{x} = (x_i, x_{-i})$ which must satisfy $V_i = BRV_{i,\sigma}(x_{-i})$ and $(x_i, x_{-i}) = \mathbf{x}^*_\sigma(V_i, V_{-i})$. Hence, Markovian thresholds must satisfy $\mathbf{x} = \xi_\sigma(\mathbf{x})$. Inversely, if a vector x satisfies $\mathbf{x} = \xi_\sigma(\mathbf{x})$, then the values $\mathbf{V} = (V_i, V_{-i})$ defined by $V_i = BRV_{i,\sigma}(x_{-i})$ must satisfy, $\mathbf{V} = \phi(\sigma, \mathbf{V})$ and hence, using Theorem 2, values $\mathbf{V}$ support a Markovian equilibrium with thresholds $\mathbf{x}$. This gives us the second part of the lemma.

The third part of the lemma is an almost immediate consequence of Theorem 3 of Chassang (2006). One only needs to show that $BRV_{i,\sigma}(\cdot)$ converges uniformly to $BRV_{i,0}(\cdot)$ as σ goes to 0. Indeed, because states of the world have a bounded density and payoffs are Lipschitz, the best response when $\sigma = 0$ is almost optimal when $\sigma > 0$ and small and vice-versa. Hence, there exists $K > 0$ such that for all $x \in \mathbb{R}$, $|BRV_{i,0}(x) - BRV_{i,\sigma}(x)| < K\sigma$.

The fourth part of the lemma is a consequence of the fact that $x^{RD}(\mathbf{V})$ is decreasing in $\mathbf{V}$ and Assumption 5 which implies that $BRV_{i,0}(x)$ is decreasing in x. ■

Proof of Proposition 2: A direct proof can be given but it is faster to use the local dominance solvability property that will be proven in Theorem 3. For any $x \in \mathbb{R}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_x)$ takes a threshold form, $s_{x'}$. Define $\chi_\sigma(\cdot)$ by $\chi_\sigma(x) = x'$. For σ small enough, Lemma 6 and Proposition 1 imply that χ_σ is continuous and increasing. By definition of χ_σ, s_x is a threshold form Markovian equilibrium of Γ_σ if and only if $\chi_\sigma(x) = x$.

Consider a non singular fixed point of ξ denoted by x. Indeed, Assume that x is a stable fixed point – that is ξ cuts the 45^o line from below – then Theorem 3 implies that, for all $\tau > 0$, there exists $\overline{\sigma} > 0$ and $\eta \in (0, \tau)$ such that for all $\sigma \in (0, \overline{\sigma})$, the interval $[x - \eta, x + \eta]$ is stable by χ_σ. Since χ_σ is continuous and increasing, this implies that it has a fixed point belonging to $[x - \eta, x + \eta]$. This proves the lower hemicontinuity of stable fixed points of ξ.

Assume that x is unstable. Then for any $\tau > 0$, there exists $\eta \in (0, \tau)$ such that $x - \eta$ and $x + \eta$ respectively belong to the basins of attraction of a lower and a higher fixed point.

Proposition 3 implies that there exist η' and η'' in $(0, \eta)$ such that $\chi_\sigma(x - \eta') < x - \eta'$ and $\chi_\sigma(x + \eta'') > x + \eta''$. Since χ_σ is continuous, this implies that it admits a fixed point within $[x - \eta', x + \eta'']$. This proves the lower hemicontinuity of unstable non-singular points of ξ. ■

Lemma 16 *Consider x a fixed point of ξ. Then there exists $\eta > 0$ and $\overline{\sigma} > 0$ such that for all $\sigma \in [0, \overline{\sigma})$, $x' \in [x - \eta, x + \eta]$, and $i \in \{1, 2\}$ there exists $x'' \in \mathbb{R}$ such that $BR_{i,\sigma}(s'_x) = \{s_{x''}\}$ and $|x'' - x'| < 2\sigma$.*

Proof of Lemma 16: Since, x is a fixed point of ξ, it must be that at $w = x$, both (E, E) and (S, S) are strict Nash equilibria of the game $G(BRV_{i,0}(x), BRV_{-i,0}(x), w)$. Since $BRV_{i,\sigma}(x')$ is continuous in σ and x', and payoffs are continuous in w, there exist $\eta > 0$ and $\overline{\sigma} < \eta/4$ such that for all $\sigma < \overline{\sigma}$ and $x' \in [x - \eta, x + \eta]$, then for all $w \in [x' - \sigma, x' + \sigma]$, both (E, E) and (S, S) are strict Nash equilibria of $G(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'), w)$.

Take $\sigma < \overline{\sigma}$ and $x' \in [x - \eta/2, x + \eta/2]$. Assumption 3 implies that for any σ, the best reply to a threshold form strategy is also a threshold for strategy. This implies that indeed $BR_{i,\sigma}(x')$ takes the form $s_{x''}$. Let us show that$|x'' - x'| < 2\sigma$. When she gets a signal $x_{i,t} < x' - 2\sigma$, player i knows for sure that player $-i$ will be playing E. From the definition of η, we know that (E, E) is an equilibrium of $G(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'), w)$ for all values of w consistent with a signal value $x_{i,t}$. Thus, it must be that player i's best reply is to play E as well. Inversely, when she gets a signal $x_{i,t} > x' + 2\sigma$, player i knows that player $-i$ will play S, and her best reply is to Stay as well. This implies that $|x'' - x'| < 2\sigma$. ■

Lemma 17 *Define the function $\zeta : \mathbb{R} \mapsto \mathbb{R}$ by, $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)}\mathbf{E}\left[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}\right]$ is the value player i obtains when both players naively follow the threshold strategy s_x.*

Then, x is a fixed point of ξ if and only if it is a fixed point of ζ. Furthermore, for any fixed point x, there exists $\eta > 0$ such that for all $x' \in [x - \eta, x + \eta]$, $\zeta(x') = \xi(x')$.

Proof of Lemma 17: Lemma 16 implies that for $\sigma = 0$, whenever x is a fixed point of ξ, then there exists $\eta > 0$ such that for all $x' \in [x - \eta, x + \eta]$, $BR_i(s_{x'}) = s_{x'}$. Hence, for all $x' \in [x - \eta, x + \eta]$, $BRV_i(x') = NV_i(x')$ and thus, $\xi(x') = \zeta(x')$.

Moreover whenever x satisfies $\zeta(x) = x$, then s_x is a threshold form Markovian equilibrium of the full information game Γ_0, which implies that $BRV_i(x) = NV_i(x)$. Thus $\xi(x) = \zeta(x) = x$ and x is also a fixed point of ζ. ■

Proof of Lemma 12: Let us first show that the set P of payoff structures such that ξ has a finite number of fixed points and has a derivative different from 1 at each of these fixed points is open in Π_1. From Lemma 17, we know that x is a non-singular fixed point of ξ if and only if it is a non-singular fixed point of ζ. One can compute ζ explicitly : $\zeta(x) = x^{RD}(NV_i(x), NV_{-i}(x))$, where $NV_i(x) \equiv \frac{1}{1-\beta Prob(w>x)}\mathbf{E}\left[g^i + (W^i_{22} - g^i)\mathbf{1}_{x>w}\right]$. Since

$x^{RD}(V_i, V_{-i})$ is defined as the locally unique solution of the C^1 equation

$$Q_\pi(x, V_i, V_{-i}) \equiv \prod_{i\in\{1,2\}} (g^i + \beta V_i - W_{21}^i) - \prod_{i\in\{1,2\}} (W_{22}^i - W_{12}^i) = 0$$

the implicit function theorem implies that $\frac{\partial x^{RD}}{\partial V_i}$ exists and is a continuous expression of the derivatives of the payoff functions $(g^i, W_{12}^i, W_{21}^i, W_{22}^i)_{i\in\{1,2\}}$. Hence ζ admits a derivative,

$$\begin{aligned}\frac{\partial \zeta}{\partial x} = \quad \textstyle\sum_{i\in\{1,2\}} \quad & \frac{\partial x^{RD}}{\partial V_i}\Bigg(\frac{1}{1-\beta Prob(w>x)} f_w(x^{RD})\left(W_{22}^i(x^{RD}) - g^i(x^{RD})\right) \\ & - \frac{\beta f(x)}{(1-\beta Prob(w>x))^2}\mathbf{E}[g^i + (W_{22}^i - g^i)\mathbf{1}_{x>w}]\Bigg).\end{aligned}$$

This derivative is continuous with respect to x and continuous in the payoff structure with respect to $||\cdot||_{\Pi_1}$. Assume that for a payoff structure π, the mapping ξ has a finite number of fixed points and has a derivative that is different from 1 at all it's fixed points. Then there exists $\eta > 0$ such that for any fixed point x, $\frac{\partial \xi}{\partial x}$ is either less than $1-\eta$ over $[x-\eta, x+\eta]$ or greater than $1+\eta$ over $[x-\eta, x+\eta]$. A payoff structure $\tilde{\pi}$ close enough to π, is associated with a mapping $\tilde{\xi}$ such that all fixed points $\tilde{x}$ of $\tilde{\xi}$ belong to $[x-\eta, x+\eta]$ and such that its derivative over $[x-\eta, x+\eta]$ is either greater than $1+\eta/2$ or lower than $1-\eta/2$. This implies that all payoff structures close enough to π are also associated with mappings ξ that have a finite number of fixed points and have a derivative different from 1 at each of these fixed points.

Let us now show that P is dense in Π_1. Consider a payoff structure π and $\nu > 0$. We know from Assumption 2 that fixed points of ξ are restricted to a compact region $[\underline{x}, \overline{x}]$. By Weierstrass's Theorem, there exist uniform polynomial approximations of the derivative of the vector of functions π over $[\underline{x}-1, \overline{x}+1]$. Hence, one can find a payoff structure $\tilde{\pi}$ such that $||\pi - \tilde{\pi}||_{\Pi^1} < \nu/2$, π, $\tilde{\pi}$ coincide over the complementary of $[\underline{x}, \overline{x}]$, and $\tilde{\pi}$ is polynomial over $[\underline{x}, \overline{x}]$. By Lemma 18, this implies that the mapping ζ is analytic over $[\underline{x}, \overline{x}]$. Let us now define the family of payoffs $\tilde{\pi}^\delta$ by,

$$\begin{aligned}\forall w \in \mathbb{R}, \qquad & \tilde{g}^{i,\delta}(w) \equiv \tilde{g}^i(w) \\ & \tilde{W}_{22}^{i,\delta}(w) = \tilde{W}_{22}^i(w) \\ & \tilde{W}_{12}^{i,\delta}(w) = \tilde{W}_{12}^i(w-\delta) + \tilde{g}^i(w) - \tilde{g}^i(w-\delta) \\ & \tilde{W}_{21}^{i,\delta}(w) = \tilde{W}_{21}^i(w-\delta) + \tilde{W}_{22}^i(w) - \tilde{W}_{22}^i(w-\delta)\end{aligned}$$

This new payoff structure is such that for any δ, and any $x \in \mathbb{R}$, $\tilde{\zeta}^\delta(x) = \tilde{\zeta}(x) + \delta$. Note that for δ small enough, $\tilde{\pi}^\delta$ is arbitrarily close to $\tilde{\pi}$. More over $\tilde{\zeta}^\delta$ is analytic in x. Whenever $\delta_1 \neq \delta_2$ then $\tilde{\zeta}^{\delta_1}$ and $\tilde{\zeta}^{\delta_2}$ have strictly different fixed points. Assume that for every $\delta \in (0, \nu)$, there exists a fixed point x^δ of $\tilde{\zeta}^\delta$ such that $\tilde{\zeta}^\delta$ is singular at x^δ. this implies that the derivative of ζ is equal to 1 an infinte number of times in a compact set. Since the derivative of ζ is analytic, this would imply that it is identically equal to 1 over $[\underline{x}-1, \overline{x}+1]$. Since ζ has a fixed point in $[\underline{x}, \overline{x}]$, this would imply that ζ is equal to the identity over $[\underline{x}-1, \overline{x}+1]$ which contradicts the fact that fixed points of ζ belong to $[\underline{x}, \overline{x}]$. ■

Lemma 18 (Analyticity of ζ) *Whenever functions of the payoff structure*

$$\pi = \times_{i\in\{1,2\}}(g^i, W_{12}^i, W_{21}^i, W_{22}^i) \times F_w$$

are polynomial over the range $[\underline{w}, \overline{w}]$ then the mapping ζ is analytic.

Proof: We give the proof for the stationary case. First, note that $x^{RD}(V_i, V_{-i})$ is a simple root of the polynomial $Q(w) = \Pi_{i\in\{1,2\}}(g^i(w) + \beta V_i - W_{21}^i(w)) - \Pi_{i\in\{1,2\}}(W_{22}^i(w) - W_{12}^i(w))$. Indeed, $Q(w)$ is strictly decreasing at x^{RD}. A simple root of a polynomial is jointly analytic in the polynomial's coefficients. This implies that $x^{RD}(V_i, V_{-i})$ is analytic in (V_i, V_{-i}). Further more the functions $V_i(x)$ and $V_{-i}(x)$ can be computed explicitly:

$$V_i(x) = \frac{1}{1-\beta P(w>x)}\mathbf{E}[W_{22}^i(x) + (g^i - W_{22}^i)\mathbf{1}_{w>x}].$$

Clearly, $V_i(x)$ is analytic in x. Since the composition of analytic functions is analytic, this implies that ζ is analytic in x. ■

Proof of Proposition 3: Let us prove the first part of the proposition. Define $ba_-(x) = \inf\{\tilde{x} < x \mid \forall y \in [\tilde{x}, x], \xi(y) > y\}$. Because x is stable, $ba_-(x)$ is well defined. We distinguish two cases, either $ba_-(x) = -\infty$ or $ba_-(x) \in \mathbb{R}$.

If $ba_-(x) = -\infty$, any $x' < x$ belongs to the basin of attraction of x. Assumption 2 implies that there exists $\underline{x}$ such that for all $\sigma < 1$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{-\infty}) \preceq s_{\underline{x}}$. Pick any $x' < \min\{x, \underline{x}\}$. Using the monotonicity implied by Proposition 1, we conclude that there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) \preceq BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{-\infty}) \preceq s_{x'}$.

Consider now the case where $ba_-(x) \in \mathbb{R}$. Then by continuity of ξ, we have that $\xi(ba_-(x)) = ba_{-x}$. From Lemma 16 we know that there exist $\eta > 0$ and $\overline{\sigma}$ such that for all $x' \in [ba_-(x) - \eta, ba_-(x) + \eta]$, and $i \in \{1,2\}$, $BR_{i,\sigma}(s_{x'}) = s_{x_i''}$ with $|x_i'' - x'| < 2\sigma$. By definition, we must have $y > ba_-(x)$. Thus we can pick $x' \in (ba_-(x), ba_-(x)+\eta)$ such that $x' < \min\{x, y\}$. We have that $\xi(x') > x'$. By continuity of ξ there exists $\tilde{x}'$ such that $\tilde{x}' < x'$ and $\xi(\tilde{x}') > x'$. Using the notation $BR_{i,\sigma}(a, V)$ to denote the best reply of player i to a one shot action profile a and continuation value V, and using the fact that one-shot action profile are identical to Markovian strategies, we obtain,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) = BR_{i,\sigma}\left(BR_{-i}(s_{x'}, BRV_{-i,\sigma}(x')), BRV_{i,\sigma}(BR_{-i,\sigma}(s_{x'}))\right) \tag{13}$$

We know that $|x_{-i}'' - x'| \leq 2\sigma$. Thus there exists $\overline{\sigma}$ small enough such that $BR_{-i,\sigma}(s_{x'}) \preceq s_{\tilde{x}'}$. Joint with Assumption 5, this implies that, $BRV_{i,\sigma}(BR_{-i,\sigma}(s_{x'})) \leq BRV_{i,\sigma}(\tilde{x}')$. Furthermore, $\tilde{x}' < x'$ implies that $BRV_{i,\sigma}(x') \leq BRV_{i,\sigma}(\tilde{x}')$. Hence, using inequality (13), and the fact that for $i \in \{1,2\}$, $BR_i(a,V)$ is increasing in a and V with respect to $\preceq$, we obtain

$$
\begin{aligned}
BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'}) &\preceq BR_{i,\sigma}\left(BR_{-i}(s_{x'}, BRV_{-i,\sigma}(\tilde{x}')), BRV_{i,\sigma}(\tilde{x}')\right) \\
&\preceq BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i,\sigma}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)(s_{x'})
\end{aligned}
\tag{14}
$$

We know from Theorem 2 of Chassang (2006) that there exists $\overline{\sigma}$ small enough such that for all $\sigma \in (0, \overline{\sigma})$ and all $(V_i, V_{-i}) \in [m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $\Psi_\sigma(V_i, V_{-i})$ has a unique rationalizable pair of strategies $\mathbf{x}^*_\sigma(V_i, V_{-i})$.

We know from Theorem 3 of Chassang (2006) that $\mathbf{x}^*_\sigma(V_i, V_{-i})$ converges uniformly to $x^{RD}(V_i, V_{-i})$ as σ goes to 0. This implies that $\mathbf{x}^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$ converges to $(\xi(x'), \xi(x'))$ as σ goes to 0. Since $x' < \xi(\tilde{x}')$, it implies there exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, $x' < x^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$.

The fact that $\Psi_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$ has a unique rationalizable strategy and the monotonicity of Lemma 3 imply that the sequence of threshold form strategies

$$
\left(BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)\right)^n (s_{x'}), \quad \text{for } n \in \mathbb{N},
$$

converges monotonously to the Markovian strategy of threshold $x^*_\sigma(BRV_{i,\sigma}(x'), BRV_{-i,\sigma}(x'))$. Since $x' < x^*_\sigma(BRV_{i,\sigma}(\tilde{x}'), BRV_{-i,\sigma}(\tilde{x}'))$, the sequence must be decreasing with respect to $\preceq$. Thus $BR_{i,\sigma}\left(\cdot, BRV_{i,\sigma}(\tilde{x}')\right) \circ BR_{-i,\sigma}\left(\cdot, BRV_{-i,\sigma}(\tilde{x}')\right)(s_{x'}) \preceq s_{x'}$. Using inequality (14), this yields that indeed $BR_{i,\sigma} \circ BR_{-i,\sigma}(s'_x) \preceq s_{x'}$.

The second part of the lemma results from a symmetric reasoning, switching all inequalities. ■

Proof of Lemma 13: Set $w_0 = 0$. Denote m_i and M_i the bounds on value implied by Assumption 1. For $i \in \{1,2\}$, define $\lambda_i = \min\{1, \frac{\gamma^i_{22}(0) - \gamma^i_{12}(0)}{\gamma^i_{21}(0) - m_i - \gamma^i_{11}(0)}\}$. Note that $\lambda_i > 0$. For any $w \in \mathbb{R}$ and $\imath \in \{1,2\}$, define $\gamma^i(w)$ by,

$$
\gamma^i_{21}(w) = \gamma^i_{21}(0)\ ; \quad \gamma^i_{22}(w) = \gamma^i_{22}(0)\ ; \quad \gamma^i_{11}(w) = \gamma^i_{11}(0) + w
$$

$$
\gamma^i_{12}(w) = \begin{cases} \gamma^i_{12}(0) + \lambda_i w & \text{when } w > 0 \\ \gamma^i_{12}(0) + \lambda_i^{-1} w & \text{when } w < 0 \end{cases}
$$

Assume that Assumption 5 holds strictly, more precisely, that, $A(0, m_i) > 0$. Then whenever, f_n is close enough to a Dirac mass at 0, there exists a lower bound m^n_i arbitrarily close to m_i. Pick any sequence $\{f_n\}_{n\in\mathbb{N}}$ with support $\mathbb{R}$ and weakly converging to a Dirac mass at 0 as n goes to ∞. Then there exists N such that for all $n > N$, $A(0, m^n_i) > 0$. Since for all w, $A(w, m^n_i) \geq A(0, m^n_i)$, this implies that for all n, $\pi_n = (\gamma, f_n)$ satisfies Assumption 5. Assumptions 1, 4, 3 and 2 are easily checked. Hence $\{(\gamma, f_n)\}_{n>N}$ is a global game extension of $\gamma(0)$.

When $A(0, m_i) = 0$, then the sequence f_n has to be chosen appropriately skewed to the right so that $m^n_i \geq m_i$. This can clearly be done, since by skewing f_n to the right, we can give value to staying by guaranteeing future cooperation in dominant states. This essentially puts us in the former case, and for such a sequence $\{f_n\}_{n\in\mathbb{N}}$, $\{(\gamma, f_n)\}_{n>N}$ is a global game extension of $\gamma(0)$. ■

Proof of Proposition 5: We have that,

$$\frac{\partial \log g(I)}{\partial I} = \frac{u'(I+G)-u'(I+\Pi)}{u(I+G)-u(I+\Pi)} - \frac{u'(I+\Pi)-u'(I)}{u(I+\Pi)-u(I)} \tag{15}$$
$$= \frac{\int_{I+\Pi}^{I+G} u''(x)dx}{\int_{I+\Pi}^{I+G} u'(x)dx} - \frac{\int_{I}^{I+\Pi} u''(x)dx}{\int_{I}^{I+\Pi} u'(x)dx}$$

Consider the following lemma.

Lemma 19 *For any $n \in \{1, \ldots, +\infty\}$, consider sequences $\{a_1, b_1, \ldots, a_n, b_n\}$ and $\{a'_1, b'_1, \ldots, a'_n, b'_n\}$ such that for all $k \in \{1, \ldots, n\}$, $b_k > 0$, $b'_k > 0$, and*

$$\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \cdots \frac{a_n}{b_n} \le \frac{a'_1}{b'_1} \le \frac{a'_2}{b'_2} \le \cdots \le \frac{a'_n}{b'_n}$$

then we have that

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \le \frac{a'_1 + \cdots + a'_n}{b'_1 + \cdots + b'_n}.$$

For any $n \ge 1$, and $k \in \{0, \ldots, n\}$, consider the wealth levels $x_k^n = I + \Pi + \frac{k}{n}(G - \Pi)$ and $y_k^n = I + \frac{k}{n}\Pi$. Lemma 19 applies to the numbers $a_k = u''(y_k^n)$, $b_k = u'(y_k^n$, $a'_k = u''(x_k^n)$, and $b'_k = u'(x_k^n)$. This yields that,

$$\frac{\frac{G-\Pi}{n}\sum_{k=0}^{n} u''(x_k^n)}{\frac{G-\Pi}{n}\sum_{k=0}^{n} u'(x_k^n)} \ge \frac{\frac{\Pi}{n}\sum_{k=0}^{n} u''(y_k^n)}{\frac{\Pi}{n}\sum_{k=0}^{n} u'(y_k^n)}.$$

Letting n go to infinity and using equation (15) yields that $\frac{\partial \log g(I)}{\partial I} \ge 0$. The proof can be easily adapted to show that the inequality holds strictly whenever u exhibits strictly diminishing absolute risk aversion. ■

Proof of Lemma 19: The property obviously holds for $n = 1$. Let us show it holds for $n = 2$. $\frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a'_1}{b'_1} \le \frac{a'_2}{b'_2}$ implies the four inequalitites $a_k b'_l \le a'_k b_l$ for $(k, l) \in \{1, 2\}^2$. Summing these inequalities and dividing both sides of the resulting inequality by $(b_1 + b_2)(b'_1 + b'_2)$ yields the result.

We prove by induction the property for $n \ge 2$. Assume it holds for $n - 1$, then by applying it to the subsequences $(a_1, b_1, \ldots, a_{n-1}, b_{n-1})$ and $(a'_2, b'_1, \ldots, a'_n, b'_n)$ yields that

$$\frac{a_1 + \cdots + a_{n-1}}{b_1 + \cdots + b_{n-1}} \le \frac{a_n}{b_n} \le \frac{a'_1}{b'_1} \le \frac{a'_1 + \cdots + a'_n}{b'_1 + \cdots + b'_n}.$$

We can again apply the property for $n = 2$ to this last inequality. It yields that

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \frac{a'_1 + \cdots + a'_n}{b'_1 + \cdots + b'_n}$$

which concludes the proof. ■

Proof of Proposition 6: We have,

$$\frac{\partial \log h(I)}{\partial I} = \frac{u'(I+G) - u'(I-L)}{u(I+G) - u(I-L)} - \frac{u'(I+\Pi) - u'(I)}{u(I+\Pi) - u(I)}.$$

Define

$$d(I) \equiv \frac{u'(I+G) - u'(I-L)}{u(I+G) - u(I-L)} = \frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3}$$

where

$$\begin{array}{ll} A_1 = u'(I+G) - u'(I+\Pi) & B_1 = u(I+G) - u(I+\Pi) \\ A_2 = u'(I+\Pi) - u'(I) & B_2 = u(I+\Pi) - u(I) \\ A_3 = u'(I) - u'(I-L) & B_3 = u(I) - u(I-L) \end{array}$$

Proving that $\frac{\partial \log h(I)}{\partial I} < 0$ boils down to showing that $\frac{A_1+A_2+A_3}{B_1+B_2+B_3} < \frac{A_2}{B_2}$. We have

$$\frac{A_1 + A_2 + A_3}{B_1 + B_2 + B_3} = \frac{A_1}{B_1} \frac{B_1}{B_1 + B_2 + B_3} + \frac{A_2}{B_2} \frac{B_2}{B_1 + B_2 + B_3} + \frac{A_3}{B_3} \frac{B_3}{B_1 + B_2 + B_3}. \tag{16}$$

We know from the proof of Proposition 5, that $\frac{A_1}{B_1} \geq \frac{A_2}{B_2} \geq \frac{A_3}{B_3}$. Since by assumption $L \geq G$, we have $B_3 \geq B_1$. These last two inequalities and equation (16) imply that to prove $\frac{A_1+A_2+A_3}{B_1+B_2+B_3} \leq \frac{A_2}{B_2}$, it is sufficient to show that $\frac{1}{2}\left(\frac{A_1}{B_1} + \frac{A_3}{B_3}\right) < 2\frac{A_2}{B_2}$. We know from the proof of Proposition 5 that

$$\frac{A_2}{B_2} \geq \frac{u''(I)}{u'(I)}. \tag{17}$$

Consider the following lemma.

Lemma 20 *For any $n \in \mathbb{N}$, consider a sequence of numbers $\{a_1, b_1, a_2, b_2, \ldots, a_{2n+1}, b_{2n+1}\}$ such that $b_1 \geq b_2 \geq \cdots \geq b_{2n+1} > 0$,*

$$\frac{a_1}{b_1} \leq \cdots \leq \frac{a_{2n+1}}{b_{2n+1}}$$

and for all $i \in \{1, \ldots, k\}$ then, we have that

$$\frac{a_1 + a_2 + \cdots + a_{2n+1}}{b_1 + b_2 + \cdots + b_{2n+1}} \leq \frac{a_n}{b_n}.$$

By considering integrals as the limit of sums as in the proof of Proposition 5, this lemma and the concavity of $\frac{u''}{u'}$ imply that

$$\frac{A_1}{B_1} = \frac{u'(I+G)-u'(I+\Pi)}{u(I+G)-u(I+\Pi)} \leq \frac{u''\left(I+\frac{G-\Pi}{2}\right)}{u'\left(I+\frac{G-\Pi}{2}\right)} \tag{18}$$

$$\frac{A_3}{B_3} = \frac{u'(I)-u'(I-L)}{u(I)-u(I-L)} \leq \frac{u''\left(I-\frac{L}{2}\right)}{u'\left(I-\frac{L}{2}\right)}. \tag{19}$$

Hence, by using inequalities (17), (18), and (19), we obtain that,

$$\frac{A_1}{B_1}+\frac{A_3}{B_3}-2\frac{A_2}{B_2} \leq 2r(I) - r\left(I+(G-\Pi)/2\right) - r\left(I-L/2\right).$$

Since $L \geq G > G-\Pi$, and r is strictly convex, this implies that indeed

$$\frac{A_1}{B_1}+\frac{A_3}{B_3}-2\frac{A_2}{B_2} < 0.$$

This implies that $\frac{\partial \log h(I)}{\partial I} < 0$, and concludes the proof. ■

Proof of Lemma 20: We can write,

$$\frac{a_1+a_2+\cdots+a_{2n+1}}{b_1+b_2+\cdots+b_{2n+1}} = \sum_{i=1}^{n} \frac{b_{n-i}+b_{n+i}}{\sum_{j=1}^{n} b_{n-j}+b_{n+j}} \left[\frac{a_{n-i}}{b_{n-i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} + \frac{a_{n+i}}{b_{n+i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}}\right]. \tag{20}$$

By assumption, we know that $\frac{a_{n-i}}{b_{n-i}} \leq \frac{a_{n+i}}{b_{n+i}}$, and $b_{n-i} \geq b_{n+i} > 0$. This yields that

$$\frac{a_{n-i}}{b_{n-i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} + \frac{a_{n+i}}{b_{n+i}}\frac{b_{n-i}}{b_{n-i}+b_{n+i}} \leq \frac{1}{2}\left(\frac{a_{n-i}}{b_{n-i}}+\frac{a_{n+i}}{b_{n+i}}\right).$$

Using the assumption that $\frac{1}{2}\left(\frac{a_{n-i}}{b_{n-i}}+\frac{a_{n+i}}{b_{n+i}}\right) \leq \frac{a_n}{b_n}$ and reinjecting in expression (20) yields that indeed,

$$\frac{a_1+a_2+\cdots+a_{2n+1}}{b_1+b_2+\cdots+b_{2n+1}} \leq \frac{a_n}{b_n}$$

which concludes the proof. ■

Appendix B: Extension to non-stationary games

From a methodological perspective, this paper shows how selection results holding for one-shot global games can be exploited to derive insights on the impact of a global game information structure in dynamic games. Because the key step of the approach is to recognize that actions in dynamic subgame perfect equilibria must be Nash equilibria in a one shot global game with augmented payoffs, there is hope that this methodology can be scaled – at least in part – to study the impact of a global game information structure on a variety of other games. This appendix extends the results of Section 3 to non-stationary exit games.

B.1 The setup

There are two players $i \in \{1, 2\}$, time is discrete $t \in \{1, \ldots, \infty\}$, players have discount rate β and there are two actions $A = \{Stay, Exit\}$. In addition, payoffs are indexed by a state of the world $w_t \in \mathbb{R}$, which is independently drawn each period, and by a state variable $k_t \in K \subset \mathbb{R}^d$. We will discuss different processes for k_t. Given the state of the world, player i expects flow payoffs,

	S	E
S	$g^i(w_t, k_t)$	$W^i_{12}(w_t, k_t)$
E	$W^i_{21}(w_t, k_t)$	$W^i_{22}(w_t, k_t)$

where i is the row player. States of the world $\{w_t\}_{t \in \{1,\ldots,\infty\}}$ form an i.i.d. sequence of real numbers drawn from a distribution with density f_w, c.d.f. F and convex support $I \subset \mathbb{R}$. All payoffs, $g^i, W^i_{12}, W^i_{21}, W^i_{22}$ are continuous in w_t and k_t.

The state variable k_t is perfectly observable at the beginning of period t and common knowledge. The state of the world w_t is unknown but players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$, where $\{\varepsilon_{i,t}\}_{i,t}$ is a sequence of independent random variables lying in the interval $[-1, 1]$.

We allow for the possibility of players' final payoffs to be shifted by some idiosyncratic noise $\eta_{i,t}$ independent of everything else. That makes the true state of the world unobservable ex-post, but it is also more realistic and adds no difficulty. Let us denote $r_{i,t} = g(w_{i,t}) + \eta_{i,t}$ the realized payoff obtained when both players stay.

Whenever there is an exit, the game stops and we assume without loss of generality that players get a zero continuation value. Because of exit, any history $h_{i,t}$ is characterized by a sequence of past signals and past outcomes: $h_{i,t} \equiv \{x_{i,1}, \ldots, x_{i,t}\ ;\ r_{i,1}, \ldots, r_{i,t-1}\ ,\ k_1, \ldots, k_t\}$. Let us denote $\mathcal{H}$ the set of all such sequences. We will denote $V_i(h_{i,t})$ the value of playing the game starting at history $h_{i,t}$ from the perspective of player i. Note that these continuation values are endogenously determined and will depend on players strategies.

Assumption 6 (control) *There are finite bounds on the value of continuation* $V_i \in [m_i, M_i]$.

For instance one could take the max-min and maximum values. Those bounds have to be proven for each particular case. In the partnership game of Section 2, we had $m_i = 0$ and $M_i = \frac{1}{1-\beta}\mathbf{E}\max\{V_E, w_t\}$. The tighter bounds, the easier it will be to show that the selection results of Section 4 apply, however these bounds are mainly needed to insure compactness.

Definition 13 *For any pair of functions* $(V_i, V_{-i}) : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^2$, *we denote* $G(V_i, V_{-i}, w_t, k_t)$ *the full information one-shot game,*

	S	E
S	$g^i(w_t, k_t) + \beta V_i(w_t, k_t)$	$W^i_{12}(w_t, k_t)$
E	$W^i_{21}(w_t, k_t)$	$W^i_{22}(w_t, k_t)$

We denote $\Psi_\sigma(V_i, V_{-i})$ the corresponding global game in which players get signals $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 7 (symmetry) *For all states of the world w_t, and capital stock k_t, $G(m_i, m_{-i}, w_t, k_t)$ has a pure strategy equilibrium. All pure strategy equilibria belong to $\{(S,S),(E,E)\}$.*

Note that if Assumption 7 is satisfied, then for any $\mathbf{V}$ taking values in $[m_i, M_i] \times [m_{-i}, M_{-i}]$, the game $G(V_i, V_{-i}, w_t, k_t)$ also has a pure strategy equilibrium, and its pure strategy equilibria also belong to $\{(S,S),(E,E)\}$.

Assumption 8 (increasing differences in the state of the world) *For all $k \in \mathbb{R}^d$ and $i \in \{1,2\}$, $g^i(w_t,k) - W^i_{21}(w_t,k)$ and $W^i_{12}(w_t,k) - W^i_{22}(w_t,k)$ are strictly increasing in w_t with a rate greater than some $r > 0$ independent of k.*

Together, Assumptions 7 and 8 respectively insure that at any state of the world, either (S,S) or (E,E) is a risk-dominant equilibrium and that there is a unique risk-dominant threshold x^{RD} – (S,S) being risk-dominant above this threshold and (E,E) being risk-dominant below. This is the unidimensional version of Carlsson and van Damme's assumption that states of the world should be connected to dominance regions by a path that is entirely contained in the risk-dominance region of either of the equilibria.

Assumption 9 (dominance) *There exists $\underline{w}$ such that for all k, $g^i(\underline{w},k) + \beta M_i - W^i_{21}(\underline{w},k) < 0$ and $\overline{w}$ such that $W^i_{12}(\overline{w},k) - W^i_{22}(\overline{w},k) > 0$*

Definition 14 *For any function $V : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $w \in \mathbb{R}$, we define $A_i(V,w,k)$ and $B_i(w,k)$ by,*

$$A_i(V_i,w,k) = g^i(w,k) + \beta V_i(w,k) - W^i_{12}(w,k) \quad \textit{and} \quad B_i(w,k) = W^i_{21}(w,k) - W^i_{22}(w,k)$$

Take as given the strategy s_{-i} of player $-i$ and let V_i be some continuation value for player i. When choosing to stay or exit, player i expects payoffs,

$$\begin{aligned} \Pi^i_S(V_i,k) &= \mathbf{E}\Big[W^i_{12}(w,k) + \{\underbrace{g^i(w,k) + \beta V_i(h_{i,t},w,k) - W^i_{12}(w,k)}_{\equiv A_i(V_i,w,k)}\}\mathbf{1}_{s_{-i}=S} | h_{i,t}, s_{-i}\Big] \\ \Pi^i_E(k) &= \mathbf{E}\Big[W^i_{22}(w,k) + \{\underbrace{W^i_{21}(w,k) - W^i_{22}(w,k)}_{\equiv B_i(w,k)}\}\mathbf{1}_{s_{-i}=S} | h_{i,t}, s_{-i}\Big] \end{aligned}$$

Player i's best response is to choose S if and only if $\Pi_S > \Pi_E$.

Assumption 10 (staying is good) *For all players $i \in \{1,2\}$, all states of the world w and all capital stocks k, we have, $A_i(m_i,w,k) \geq 0$ and $B_i(w,k) \geq 0$.*

Finally we make a compactness assumptions for technical reasons.

Assumption 11 *All exogenously given payoffs, $g^i(w,k), W^i_{12}(w,k), W^i_{21}(w,k)$ and $W^i_{22}(w,k)$ are Lipschitz in w with a rate r independent of k.*

By the Azrelà-Ascoli theorem, this assumption guarantees that the set of payoff functions indexed by k and mapping w to real numbers is compact. Such compactness is required for global game selection to occur at a speed that is independent of the state variable k_t.

B.2 Markovian state variables

In this section we consider the case where k_t follows a Markov chain over a countable set of states $\mathcal{S} \subset K \subset \mathbb{R}^d$, where K is compact. Denote by $h(\cdot, k_t)$ the distribution of k_{t+1} conditional on k_t. We assume that for all $k \in K$, $h(\cdot, k)$ is bounded and continuous in k with respect to the supremum norm over functions $||\cdot||_\infty$.

B.2.1 General results

Lemma 21 *There exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$, whenever s_{-i} is a Markovian strategy, then, for* ***all*** *strategies s'_{-i},*

$$s'_{-i} \preceq s_{-i} \Rightarrow BR_i(s'_{-i}) \preceq BR_i(s_{-i}) \quad \textit{and} \quad s_{-i} \preceq s'_{-i} \Rightarrow BR_i(s_{-i}) \preceq BR_i(s'_{-i})$$

Lemma 22 (extreme strategies) *Under Assumptions 6, 7, 8 and 10, there exists $\overline{\sigma} > 0$ small enough such that for all $\sigma < \overline{\sigma}$, rationalizable strategies of Γ_σ are bounded by extreme Markovian Nash equilibria. Those equilibria take threshold forms : for any state variable k, there exists threshold $\{x_{i,k}\}_{k \in \mathcal{S}}$ such that player i chooses to stay at when the state variable is k if and only if her signal is above $\{x_{i,k}\}$.*

Let us denote by x^H_σ and x^L_σ the thresholds associated with the highest and lowest equilibria of Γ_σ. From Assumption 10 we obtain that x^H_σ and x^L_σ are respectively associated with the highest and lowest possible pairs of rationalizable value functions, V^H_σ and V^L_σ. More precisely, if s_{-i} is a rationalizable strategy, the value function V_i associated with player i's best reply is such that at all histories $h_{i,t}$, $V^L_i(h_{i,t}) < V_i(h_{i,t}) < V^H_i(h_{i,t})$.

Theorem 4 *Under Assumptions 6, 7, 8, 9, 10, and 11 there exists $\overline{\sigma} > 0$ such that for all $\sigma \in (0, \overline{\sigma})$, there exists a continuous operator $\phi_\sigma(\cdot)$ mapping value functions onto value functions such that,*

(i) $\mathbf{V}^L_\sigma(\cdot)$ and $\mathbf{V}^H_\sigma(\cdot)$ are the lowest and highest fixed points of $\phi_\sigma(\cdot)$.

(ii) A vector of continuation value functions is supported by a Markovian equilibrium if and only if it is a fixed point of $\phi_\sigma(\cdot)$.

(iii) As σ goes to 0, the family of operators $\phi_\sigma(\cdot)$ converges uniformly over any bounded family of functions to an increasing operator Φ defined by

$$\Phi(V_i, V_{-i})(k_t) = \begin{pmatrix} \mathbf{E}\left[W^i_{22}(w) + (g^i_{11} + \beta V_i(k_{t+1} - W^i_{22}(w)))\,\mathbf{1}_{w > x^{RD}(V_i(k_{t+1}, V_{-i}(k_{t+1}))} \mid k_t\right] \\ \mathbf{E}\left[W^{-i}_{22}(w) + \left(g^{-i}_{11} + \beta V_{-i}(k_{t+1} - V^{-i}_{22}(w))\right)\mathbf{1}_{w > x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}))} \mid k_t\right] \end{pmatrix}$$

Where $x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}), k_t)$ *is the risk dominant threshold of the* 2×2 *global game* $\Psi(V_i(k_{t+1}), V_{-i}(k_{t+1}, k_t))$.

Lemma 23 (upper hemicontinuity) *Denote* $\mathbf{V}^H$ *and* $\mathbf{V}^L$ *the highest and lowest fixed points of* Φ. *Consider any family* $\{\mathbf{V}_\sigma\}_{\sigma>0}$ *of fixed points of* $\phi(\sigma, \cdot)$. *Then,*

$$\limsup_{\sigma\to 0} \mathbf{V}_\sigma \ll \mathbf{V}^H \quad \text{and} \quad \liminf_{\sigma\to 0} \mathbf{V}_\sigma \gg \mathbf{V}^L.$$

Where the lim sup *and* lim inf *are taken component by component.*

The definition of ξ and ζ for real numbers is extended to mappings $x : K \mapsto \mathbb{R}$.

Definition 15 *For any mapping* $x : K \mapsto \mathbb{R}$ *we define the mappings* ξ *and* ζ *by,*

$$\begin{aligned} \forall k_t \in K, \quad \xi(x)(k_t) = &\quad x^{RD}(BRV_i(x, k_{t+1}), BRV_{-i}(x, k_{t+1}), k_t) \\ \zeta(x)(k_t) = &\quad x^{RD}(NV_i(x, k_{t+1}), NV_{-i}(x, k_{t+1}), k_t). \end{aligned}$$

Definition 16 (non-singular extreme fixed points) *An extreme fixed point* x *of an increasing mapping* $g : \mathbb{R}^{\mathcal{S}} \mapsto \mathbb{R}^{\mathcal{S}}$ *is said to be non-singular if and only if,*

1. *It is strongly isolated among fixed points of* g *in the sense that there exists* $\delta > 0$ *such that whenever, for all* $n \in \mathbb{N}$, $y_n \in \mathbb{B}_{\delta,||\cdot||_\infty}(x)$ *and* $\lim_{n\to\infty} ||g(y_n) - y_n||_\infty = 0$, *then* $\lim_{n\to\infty} ||y_n - x||_\infty = 0$

2. *For all* $\delta > 0$ *there exists* $\eta \in (0, \delta)$, *and* $v \in R^{\mathcal{S}}$ *such that for all* $k \in \mathcal{S}$, $v(k) \in (\eta, \delta)$ *and for all* $k \in \mathcal{S}$, *we have:*

$$g(x + v)(k) < x + v - \eta/2 \quad \text{and} \quad g(x - v)(k) > x - v + \eta/2$$

Lemma 24 (lower hemicontinuity) *Assume the extreme fixed points of* ξ, x^H *and* x^L *(by convention* $x^H \ll x^L$*), are non-singular. Denote by* $\mathbf{x}^H_\sigma$ *(resp.* $\mathbf{x}^L_\sigma$*) the threshold function associated to the highest (resp. lowest) equilibrium of* Γ_σ. *Then, we have,*

$$\lim_{\sigma\to 0} ||\mathbf{x}^H_\sigma - (x^H, x^H)||_\infty = 0 \quad \text{and} \quad \lim_{\sigma\to 0} ||\mathbf{x}^L_\sigma - (x^L, x^L)||_\infty = 0.$$

Theorem 5 (ALSR of extreme equilibria) *Whenever the extreme fixed points* x^H *and* x^L *of* ξ *are non-singular, the strategies* (s_{x^H}, s_{x^H}) *and* (s_{x^L}, s_{x^L}) *are ALSR.*

Lemma 25 *Assume that* $\mathcal{S}$ *is finite and consider a* C^1 *payoff structure* π. *Then at an extreme fixed point* x *of* ξ, *the greatest eigenvalue* λ_{max} *of* $d\xi$ *is weakly less than one.*

Whenever λ_{max} *is strictly less than one, then* x *is non-singular.*

Lemma 26 (generic non-singularity) *Whenever* $\mathcal{S}$ *is finite, there exists a subset* P *of* C^1 *payoff structures that is open and dense in* Π_1 *with respect to* $||\cdot||_{\Pi_1}$, *and such that for any* $\pi \in P$, *the extreme fixed points of the associated mapping* ξ *are non-sigular.*

B.3 Auto-correlation

Because we might want to introduce autocorrelation in the states of the world, we may want to consider state variables following a recurrence equation of the form $k_{t+1} = f(k_t, w_t)$, where f is a deterministic function. In this formulation, w_t is the innovation and k_t is an observable sufficient statistic for past innovations.

Compared with the previous section, the main difficulty comes from the fact that because next period's capital stock depends on the realization of the state of the world, the continuation values at time t will now depend on the state of the world w_t. Because the uniform global games selection results we use require that all payoff functions be continuous and share a common modulus of continuity (see Chassang (2006) for more details), we must prove that equilibrium value functions indexed by the current capital stock are equicontinuous in w_t. More precisely we will to show they are increasing in w_t and Lipschitz with a rate independent of k. Let us denote by $\Psi_\sigma(V_i, V_{-i}, k_t)$ the global game,

	S	E
S	$g_i(w_t, k_t) + \beta V_i(k_{t+1})$	$W^i_{12}(w_t, k_t)$
E	$W^i_{21}(w_t, k_t)$	$W^i_{22}(w_t, k_t)$

where i is the row player. In addition to the assumptions of Section B.1, we need to make a few more technical assumptions. These assumptions make the statement of theorems somewhat tedious, but they are fairly general so that our selection result will in fact be easily applicable.

Assumption 12 (increasing differences in capital stock) *For all $w_t \in \mathbb{R}$ and $i \in \{1, 2\}$, $g^i(w_t, k) - W^i_{21}(w_t, k)$ and $W^i_{12}(w_t, k) - W^i_{22}(w_t, k)$ are increasing in k.*

Assumption 13 (capital is good) *For $i \in \{1, 2\}$, and $w \in \mathbb{R}$ we assume that g^i, V^i_{12}, V^i_{21} and V^i_{22} are weakly increasing in k.*

Definition 17 (iterated capital stock) *For all $n \in \mathbb{N}$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n$, we define by induction the iterated capital stock $f_n(k, \mathbf{w})$ as follows,*

$$f_1(k, w_1) = f(k, w_1) \quad \text{and} \quad f_n(k, \mathbf{w}) = f\left(f_{n-1}\left(k, (w_1, \ldots, w_{n-1})\right), w_n\right)$$

In other words, $f_n(k_t, \mathbf{w}) = k_{t+n} \mid \mathbf{w}$.

Assumption 14 *We make four assumptions on the process of k_t and how it affects payoffs.*

1. *f is increasing in both arguments.*
2. *there exists $H \in \mathbb{R}$ such that*

$$\left|\frac{\frac{\partial f}{\partial k}}{\frac{\partial f}{\partial w}}\right| < H\ , \quad \left|\frac{\frac{\partial(g^i - W^i_{21})}{\partial k}}{\frac{\partial(g^i - W^i_{21})}{\partial w}}\right| < H \quad \text{and} \quad \left|\frac{\frac{\partial(W^i_{12} - W^i_{22})}{\partial k}}{\frac{\partial(W^i_{12} - W^i_{22})}{\partial w}}\right| < H$$

3. There exists $n* \in \mathbb{N}$ and $\mu < \frac{1}{\beta}$ such that,

$$\forall k \in \mathbb{R}^d, \quad \mathbf{E}_w \left[\frac{\partial f_{n*}}{\partial k} \right] < \mu^{n*}$$

4.

$$\forall n \in \mathbb{N}, k \in \mathbb{R}^d, \quad \mathbf{E}_w \left[\frac{\partial f_n}{\partial k} \right] < +\infty$$

Lemma 27 (joint selection) *Define $\mathcal{V}_R$ the set of functions mapping $\mathbb{R}^d$ into $\mathbb{R}$ that are weakly increasing and Lipschitz continuous with rate R. For all R, there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, for any $\mathbf{V} = (V_i, V_{-i}) \in \mathcal{V}_R^2$, the global game $\Psi_\sigma(V_i, V_{-i}, k_t)$ has a unique pair of rationalizable strategies $(x_i(\mathbf{V}, k_t), x_{-i}(\mathbf{V}, k_t))$.*

Proof: This result is a direct application of Theorem 2 of Chassang (2005). ■

Given a class $\mathcal{V}_R$, for σ small enough, Lemma 27 allows us to define a mapping $\phi(\mathbf{V}, \sigma)$ which maps any vector of value functions from $\mathcal{V}_R$ into the value of playing game $\Psi_\sigma(V_i, V_{-i}, \cdot)$ for each player. Function ϕ maps value functions into value functions. In order to combine the Abreu, Pearce and Stacchetti (1990) approach and a global games selection argument, we need to find a class $\mathcal{V}_R$ stable by ϕ. Lemmas 28 and 29 show that $\phi_\sigma(\cdot)$ maps $\mathcal{V}_R$ into $\mathcal{V}_{R^\#}$ for some $R^\#$ independent of σ. Finally Lemma 30 shows that for some R big enough, all iterated images of $\mathcal{V}_R$ by $\phi_\sigma(\cdot)$ are subsets of some fixed $\mathcal{V}_{R^\#}$ with $R^\#$ independent of σ.

Lemma 28 (Lipschitz continuity in k of the selected equilibrium) *Pick $\mathbf{V} \in \mathcal{V}_R^2$ and $\overline{\sigma}$ such that Lemma 27 applies. Then there exists $\rho > 0$, independent of R, such that for all $\sigma < \overline{\sigma}$, the uniquely selected equilibrium of $\Psi_\sigma(\mathbf{V})$, $(x_i(\mathbf{V}, k), x_{-i}(\mathbf{V}, k))$ is Lipschitz in k with rate ρ.*

Lemma 29 (stability of Lipschitz continuity) *If $\mathbf{V}$ belongs to $\mathcal{V}_R$ for some $R > 0$, then $\mathbf{V}^\#$, defined by $\mathbf{V}^\#(k) = \phi(\mathbf{V}, k, \sigma)$, belongs to $\mathcal{V}_{R^\#}$ for some $R^\# > 0$.*

Lemma 29 allows us to define iterations of $\phi(\cdot, \sigma)$ as follows:

Definition 18 *Pick $n \in \mathbb{N}$. For $\mathbf{V}$ and σ small enough, we define by induction $\phi_n(\mathbf{V}, k, \sigma)$ by,*

$$\phi_1(\mathbf{V}, k, \sigma) = \phi(\mathbf{V}, k, \sigma) \quad \textit{and} \quad \phi_n(\mathbf{V}, k_t, \sigma) = \phi(\mathbf{E}\phi_{n-1}(\mathbf{V}, k_{t+1}, \sigma), k_t, \sigma)$$

Using Lemmas 27 and 29 we know that for any $R > 0$ and $n \in \mathbb{N}$ there exists $\overline{\sigma}_{n,R}$ such that for all $\sigma < \overline{\sigma}_{n,R}$, $\phi_n(\mathbf{V}, k, \sigma)$ is well defined for all $\mathbf{V} \in \mathcal{V}_R^2$ and all $k \in \mathbb{R}^d$.

Lemma 30 (stable Lipschitz class) *Take the integer $n*$ defined in Assumption 14. There exists $\overline{\sigma} > 0$ and R such that for all $\sigma < \overline{\sigma}$, $\phi_{n*}(\mathcal{V}_R^2, \sigma) \subset \mathcal{V}_R^2$*

Moreover, for all $n \in \mathbb{N}$ and $\sigma < \overline{\sigma}$, $\phi_n(\mathbf{V}, k, \sigma)$ is well defined for $\mathbf{V} \in \mathcal{V}_R$ and that there exists $R^\# > 0$ such that for all $n \in \mathbb{N}$, $\phi_n(\mathcal{V}_R^2, k, \sigma) \subset \mathcal{V}_{R^\#}^2$

The last tricky step is to prove that to obtain tight bounds on the set of rationalizable strategies, it is enough to study strategies corresponding to value functions in $\mathcal{V}_R$. To prove this, we prove that the set of rationalizable strategies is bounded by extreme Markovian Nash equilibria that can be obtained by iteratively applying the mapping $\phi_\sigma(\cdot)$ to vectors of constant value functions.

Lemma 31 *Pick $\overline{\sigma}$ and R such that Lemma 30 holds. Denote $\mathcal{V} = \mathcal{V}_i \times \mathcal{V}_{-i}$ the set of rationalizable continuation values of Γ_σ. Pick any vectors of value functions $\mathbf{V}^L$ and $\mathbf{V}^H$ in $\mathcal{V}_R$, then for all $\sigma < \overline{\sigma}$,*

$$\left\{\mathcal{V} \subset [\mathbf{V}^L, \mathbf{V}^H]\right\} \Rightarrow \left\{\mathcal{V} \subset [\phi(\mathbf{V}^L, \sigma), \phi(\mathbf{V}^H, \sigma)]\right\}$$

Theorem 6 *Under Assumptions 6, 7, 8, 9, 10, 11, 12, 13 and 14, there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, the rationalizable strategies of game Γ_σ are bounded by extreme Nash equilibria associated with extreme value functions $\mathbf{V}^H_\sigma(\cdot)$ and $\mathbf{V}^L_\sigma(\cdot)$. Moreover, there exists a continuously increasing operator $\phi_\sigma(\cdot)$ mapping value functions into value functions, such that,*

1. *$\mathbf{V}^L_\sigma(\cdot)$ and $\mathbf{V}^H_\sigma(\cdot)$ are the lowest and highest fixed points of $\phi_\sigma(\cdot)$.*

2. *As σ goes to 0, the family of operators $\phi_\sigma(\cdot)$ converges uniformly over any class $\mathcal{V}_R$ to a function Φ defined by*

$$\Phi(V_i, V_{-i})(k_t) = \begin{pmatrix} \mathbf{E}\left[W^i_{22}(w) + (g^i_{11} + \beta V_i(k_{t+1}) - W^i_{22}(w))\, \mathbf{1}_{w > x^{RD}(V_i(f(k,w)), V_{-i}(f(k,w)))}\right] \\ \mathbf{E}\left[W^{-i}_{22}(w) + \left(g^{-i}_{11} + \beta V_{-i}(k_{t+1}) - W^{-i}_{22}(w)\right) \mathbf{1}_{w > x^{RD}(V_i(k_{t+1}), V_{-i}(k_{t+1}))}\right] \end{pmatrix}$$

Where $x^{RD}\left(V_i(k_{t+1}), V_{-i}(k_{t+1})\right)$ is the risk-dominant threshold of the 2×2 global game $\Psi\left(V_i(k_{t+1}), V_{-i}(k_{t+1})\right)$.

Whenever the fixed points of Φ are isolated with respect to the uniform norm, then as σ goes to 0, uniform convergence of $\phi_\sigma(\cdot)$ implies that $\mathbf{V}^H_\sigma(\cdot)$ and $\mathbf{V}^L_\sigma(\cdot)$ converge to the highest and lowest fixed points of Φ with respect to the uniform norm.

B.4 Proofs for Appendix B

Proof of Lemma 21: Consider s_{-i} a Markovian strategy and s'_{-i} such that $s'_{-i} \preceq s_{-i}$. Define V_i and V'_i the continuation value functions respectively associated to player i's best response to s_{-i} and s'_{-i}. Assumption 5, that "staying is good", implies that at all histories $h_{i,t}$, $V'_i(h_{i,t}) < V_i(h_{i,t})$. At any history $h_{i,t}$, the best-reply action profiles of player i are $BR_i(a_{-i}, V_i(h_{i,t}), \sigma)$ and $BR_i(a'_{-i}, V'_i(h_{i,t}), \sigma)$. From Lemma 4, we have that

$$BR_i(a'_{-i}, V'_i(h_{i,t}), \sigma) \preceq BR_i(a'_{-i}, V_i(h_{i,t}), \sigma) \tag{21}$$

Since s_{-i} is Markovian, $V_i(h_{i,t})$ is constant. Thus Lemma 3 implies that

$$BR_i(a'_{-i}, V_i(h_{i,t}), \sigma) \preceq BR_i(a_{-i}, V_i(h_{i,t}), \sigma) \tag{22}$$

Combining equations (21) and (22) we obtain that indeed, $BR_i(s'_{-i}) \preceq BR_i(s_{-i})$. An identical proof holds for the other inequality. $\blacksquare$

Proof of Lemma 22: This a corollary of Lemma 21. The proof, again drawing on the methodology of Milgrom and Roberts (1990) and Vives (1990) is identical to that of Theorem 1. $\blacksquare$

Proof of Theorem 4: The proof is almost identical to that of Theorem 2. Selection in one-shot global games is applied to the augmented game associated with each capital stock. See the proof of Theorem 2 for more details. $\blacksquare$

Proof of Lemma 23: There exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ going to 0, such that $\{\mathbf{V}_{\sigma_n}\}_{n\in\mathbb{N}}$ converges weakly to $\limsup_{\sigma\to 0}\mathbf{V}_\sigma \equiv \mathbf{V}^*$. Let us show that V^* is a fixed point of Φ. Indeed for every k_t, $\mathbf{V}_\sigma$ satisfies $\mathbf{V}_\sigma(k_t) = \phi(k_t, \mathbf{V}_\sigma(k_{t+1}), \sigma)$. Since $\phi_\sigma(\cdot)$ converges to Φ, and Φ is continuous in $\mathbf{V}$, the equation must hold at the limit. This implies that indeed $\mathbf{V}^*$ is a fixed point of Φ which proves the right side of the inequality. An symmetric proof gives the left side. $\blacksquare$

Proof of Lemma 24: This is a direct implication of Theorem 5. Indeed a ball centered on a threshold form strategy and of radius ρ with respect to the topology on strategies corresponds to a ball centered on the threshold x and radius ρ with respect to the supremum distance. $\blacksquare$

Proof of Theorem 5: The proof is very similar to that of Proposition 3 and Theorem 3. Let x denote an extreme fixed point of ξ. By assumption, for any $\delta > 0$, there exists $\eta > 0$ and $x'' \in \mathcal{B}_\delta(x)$ such that for all $k \in \mathcal{S}$, $x''(k) > x(k)+2\eta$ and $\xi(x'')(k) \ll x''(k)-\eta''$. This implies there exists $x' \in [\xi(x''), x'']$ such that for all $k \in \mathcal{S}$, $\xi(x'')(k) + \eta/2 < \xi(x')(k) < x''(k) - \eta/2$. Let us now show that for σ small enough, for all $k \in \mathcal{S}$, $BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec s_{x'}(k)$. We have,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) = BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(BR_{-i,\sigma}(x'), k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'), k)(x').$$

For η small enough, we know that $|BR_{-i,\sigma}(x')(k) - x'(k)| < 2\sigma$. Hence, for σ small enough, $\forall k \in \mathcal{S}$, $BR_{-i,\sigma}(x')(k) \prec x''(k)$. This implies that,

$$BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(x'', k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'', k), k)(x').$$

From Theorem 3 of Chassang (2006), we know that $x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k)$ converges uniformly to $\xi(x'')(k)$, hence for σ small enough,

$$x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k) > x'(k). \tag{23}$$

From Theorem 2 of Chassang (2006), we know that for σ small enough and for all $k \in \mathcal{S}$, the sequence $[BR_{i,\sigma}(\cdot, BRV_{i,\sigma}(x'', k), k) \circ BR_{i,\sigma}(\cdot, BRV_{-i,\sigma}(x'', k), k)]^p (x')(k)$ converges monotonously to $x^*_\sigma(BRV_{i,\sigma}(x'', k), BRV_{-i,\sigma}(x'', k), k)$ as p goes to infinity. Using 23, this implies that

$$\forall k \in \mathcal{S}, \quad BR_{i,\sigma} \circ BR_{-i,\sigma}(s_{x'})(k) \prec s_{x'}(k). \tag{24}$$

Since $BR_{i,\sigma} \circ BR_{-i,\sigma}$ is monotonous, this implies that as q goes to infinity, the sequence $\{BR_{i,\sigma} \circ BR_{-i,\sigma}\}^q(s_{x'})$ converges weakly to a strategy of threshold $x^*_{i,\sigma}$ such that $x^*_{i,\sigma} \in \mathcal{B}_{\delta,||\cdot||_\infty}(x)$. Note that $(x^*_{i,\sigma}, BR_{-i,\sigma}(x^*_{i,\sigma}))$ is a fixed point of ξ_σ. Let us now show that $x^*_{i,\sigma}$ converges to x as σ goes to 0.

We know that ξ_σ converges uniformly to (ξ, ξ). Indeed, $BRV_{i,\sigma}(y, k)$ converges uniformly over $(y, k) \in \mathbb{R}^{\mathcal{S}} \times \mathcal{S}$ to $BRV(y, k)$ as σ goes to 0, and Theorem 4 of Chassang (2006) implies that $x^*_\sigma(V_i, V_{-i}, k)$ is Lipschitz-continuous in $\mathbf{V}$ with a rate independent of k. This implies that for any $\nu > 0$, there exists $\overline{\sigma}$ such that for all $\sigma \in (0, \overline{\sigma})$,

$$||\xi(x^*_\sigma) - x^*_\sigma||_\infty \leq ||\xi(x^*_\sigma) - \xi_\sigma(x^*_\sigma) + \xi_\sigma(x^*_\sigma) - x^*_\sigma||_\infty \leq ||\xi(x^*_\sigma) - \xi_\sigma(x^*_\sigma)||_\infty \leq \nu.$$

Since x is an isolated fixed point of x, this implies that $\lim_{\sigma \to 0} ||x^*_\sigma - x||_\infty = 0$.

To prove ALSR, note that one can construct $\tilde{x}$ with the same properties as x', but strictly below x rather than strictly above. Then $[s_{x'}, s_{\tilde{x}}]$ is a neighborhood of s_x for which

$$\lim_{\sigma \to 0} \lim_{n \to \infty} [BR_{i,\sigma} \circ BR_{-i,\sigma}]^n([s_{x'}, s_{\tilde{x}}]) = s_x.$$

This shows that indeed, (s_x, s_x) is ALSR. ■

Proof of Lemma 25: ξ and ζ coincide around their fixed points. Since $\mathcal{S}$ is finite, ζ is really a mapping from $\mathbb{R}^p$ to $\mathbb{R}^p$ with $p \in \mathbb{N}$. Whenever $\pi \in \Pi_1$, ζ will be differentiable. Because ξ and ζ coincide around their fixed points, this implies that ξ is differentiable around its fixed points. Denote by x the highest fixed point of ξ. Since ξ is strictly increasing around its fixed points, the Perron-Frobenius theorem applies to $\mathrm{d}\xi_{|x}$. It states that $\mathrm{d}\xi_{|x}$ admits a greatest eigenvalue $\lambda_{max} > 0$ associated with an eigenvector v with strictly positive components. Assume that $\lambda_{max} > 1$. Then, for some $\delta > 0$ small enough, we will get that $\xi(x + \delta v) \gg x + \delta v$. This implies that ξ admits a fixed point $x' \gg x + \delta v$, which contradicts the fact that x is the highest fixed point of ξ. Hence, it must be that $\lambda_{max} \leq 1$.

For the second part of the lemma, $\lambda_{max} < 1$ implies that, there exists $\delta > 0$ such that for all $\eta \in (0, \delta)$, $\xi(x+\eta v) \ll x+\eta v - \frac{1-\lambda_{max}}{2}\eta v$ and $\xi(x-\eta v) \gg x-\eta v + \frac{1-\lambda_{max}}{2}\eta v$. Now consider any y such that $y \in [x - \delta v, x + \delta v]$. There exists $\eta \in (-\delta, \delta)$ such that y is weakly less than $x + \eta v$, and y and $x + \eta v$ share one strictly positive coordinate $y(k)$. Assume without loss of generality that $\eta > 0$. Since $\xi(x + \eta v) \ll x + \eta v - \frac{1-\lambda_{max}}{2}\eta v$, and ξ is increasing, this implies that $\xi(y)(k) < y(k)$. Hence, y cannot be a fixed point of ξ, proving that x is isolated. This concludes the proof. ■

Proof of Lemma 26: Let us consider the set off payoff functions such that at its extreme fixed points, ζ admits an eigenvalue $\rho \in (0, 1)$ associated with an eigenvector with strictly positive coordinates. Note that since ζ is increasing around its fixed points, this property clearly implies that ζ is contracting around its extreme fixed points. Denote $\mathcal{P}$ the set of payoff structures satisfying this property.

Denote x the highest fixed point of ζ. From Lemma 25, we already know that the Jacobian of ζ at x admits a largest eigenvalue $\lambda_{max} \in (0, 1]$, associated with an eigenvector v whose components are strictly positive.

Let us show that $\mathcal{P}$ is open. Pick $\pi \in \mathcal{P}$. At x^*, $\det(\mathrm{d}\zeta(x^*) - Id)$ is strictly different from 0. By continuity of $\mathrm{d}\zeta$, this implies that there exists a ball of center x^* and radius $\eta > 0$ and $\nu > 0$ such that for all $x \in \mathcal{B}_\eta(x^*)$, $|\det(\mathrm{d}\zeta - Id)| > \nu$. There exists $\mu > 0$ such that for all payoff structures $\tilde{\pi}$ within distance μ of π, the extreme fixed points of $\tilde{\zeta}$ are within distance η of x^* and $|\det(\mathrm{d}\zeta - Id)| > \nu/2$ over $\mathcal{B}_\eta(x^*)$. We already know that the greatest eigenvalue of $\tilde{\zeta}$ is weakly less than 1. This implies it is strictly less than one and proves that $\mathcal{P}$ is open.

We now show that $\mathcal{P}$ is dense. Since the intersection of dense open sets is dense and open, we proceed separately for the highest and lowest fixed points. The set of C^2 payoff structures in Π_1 that strictly satisfy Assumption 10 is dense in Π_1. Pick such a payoff structure π, and denote by x the highest fixed point of the associated mapping ζ. For any vector $u \in \mathbb{R}^{\mathcal{S}}$, consider the payoff structure π^u defined by,

$$\begin{aligned} \forall (w,k) \in \mathbb{R} \times K, \qquad & \tilde{g}^{i,u}(w,k) \equiv g^i(w,k) \\ & W_{22}^{i,u}(w,k) = W_{22}^i(w,k) \\ & W_{12}^{i,u}(w,k) = W_{12}^i(w - u(k), k) + g^i(w,k) - g^i(w - u(k), k) \\ & W_{21}^{i,u}(w) = W_{21}^i(w - u(k), k) + W_{22}^i(w,k) - W_{22}^i(w - u(k), k) \end{aligned}$$

There exists $\delta > 0$ such that for all u satisfying $||u||_\infty < \delta$, π^u satisfies Assumptions 6, 7, 8, 9, and 10. Moreover, for $||u||_\infty$ small enough, π^u is arbitrarily close to π in the sense of $||\cdot||_{\Pi_1}$. Also, for any u, $\zeta^u = \zeta + u$. Assume that there exists $\eta > 0$ such that for all u satisfying $||u||_\infty < \eta$, at the highest fixed point of ζ^u, $\mathrm{d}\zeta$ has a largest eigenvalue equal to 1. We will show it leads to a contradiction. More precisely, let us show that if this is true, then x cannot be the highest fixed point of ζ. First note that ζ is C^2 and that Assumption 2 implies potential fixed points of ζ^u belong to a compact L. Hence, there exists $H > 0$ such that for all $u \in \mathbb{R}^{\mathcal{S}}$, and $y \in L$, $| < u, \mathrm{d}^2\zeta u > | \leq H < u, u >$. For all $n \in \mathbb{N}$, define the sequences $\{x_m^n\}_{m \in \{0,\ldots,n\}}$ and $\{u_m^n\}_{m \in \{0,\ldots,n\}}$ such that for all m, x_m^n is the highest fixed point of $\zeta^{u_m^n}$ and $||u_m^n||_\infty < \eta$, as follows:

1. $x_0^n \equiv x$ and $u_0^n = 0$
2. For all $m \in \{0, \ldots, n-1\}$, by assumption, at x_m^n, $\mathrm{d}\zeta$ has a largest eigenvalue equal to 1. By the Perron-Frobenius theorem, this largest eigenvalue is associated to an eigenvector v with strictly positive coordinates. Pick a representant such that $||v||_\infty = 1$ and define $u_{m+1}^n = \zeta^{u_m^n}(x_m^n + \eta v/n) - x_m^n - \eta v/n + u_m^n$.
3. Define x_{m+1}^n as the highest fixed point of $\zeta^{u_{m+1}^n}$.

First note that $x_m^n + \eta v/n$ is a fixed point of $\zeta^{u_{m+1}^n}$, hence, $x_{m+1}^n \gg x_m^n + \eta v/n$. Second, since v was picked as an eigenvector associated to 1 at each stage, we have that $||u_{m+1}^n - u_m^n||_\infty =$

$||\zeta^{u^n_m}(x^n_m + \eta v/n) - x^n_m - \eta v/n||_\infty \leq H\eta^2/n^2$. Hence we obtain,

$$\begin{aligned}
\zeta(x^n_n) - \zeta(x) &= \sum_{m\in\{0,\dots,n-1\}} \zeta(x^n_{m+1}) - \zeta(x^n_m) \\
&= \sum_{m\in\{0,\dots,n-1\}} \zeta^{u^n_{m+1}}(x^n_{m+1}) - \zeta^{u^n_m}(x^n_m) + \sum_{m\in\{0,\dots,n-1\}} u^n_m - u^n_{m+1} \\
&= x^n_n - x + \sum_{m\in\{0,\dots,n-1\}} u^n_m - u^n_{m+1}
\end{aligned}$$

which yields

$$||\zeta(x^n_n) - x^n_n||_\infty \leq ||\sum_{m\in\{0,\dots,n-1\}} u^n_m - u^n_{m+1}||_\infty \leq H\eta^2/n.$$

Consider $e \in \mathbb{R}^{\mathcal{S}}$ the vector whose components are all equal to 1. By construction, $x^n_n \in [x, x+\eta e] \setminus [x, x+\frac{\eta e}{|\mathcal{S}|}]$. Extract a converging sequence from $\{x^n_n\}_{n\in\mathbb{N}}$. Its limit x' is such that $x' \gg x + \frac{\eta e}{2|\mathcal{S}|}$ and satisfies $\zeta(x') = x'$. This contradicts the fact that x is the greatest fixed point of ζ. Hence for any $\delta > 0$, there exists u satisfying $||u||_\infty < \delta$ such that at the highest fixed point of ζ^u, $\mathrm{d}\zeta^u$ has a largest eigenvalue strictly less than 1. ■

Proof of Lemma 28: This is a direct application of Theorem 5 from Chassang (2006). We refer to that paper for details. This theorem holds under conditions which, in this particular case, boil down to showing there exists a constant C such that for all k, w, k', w',

$$\frac{\Delta_{k,k'} V_i(f(k,w))}{\Delta_{w,w'} V_i(f(k,w))} < C$$

Where for any function u, $\Delta_{s,s'} u(s) \equiv \frac{||u(s')-u(s)||}{||s'-s||}$.

Assumption 14 was specifically introduced to prove this inequality. Assume temporarily that V_i is differentiable. Then using the fact that $(V \circ f)' = V' \circ f \times f'$, we get that

$$\left| \frac{\frac{\partial V_i(f(k,w))}{\partial k}}{\frac{\partial V_i(f(k,w))}{\partial w}} \right| < H \tag{25}$$

Noting that we have $\frac{\partial V_i(f(k,w))}{\partial w} > 0$ and using the inequality

$$\forall a, b, c, d > 0, \quad \frac{a}{b} < m \ , \ \ \frac{c}{d} < m \Rightarrow \frac{a+c}{b+d} < m$$

we get by integration of the numerator and denominator of equation (25) that,

$$\frac{\Delta_{k,k'} V_i(f(k,w))}{\Delta_{w,w'} V_i(f(k,w))} < H$$

This result does not depend on the smoothness of V_i. Thus using the density of differentiable functions, we know it holds for any V_i in $\mathcal{V}_R$. Thus Theorem 5 of Chassang (2005) applies. The uniquely selected equilibrium $(x_i(\mathbf{V},k), x_{-i}(\mathbf{V},k))$ is Lipschitz in k with a rate ρ independent of R. ■

Proof of Lemma 29: $\mathbf{V}^\#(k) = (V_i^\#(k), V_{-i}^\#(k))$. Assume – temporarily – that $V_i^\#$ and V_i are differentiable, denoting x_i and x_{-i} the equilibrium strategies, we have

$$\begin{aligned}
\frac{\partial V_i^\#}{\partial k} &= \mathbf{E}\left[\left(\frac{\partial g_{11}^i}{\partial k} + \beta \frac{\partial V_i}{\partial k}\frac{\partial f}{\partial k}\right)\mathbf{1}_{s_i>x_i}\mathbf{1}_{s_{-i}>x_{-i}} + \frac{\partial W_{12}^i}{\partial k}\mathbf{1}_{s_i>x_i}\mathbf{1}_{s_{-i}<x_{-i}} + \frac{\partial W_{21}^i}{\partial k}\mathbf{1}_{s_i<x_i}\mathbf{1}_{s_{-i}>x_{-i}} + \frac{\partial W_{22}^i}{\partial k}\mathbf{1}_{s_i<x_i}\mathbf{1}_{s_{-i}<x_{-i}}\right] \\
&+ \mathbf{E}\left[\frac{\partial x_i}{\partial k} f_{s_i}(x_i)\left((W_{22}^i - W_{12}^i)\mathbf{1}_{s_{-i}>x_{-i}} + (W_{21}^i - g_{11}^i - \beta W_i)\mathbf{1}_{s_{-i}>x_{-i}}\right)\right] \\
&+ \mathbf{E}\left[\frac{\partial x_{-i}}{\partial k} f_{s_{-i}}(x_{-i})\left((W_{22}^i - W_{12}^i)\mathbf{1}_{s_i>x_i} + (W_{21}^i - g_{11}^i - \beta V_i)\mathbf{1}_{s_i>x_i}\right)\right]
\end{aligned}$$

Using Assumption 11, Lemma 28 and the fact that $V_i \in \mathcal{V}_R$, we conclude there exist absolute constants C_1, C_2 such that,

$$\left|\frac{\partial V_i^\#}{\partial k}\right| \leq C_1 + C_2 R$$

This inequality doesn't depend on the smoothness of either V_i or $V_i^\#$. Using the density of smooth functions we conclude it holds generally.

Finally, note that $V_i^\#$ is increasing in k. This results directly from Assumptions 12 and 13. Increasing k increases cooperation directly because of Assumption 12, and indirectly because Assumption 13 implies that more capital increases continuation values. ■

Proof of Lemma 30: We know that weak monotonicity is maintained, the difficulty is to show that Lipschitz continuity is maintained with a stable rate. Pick $\mathbf{V} \in \mathcal{V}_R$. We can express ϕ_{n*} explicitly.

$$\begin{aligned}
\phi_{n*}^i(\mathbf{V},k,\sigma) &= \mathbf{E}\Bigg[\sum_{t=1}^{n*-1} \beta^t \prod_{q=1}^{t-1} \mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}}\big(g^i\mathbf{1}_{s_{i,t}>x_{i,t}}\mathbf{1}_{s_{-i,t}>x_{-i,t}} \\
&+ W_{12}^i\mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}<x_{-i,q}} + W_{21}^i\mathbf{1}_{s_{i,q}<x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}} + W_{12}^i\mathbf{1}_{s_{i,q}<x_{i,q}}\mathbf{1}_{s_{-i,q}<x_{-i,q}}\big)\Bigg] \qquad (26) \\
&+ \mathbf{E}\left[\beta^{n*}\prod_{q=1}^{n*}\mathbf{1}_{s_{i,q}>x_{i,q}}\mathbf{1}_{s_{-i,q}>x_{-i,q}} V(f_{n*}(k,\mathbf{w}))\right]
\end{aligned}$$

Assume temporarily, that all functions involved are differentiable with respect to k. Using Assumptions 11, 14, Lemma 28, and the fact that $\mathbf{V} \in \mathcal{V}_R^2$, equation (26) yields after some manipulation an inequality of the form,

$$\left|\frac{\partial \phi_{n*}}{\partial k}\right| \leq C + \beta^{n*}\mathbf{E}\left[\frac{\partial f_{n*}}{\partial k}\right] R \leq C + (\beta\mu)^{n*} R$$

Therefore, if we pick $R \geq \frac{C}{1-(\beta\mu)^{n*}}$, then $\mathcal{V}_R$ is stable via ϕ_{n*}. The second part of the lemma follows directly using this result and Lemma 29. ■

Proof of Lemma 31: Consider a maximal rationalizable action profile $a^i_{m,1}$. It is a best response to some action profile a^{-i} and some rationalizable continuation value V_i. This implies that $a^i_{m,1} \preceq BR_i(a^{-i}, V_i^H)$. Moreover, since $V_i^H \in \mathcal{V}_R$, we know from Lemma 3 of Chassang (2005) that $BR_i(\cdot, V_i^H)$ is monotone in strategies. Thus there exists a maximal rationalizable action $a^{-i}_{m,1}$ such that $a^i_{m,1} \preceq BR_i(a^{-i}_{m,1}, V_i^H)$. For $i \in \{1,2\}$, we define $\overline{BR}_i(\cdot) \equiv BR_i(\cdot, V_i^H)$. By iterating the former reasoning, we get a sequence of maximal actions $\{a^i_{m,q}\}_{q \in \mathbb{N}}$, such that $a^i_{m,1} \preceq (\overline{BR}_i \circ \overline{BR}_{-i})^q(a^i_{m,q})$. Taking q to infinity, this implies that rationalizable actions are smaller than the unique rationalizable strategy of $\Psi_\sigma(\mathbf{V}^H)$. Because of Assumption 10, this also implies that the value associated with any rationalizable action is less than the value of playing the unique equilibrium of $\Psi_\sigma(\mathbf{V}^H)$. This shows that $\mathcal{V} \preceq \phi(\mathbf{V}^H, \sigma)$. An identical proof holds for the lower bound. ■

Proof of Theorem 6: To prove the existence of extreme equilibria, we use Lemma 31 iteratively. Pick R and $\overline{\sigma}$ such that Lemma 30 holds. Denote $\mathcal{V}$ the set of rationalizable value functions. Begin by setting $\mathbf{V}^H_{\sigma,0} = (M_i, M_{-i})$ and $\mathbf{V}^L_{\sigma,0} = (m_i, m_{-i})$. We must have $\mathcal{V} \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$. Since $\mathbf{V}^H_{\sigma,0}$ and $\mathbf{V}^L_{\sigma,0}$ belong to $\mathcal{V}_R^2$, Lemma 31 implies that

$$\mathcal{V} \subset [\phi(\mathbf{V}^L_{\sigma,0}, \sigma), \phi(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$$

From Lemma 30, we know that all functions $\phi_n(\mathbf{V}^L_{\sigma,0}, \sigma)$ and $\phi_n(\mathbf{V}^H_{\sigma,0}, \sigma)$ are Lipschitz with rate $R^{\#}$ so that we can keep applying $\phi_\sigma(\cdot)$ iteratively. Using Lemma 31 and the monotonicity of $\phi(\cdot, \sigma)$ at each step, we get, that for all $q \in \mathbb{N}$,

$$\mathcal{V} \subset [\phi_q(\mathbf{V}^L_{\sigma,0}, \sigma), \phi_q(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset [\phi_{q-1}(\mathbf{V}^L_{\sigma,0}, \sigma), \phi_{q-1}(\mathbf{V}^H_{\sigma,0}, \sigma)] \subset \cdots \subset [\mathbf{V}^L_{\sigma,0}, \mathbf{V}^H_{\sigma,0}]$$

The sequences $\left\{\phi_q(\mathbf{V}^L_{\sigma,0}, \sigma)\right\}_{q \in \mathbb{N}}$ and $\left\{\phi_q(\mathbf{V}^H_{\sigma,0}, \sigma)\right\}_{q \in \mathbb{N}}$ are respectively increasing and decreasing. Moreover these are sequences of bounded functions with a fixed Lipschitz rate $R^{\#}$. Thus, by Ascoli's Theorem, they converge uniformly to value functions $\mathbf{V}^L_{\sigma,\infty}$ and $\mathbf{V}^H_{\sigma,\infty}$ with Lipschitz rate $R^{\#}$. Using Theorem 4 of Chassang (2005), we know that $\phi_\sigma(\cdot)$ is continuous over $\mathcal{V}_{R^{\#}}$ endowed with the uniform norm. This implies that $\mathbf{V}^H_{\sigma,\infty}$ and $\mathbf{V}^L_{\sigma,\infty}$ satisfy,

$$\mathbf{V}^H_{\sigma,\infty} = \phi(\mathbf{V}^H_{\sigma,\infty}, \sigma) \quad \text{and} \quad \mathbf{V}^L_{\sigma,\infty} = \phi(\mathbf{V}^L_{\sigma,\infty}, \sigma)$$

This implies that $\mathbf{V}^L_{\sigma,\infty}$ and $\mathbf{V}^H_{\sigma,\infty}$ sustain extreme Markovian Nash equilibria in which players respectively play the unique equilibria of $\Psi_\sigma(\mathbf{V}^L_{\sigma,\infty})$ and $\Psi_\sigma(\mathbf{V}^H_{\sigma,\infty})$. Finally, we know from Theorem 3 of Chassang (2005) that $\phi_\sigma(\cdot)$ converges uniformly towards Φ over $\mathcal{V}_{R^{\#}}$. ■

Appendix C: Alternative assumptions

This section describes assumptions generalizing those of Section 2.3, and under which the analysis of Section 3 still holds step by step. The analysis is not repeated here. Note that these assumptions accommodate the possibility of exit payoffs also being indexed by σ as in Section 4.4.

Consider an exit game with flow payoffs γ_σ (now indexed by σ)

	S	E
S	$g^i_\sigma(w_t)$	$W^i_{12,\sigma}(w_t)$
E	$W^i_{21,\sigma}(w_t)$	$W^i_{22,\sigma}(w_t)$.

Denote by $G(\mathbf{V}, w, \sigma)$ the associated one-shot full-information game augmented with continuation $\mathbf{V}$, and by Γ_σ the exit game with payoffs indexed by σ and information $x_{i,t} = w_t + \sigma\varepsilon_{i,t}$.

Assumption 0' (Compactness of payoff structures) *There exists $\overline{\sigma} > 0$ such that for all $\sigma \in [0, \overline{\sigma}]$ all payoff structures γ_σ share a common modulus of continuity in w and converge to γ_0 with respect to the supremum norm $|| \cdot ||_\infty$ as σ goes to 0.*

Assumption 1' (Bounded values) *Denote by $m_{i,\sigma}$ and $M_{i,\sigma}$ the min-max and maximum values of player i in game Γ_σ. There exist finite bounds m and M such that for all σ, $m \leq m_{i,\sigma}$, and $M_{i,\sigma} \leq M$.*

Assumption 2' (Dominance) *There exist $\overline{\sigma} > 0$, $\underline{w}$ and $\overline{w}$ such that for all $\sigma \in [0, \overline{\sigma}]$ and all $i \in \{1, 2\}$,*

$$g^i_\sigma(\underline{w}) + \beta M_{i,\sigma} - W^i_{21,\sigma}(\underline{w}) < 0 \quad \textit{and} \quad W^i_{12,\sigma}(\underline{w}) - W^i_{22,\sigma}(\underline{w}) < 0$$
$$\textit{and} \quad W^i_{12,\sigma}(\overline{w}) - W^i_{22,\sigma}(\overline{w}) > 0 \quad \textit{and} \quad g^i_\sigma(\overline{w}) + \beta m_{i,\sigma} - W^i_{21,\sigma}(\overline{w}) > 0.$$

Assumption 3' (Increasing differences in the state of the world) *There exists $\overline{\sigma}$ such that for all $\sigma \in [0, \overline{\sigma}]$ and all $i \in \{1, 2\}$, $g^i_\sigma(w_t) - W^i_{21,\sigma}(w_t)$ and $W^i_{12,\sigma}(w_t) - W^i_{22,\sigma}(w_t)$ are strictly increasing over $w_t \in [\underline{w}, \overline{w}]$, with a slope greater than some real number $r > 0$*

Assumption 4' (Equilibrium symmetry) *For all states of the world w_t, $G(m_{i,\sigma}, m_{-i,\sigma}, w_t, \sigma)$ has a pure strategy equilibrium. All pure equilibria belong to $\{(S, S), (E, E)\}$.*

Assumption 5' (Staying is good) *For all players $i \in \{1, 2\}$ and all states of the world $w \in [\underline{w}, \overline{w}]$, $A_i(m_{i,\sigma}, w, \sigma) \geq 0$ and $B_i(w, \sigma) \geq 0$.*

Under these assumptions, the analysis of Section 3 goes through step by step.

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