Incomplete Markets and the Yield Curve

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Abstract

We present a general equilibrium model with incomplete markets and borrowing constraints to study the interest rate term structure. Agents face both an aggregate risk and an idiosyncratic risk of unemployment, which is not insurable. We derive analytical expressions for the bond prices, whatever their maturities. We also exhibit the effects of credit constraints on the whole price structure.

The bond supply affects the yield curve through a wealth effect. We prove notably that a larger volume of titles shifts the level of the yield curve downward and increases its slope. Finally, credit constraints allow idiosyncratic and aggregate risks to interact and thus make interest rates more volatile.

Keywords: incomplete markets, yield curve, credit constraints.

JEL codes: E21, E43, G12.

1 Introduction

Incomplete markets are a potential explanation for the empirical limits of asset pricing models based on complete markets. When markets are incomplete, agents need indeed to hedge themselves against uninsurable risks and they buy therefore more securities than the amount implied by the traditional smoothing motive. In addition, uninsurable idiosyncratic risks are likely to affect in a different manner the demand for titles with various maturities. For this reason, incomplete market models have been used to analyze the yield curve, as a simple departure from the standard Cox, Ingersoll, and Ross (1986) complete market model. But due to the implied heterogeneous agents structure, only few analytical results have been proposed in such frameworks, and most results rely on simulations (Seppala, 2004 among others).

We present a production economy with heterogeneous agents, incomplete markets and credit constraints. Agents switch randomly between an employed and an unemployed status on the labor market. They face both an uninsurable employment risk and an aggregate risk, which affects their wages if they are employed. We are able to derive closed-form solutions for the prices of the zero-coupon bonds of any maturity and to study the effect of the idiosyncratic risk on the level and the slope of the yield curve. This analytical solution allows us to exhibit new effects that have not been put forward in the literature yet. In particular, we show that the volume of titles affect the level and the slope of the yield curve.

The intuition for the effect of credit constraints on the yield curve can be simply explained. If an agent buys at period t a zero coupon bond of maturity $k+1$, he may have to liquidate at period $t+1$ his complete portfolio of titles of maturity k because of a bad idiosyncratic shock and credit constraints. The demand and thus the price of a title of maturity $k+1$ at period t will depend on the expected liquidation price of titles of maturity k at period $t+1$. Moreover, the liquidated portfolio is valued with a high marginal utility because of the bad idiosyncratic shock. We show that the standard Euler equation is modified to include a new term, which reflects this portfolio liquidation risk. Moreover, the larger the security volumes, the more volatile the liquidation value of the portfolio. As a consequence, this portfolio liquidation risk makes the term premium of long run over short run titles increase with the volume of titles.

Investigating the model, we prove the following new results. First, an increase in the probability of being credit constrained widens the volatility of short run interest rate. When agents are more likely to become credit constrained, their demand for insurance is larger when a good aggregate shock occurs than in case of a bad one. Second, larger volumes of titles of any maturity decrease the price of titles of all maturities. These larger volumes increase the liquidation value of agents' portfolio and make them richer on average. The self-insurance motive weakens and so does the bond demand. Third, an increase in the volume of titles steepens the yield curve. Larger bond quantities induce a liquidation risk for bonds, which widens with their maturities. Relative to short bonds, the term premium for long bonds is larger, which commands a curve steepening.

We derive analytical solutions in this framework with credit constraints and incomplete markets. Two key hypotheses drive this tractability and decrease the heterogeneity between agents. They have been used in the Bewley-Hugget literature with incomplete markets, but never together yet. The first one states that the labor supply is infinitely elastic. This assumption is made by Scheinkman and Weiss (1986) in a monetary economy. They obtain that all agents for which the credit constraint does not bind, hold the same quantity of titles. The second is made by Kehoe, Levine and Woodford (1991). In their model, agents with low income are credit constrained and liquidate all their titles. As a consequence, they do not sell progressively their assets to smooth their consumption. With this two hypotheses, the heterogeneity of the model collapse to only four different agents' types, which make the model tractable, even when an arbitrarily large number of titles with different maturities is available. The contribution of this paper is thus the derivation of this parsimonious heterogeneous equilibrium and its application to the yield curve.

This article belongs to the literature on the equilibrium effect of market incompleteness (Hugget, 1993; Aiyagari, 1994 and the vast heterogeneous agents literature), which was first concerned by monetary issues (Bewley, 1983; Scheinkman and Weiss, 1986; Kehoe, Woodford and Levine, 1991). More precisely, it is in line with papers, which focus on effects of incomplete markets and agents heterogeneity on asset pricing (see Heaton and Lucas, 1995 for a survey of previous works). Those papers were mainly concentrated on issues regarding the market price of risk (Constantines and Duffie, 1996 on the theoretical side and Krussel and Smith, 1997; Heaton and Lucas, 1998 for quantitative exercise, among others), and barely on the term structure of the yield curve. Backus, Foresi and Telmer (1998) present notably a survey, which includes models with latent factors. Exceptions are Heaton and Lucas (1992), who study a three period economy and Seppala (2004), who performs mainly a quantitative exercise. To our knowledge, our paper is the first to provide analytical solutions and to exhibit volume effects on the yield curve.

The model is presented in six other sections. The second section presents the model and the definition of the equilibrium. The third one proves the existence of the equilibrium in the general case. The fourth section is a simplified model where the effects of credit constraints and of title volumes can be easily exhibited. Section five presents some results in the case where assets are in zero net supply. Section 6 presents the volume effects on the level and slope of the yield curve. Section 7 presents a quantitative estimation of volume effects implied by the model.

2 The model

The economy is populated by a unit mass of households and firms who interact in perfectly competitive markets.

2.1 Firms

Firms produce output y_t out of a single input, the labor l_t. They have access to a constant return-to-scale production function $y_t = z_t l_t$. Productivity depends on an aggregate state variable s_t which is a Markov process between two states $\{h, l\}$ and with a transition matrix T:

$$T = \begin{bmatrix} \pi^h & 1 - \pi^h \\ 1 - \pi^l & \pi^l \end{bmatrix}$$

We make the following assumption

Assumption A. $\pi^h + \pi^l > 1$.

The previous inequality stipulates that the persistence of aggregate shock is high enough. This assumption is the most relevant one as will be shown in the empirical discussion below. Although it is not necessary for all results of the model, we make assumption **A** to avoid the discussion of irrelevant cases.

Current aggregate state is denoted s_t and $S_t = \{s_0, \ldots, s_t\}$ is the history of all past aggregate states. The productivity levels in state h and l are respectively z^h and z^l where $z^h > z^l$.

Firms' profit maximization under perfect competition implies that the real wage is equal to the marginal productivity of labor, i.e. $w_t = z_t$.

2.2 Households

The economy is populated by a unit mass of households. Each agent i has preferences over consumption and labor. We suppose that intertemporal preferences are time–separable and c_t^i (resp. l_t^i) is the consumption (labor) of the agent i at date t. All agents discount the utility at the same rate β. The agent i's objective is therefore to maximize his intertemporal utility:

$$E_t \sum_{j=0}^{\infty} \beta^j \left(u\left(c_{t+j}^i\right) - \phi l_{t+j}^i \right) \tag{1}$$

u is a one period utility function with the standard regularity properties: it is twice derivable, strictly increasing and strictly concave. The assumption of infinite labor elasticity will be discussed later.

Each agent can transfer wealth from one period to another using a wide variety of riskless bonds, whose maturities vary between 1 and $n \geq 1$ periods. $b_{t,k}^i$ is the quantity of k–period ($k \in \{1, \ldots n\}$) bonds the consumer i buys at period t. This bonds costs at t $p_{t,k}$ and pays off at date $t+k$ one unit of goods. Note that the k period bond is sold at period $t+1$ as a $k-1$ period bond at price $p_{t+1,k-1}$. By convention, $p_{t,0}$ is equal to 1 for all t.

In addition to aggregate shocks, agents face idiosyncratic shocks, which are not insurable: no agent can issue or buy a title contingent on his idiosyncratic state. Although this assumption is of course extreme, it provides a natural basis for analyzing the effect of market incompleteness on the yield curve. The agents can exclusively be in one of the two possible states: they are either *employed* or *unemployed*. We use the dummy ξ_t^i to characterize the state of the agent i at date t: $\xi_t^i = 1$ if the agent is employed and $\xi_t^i = 0$ otherwise. Employed agents work l_t^i and earn a labor income $w_t l_t^i$, where w_t is the hourly wage. Unemployed agents only get the monetary equivalent δ of their work disutility.

Agents switch randomly between these two states, $(\alpha, \rho) \in (0,1)^2$ being the probabilities of staying employed and unemployed, respectively. We denote h_t^i the state of agent i at date t ($h_t^i = e$ or u whether this agent is employed or not), and $H_t^i = \{h_0^i, h_1^i, \ldots, h_t^i\}$ is the history of agent i's states from date 0 to date t.

The budget constraint of the consumer becomes with these notations:

$$c_t^i + \sum_{k=1}^{n} p_{t,k}\, b_{t,k}^i = \xi_t^i w_t l_t^i + \left(1 - \xi_t^i\right)\delta + \sum_{k=1}^{n} p_{t,k-1}\, b_{t-1,k}^i \tag{2}$$

$$c_t^i, \quad l_t^i, \quad b_{t,s}^i \quad \geq 0 \tag{3}$$

The assumption $b_{t,s}^i \geq 0$ is the expression of credit constraints in its simplest form. Agent can not issue titles of any maturity to borrow. We denote $\left[b_{t,k}^i\right]_{k=1,\ldots,n}$ as the $n-$vector of bonds held at the end of period t.

Finally, we make the following assumption:

Assumption B. $\phi/z^l < u'(\delta)$

The previous assumption provides an upper bound to δ compared utility of one unit of labor. It is necessary for the unemployed to be worse-off compared to employed agents in all states of the world.

We solve the problem of the two types of agents in turn. By standard convexity arguments, we use the Bellman equations to derive analytical solutions. As the wealth of each agent is bounded, the transversality conditions will be fulfilled.

Employed agents. Call V^e the value function of employed agents and V^u that of unemployed agents. At period t, the optimal allocation of the agent depends on three main determinants: (i) its wealth, which is the the amount of bonds of all maturities he holds $\left[b_{t-1,k}^i\right]_{k=1,\ldots,n}$, (ii) its individual history until period t, H_t^i, and (iii) the history of aggregate states S_t. Because of the aggregate markovian structure, the knowledge of current aggregate shock s_t summarizes all the relevant information. As a consequence, the problem of an agent i when employed can be written in recursive form:

$$\begin{aligned} V^e\left(\left[b_{t-1,k}^i\right]_{k=1,\dots,n}, H_t^i, s_t\right) &= \max_{c_t^i, l_t^i, \left[b_{t,k}^i\right]_{k=1,\dots,n}} \left\{u\left(c_t^i\right) - \phi l_t^i\right. \\ &\left.+\beta\left(\alpha E_t V^e\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right) + (1-\alpha) E_t V^u\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right)\right)\right\} \\ s.t. \quad & c_t^i + \sum_{k=1}^n p_{t,k}\, b_{t,k}^i = z_t l_t^i + \sum_{k=1}^n p_{t,k-1}\, b_{t-1,k}^i \end{aligned}$$

and subject to non-negativity constraints (3). In the budget constraint, we have used the equality $w_t = z_t$ to substitute for the real wage. Substituting for the expression of c_t^i, the program reduces to the following form:

$$\begin{aligned} V^e\left(\left[b_{t-1,k}^i\right]_{k=1,\dots,n}, H_t^i, s_t\right) &= \max_{l_t^i, \left[b_{t-1,k}^i\right]_k} \left\{u\left(z_t l_t^i + \sum_{k=1}^n (p_{t,k-1}\, b_{t-1,k}^i - p_{t,k}\, b_{t,k}^i)\right) - \phi l_t^i\right. \\ &\left.+\alpha\beta E_t V^e\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right) + (1-\alpha)\beta E_t V^u\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right)\right\} \end{aligned}$$

Derivatives respective to l_t, and $b_{t,k}^i$ as well as envelop conditions relative to $b_{t-1,k}^i$ provide the following conditions for all $k = 1, \dots, n$

$$u'\left(c_t^i\right) = \phi / z_t, \tag{4}$$

$$\begin{aligned} p_{t,k} u'\left(c_t^i\right) &\underset{(>)}{=} \alpha\beta E_t\left[V_{b_k}^e\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right)\right] \\ &+ (1-\alpha)\beta E_t\left[V_{b_k}^u\left(\left[b_{t,k}^i\right]_{k=1,\dots,n}, H_{t+1}^i, s_{t+1}\right)\right] \text{ and } b_{t,k}^i \underset{(=)}{>} 0 \end{aligned} \tag{5}$$

$$p_{t,k-1} u'\left(c_t^i\right) = V_{b_k}^e\left(\left[b_{t,k-1}^i\right]_{k=1,\dots,n}, H_t^i, s_t\right) \tag{6}$$

Equation (4) is the optimal labor supply of employed agents, while (5) is the consumption Euler equation between period t and period $t+1$ for titles of maturity k. Whether the latter holds with equality or not depends on the slackness of the borrowing constraint. When the borrowing constraint is slack (binding), then the solution to agent i's consumption/saving choice is interior (corner), and the Euler equations hold with equality (inequality) and $b_{t,k}^i > 0\,(=0)$. The last equation is the envelop condition for titles of maturity k.

Unemployed agents. In a similar way, the problem of any unemployed agent i expresses as:

$$\begin{aligned} V^u\left(\left[b_{t-1,k}^i\right]_k, H_t^i, s_t\right) &= \max_{c_t^i, l_t^i, \left[b_{t,k}^i\right]_k} \left\{u\left(c_t^i\right) - \phi l_t^i\right. \\ &\left.+\beta\left((1-\rho) E_t V^e\left(\left[b_{t,k}^i\right]_k, H_{t+1}^i, s_{t+1}\right) + \rho E_t V^u\left(\left[b_{t,k}^i\right]_k, H_{t+1}^i, s_{t+1}\right)\right)\right\} \\ s.t. \quad & c_t^i + \sum_{k=1}^n p_{t,k}\, b_{t,k}^i = \delta + \sum_{k=1}^n p_{t,k-1}\, b_{t-1,k}^i \end{aligned}$$

The optimal solution to this problem is such that $l_t^i = 0$ and:

$$p_{t,k} u'\left(c_t^i\right) \underset{(>)}{=} E_t\left[(1-\rho)\beta V_{b_1}^e\left(\left[b_{t,k}^i\right]_k, H_{t+1}^i s_{t+1}\right)\right] + E_t\left[\rho\beta V_{b_1}^u\left(\left[b_{t,k}^i\right]_k, H_{t+1}^i, s_{t+1}\right)\right] \quad (7)$$

$$\text{and} \qquad b_{t,1}^i \underset{(=)}{>} 0$$

$$p_{t,k-1} u'\left(c_t^i\right) = V_{b_k}^u\left(\left[b_{t,k-1}^i\right]_k, H_t^i, s_t\right) \quad (8)$$

We derive therefore analogous equations for the constrained agent except for the labor supply equation. He is indeed unemployed and does not work.

2.3 Invariant distribution of agents' types

In the general case, heterogeneous agents' models such as the one sketched above, generate an infinite-dimensional distribution of agents' types. Each individual characteristics (i.e. wealth and implied optimal choices) depend indeed on the personal history of all individual agents. The conventional approach consists in solving the model computationally in order to approximate the invariant distribution of agents' types (e.g., Imrohoglu (1992), Aiyagari (1994), Krussel and Smith (1998))[1]. In this paper, we adopt an alternative approach by deriving a closed-form solution with a finite number of types, so that the distribution of wealth, consumption and labor choices can be solved analytically. We derive our closed-form equilibrium solutions with borrowing constraints in three steps. First, we conjecture the general shape of the solution; second, we identify the conditions under which the conjectured solution exists; and third, we prove that these conditions always hold along the equilibrium.

2.3.1 Conjectured equilibrium

We conjecture the existence of an equilibrium where all unemployed agents are borrowing constrained while no employed agent is. As a consequence, unemployed agent does not hold any asset of any maturity, whereas employed agents hold bonds of all maturities.

Unemployed Agents. An agent i, who falls into unemployment at date t after having been employed at date $t-1$, liquidate his asset portfolio because of credit constraints. He therefore chooses $b_{t,k}^i = 0$ for all maturities k. His budget constraint implies:

$$c_t^i\left(H_t^i = \left\{H_{t-2}^i, e, u\right\}, z_t\right) = \delta + \sum_{k=1}^{n} p_{t,k-1}\, b_{t-1,k}^i \quad (9)$$

[1] To our knowledge, this computational exercise has not been done for the simple model sketched above, because of the presence of aggregate shocks and of the high number of state variables.

On the other hand, an agent staying unemployed from date $t-1$ to date t does not have neither labor income nor financial revenues, because he does not hold any asset. He consumes only the monetary equivalent δ of his labor disutility:

$$c_t^i\left(H_t^i=\left\{H_{t-2}^i,u,u\right\},z_t\right)=\delta\equiv c_t^{uu}\qquad\forall H_{t-2}^i. \tag{10}$$

Employed Agents. Since labor supply is infinitely elastic (see (1)), all employed agents are willing to work as hard as necessary to secure the consumption level implied by the optimal labor supply equation (4). Thus, their consumption does not depend on their past history and expresses as:

$$c_t^i\left(H_t^i=\left\{H_{t-1}^i,e\right\},z_t\right)=u'^{-1}\left(\phi/z_t\right)\equiv c_t^e\qquad\forall H_{t-1}^i, \tag{11}$$

where we replace the wage w_t with the productivity z_t.

Since employed agents are not borrowing-constrained, the Euler equation (5) holds with equality. Using the consumption expressions of unemployed (9) and employed (11), together with envelope conditions (6) and (8), we simplify this Euler equation to obtain the following price definitions:

$$\begin{aligned} p_{t,k+1}/z_t &= \alpha\beta E_t\left(p_{t+1,k}/z_{t+1}\right)+(1-\alpha)\beta E_t\left(p_{t+1,k}\,u'_{t+1}\right)\ \text{ for } k=1,\ldots,n-1 \\ p_{t,1}/z_t &= \alpha\beta\, E_t\left(1/z_{t+1}\right)+(1-\alpha)\beta E_t\left(u'_{t+1}\right) \\ \text{with:}\quad u'_{t+1} &\equiv u'\left(\delta+\sum_{k=1}^{n}p_{t+1,k-1}\,b_{t,k}^i\right)/\phi \end{aligned} \tag{12}$$

The choice of titles of maturity k by the employed agent i that we note $b_{t,k}^i$ depends only on aggregate variables such as prices and the aggregate technology shock. As a consequence, all employed agents hold the same quantity of each title, regardless of their personal history. Denoting $b_{t,k}^e$ the assets of employed agents, the consumption of agents with history $\left\{H_{t-2}^i,e,u\right\}$ is identical across agents, because they liquidate the same portfolio. From (9), this consumption expresses as:

$$c_t^i\left(H_t^i=\left\{H_{t-2}^i,e,u\right\}\right)=\delta+\sum_{k=1}^{n}p_{t,k-1}\,b_{t-1,k}^e\equiv c_t^{eu}\qquad\forall H_{t-2}^i. \tag{13}$$

Finally, price equations (9) to (13) as well as the budget constraint (2) imply that the labor supplies of agents with histories $\left\{H_{t-2}^i,e,e\right\}$ and $\left\{H_{t-2}^i,e,u\right\}$ are respectively:

$$l_t^i\left(H_t^i=\left\{H_{t-2}^i,e,e\right\}\right)=\left(c_t^e+\sum_{k=1}^{n}\left(p_{t,k}\,b_{t,k}^e-p_{t,k-1}\,b_{t-1,k}^e\right)\right)/z_t\equiv l_t^{ee}\qquad\forall H_{t-2}^i, \tag{14}$$

$$l_t^i\left(H_t^i=\left\{H_{t-2}^i,u,e\right\}\right)=\left(c_t^e+\sum_{k=1}^{n}p_{t,k}\,b_{t,k}^e\right)/z_t\equiv l_t^{ue}\qquad\forall H_{t-2}^i. \tag{15}$$

Because all unemployed agents are borrowing-constrained and no employed agent is, agents can only be of four different types, which depend on their employment states in the current and past periods. Their personal history before date $t-1$ becomes then irrelevant. In the following, we denote these types ee, eu, ue, uu, where the first and second letters refer respectively to the last and current employment states. Given the transition probabilities α and ρ, the asymptotic proportions of agents of each type are:

$$\omega^{ee} = \frac{\alpha(1-\rho)}{2-\alpha-\rho},\ \omega^{eu} = \omega^{ue} = \frac{(1-\alpha)(1-\rho)}{2-\alpha-\rho},\ \omega^{uu} = \frac{\rho(1-\alpha)}{2-\alpha-\rho}, \tag{16}$$

while the employment and unemployment rates are $\omega^{e} = \omega^{ee} + \omega^{ue}$ and $\omega^{u} = \omega^{uu} + \omega^{eu}$, respectively.

2.3.2 Conditions for this equilibrium to exist

The stationary distribution of agents' type was constructed under the assumption that unemployed agents are borrowing-constrained. We now derive the conditions under which this is actually the case in our 4-type equilibrium.

The intertemporal optimality condition (7) must hold with strict inequality for both uu and eu agents.

For *uu agents*, their consumption is equal at $c^{uu}_{t+1} = \delta$. The condition (7) implies that these agents are credit constrained if and only if:

$$\forall k = 1,\ldots,n \quad p_{t,k}\, u'(\delta) > \beta(1-\rho)\,\phi\, E_t\left(p_{t+1,k-1}/z_{t+1}\right) + \beta\rho\, u'(\delta)\ E_t\left(p_{t+1,k-1}\right) \tag{17}$$

Agents eu liquidate their asset portfolio and consume everything, such that their consumption c^{eu}_{t+1} at date $t+1$ is: $c^{eu}_{t+1} = \delta + \sum_{k=1}^{n} p_{t+1,k-1}\, b^{e}_{t+1,k}$. The condition (7) expresses as:

$$\forall k \quad p_{t,k}\, u'\left(\delta + \sum_{j=1}^{n} p_{t,j-1}\, b^{e}_{t-1,j}\right) > \beta(1-\rho)\,\phi\, E_t\left(p_{t+1,k-1}/z_{t+1}\right) + \beta\rho\, u'(\delta)\ E_t\left(p_{t+1,k-1}\right) \tag{18}$$

Because agents hold a positive quantity of assets, the marginal utility of an agent uu is larger than the one of an agent eu. The first type can only consume his private revenue, whereas the last type liquidate his portfolio to increase its revenue and its consumption. As soon as the condition (18) insuring that agents eu are credit constrained, is verified, the condition for agent uu is also true.

This condition states that neither the private income δ nor the bond supply is too large. Agents liquidating their asset portfolio as well as agents without any earning must be constrained and would like to borrow to smooth intertemporally their consumption. The condition on the private income

δ is natural, since without it agents would not have any incentive neither to borrow nor to work. The other condition insures that the agent is constrained to liquidate in one shot his complete asset portfolio. Otherwise the equilibrium would not be so simple and the agents' distribution would not sum up to four types.

2.4 Market clearing

Only employed agents have access to financial markets and hold securities. All these non-constrained agents hold the same quantity $b^e_{t,k}$ of k period bonds. As a consequence, the aggregate demand for the bond of maturity k reaches $\omega^e b^e_{t,k}$.

At each date t, a net quantity $A_{t,k}$ of zero coupon bonds yielding one unit of good at period $t+k$ is issued on the market. The aggregate supply of securities for a given maturity is composed of newly issued bonds, but also of longer bonds issued earlier and becoming closer to maturity. At date t, a quantity $B_{t,k} \equiv \sum_{j=k}^{n} A_{t-j,k+j}$ of bonds with maturity k is available on the market.

The equilibrium condition on the bond market between supply and demand implies the following equalities for each maturity and at each date t:

$$\forall t \quad \forall k \in \{1, \ldots, n\}, \quad \omega^e \, b^e_{t,k} = B_{t,k} \text{ where } B_{t,k} \equiv \sum_{j=k}^{n} A_{t-j,k+j}$$

2.5 Conjectured asset price structure

Since all employed agents hold the same quantity of securities, we can derive the Euler pricing equation. Using the first order condition as well as the envelop one, we obtain the following expressions:

$$\forall t \quad \forall k \in \{1, \ldots, n\}$$
$$p_{t,k}/z_t = \alpha \, \beta \, E_t \left[p_{t+1,k-1}/z_{t+1} \right] + (1-\alpha) \, \beta \, E_t \left[p_{t+1,k-1} \, u' \left(\delta + \sum_{j=1}^{n} p_{t+1,j-1} B_{t,j}/\omega^e \right) /\phi \right] \quad (19)$$

The previous equations are the pricing equations, which pins down the price of any bond as a function of the current and next aggregate states and of all future prices. The price of a k period bond expresses as the sum of two terms: (i) a smoothing one and (ii) a liquidation value where a wealth effect intervenes. The first term reflects the standard Euler pricing equation and values the bond through the marginal utility of an employed agent. The second term is due to credit constraints and values the security with the marginal utility of an agent becoming unemployed and thus forced to liquidate its asset portfolio.

From the literature on asset pricing with finite state space, we can conjecture a simple form of asset prices. If we suppose that:

$$\forall t \geq 0, \forall k \in \{1, \ldots, n\}, \forall s \in \{h, l\} \quad p^s_{t,k} = C^s_k \, z^s \tag{20}$$

..., then the pricing equations (19) expressed in both states h and l provide for $k = 2, \ldots, n$:

$$\left\{ \begin{array}{l} C^h_k = \alpha\beta \left(\pi^h C^h_{k-1} + \left(1-\pi^h\right) C^l_{k-1}\right) + (1-\alpha)\beta \left(\pi^h C^h_{k-1} \, z^h \, u'^h_{t+1} + \left(1-\pi^h\right) C^l_{k-1} z^l \, u'^l_{t+1}\right) \\ C^l_k = \alpha\beta \left(\pi^l C^l_{k-1} + \left(1-\pi^l\right) C^h_{k-1}\right) + (1-\alpha)\beta \left(\pi^l C^l_{k-1} z^l \, u'^l_{t+1} + \left(1-\pi^l\right) C^h_{k-1} \, z^h \, u'^h_{t+1}\right) \end{array} \right. \tag{21}$$

$$\text{and} \left\{ \begin{array}{l} C^h_1 = \alpha\beta \left(\pi^h / z^h + \left(1-\pi^h\right) / z^l\right) + (1-\alpha)\beta \left(\pi^h \, u'^h_{t+1} + \left(1-\pi^h\right) \, u'^l_{t+1}\right) \\ C^l_1 = \alpha\beta \left(\pi^l / z^l + \left(1-\pi^l\right) / z^h\right) + (1-\alpha)\beta \left(\pi^l \, u'^l_{t+1} + \left(1-\pi^l\right) \, u'^h_{t+1}\right) \end{array} \right.$$

with the notations:

$$\left\{ \begin{array}{l} u'^h_{t+1} \equiv u' \left(\delta + \sum_{i=0}^{n-1} C^h_i \, z^h B_{t+1,i}\right) / \phi \\ u'^l_{t+1} \equiv u' \left(\delta + \sum_{i=0}^{n-1} C^l_i \, z^l B_{t+1,i}\right) / \phi \end{array} \right.$$

Because prices of bonds with maturity 0 is equal to 1, we define:

$$\left[\begin{array}{c} C^h_0 \\ C^l_0 \end{array} \right] = \left[\begin{array}{c} 1/z^h \\ 1/z^l \end{array} \right], \tag{22}$$

The price structure can be written compactly in a recursive form for $k = 1, \ldots, n$:

$$\left[\begin{array}{c} C^h_k \\ C^l_k \end{array} \right] = \beta T \cdot \left[\begin{array}{cc} \alpha + (1-\alpha) w^h \, u'^h_{t+1} & 0 \\ 0 & \alpha + (1-\alpha) w^l \, u'^l_{t+1} \end{array} \right] \cdot \left[\begin{array}{c} C^h_{k-1} \\ C^l_{k-1} \end{array} \right] \tag{23}$$

This system provides $2 \times n$ equations that determine the $2 \times n$ coefficients $\{C^h_k, C^l_k\}_{k=1,\ldots,n}$. This system is not linear because the whole price structure appears in each coefficient u'^h_{t+1} and u'^l_{t+1}. We prove the existence of the equilibrium below.

3 Existence of the equilibrium in the general case

3.1 Existence with zero net supply and no aggregate shock

Equilibrium Yield Curve with zero net supply. We skip the subscript t when values are constant. We assume that $z^h = z^l = 1$. The equilibrium on bond markets simplifies into:

$$\forall k \in \{1, \ldots, n\}, \quad \omega^e \, b^e_k = 0$$

In this case $u'^h = u'^l = u'(\delta) / \phi$. The conjectured price structure (20) is simply $p_k = C_k$ and the system (23) reduces to a simple equation:

$$C_k = \beta \left(\alpha + (1-\alpha) \, u'(\delta) / \phi\right) C_{k-1} \text{ for } k = 1, \ldots, n$$

with $C_0 = 1$. The price structure follows directly:

$$p_k = \beta^k \left(\alpha + (1-\alpha) u'(\delta) / \phi\right)^k \tag{24}$$

The yield to maturity of a bond of maturity k is defined by the standard expression $r_k = -\frac{1}{k} \ln p_k$. Interest rates are supposed to be continuously compounded. The yield structure is thus:

$$r_k = -\ln \beta - \ln \left(\alpha + (1-\alpha) u'(\delta) / \phi\right)$$

All yields are constant and the yield curve is flat because there is no aggregate risk. Note that even in this simple case, credit constraints affect the yield curve. Indeed, assumption **B** means $u'(\delta)/\phi > 1$. As a consequence, an increase in the risk of becoming credit constrained shifts globally the whole yield curve downward. Households self-insure themselves more, when they are more likely to become credit constrained. The demand for titles increases, which pushes bond prices uniformly upward, because all titles are perfect substitutes without aggregate risk.

As a consequence, an equilibrium with only an idiosyncratic risk does not provide any rich pattern. The short yield sums indeed up the information contained in the whole yield curve. Longer maturities does not play any role. The intuition behind this is quite simple. Agents want to hedge against the risk of unemployment, but they only have access to the financial market when they are employed. Moreover, the market provides them an imperfect hedge for next period risk: credit constraints are reflected in the price of short bond p_1, while more time-distant risks are be covered and do not appear in the pricing relationship.

Existence of the Equilibrium with zero net supply and no aggregate shock Using the zero net supply assumption and prices given by (24) in the condition (18), one finds that credit constraints bind for unemployed agent if:

$$\left(\alpha + (1-\alpha)\frac{u'(\delta)}{\phi}\right)\frac{u'(\delta)}{\phi} > (1-\rho) + \rho\frac{u'(\delta)}{\phi}$$

Because of assumption **B**, which implies $\frac{u'(\delta)}{\phi} > 1$, this condition is fulfilled as soon[2] as $\alpha < 1$. In zero net supply, unemployed agents are always credit constrained as soon as there are idiosyncratic shocks.

Continuity of the yield curve as a function of the supply of titles and shocks. The following proposition summarizes the regularity property of the yield curve, which will be extensively

[2] The RHS reaches its maximum $\frac{u'(\delta)}{\phi}$ when $\rho = 1$ and when $\alpha < 1$, we have $\left(\alpha + (1-\alpha)\frac{u'(\delta)}{\phi}\right)\frac{u'(\delta)}{\phi} > \frac{u'(\delta)}{\phi}$.

used below. We introduce the following vector notations. B is the vector of bond quantities for the n maturities: $B = [B_1 \dots B_n]^\top$, Z the vector of wages (or equivalently productivities) $Z = \left[z^l, z^h\right]^\top$and C is the vector of coefficients for both states h and l and the n maturities: $C = \left[C_1^h \quad C_1^l \dots C_n^h \quad C_n^l\right]^\top$

Proposition 1 *If B is in the neighborhood of* 0 *and Z in the neighborhood of* $(1,1)$ *, then C is a $\mathcal{C}^1$ function of B and of Z .*

The proof of the proposition is left in appendix. This proposition states simply that the introduction of positive bond quantities as the existence aggregate risk does not make the yield curve jump.

Proof of the existence of the equilibrium in the general case. The system (23) with initial conditions (22) defines the vector C as a continuous function of B and Z, when $\left[B^\top, Z^\top\right]$ is in a neighborhood V_1 of $\left[O_n^\top, 1_2^\top\right]$. Moreover, if $\left[B^\top, Z^\top\right] = \left[O_n^\top, 1_2^\top\right]$, the equilibrium vector C satisfies conditions (18). By continuity, there exists a neighborhood V_2 of $\left[O_n^\top, 1_2^\top\right]$, and $V_2 \subset V_1$ such that conditions (18) is fulfilled if $\left[B^\top, Z^\top\right] \in V_2$.

In words, if the supply of titles of any maturities, and if the variance of the aggregate shock is small enough around the mean 1, then the conjectured equilibrium exists.

In the general case, the average yield curve is the sum of the yield curves in states h and l weighted by the average frequency of aggregate state h and l (given by the matrix T). The average yield r_k of maturity k is

$$r_k = \frac{1-\pi^l}{2-\pi^l-\pi^h} r_k^h + \frac{1-\pi^h}{2-\pi^l-\pi^h} r_k^l \tag{25}$$

4 The Simple Model

We simplify the preceding model and suppose that agents can only invest in two securities, whose mature respectively in one and two periods. We suppose that z is a process with value $z^h = 1+\varepsilon$ with probability 1/2 and $z^l = 1-\varepsilon$ with probability 1/2. The variation ε is small and we consider only second order approximations in ε. The utility is quadratic and the marginal utility is linear in consumption: $u'(c) = u_1 - u_2\, c$ with $u_1, u_2 > 0$. Constants u_1 and u_2 are such that standard assumptions regarding growth and concavity of the utility function are fulfilled. The explicit conditions on u_1 and u_2 are given in appendix. The period t prices of both assets $p_{1,t}$ and

$p_{2,t}$ satisfy the following equations:

$$p_{1,t}/z_t = \alpha\beta E_t\left[z_{t+1}^{-1}\right] + (1-\alpha)\,\beta E_t\left[u'\left(\delta + B_1 + B_2\, p_{1,t+1}\right)/\phi\right]$$
$$p_{2,t}/z_t = \alpha\beta E_t\left[p_{1,t+1}/z_{t+1}\right] + (1-\alpha)\,\beta E_t\left[p_{1,t+1}\, u'\left(\delta + B_1 + B_2\, p_{1,t+1}\right)/\phi\right]$$

Because of the $i.i.d$ assumption, both prices are proportional to the level of aggregate shock: $p_{i,t} = C_i\, z_t$ and as $Ez_t = 1$, the average price of a one-period bond is $p_1 = Ep_{1,t} = C_1$, and the average price of a two period bond is $p_2 = Ep_{2,t} = C_2$. The previous equations imply the following prices:

$$p_1 = \frac{\alpha\beta\left(1+\varepsilon^2\right) + (1-\alpha)\,\beta(u_1 - u_2(\delta + B_1))}{1 + (1-\alpha)\,\beta u_2\, B_2}$$
$$\frac{p_2}{p_1} = \alpha\beta + (1-\alpha)\,\beta\left(u_1 - u_2\left(\delta + B_1 + p_1 B_2\,\left(1+\varepsilon^2\right)\right)\right)/\phi$$

Title volumes affect both prices. An increase in either B_1 or B_2 decreases the price p_1. Short run assets are indeed bought to smooth consumption and to self-insure against the risk of unemployment. When volumes of titles widen, the liquidation value of the portfolio becomes higher. As a consequence, this wealth effect softens the self-insurance motive. Volumes have an additional effect on p_2 through p_1. As p_1 decreases, the value of the liquidation of long run titles tomorrow $p_1 B_2$ decreases. This tends to diminish the liquidation value of the portfolio and to sharpen the demand for insurance.

To analyze the volume effects on the prices, we investigate the change of the yield curve with credit constraints when the probability of becoming credit constrained is low (α close to 1). This is the relevant case for all developed countries, as the empirical discussion below will show. The average slope of the yield curve is $S = -\frac{1}{2}\log\frac{p_2^2}{p_1}$. After few calculations, the slope expression simplifies to:

$$S/\varepsilon^2 = 1 - \left(u_1 - u_2\left(B_1 + \delta\right) - u_2\,\frac{2B_2\beta}{1-\varepsilon^2}\right)(1-\alpha) + O\left(1-\alpha\right)^2$$

We split our discussion according the two relevant aspects: (i) the credit constraints effects and (ii) the volumes effects.

Effect of credit constraints. If the term under bracket $u_1 - u_2\left(B_1 + \delta\right) - u_2\left(\frac{2B_2\beta}{1-\varepsilon^2}\right)$ is negative, then the slope increases when α decreases, or equivalently when the probability of being credit constrained increases. As the marginal utility of unemployed agents is positive, we necessarily have $u_1 - u_2\left(\delta + B_1\right) > 0$. The preceding term under bracket is negative if B_2 is large enough: If the long run security supply is large enough, the yield curve steepens with the probability of credit constraints.

The intuition is the following. Bond volumes affect agents' behavior because of the wealth effect in case of portfolio liquidation. The long run titles bought at period t have a specific risk, because their $t+1$ price equal to $P_{1,t+1} = C_1\, z_{t+1}$ varies with the aggregate state. If B_2 is large, this risk concerns an important volume and thus commands a high premium of long run titles over short run ones. As a consequence, the term premium increases with the probability of credit constraints $1-\alpha$, if the risk concerns a large enough amount of titles.

If the supply of long run titles is low (and in particular in the zero net supply case), the liquidation risk is low and the slope of the yield curve decreases with the probability of facing credit constraints. On the one hand, the marginal utility in case of unemployment is high, but constant. The covariance between returns on both long run and short run titles and the marginal utility is therefore 0 for unemployed agents, who thus have the same appreciation for the risk of both assets. On the other hand, employed agents have a countercyclical marginal utility, whereas the one period return on long run bonds is procyclical. These agents ask a term premium, which is increasing in the probability of staying employed[3]. Without volume effects, the average term premium decreases with the probability of staying employed. To sum these effects up, the yield curve steepens with the probability of facing credit constraints if the liquidation risk is high enough. In appendix, we prove that, when the equilibrium exists, credit constraints either increase or decrease the slope of the yield curve.

Volume Effects. First, if $\alpha = 1$, bond volumes do not affect the slope of the yield curve. This is the standard result in complete markets. If $\alpha < 1$ (but close enough to 1), then larger volumes, either B_1 or B_2, increase the slope of the yield curve. The effect through B_2 has been explained in the previous paragraph and concerns the volatility of the portfolio value in case of liquidation. One can check that the effect of B_2 on the slope increases with the variance of the aggregate shock ε^2. When B_1 becomes larger, the demand for both assets diminishes, because unemployed agents are richer demand less self-insurance is lower. However, the demand for long assets decreases more than the demand for short ones. When unemployed agents become richer, their marginal utility decreases and the credit constraints have less impact in pricing equations. The covariance between returns of short and long titles and the marginal utility of employed agents, has, on the contrary, a higher weight in the pricing equation. Because this covariance is negative for long assets, their relative demand decreases more than the demand for short ones. Note that if $\beta > 0.5$ which is the

[3] This is the standard term structure in complete market models. One can easily check that this term premium is ε^2 in this model.

standard assumption, then a larger B_2 increases more the slope of the yield curve than a larger B_1.

5 Credit Constraints and Aggregate shock

In this section, we derive the effect of credit constraints on the yield curve, when assets of all maturities are in zero net supply. We find closed form expressions for the yield curve in each state of the world. The system (23) defining bond prices can be written as:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \beta M. \begin{bmatrix} C_{k-1}^h \\ C_{k-1}^l \end{bmatrix} \tag{26}$$

where

$$M = T. \begin{bmatrix} \alpha + (1-\alpha) z^h\, u'(\delta) & 0 \\ 0 & \alpha + (1-\alpha) z^l\, u'(\delta) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix} = \begin{bmatrix} 1/z^h \\ 1/z^l \end{bmatrix}$$

We prove in appendix that the matrix M can be diagonalized. There exists an invertible square matrix Q and a diagonal matrix D such that $M = Q.D.Q^{-1}$. The expression of the prices simplifies then in:

$$\begin{bmatrix} p_k^h \\ p_k^l \end{bmatrix} = \beta^k \begin{bmatrix} z^h & 0 \\ 0 & z^l \end{bmatrix} Q.D^k.Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix}$$

With this expression, we deduce the value of the yield curve $r_k^s = -\frac{1}{k} \ln p_k^s$ for $k = 1, \ldots, n$ and $s = h, l$. We analyse properly these expressions in appendix and main results are summed up in the following proposition.

Proposition 2 *When $B = 0$:*

1) The average short run interest rate r_1 increases with α.

2) $\lim_{k\to\infty} r_k^l = \lim_{k\to\infty} r_k^h = r^{\lim}$ and $r^{\lim}$ increases with α.

3) $r_k^h < r_k^l \quad k = 1, \ldots, n$.

4) If α is close to 1, the variance of all yields r_k decreases with α (increases with credit constraints), $k = 1, \ldots, n$.

The first two results prove that the average yield curve shifts downward, when agents are more likely to be credit constrained. Their demand for self insurance widens, what raises bond prices. The yield curve in case of bad aggregate shock lies uniformly above the yield curve in case of good state of the world (see Seppala (2004) for some evidence on the real yield curve in UK). If productivity is high, the demand for titles is high, since agents save more in order to transfer resources in worst states of the world, where the wage is lower. This raises prices and shifts globally the yield curve downward.

The last result proves that the variance of the short run yield increases with credit constraints. The demand for insurance becomes larger when it is more probable to be credit constrained. It is then less costly to self-insure, when the wage is high, because the marginal utility of consumption is low. As a consequence, the higher the productivity, the larger the demand for bonds. The short run yield decreases more when the productivity is high, and the gap between the two yield curves widen with credit constraints, what increases the variance of the short run yield.

This section has presented some results concerning the effect of credit constraints on the yield curve when all assets are in zero net supply. By continuity, all of these results remain when the supply of assets is small enough.

6 Volume Effects on The Yield Curve

When agents face a positive probability of being credit constrained ($\alpha < 1$), the volume of titles affect both the level and the slope of the yield curve, and the variance of short run titles.

The average slope of the yield curve Δ is the difference between the long run yield which is denoted $\widetilde{r}^{\lim}$ and the short run yield r_1 given by (25): $\Delta = \widetilde{r}^{\lim} - r_1$.

The following proposition summarizes the results.

Proposition 3 *If B is small enough and $\alpha < 1$:*

1) $\dfrac{\partial p_k^s}{\partial B_i} < 0$ *for $i, k = 1, \ldots, n$ et $s = h, l$.*

2) $\lim\limits_{k\to\infty} r_k^l = \lim\limits_{k\to\infty} r_k^h = \widetilde{r}^{\lim}$.

Moreover if α is close to 1, then

3) $\dfrac{\partial \Delta}{\partial B_i} > 0$ *for $i = 1, \ldots, n$.*

4) The variance of all yields r_k decreases with B_i, $i = 1, \ldots, n$.

The first point 1) proves that a larger supply of any asset (whatever its maturity) decreases the bond prices for all maturities in all states of the world. As shown in the simple model, agents *eu* who liquidate their portfolio are richer when the volume of titles is higher. As a consequence, they demand less titles of any maturity. The yield curve shifts then upward in both state of the world h and l.

The item 2) states that the yield curves in state h or j converge to a common value, if maturities are high enough. The next two items are valid if the probability of facing credit constraints is low enough, which is the relevant case. The third item summarizes the volume effect on the slope of the yield curve. A larger bond volume, whatever its maturity, steepens the yield curve. A larger bond volume makes indeed the portfolio liquidation value more volatile. As shown in the simple model, long run assets become then as relatively riskier than short ones, because their price volatility concerns a more important volume. Item 4) proves that for a given α small enough, the variance of all yields, whatever their maturities, decreases with the volume of titles. Since agents become richer in case of liquidation, they demand less titles to self-insure. But, as stated in the previous proposition, agents self-insure more in state h (relative to state l), because it is cheaper. The slowdown in demand concerns more titles in state h than in state l: The price p_1^h decreases more (the yield r_1^h increases more) than the price p_1^l (the yield r_1^l). The difference between both yields diminishes and the variance decreases.

7 Estimation of volume effects

We investigate the empirical size of volume effects on the yield curve. Before empirical results, we present our calibration strategy.

7.1 Model calibration

In order to assess the empirical behavior for our model , we need to calibrate four types of parameters: (i) composition of the population, which characterizes the model heterogeneity, (ii) the preference parameters of all individuals, (iii) the wage process and (iv) the bond supply.

For all data, our time step is the quarter.

Heterogeneity. We use Imrohoroglu (1992) values to calibrate the model heterogeneity and in particular the matrix T driving the idiosyncratic shock of unemployment. However, since her time baseline is 6 weeks, we need to convert her data into quarterly ones. We obtain then:

Parameters	α	ρ
Values	0.9366	0.2717

Preference parameters. We suppose that the consumption utility is CRRA, so that the per period utility of a given agent consuming c and working l is:

$$u(c,l) = \frac{c^{1-\sigma}}{1-\sigma} - \phi\, l$$

We chose for preference parameters the following ones:

Parameters	σ	ϕ
Values	3	1

Wage process. Instead of calibrating directly the wage process z when agents are employed, we have chosen that our model replicates the US real GDP per capita. The GDP in our model is indeed endogenously determined as the sum of the consumption of the four agents' types:

$$\begin{aligned} GDP_t &= \omega^e\, c_t^e + \omega^{eu}\, c_t^{eu} + \omega^{uu}\, c_t^{uu} \\ &= \omega^e \left[\frac{z^{s_t}}{\phi}\right]^{\frac{1}{\sigma}} + (\omega^{eu} + \omega^{uu})\,\delta + \omega^{eu}\, z^{s_t} \sum_{k=1}^{n} C_{k-1}\, b_k^e \end{aligned}$$

The process s_t characterizes the aggregate state at date t.

We use US real GDP per capita, from Q1 1947 to Q3 2006. We assume that at each date, the GDP per capita is the product of a structural term and a cyclical one. We compute the structural term using the Hodrick and Prescott filter (1981) with a penalty of 1600, which is standard for quarterly data. We finally focus on the cyclical term y_t, from which we derive our wage process. It is supposed to be a Markov switching mean constant process, equal to GDP^h with probability π^h and to GDP^l with probability π^l

$$\begin{aligned} y_t &= GDP^{s_t} + \varepsilon_t \\ s_t &= \begin{cases} h & \text{with probability } \pi^h \\ l & \text{with probability } \pi^l \end{cases} \\ \varepsilon_t &\rightsquigarrow \mathcal{N}(0,\sigma^2) \end{aligned}$$

We use Hamilton (1989) procedure to compute the likelihood. Its maximization leads to the following results:

	π^l	π^h	GDP^l	GDP^h	σ
Values	0.81313	0.94470	0.99358	1.02347	0.01195
Std Err.	0.00131	0.00105	0.03175	0.06943	0.00062

In good times, the cyclical component is 2.3% higher than the trend. The probability that a good quarter follows another good quarter is 0.95. A wealthy period persists on average for $1/(1-0.95) \approx 18.1$ quarters, whereas a periods below trend persists for 5.4 quarters. Hard episodes are typically 1% percent GDP below trend.

Both types of episodes are highly persistent and our assumption regarding probabilities π^h and π^l is largely fulfilled: $\pi^h + \pi^l = 1.8 >> 1$.

We deduce then the values for the wage process z with bond volumes and without:

	$B = 0$		$B > 0$	
	z^l	z^h	z^l	z^h
Values	0.86	0.94	0.965	0.94

Bond supply. Our model is essentially a real one and does not account for nominal assets. However, we need a proxy for (i) the amount of bond holdings by households and (ii) their maturity structure. For the amount, we use the Flow of Funds Accounts provided by the Federal Reserve. The balance sheet of households in Q3 2006 states that households hold 2 674 billions of dollars in credit market instruments, from which we have excluded mortgages and open market instruments. Other credit market instruments include indeed various types of bonds, like Treasury securities, municipal securities, corporate and foreign bonds. At the same date, the yearly disposable income of households reach 9 588.4 bn \$. It means that bond holdings correspond to 6.9% of the quarterly disposable income.

However, this figure is silent about the maturity structure of household holdings. As a proxy, we use marketable public debt maturities, because we do not have any data concerning the maturity structure of corporate bonds. It is nonetheless noteworthy that the maturity structure has little quantitative impact on our results. More precisely, the US treasury debt service publishes each month a *Monthly Statement of the Public Debt*[4] and we focused on the one of September 2006. This file provides a picture at a given date of the situation of the public debt. In particular, for the public debt, one can find for T–Bills, T–notes and T–bonds, the outstanding amount, the coupon, the issue date and the maturity. We have transformed these raw data into zero coupon bonds using times-to-maturity, outstanding amounts and coupons. We obtained finally a profile of bond supply per maturities. Since we only focus on the 10 first years of the yield curve, we gathered all amounts after 10 years in the last one.

[4] `http://www.treasurydirect.gov/govt/reports/pd/mspd/mspd.htm`

Finally, we have been silent about a last parameter β, which is the level of the yield curve. As is standard in this literature which departs from perfect financial markets, we calibrate it to get a reasonable value for the level of the yield curve with bond supply, which leads to $\beta = 0.78$.

7.2 Yield curve simulation

The following graphs represent the yield curve as well as the variance as function of the maturity. Every one of them plots two lines, which describe exactly the same model with the same parameters except for the bond volumes. One of the curve reflects the zero net supply case, whereas the other accounts for the positive supply case.

The figure (FIG. 1) plots the average yields r_k as a function of the maturity k, for maturities varying between 1 and 40 quarters. The two lines correspond respectively to the zero net supply case and to the volume effects one. This graph is a relevant illustration of points 1) and 3) of our proposition 3. First, it shows quantitatively how the yield curve level shifts when bond volumes are taken into account. With our calibration, the yield curve jumps from -36% to 1.9%: the introduction of bond volumes produces a shift in yields with a magnitude of almost 40 points. Second, the graph also shows to what extent the curve steepens with bond volumes. Whereas in the zero net supply case, the curve is globally decreasing with a slope of -10 basis points, the volumes increase sensibly the slope, which becomes positive, around 5 bp. As a conclusion, the impact of bond supply on the curve is quantitatively important, for the level as well as for the slope.

It is noteworthy that our empirical exercise is mainly illustrative. We have not tried to estimate the yield curve and to reproduce the 'true' shape for the real curve. There are at least two reasons for that. First, as our goal is to exhibit new channels, we kept away from extensions in several dimensions, which could be useful to confront the model to the data Second, data available on inflation-linked bonds do not allow to build up a consensus regarding the true shape of the curve, at least in the UK market. We do not want to take part in it. Our figures illustrate however the variety of shapes it is possible to obtain with this model. It is notably able to replicate a downward sloping curve.

The other figure (FIG. 2) draws the variances of yields $Var(r_k)$ as a function of the maturity, with volumes effects and in the zero net supply case. In both cases, the variance is is decreasing function of the maturity. This graph illustrates the last item of the proposition 3, which states that the variance of yields decreases with bond supply.

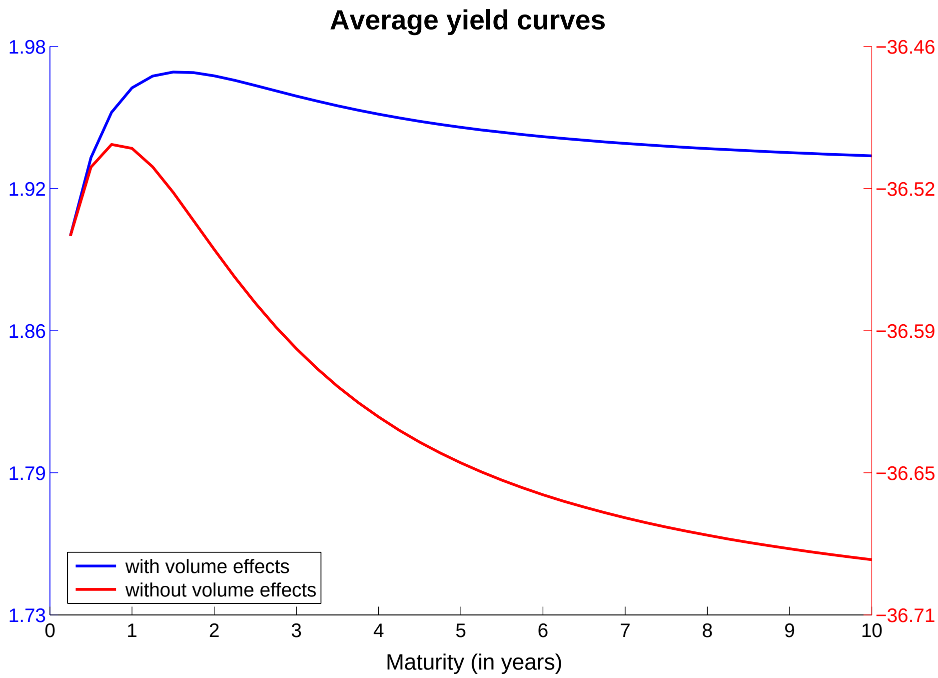

Figure 1: Average Yield Curves

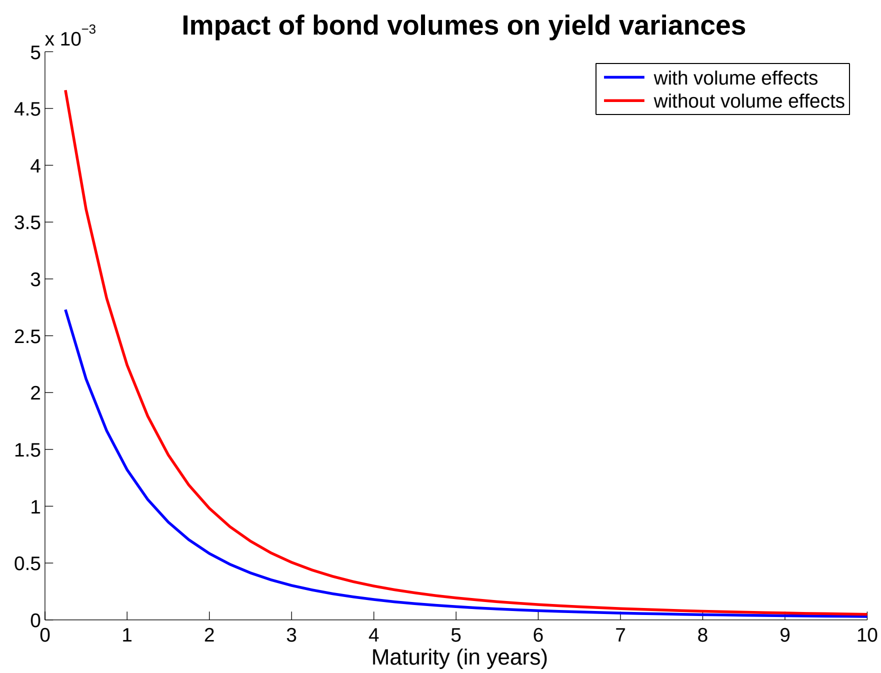

Figure 2: Yield Variances

8 Conclusion

We analyze the term structure of interest rates in a general equilibrium model with incomplete markets and credit constraints. The two main findings of this paper consist in: (i) a tractable framework with parsimonious heterogeneity leading to analytical expressions for bond prices and (ii) proposing two new channels likely to influence the yield curve, credit constraints and volume effects. Up to our knowledge, this model is the first one to exhibit in an analytical setting volume effects on the curve. Briefly, the introduction of bond volume shifts the yield curve upward and increases the slope. Our quantitative exercise illustrates the magnitude of preceding effects for a reasonable calibration.

A route for further research consists in enlarging our existent model to confront it to the data. The first way to improve the quantitative performances of the model is to define the wage z as a Markovian process with more than two states. We can also add a covariance between the aggregate shock and the idiosyncratic one. Because we mainly focused on the simplicity of the model, we supposed in the theoretical model that both were independent but it could be of great interest for quantitative purpose to add it. Finally, a last solution for improving the model consists in defining Markovian processes for the probability of being credit constrained α as well as for the monetary equivalent of labor disutility δ. These processes could in general be correlated to the aggregate and idiosyncratic shocks. In the quantitative exercise, we only aimed at quantifying the impact of bond volumes on the yield curve and not at measuring its empirical performances. However, to confront properly the augmented model to the data, we should estimate model parameters rather than calibrate them as we did.

A Equilibrium of the simple model

We derive sufficient conditions for the existence of the simple model equilibrium. For this equilibrium to exist, unemployed must be credit constrained in all states. Using directly general conditions (18), we obtain for all $t > 0$:

$$C_1 z_t \, u' \left(\delta + B_1 + C_1 z_t B_2\right) > \beta \left(1-\rho\right) \phi \, E_t 1/z + \beta \rho \, u' \left(\delta\right)$$
$$C_2 z_t \, u' \left(\delta + B_1 + C_1 z_t B_2\right) > \beta \left(1-\rho\right) \phi \, C_1 + \beta \rho \, u' \left(\delta\right) C_1$$

Let us define:

$$A = \beta \left(1-\rho\right) \phi \, E \left(1/z\right) / z^l + \beta \rho \, \left(u_1 - u_2 \left(\delta\right)\right) / z^l$$

With this notation, the sufficient conditions for the equilibrium existence express as:

$$\frac{C_2}{C_1} \left(u_1 - u_2 \left(\delta + B_1 + C_1 z^h B_2\right)\right) > A$$
$$C_1 \left(u_1 - u_2 \left(\delta + B_1 + C_1 z^h B_2\right)\right) > A$$

The following simple numerical example illustrates that when the equilibrium exists, credit constraints can either increase or decrease the slope of the yield curve. If $\varepsilon = 0.01$, $\alpha = 0.1, \delta = 0.2, \beta = 0.96, u_1 = 2, u_2 = 0.2$, $\rho = 0.5$ the existence conditions are fulfilled for $B_1 = 0$ and either $B_2 = 0$ or $B_2 = 1$. When $B_2 = 0$ the slope of the yield curve decreases with credit constraints, when $B_2 = 1$, the slope increases.

B Proof of Proposition 1

We prove that C_k^s are $\mathcal{C}^1$ functions of B_i and z^h, z^l for $s = h, l$ and $k, i = 1, \ldots, n$. For reading convenience, we rewrite the pricing definitions:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \beta \underbrace{\begin{bmatrix} \pi^h \left(\alpha + (1-\alpha) \, z^h \, u'^h_{t+1}\right) & \left(1-\pi^h\right) \left(\alpha + (1-\alpha) \, z^l \, u'^l_{t+1}\right) \\ \left(1-\pi^l\right) \left(\alpha + (1-\alpha) \, z^h \, u'^h_{t+1}\right) & \pi^l \left(\alpha + (1-\alpha) \, z^l \, u'^l_{t+1}\right) \end{bmatrix}}_{=M(C,X)} \begin{bmatrix} C_{k-1}^h \\ C_{k-1}^l \end{bmatrix}$$

with the notations:

$$u'^h_{t+1} \equiv u' \left(\delta + \sum_{k=0}^{n-1} C_k^h \, z^h B_{t+1,k+1}\right) / \phi$$
$$u'^l_{t+1} \equiv u' \left(\delta + \sum_{k=0}^{n-1} C_k^l \, z^l B_{t+1,k+1}\right) / \phi$$
$$C_0^k = 1/z^h$$
$$C_0^l = 1/z^l$$

We have defined the following vectors:

$$
\begin{aligned}
C &= \begin{bmatrix} C_n^h & C_n^l & \dots & C_1^h & C_1^l & C_0^h & C_0^l \end{bmatrix}^\top \\
B &= [B_n \quad \dots \quad B_1]^\top \\
X &= [z^h \quad z^l \quad B^\top]
\end{aligned}
$$

Skipping the time subscripts, and rewriting the definition of C as a function of B and X in a matrix form, we obtain:

$$
\begin{aligned}
0_{(2n+2)\times 1} &= C - \begin{bmatrix} 0_{2\times 2} & M(C,X) & 0_{2\times 2} & \dots & 0_{2\times 2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & M(C,X) \\ 0_{2\times 2} & & \dots & & 0_{2\times 2} \end{bmatrix} C - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/z^h \\ 1/z^l \end{bmatrix} \Leftrightarrow \\
0_{(2n+2)\times 1} &= f(X,C)
\end{aligned}
$$

where $0_{m\times n}$ is the $m \times n$ null matrix.

Since u' is a $\mathcal{C}^1$ function from $\mathbb{R}$ to $\mathbb{R}$, M and f are also $\mathcal{C}^1$ in (C, X).

In order to apply the implicit function theorem to show that C is a $\mathcal{C}^1$ function of B and X, one only needs to check that the Jacobian Df_Y of f relative to C is invertible. We denote:

$$
Df_Y = \left(\partial f_i / \partial C_j^*\right)_{1\le i\le 2n, 1\le j\le n, *=h,l}
$$

The partial derivatives of f relative to C_i^h and C_i^l express as:

$$
\frac{\partial f}{\partial C_i^h} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\begin{matrix} \\ \\ \longleftarrow \quad \textit{Rank } i \quad \longrightarrow \\ \\ -(\alpha + (1-\alpha) z^h u'^h) \\ \\ \\ \\ \\ \end{matrix}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \pi^h \\ (1-\pi^l) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
- (z^h)^2 u''^h B_{k+1}
\begin{bmatrix} 0 \\ \pi^h C_n^h \\ (1-\pi^l) C_n^l \\ \vdots \\ \\ \\ \vdots \\ \pi^h C_0^h \\ (1-\pi^l) C_0^l \end{bmatrix}
$$

$$\frac{\partial f}{\partial C_i^l} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \longleftarrow \; Rank\ i+1 \; \longrightarrow \\ -(\alpha + (1-\alpha) z^l u'^l) \\ \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1-\pi^h \\ \pi^l \\ 0 \\ \vdots \\ 0 \end{bmatrix} - (z^l)^2 u''^l B_{k+1} \begin{bmatrix} 0 \\ (1-\pi^h) C_n^l \\ \pi^l C_n^h \\ \vdots \\ \\ \\ \vdots \\ (1-\pi^h) C_0^l \\ \pi^l C_0^h \end{bmatrix}$$

The Jacobian Df_Y of f relative to C is the $2n \times 2n$ matrix $(\frac{\partial f}{\partial C_1^h}, \frac{\partial f}{\partial C_1^l}, \ldots, \frac{\partial f}{\partial C_i^h}, \frac{\partial f}{\partial C_i^l}, \ldots, \frac{\partial f}{\partial C_n^h}, \frac{\partial f}{\partial C_n^l})$. It expresses as the sum of a triangular matrix with only 1 on its diagonal and a matrix which is equal to 0 when $B = 0$. At the neighborhood of $B = 0$ (zero net supply) the Jacobian is therefore invertible and C is a $\mathcal{C}^1$ function of (X and thus of) B and of $\{z^h, z^l\}$.

C Proof of Proposition 2

C.1 Average short run interest rate

We prove that the average short yield r_1 decreases with credit constraints and then increases with α. We rewrite the short run coefficients C_1^h and C_1^l:

$$\begin{aligned} C_1^h &= \alpha\beta\, E_h\,(1/z) + (1-\alpha)\beta\, u'\,(\delta)\,/\phi \\ C_1^l &= \alpha\beta\, E_l\,(1/z) + (1-\alpha)\beta\, u'\,(\delta)\,/\phi \end{aligned}$$

Since short run interest rates are in both states $r^s = -\log\left(C_1^s z^s\right)$, $s = h, l$, the average short run interest rate expresses as:

$$\begin{aligned} Er &= \frac{1-\pi^l}{2-\pi^l-\pi^h}\left(-\log C_1^h z^h\right) + \frac{1-\pi^h}{2-\pi^l-\pi^h}\left(-\log C_1^l z^l\right) \\ -\left(2-\pi^l-\pi^h\right) Er &= \left(1-\pi^l\right)\log z^h + \left(1-\pi^h\right)\log z^l \\ &+ \left(1-\pi^l\right)\log\left(\alpha\beta\left(\frac{z^h\left(1-\pi^h\right)+\pi^h z^l}{z^h z^l} - u'\,(\delta)\,/\phi\right) + \beta u'\,(\delta)\,/\phi\right) \\ &+ \left(1-\pi^h\right)\log\left(\alpha\beta\left(\frac{z^l\left(1-\pi^l\right)+\pi^l z^h}{z^h z^l} - u'\,(\delta)\,/\phi\right) + \beta u'\,(\delta)\,/\phi\right) \end{aligned}$$

Moreover, the assumption **A** stating that $\phi/z^l < u'(\delta)$ implies the following inequalities:

$$\frac{z^h\left(1-\pi^h\right)+\pi^h z^l}{z^h z^l} - \left(u'(\delta)/\phi\right) \leq \frac{1}{z^l} - u'(\delta)/\phi < 0$$
$$\frac{z^l\left(1-\pi^l\right)+\pi^l z^h}{z^h z^l} - \left(u'(\delta)/\phi\right) \leq \frac{1}{z^l} - u'(\delta)/\phi < 0$$

Using these inequalities and the price definition, one deduces that the average short rate increases with α and thus decreases with credit constraints. QED.

C.2 Variance of yields

We prove by inference that the variance of yields of any maturity increases with credit constraints. Showing that the variance increases with credit constraints is equivalent to:

$$\begin{aligned}
& \frac{\partial Var(r_j)}{\partial \alpha} < 0 \\
\Leftrightarrow\ & \frac{(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\frac{\partial (r_j^h - r_j^l)^2}{\partial \alpha} < 0 \\
\Leftrightarrow\ & \underbrace{(r_j^h - r_j^l)}_{\leq 0}\frac{\partial (r_j^h - r_j^l)}{\partial \alpha} < 0 \\
\Leftrightarrow\ & \forall k \geq 1, \forall j \geq 1, \quad \frac{1}{C_j^l}\frac{\partial C_j^l}{\partial \alpha} - \frac{1}{C_j^h}\frac{\partial C_j^h}{\partial \alpha} > 0
\end{aligned}$$

We will concentrate on proving the last inequality. For reading convenience, we rewrite the price definitions in zero net supply:

$$\begin{cases} C_k^h = \alpha\beta\left(\pi^h C_{k-1}^h + \left(1-\pi^h\right)C_{k-1}^l\right) + (1-\alpha)\beta\left(\pi^h C_{k-1}^h z^h + \left(1-\pi^h\right)C_{k-1}^l z^l\right) u'(\delta) \\ C_k^l = \alpha\beta\left(\pi^l C_{k-1}^l + \left(1-\pi^l\right)C_{k-1}^h\right) + (1-\alpha)\beta\left(\pi^l C_{k-1}^l z^l + \left(1-\pi^l\right)C_{k-1}^h z^h\right) u'(\delta) \end{cases}$$

$$\text{and} \begin{cases} C_1^h = \alpha\beta\left(\pi^h/z^h + \left(1-\pi^h\right)/z^l\right) + (1-\alpha)\beta\, u'(\delta) \\ C_1^l = \alpha\beta\left(\pi^l/z^l + \left(1-\pi^l\right)/z^h\right) + (1-\alpha)\beta\, u'(\delta) \end{cases}$$

First step: Yields of maturity 1. We begin with proving the result for yields of maturity 1. For small amounts of debt, we have:

$$\frac{1}{\beta}\frac{\partial C_1^h}{\partial \alpha} = \pi^h/z^h + (1-\pi^h)/z^l - u'(\delta)$$
$$\frac{1}{\beta}\frac{\partial C_1^l}{\partial \alpha} = \pi^l/z^l + (1-\pi^l)/z^h - u'(\delta)$$

We can now compute the impact on the variance:

$$\frac{1}{\beta}\left[\frac{1}{C_1^l}\frac{\partial C_1^l}{\partial \alpha} - \frac{1}{C_1^h}\frac{\partial C_1^h}{\partial \alpha}\right] = \frac{\left(\pi^h+\pi^l-1\right) z^h\, z^l\, (z^h - z^l)\, u'(\delta)}{C_1^l\, C_1^h} > 0$$

Because $z^h > z^l$, we have $\frac{1}{C_1^l}\frac{\partial C_1^l}{\partial B_{k+1}} - \frac{1}{C_1^h}\frac{\partial C_1^h}{\partial B_{k+1}} > 0$. The variance of the short rate r_1 increases therefore with credit constraints.

Second step: Yields with an unspecified maturity We prove now the result for a yield of an unspecified maturity j. Partial derivatives of C_j^h and C_j^l express as follows:

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial C_j^h}{\partial \alpha} &= \pi^h C_{j-1}^h (1 - z^h\, u'(\delta)) + (1-\pi^h) C_{j-1}^l (1 - z^l\, u'(\delta)) \\
&+ \pi^h \left(\alpha + (1-\alpha) z^h u'(\delta)\right) \frac{\partial C_{j-1}^h}{\partial \alpha} \\
&+ (1-\pi^h) \left(\alpha + (1-\alpha) z^l u'(\delta)\right) \frac{\partial C_{j-1}^l}{\partial \alpha} \\
\frac{1}{\beta}\frac{\partial C_j^l}{\partial \alpha} &= \pi^l C_{j-1}^l (1 - z^l\, u'(\delta)) + (1-\pi^l) C_{j-1}^h (1 - z^h\, u'(\delta)) \\
&+ \pi^l \left(\alpha + (1-\alpha) z^l u'(\delta)\right) \frac{\partial C_{j-1}^l}{\partial \alpha} \\
&+ (1-\pi^l) \left(\alpha + (1-\alpha) z^h u'(\delta)\right) \frac{\partial C_{j-1}^h}{\partial \alpha}
\end{aligned}$$

The impact of bond volumes on the variance depends on the sign of:

$$\begin{aligned}
\frac{1}{\beta}\left[\frac{1}{C_j^h}\frac{\partial C_j^h}{\partial \alpha} - \frac{1}{C_j^l}\frac{\partial C_j^l}{\partial \alpha}\right] &= \left(\frac{\pi^h}{C_j^h} - \frac{1-\pi^l}{C_j^l}\right) C_{j-1}^h\, (1 - z^h\, u'(\delta)) \\
&- \left(\frac{\pi^l}{C_j^l} - \frac{1-\pi^h}{C_j^h}\right) C_{j-1}^l\, (1 - z^l\, u'(\delta)) \\
&+ \left(\frac{\pi^h}{C_j^h} - \frac{1-\pi^l}{C_j^l}\right) C_{j-1}^h \left(\alpha + (1-\alpha) z^h u'(\delta)\right) \frac{1}{C_{j-1}^h}\frac{\partial C_{j-1}^h}{\partial \alpha} \\
&- \left(\frac{\pi^l}{C_j^l} - \frac{1-\pi^h}{C_j^h}\right) C_{j-1}^l \left(\alpha + (1-\alpha) z^l u'(\delta)\right) \frac{1}{C_{j-1}^l}\frac{\partial C_{j-1}^l}{\partial \alpha}
\end{aligned}$$

In order to determine this sign, we proceed in two steps and consider: (i) the first two terms and (ii) the last two ones. First, remark that:

$$\begin{aligned}
&\left(\frac{\pi^h}{C_j^h} - \frac{1-\pi^l}{C_j^l}\right) (1 - z^h\, u'(\delta))\, C_{j-1}^h - \left(\frac{\pi^l}{C_j^l} - \frac{1-\pi^h}{C_j^h}\right) (1 - z^l\, u'(\delta))\, C_{j-1}^l = \\
&\qquad - \frac{\beta C_{j-1}^h C_{j-1}^l (\pi^h + \pi^l - 1)(z^h - z^l) u'(\delta)}{C_j^h C_j^l} < 0
\end{aligned}$$

The two first terms are then negative.

Second, we have:

$$\begin{aligned}
&\left(\frac{\pi^h}{C_j^h} - \frac{1-\pi^l}{C_j^l}\right) C_{j-1}^h \left(\alpha + (1-\alpha) z^h u'(\delta)\right) = \left(\frac{\pi^l}{C_j^l} - \frac{1-\pi^h}{C_j^h}\right) C_{j-1}^l \left(\alpha + (1-\alpha) z^l u'(\delta)\right) \\
&= \frac{\beta\, (\pi^h + \pi^l - 1) C_{j-1}^l\, C_j^h}{C_j^l\, C_j^h} \left(\alpha + (1-\alpha) z^l\, u'(\delta)\right) \left(\alpha + (1-\alpha) z^h\, u'(\delta)\right) > 0
\end{aligned}$$

Using the inference hypothesis $\frac{1}{C_{j-1}^h}\frac{\partial C_{j-1}^h}{\partial \alpha} < \frac{1}{C_{j-1}^l}\frac{\partial C_{j-1}^l}{\partial \alpha}$, we obtain that the last two terms are negative.

We conclude that:

$$\forall j, \forall k, \quad \frac{1}{C_j^h} \frac{\partial C_j^h}{\partial \alpha} < \frac{1}{C_j^l} \frac{\partial C_j^l}{\partial \alpha}$$

The variance of all yields increases with credit constraints. QED.

C.3 Value of the long run interest rate

We determine the common value toward which yields in both states converge.

We diagonalize the matrix defining recursively C^h and Cl^l in the system (26). Let us begin with defining the matrices Q and D as:

$$Q \equiv \begin{bmatrix} -\frac{\alpha(\pi^h-\pi^l)-(u'(\delta)/\phi)\ (\alpha-1)(z^h\pi^h-z^l\pi^l)-H}{2\left(1-\pi^l\right)\left((u'(\delta)/\phi)z^h(\alpha-1)-\alpha\right)} & -\frac{\alpha(\pi^h-\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h-z^l\pi^l)+H}{2\left(1-\pi^l\right)\left(((u'(\delta)\ \phi)z^h(\alpha-1)-\alpha)\right)} \\ 1 & 1 \end{bmatrix}$$

$$D \equiv \frac{1}{2} \begin{bmatrix} \alpha\left(\pi^h+\pi^l\right)\ +(1-\alpha)\left(u'\left(\delta\right)/\phi\right)\left(z^h\pi^h+z^l\pi^l\right)\ -H & 0 \\ 0 & \alpha\left(\pi^h+\pi^l\right)\ +(1-\alpha)\left(u'\left(\delta\right)/\phi\right)\left(z^h\pi^h+z^l\pi^l\right)+H \end{bmatrix}$$

where H is defined by:

$$\begin{aligned} H = \Big(& \left(\pi^h\left(\alpha+(1-\alpha)\left(u'\left(\delta\right)/\phi\right)z^h\right)+\pi^l\left(\alpha+(1-\alpha)\left(u'\left(\delta\right)/\phi\right)z^l\right)\right)^2 \\ & +4\left(1-\pi^h\right)\left(1-\pi^l\right)\left(\alpha+(1-\alpha)\left(u'\left(\delta\right)/\phi\right)z^h\right)\left(\alpha+(1-\alpha)\left(u'\left(\delta\right)/\phi\right)z^l\right)\Big)^{\frac{1}{2}} \end{aligned}$$

(One can check that the term under the square is always positive). We can now diagonalize the matrix defining C^h and C^l:

$$T \begin{bmatrix} \alpha+(1-\alpha)z^h\ \left(u'\left(\delta\right)/\phi\right) & 0 \\ 0 & \alpha+(1-\alpha)z^l\ \left(u'\left(\delta\right)/\phi\right) \end{bmatrix} = Q.D.Q^{-1}$$

From this diagonalization, we can deduce an expression of $[C_k^h \quad C_k^l]$ as a function of $[C_0^h \quad C_0^l]$:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \beta^k Q.D^k.Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix} \tag{27}$$

We can now easily write prices expressions:

$$\begin{bmatrix} P_k^h \\ P_k^l \end{bmatrix} = \beta^k \begin{bmatrix} z^h & 0 \\ 0 & z^l \end{bmatrix} Q.D^k.Q^{-1} \begin{bmatrix} C_0^h \\ C_0^l \end{bmatrix}$$

Developing the preceding equality allows us to obtain an analytical expression for P_k^h:

$$\begin{aligned}
P_k^h \times \left(\frac{1}{H}2^{-k-1}\beta^k\right)^{-1} & \Big(\alpha(\pi^h+\pi^l) - \big(u'(\delta)/\phi\big)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H\Big)^{-k} \\
&= \Big(\alpha(\pi^l-\pi^h) + \big(u'(\delta)/\phi\big)(\alpha-1)(z^h\pi^h-z^l\pi^l)+H\Big) \\
&\quad \times \left[\left(\frac{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)-H}{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H}\right)^k - 1\right] \\
&\quad - \frac{2z^h((u'(\delta)/\phi)z^l(\alpha-1)-\alpha)(\pi^h-1)}{z^l} \\
&\quad \times \left[\left(\frac{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)-H}{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H}\right)^k - 1\right]
\end{aligned}$$

Remarking that $H>0$, one can simplify the preceding expression using:

$$\begin{aligned}
&\frac{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)-H}{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H} < 1 \\
\Longrightarrow\ &\left(\frac{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)-H}{\alpha(\pi^h+\pi^l)-(u'(\delta)/\phi)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H}\right)^k \underset{k\to\infty}{\longrightarrow} 0
\end{aligned}$$

As a consequence, the price P_k^h verifies now for very large maturities:

$$\begin{aligned}
&\lim_{k\to\infty}\log\left[P_k^h\left(\frac{1}{H}2^{-k-1}\beta^k\right)^{-1}\Big(\alpha(\pi^h+\pi^l)-\big(u'(\delta)/\phi\big)(\alpha-1)(z^h\pi^h+z^l\pi^l)+H\Big)^{-k}\right] \\
&= \log\Big[H - \Big(\alpha(\pi^l-\pi^h)+\big(u'(\delta)/\phi\big)(\alpha-1)(z^h\pi^h-z^l\pi^l)\Big) \\
&\qquad +2(\alpha+\big(u'(\delta)/\phi\big)z^l(1-\alpha))(1-\pi^h)\frac{z^h}{z^l}\Big]
\end{aligned}$$

As $r_k^h = -\frac{1}{k}\log P_k^h$, one finally deduces the expression of r_k^h when its maturity goes to infinity:

$$\lim_{k\to\infty} r_k^h = -\log\beta - \log\frac{\alpha(\pi^h+\pi^l)+(u'(\delta)/\phi)(1-\alpha)(z^h\pi^h+z^l\pi^l)+H}{2}$$

By a simple symmetry argument (which can be checked by some algebra), one obtains the expression of r_k^l for infinite maturities:

$$\lim_{k\to\infty} r_k^h = \lim_{k\to\infty} r_k^l = -\log\beta - \log\frac{\alpha(\pi^h+\pi^l)+(u'(\delta)/\phi)(1-\alpha)(z^h\pi^h+z^l\pi^l)+H}{2} \tag{28}$$

When $z^h = z^l = 1$, the expression of the common limit yield simplifies to:

$$r^{\lim} = -\log\beta - \log\big(\alpha + \big(u'(\delta)/\phi\big)(1-\alpha)\big)\big)$$

Because $u'(\delta)/\phi > 1 = 1/z^l$ (assumption **A**), the infinite yield $r^{\lim}$ decreases with credit constraints: $\frac{\partial r^{\lim}}{\partial\alpha} > 0$. By continuity, this inequality is fulfilled for z^h, z^l close enough to 1 and the result still holds. *QED*.

C.4 Ranking of yield curves.

We show that for all maturities, yields r^h in the good state are below yields r^h in the bad state. It is equivalent to prove by inference that $C_k^h z^h \geq C_k^l z^l$ for $k = 1 \dots n$ if $\pi^h + \pi^l > 1$. We do it by inference. The system (21) provides the recursive relation between C_k^s and C_{k-1}^s $s = h, l$.

1. The result holds for $k = 0$ because $C_0^h z^h \geq C_0^l z^l$ simplifies to $1 \geq 1$.

2. For $k = 1$: Showing $C_1^h z^h \geq C_1^l z^l$ is equivalent to:
$$\alpha \left(z^h - z^l\right) \left(\left(1 - \pi^h\right) \frac{1}{z^l} + (1 - \pi^l) \frac{1}{z^h}\right) \geq (1 - \alpha) \left(z^l - z^h\right) u'(\delta)$$
which is always true because the RHS is negative and the LHS positive.

3. For a given maturity $k \geq 1$, let us suppose that the result holds for the previous maturity: $C_{k-1}^h z^h \geq C_{k-1}^l z^l$. Proving our result $C_k^h z^h \geq C_k^l z^l$ is equivalent to:
$$\begin{aligned} &\alpha \left(\left(\pi^h - (1 - \pi^l) \frac{z^l}{z^h}\right) \frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l\right) \\ &\qquad \geq (1 - \alpha) \, u'(\delta) \left(\pi^l z^l + ((1 - \pi^l) z^l - \pi^h \, z^h) \frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} - (1 - \pi^h) z^h\right) \end{aligned} \tag{29}$$
First, consider the right hand side RHS. Since $z^h \geq z^l$ and $\pi^h + \pi^l - 1 > 0$, we get:
$$(1 - \pi^l) z^l - \pi^h \, z^h = - \, z^l \left(\frac{z^h}{z^l} \pi^h + \pi^l - 1\right) < 0$$
By assumption $\frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} \geq 1$, thus the RHS verifies:
$$\begin{aligned} \text{RHS} &= \pi^l z^l - (1 - \pi^h) z^h + ((1 - \pi^l) z^l - \pi^h \, z^h) \frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} \\ &< \pi^l z^l - (1 - \pi^h) z^h + (1 - \pi^l) z^l - \pi^h \, z^h \\ &< z^l - z^h < 0 \end{aligned}$$
The RHS is thus positive.

Second, look at the LHS:
$$\text{LHS} = \left(\pi^h - (1 - \pi^l) \frac{z^l}{z^h}\right) \frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l$$
The inequality (3) gives directly that $\pi^h - (1 - \pi^l) \frac{z^l}{z^h} > 0$. By assumption, we have $\frac{z^h \, C_{k-1}^h}{z^l \, C_{k-1}^l} \geq 1$. The LHS verifies then:
$$\begin{aligned} \text{LHS} &> \pi^h - \pi^l - (1 - \pi^l) \frac{z^l}{z^h} + \frac{z^h}{z^l} (1 - \pi^h) \\ &> \left(z^h - z^l\right) \left(\frac{1 - \pi^h}{z^l} + \frac{1 - \pi^l}{z^h}\right) > 0 \end{aligned}$$

The inequality (29) is thus fulfilled.

By inference, we obtain $C_k^h z^h \geq C_k^l z^l$ for $k = 1, \ldots, n$ if $\pi^h + \pi^l > 1$. QED.

D Proof of Proposition 3

D.1 Impact of bond volumes on prices

We prove that bond volumes decrease prices and therefore increase yields.

Define $X_t^s = \alpha + (1-\alpha) z^s\, u'\left(\sum_{i=0}^{n-1} C_i^s\, z^s B_i\right)/\phi$ for $s = h, l$. Obviously, $X_t^h, X_t^l > 0$. For $k = 1, \ldots, n$, the system can be written as:

$$\begin{bmatrix} C_k^h \\ C_k^l \end{bmatrix} = \left(\beta T \cdot \begin{bmatrix} X_t^h & 0 \\ 0 & X_t^l \end{bmatrix}\right)^k \begin{bmatrix} 1/z^h \\ 1/z^l \end{bmatrix}$$

As $\beta, z^h, z^l > 0$ and as all elements of T are positive, it is easy to prove by inference that C_k^h and C_k^l are polynomials of degree k in X_t^h and X_t^l with positive coefficients (and some strictly positive). These polynomials, which we denote Φ_k^s, express as:

$$C_k^s = \Phi_k^s\left(X_t^h, X_t^l\right),\ s = h, l \quad k = 1, \ldots, n$$

As polynomial coefficients are positive and as $X_t^h, X_t^l > 0$, polynomials Φ_k^s $s = h, l$ are increasing in their both arguments[5]. One gets for $s = h, l$ and $k = 1, \ldots, n$:

$$\begin{aligned} \frac{\partial C_k^s}{\partial B_i} \frac{1}{u''\left(W_t^h\right)/\phi} \frac{1}{1-\alpha} &= \left(z^h\right)^2 \left(C_i^h + B_i \frac{\partial C_i^h}{\partial B_i}\right) \Phi_{k1}^{s\prime}\left(X_t^h, X_t^l\right) \\ &+ \left(z^l\right)^2 \left(C_i^l + B_i \frac{\partial C_i^l}{\partial B_i}\right) \Phi_{k2}^{s\prime}\left(X_t^h, X_t^l\right) \end{aligned}$$

Since C_k^s is a $\mathcal{C}^1$ function of B_i, $\frac{\partial C_k^s}{\partial B_i}$ is a continuous function of B_i. It implies that $B_i \frac{\partial C_i^h}{\partial B_i}$ and $B_i \frac{\partial C_i^l}{\partial B_i}$ are negligible quantities respectively relative to C_i^h and C_i^l, when B_i is low. Moreover, $\Phi_{k1}^{s\prime}\left(X_t^h, X_t^l\right), \Phi_{k2}^{s\prime}\left(X_t^h, X_t^l\right) > 0$. As a consequence, $\frac{\partial C_k^s}{\partial B_i} \frac{1}{u''\left(W_t^h\right)/\phi} \frac{1}{1-\alpha}$ is positive and $\frac{\partial C_k^s}{\partial B_i}$ is negative for $k = 1, \ldots, n$ and $s = h, l$. Bond supply decreases bond prices. QED.

[5] More precisely

$$\Phi_k^s\left(X_t^h, X_t^l\right) = \sum_{p=0}^{k} \sum_{q=0}^{k-p} a_{p,q}^s \left(X_t^h\right)^p \left(X_t^l\right)^q \text{ with } a_{0,0} = 0$$

One finds

$$\begin{aligned} \frac{\partial}{\partial X_t^h} \Phi_k^s\left(X_t^h, X_t^l\right) &= \sum_{p=1}^{k} \sum_{q=0}^{k-p} p \times a_{p,q}^s \left(X_t^h\right)^{p-1} \left(X_t^l\right)^q > 0 \\ \frac{\partial}{\partial X_t^l} \Phi_k^s\left(X_t^h, X_t^l\right) &= \sum_{p=0}^{k} \sum_{q=1}^{k-p} q \times a_{p,q}^s \left(X_t^h\right)^p \left(X_t^l\right)^{q-1} > 0 \end{aligned}$$

D.2 Convergence of yields

We prove that yields in both states of the world converge to a common limit $\widetilde{r}^{\text{lim}}$.

We rewrite the three following definitions:

$$\begin{aligned}
u'^h &= u'\left(\delta + B_1 + z^h \sum_{k=1}^{\infty} C_k^h \, B_{k+1}\right)/\phi \\
u'^l &= u'\left(\delta + B_1 + z^l \sum_{k=1}^{\infty} C_k^l \, B_{k+1}\right)/\phi \\
\tilde{H} &\equiv \sqrt{(\alpha(\pi^h+\pi^l)+(1-\alpha)(z^h\pi^h\,u'^h+z^l\pi^l\,u'^l))^2-4(z^h\,u'^h(1-\alpha)+\alpha)(z^l u'^l(1-\alpha)+\alpha)(\pi^h+\pi^l-1)}
\end{aligned}$$

Note that if $B=0$ then $\tilde{H}=H$ defined previously and the term under the square is strictly positive. As a consequence, if B is close enough to 0, then $\tilde{H}$ is well defined. Following the same steps as in (C.3), one finds that the long run interest rate converges in all states of the world toward:

$$\widetilde{r}^{\text{lim}} = -\log\beta - \log\left(\frac{\alpha(\pi^h+\pi^l)+(1-\alpha)(z^h\pi^h\,u'^h+z^l\pi^l\,u'^l)+\tilde{H}}{2}\right)$$

D.3 Impact of bond volumes on the slope

We prove that the bond volumes increase the slope of the curve.

Remember that the short run interest rate is:

$$\begin{aligned}
r_1 = -\log\beta &- \frac{1-\pi^l}{2-\pi^l-\pi^h}\log\left(\alpha\frac{z^h\left(1-\pi^h\right)+\pi^h z^l}{z^l}+z^h\left(1-\alpha\right)u'^h\right) \\
&-\frac{1-\pi^h}{2-\pi^l-\pi^h}\log\left(\alpha\frac{z^l\left(1-\pi^l\right)+\pi^l z^h}{z^h}+z^l\left(1-\alpha\right)u'^l\right)
\end{aligned}$$

Using the preceding expression of $\widetilde{r}^{\text{lim}}$, the average slope expresses as:

$$\begin{aligned}
\Delta = &-\log\left(\frac{\alpha(\pi^h+\pi^l)+(1-\alpha)(z^h\pi^h\,u'^h+z^l\pi^l\,u'^l)+\tilde{H}}{2}\right) \\
&+\frac{1-\pi^l}{2-\pi^l-\pi^h}\log\left(\alpha\frac{z^h\left(1-\pi^h\right)+\pi^h z^l}{z^l}+z^h\left(1-\alpha\right)u'^h\right) \\
&+\frac{1-\pi^h}{2-\pi^l-\pi^h}\log\left(\alpha\frac{z^l\left(1-\pi^l\right)+\pi^l z^h}{z^h}+z^l\left(1-\alpha\right)u'^l\right)
\end{aligned}$$

Computing the derivative of the slope relative to the bond quantity B_j, we obtain:

$$\begin{aligned}
\frac{\partial}{\partial B_j}\Delta = &(1-\alpha)\frac{(1-\pi^h)(1-\pi^l)(z^h-z^l)}{2-\pi^h-\pi^l} \\
&\times\left(\frac{z^l}{\pi^l z^h+(1-\pi^l)z^l}\frac{\partial u'^l}{\partial B_j}-\frac{z^h}{(1-\pi^h)z^h+\pi^h z^l}\frac{\partial u'^h}{\partial B_j}\right) \\
&+O\left((1-\alpha)^2\right)
\end{aligned}$$

with:

$$\frac{\partial}{\partial B_j}u'^h = z^h\left(\sum_{k=1}^{\infty}\frac{\partial C_k^h}{\partial B_j}B_{k+1}+C_j^h\right)u''\left(\delta+B_1+z^h\sum_{k=1}^{\infty}C_k^h B_{k+1}\right)/\phi$$
$$\frac{\partial}{\partial B_j}u'^l = z^l\left(\sum_{k=1}^{\infty}\frac{\partial C_k^l}{\partial B_j}B_{k+1}+C_j^l\right)u''\left(\delta+B_1+z^l\sum_{k=1}^{\infty}C_k^l B_{k+1}\right)/\phi$$

The sign of $\frac{\partial}{\partial B_j}\Delta$ depends on the sign of $A\equiv\frac{z^l}{\pi^l z^h+(1-\pi^l)z^l}\frac{\partial u'^l}{\partial B_j}-\frac{z^h}{(1-\pi^h)z^h+\pi^h z^l}\frac{\partial u'^h}{\partial B_j}$. Substituting the derivatives by their values allows to express A as:

$$A=\frac{1}{\pi^l z^h+(1-\pi^l)z^l}\left(z^l\right)^2\left(\sum_{k=1}^{\infty}\frac{\partial C_k^l}{\partial B_j}B_{k+1}+C_j^l\right)u''\left(\delta+B_1+z^l\sum_{k=1}^{\infty}C_k^l B_{k+1}\right)/\phi$$
$$-\frac{1}{(1-\pi^h)z^h+\pi^h z^l}\left(z^h\right)^2\left(\sum_{k=1}^{\infty}\frac{\partial C_k^h}{\partial B_j}B_{k+1}+C_j^h\right)u''\left(\delta+B_1+z^h\sum_{k=1}^{\infty}C_k^h B_{k+1}\right)/\phi$$

When B is close to 0 then $\left(\sum_{k=1}^{\infty}\frac{\partial C_k^s}{\partial B_j}B_{k+1}+C_j^s\right)u''\left(\delta+B_1+w^s\sum_{k=1}^{\infty}C_k^s B_{k+1}\right)/\phi$ is close to $C_j^s u''(\delta)/\phi$ for $s=h,l$. A is thus close to $\tilde{A}$, where:

$$\tilde{A}=\left(\frac{z^l}{\pi^l(z^h-z^l)+z^l}C_j^l z^l-\frac{z^h}{\pi^h\left(z^l-z^h\right)+z^h}C_j^h z^h\right)u''(\delta)/\phi$$

As $z^h>z^l$, a sufficient condition for $\tilde{A}$ to be positive is $C_j^h z^h>C_j^l z^l$, which is always true. This condition means indeed that $r_j^h<r_j^l$.

As a conclusion, bond volumes increase the slope of the curve as soon as $\pi^h+\pi^l>1$. *QED*.

D.4 Impact of bond volumes on the variance

We prove by inference that the variance of yields of any maturity decreases with the supply of bonds, whatever its maturity. Showing that the variance decreases is equivalent to:

$$\frac{\partial Var(r_j)}{\partial B_k}<0$$
$$\Leftrightarrow\frac{(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\frac{\partial(r_j^h-r_j^l)^2}{\partial B_k}<0$$
$$\Leftrightarrow\underbrace{(r_j^h-r_j^l)}_{\leq 0}\frac{\partial(r_j^h-r_j^l)}{\partial B_k}<0$$
$$\Leftrightarrow\forall k\geq 1,\forall j\geq 1,\quad\frac{1}{C_j^l}\frac{\partial C_j^l}{\partial B_k}-\frac{1}{C_j^h}\frac{\partial C_j^h}{\partial B_k}>0$$

For reading convenience, we rewrite the price definitions:

$$\begin{cases} C_k^h = \alpha\beta\, \left(\pi^h C_{k-1}^h + \left(1-\pi^h\right) C_{k-1}^l\right) + (1-\alpha)\beta \left(\pi^h C_{k-1}^h\, z^h\, u'^h + \left(1-\pi^h\right) C_{k-1}^l z^l\, u'^l\right) \\ C_k^l = \alpha\beta\, \left(\pi^l C_{k-1}^l + \left(1-\pi^l\right) C_{k-1}^h\right) + (1-\alpha)\beta \left(\pi^l C_{k-1}^l z^l\, u'^l + \left(1-\pi^l\right) C_{k-1}^h\, z^h\, u'^h\right) \end{cases}$$

and $$\begin{cases} C_1^h = \alpha\beta\, \left(\pi^h/z^h + \left(1-\pi^h\right)/z^l\right) + (1-\alpha)\beta \left(\pi^h\, u'^h + \left(1-\pi^h\right)\, u'^l\right) \\ C_1^l = \alpha\beta\, \left(\pi^l/z^l + \left(1-\pi^l\right)/z^h\right) + (1-\alpha)\beta \left(\pi^l\, u'^l + \left(1-\pi^l\right)\, u'^h\right) \end{cases}$$

with the notations:

$$\begin{cases} u'^h \equiv u'\left(\delta + \sum_{i=0}^{n-1} C_i^h\, z^h B_{i+1}\right)/\phi \\ u'^l \equiv u'\left(\delta + \sum_{i=0}^{n-1} C_i^l\, z^l B_{i+1}\right)/\phi \end{cases}$$

First step: Yields of maturity 1. We begin with rates of maturity 1. The derivatives of C_1^h and C_1^h express as:

$$\begin{aligned} \frac{1}{(1-\alpha)\beta}\,\frac{\partial C_1^h}{\partial B_{k+1}} &= \pi^h\, z^h\, \left[B_{k+1}\,\frac{\partial C_k^h}{\partial B_{k+1}} + C_k^h\right] u''^h \\ &\quad +(1-\pi^h)\, z^l\, \left[B_{k+1}\,\frac{\partial C_k^l}{\partial B_{k+1}} + C_k^l\right] u''^l \\ \frac{1}{(1-\alpha)\beta}\,\frac{\partial C_1^l}{\partial B_{k+1}} &= \pi^l\, z^l\, \left[B_{k+1}\,\frac{\partial C_k^l}{\partial B_{k+1}} + C_k^l\right] u''^l \\ &\quad +(1-\pi^l)\, z^h\, \left[B_{k+1}\,\frac{\partial C_k^h}{\partial B_{k+1}} + C_k^h\right] u''^l \end{aligned}$$

For small amounts of debt, preceding expressions simplify into:

$$\begin{aligned} \frac{1}{(1-\alpha)\beta\, u''(\delta)}\,\frac{\partial C_1^h}{\partial B_{k+1}} &= \pi^h\, z^h\, C_k^h + (1-\pi^h)\, z^l\, C_k^l \\ \frac{1}{(1-\alpha)\beta\, u''(\delta)}\,\frac{\partial C_1^l}{\partial B_{k+1}} &= \pi^l\, z^l\, C_k^l + (1-\pi^l)\, z^h\, C_k^h \end{aligned}$$

We can now compute the impact on the variance:

$$\begin{aligned} \frac{1}{(1-\alpha)\beta\, u''(\delta)}\,\left[\frac{1}{C_1^l}\,\frac{\partial C_1^l}{\partial B_{k+1}} - \frac{1}{C_1^h}\,\frac{\partial C_1^h}{\partial B_{k+1}}\right] &= \frac{1}{C_1^l}\,\left(\underbrace{\pi^l\, z^l\, C_{k-1}^l + (1-\pi^l)\, z^h\, C_{k-1}^h}_{=N_1^l}\right) \\ &\quad - \frac{1}{C_1^h}\,\left(\underbrace{\pi^h\, z^h\, C_{k-1}^h + (1-\pi^h)\, z^l\, C_{k-1}^l}_{=N_1^h}\right) \\ &= \frac{N_1^l}{C_1^l} - \frac{N_1^h}{C_1^h} \end{aligned}$$

We have for B close to 0:

$$\begin{aligned} C_1^h - C_1^l &= \alpha\beta\,(\pi^h + \pi^l - 1)\, \left(1/z^h - 1/z^l\right) \\ N_1^h - N_1^l &= (\pi^h + \pi^l - 1)\, \left(z^h\, C_{k-1}^h - z^l\, C_{k-1}^l\right) \end{aligned}$$

Because $z^h > z^l$ and $r^h_{k-1} \leq r^l_{k-1}$ (cf. proposition 2.), we have $C^h_1 < C^l_1$ and $N^h_1 \geq N^l_1$, which implies that $\frac{1}{C^l_1}\frac{\partial C^l_1}{\partial B_{k+1}} - \frac{1}{C^h_1}\frac{\partial C^h_1}{\partial B_{k+1}} > 0$. The variance of the short rate r_1 decreases therefore with bond volumes.

Second step: Yields of an unspecified maturity. We prove now the result for a given price of maturity j. The derivative of C^h_j relative to B_{k+1} expresses as:

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial C^h_j}{\partial B_{k+1}} &= \left(\alpha + (1-\alpha) z^h\, u'^h\right)\, \pi^h\, \frac{\partial C^h_{j-1}}{\partial B_{k+1}} + \left(\alpha + (1-\alpha) z^l\, u'^l\right)\, (1-\pi^h)\, \frac{\partial C^l_{j-1}}{\partial B_{k+1}} \\
&\quad + (1-\alpha)\, \pi^h\, C^h_{j-1}\, \left(z^h\right)^2 \left[B_{k+1}\frac{\partial C^h_k}{\partial B_{k+1}} + C^h_k\right] u''^h \\
&\quad + (1-\alpha)\,(1-\pi^h)\, C^l_{j-1}\, \left(z^l\right)^2 \left[B_{k+1}\frac{\partial C^l_k}{\partial B_{k+1}} + C^l_k\right] u''^l
\end{aligned}$$

For small amounts of debt, expressions of derivatives of C^h_j and C^l_j relative to B_{k+1} simplify to:

$$\begin{aligned}
\frac{1}{\beta}\frac{\partial C^h_j}{\partial B_{k+1}} &= \left(\alpha + (1-\alpha) z^h\, u'(\delta)\right)\, \pi^h\, \frac{\partial C^h_{j-1}}{\partial B_{k+1}} + \left(\alpha + (1-\alpha) z^l\, u'(\delta)\right)\, (1-\pi^h)\, \frac{\partial C^l_{j-1}}{\partial B_{k+1}} \\
&\quad + (1-\alpha)\, \left[\pi^h\, z^h\, C^h_{j-1}\, z^h\, C^h_k + (1-\pi^h)\, z^l\, C^l_{j-1}\, z^l\, C^l_k\right] u''(\delta) \\
\frac{1}{\beta}\frac{\partial C^l_j}{\partial B_{k+1}} &= \left(\alpha + (1-\alpha) z^l\, u'(\delta)\right)\, \pi^l\, \frac{\partial C^l_{j-1}}{\partial B_{k+1}} + \left(\alpha + (1-\alpha) z^h\, u'(\delta)\right)\, (1-\pi^l)\, \frac{\partial C^h_{j-1}}{\partial B_{k+1}} \\
&\quad + (1-\alpha)\, \left[\pi^l\, z^l\, C^l_{j-1}\, z^l\, C^l_k + (1-\pi^l)\, z^h\, C^h_{j-1}\, z^h\, C^h_k\right] u''(\delta)
\end{aligned}$$

The impact of bond volumes on the variance depends on the sign of the following expression:

$$\begin{aligned}
\frac{1}{\beta}\frac{1}{C^h_j}\frac{\partial C^h_j}{\partial B_{k+1}} - \frac{1}{\beta}\frac{1}{C^l_j}\frac{\partial C^l_j}{\partial B_{k+1}} &= \frac{1}{C^h_{j-1}}\frac{\partial C^h_{j-1}}{\partial B_{k+1}}\, \left(\alpha + (1-\alpha) z^h\, u'(\delta)\right) \left(\pi^h \frac{C^h_{j-1}}{C^h_j} - (1-\pi^l)\frac{C^l_{j-1}}{C^l_j}\right) \\
&\quad + \frac{1}{C^l_{j-1}}\frac{\partial C^l_{j-1}}{\partial B_{k+1}}\, \left(\alpha + (1-\alpha) z^l\, u'(\delta)\right) \left((1-\pi^h)\frac{C^l_{j-1}}{C^h_j} - \pi^l \frac{C^l_{j-1}}{C^l_j}\right) \\
&\quad + (1-\alpha)\, u''(\delta) \left[z^h\, C^h_k\, z^h\, C^h_{j-1}\, \left[\frac{\pi^h}{C^h_j} - \frac{1-\pi^l}{C^l_j}\right]\right. \\
&\qquad \left. - z^l\, C^l_k\, z^l\, C^l_{j-1}\, \left[\frac{\pi^l}{C^l_j} - \frac{1-\pi^h}{C^h_j}\right]\right]
\end{aligned}$$

We investigate this sign in two steps: (i) the last two terms and (ii) the first two ones. Let us begin with the last two terms and $z^h\, C^h_{j-1}\left[\frac{\pi^h}{C^h_j} - \frac{1-\pi^l}{C^l_j}\right] - z^l\, C^l_{j-1}\left[\frac{\pi^l}{C^l_j} - \frac{1-\pi^h}{C^h_j}\right]$. The expression simplifies into:

$$\begin{aligned}
&\frac{\alpha\, C^h_{j-1}\, C^l_{j-1}\, (\pi^h + \pi^l - 1)\, z^h\, z^l\, (z^h - z^l)}{C^l_{j-1}\, (1-\pi^h)\, \left(\alpha + (1-\alpha) u'(\delta)\, z^l\right) + C^h_{j-1}\, \pi^h\, \left(\alpha + (1-\alpha) u'(\delta)\, z^h\right)} \\
&\times \frac{1}{C^l_{j-1}\, \pi^l\, \left(\alpha + (1-\alpha) u'(\delta)\, z^l\right) + C^h_{j-1}\, (1-\pi^l)\, \left(\alpha + (1-\alpha) u'(\delta)\, z^h\right)}
\end{aligned}$$

which is positive.

Because $z^h\, C_k^h > z^l\, C_k^l$, the two last terms verify:

$$(1-\alpha)\, u''(\delta)\, \left[z^h\, C_{j-1}^h\, z^h\, C_k^h\, \left[\frac{\pi^h}{C_j^h} - \frac{1-\pi^l}{C_j^l}\right] - z^l\, C_{j-1}^l\, z^l\, C_k^l\, \left[\frac{\pi^l}{C_j^l} - \frac{1-\pi^h}{C_j^h}\right]\right] < 0$$

We consider now the two first terms. First remark the following equality:

$$\begin{aligned}[\alpha + (1-\alpha) z^h u'(\delta)] \left[\pi^h \frac{C_{j-1}^h}{C_j^h} - (1-\pi^l)\frac{C_{j-1}^l}{C_j^l}\right] &= [\alpha + (1-\alpha) z^l u'(\delta)] \left[(1-\pi^h)\frac{C_{j-1}^l}{C_j^h} - \pi^l \frac{C_{j-1}^l}{C_j^l}\right] \\ &= \frac{\beta\,(\pi^h + \pi^l - 1) C_{j-1}^l\, C_j^h}{C_j^l\, C_j^h} \Big(\alpha + (1-\alpha) z^l\, u'(\delta)\Big)\, \Big(\alpha + (1-\alpha) z^h\, u'(\delta)\Big) > 0\end{aligned}$$

Using the inference condition $\frac{1}{C_{j-1}^h}\frac{\partial C_{j-1}^h}{\partial B_{k+1}} < \frac{1}{C_{j-1}^l}\frac{\partial C_{j-1}^l}{\partial B_{k+1}}$, we conclude that the first two terms are also positive.

As a conclusion, we obtain:

$$\forall j, \forall k, \quad \frac{1}{C_j^h}\,\frac{\partial C_j^h}{\partial B_{k+1}} < \frac{1}{C_j^l}\,\frac{\partial C_j^l}{\partial B_{k+1}}$$

The variance of all yields decreases with bond supply. QED.

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