Spatial Asset Pricing: A First Step*

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Abstract

We explore the dual role of housing, as an investment vehicle and as access to a location. Our dynamic stochastic model has four classes of assets: a risk free bond, houses (in various locations), stocks, and human capital (with different productivity in different locations). Agents choose where they live and can invest in the financial market and in all real estate markets. Local rents are determined in equilibrium by the utility of the marginal residents, which in turn depends on city sizes, local labor productivities, and local shock-insulation parameters. The dividend of a house is a stochastic process that is determined endogenously by how local productivity shocks affect marginal residents. The model leads to a closed-form representation of: (i) The portfolio decisions of agents as a combination of an investment in a financial and real estate mutual fund and demand in local housing to hedge the endogenous rent risk; and (ii) The returns of financial and real estate assets in terms of the covariance matrix of dividend shocks and local productivity shocks.

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1 Introduction

Housing plays at least two important and distinct roles. First, as an investment vehicle, it is one of the largest investment most households ever make, and therefore, a key component of their asset portfolio. Second, the use of a home provides convenient access to production and consumption opportunities in its vicinity: jobs, schools, and other local amenities. The goal of this paper is to explore this dual role of housing as an investment vehicle and access to a location. To this end, we combine a model of portfolio allocation and asset pricing under uncertainty with a spatial model that allows households to choose where to locate and determines housing rents in each location. There will be important feedback between the two. On the one hand, the demand for housing resulting from location decisions affects the stochastic properties of the dividends paid by housing assets in each location. On the other hand, the characteristics of the financial returns to housing investment in each location impinge on the location decisions of households and their portfolio choice. To our knowledge, we are the first to merge classical asset pricing with spatial economics.[1]

The set-up we propose can be sketched as follows. Agents choose where to live. There are four classes of assets: a risk-free bond, stocks, residential properties, and non-transferrable human capital. As in standard asset pricing models, agents may lend and borrow at the risk-free bond rate without any constraint. Agents may also invest in stocks defined as claims over exogenous stochastic streams of dividends. The dividend stream of residential properties, however, is determined endogenously. Residential properties provide access to a stochastic production technology that is specific to the city where they are located. An agent's human capital determines the expected level of his earnings in the city and the covariance of his earnings with the city-specific production technology.[2]

Properties differ only in their location. They can be rented at the local equilibrium market rate. They can be purchased or sold (even fractionally) at the local equilibrium price without any transaction cost. Obviously, agents may buy a home in their city, in which case they are homeowners. Agents may buy residential properties, not only in the city where they live but also in the other cities.

[1] One of the local consumption opportunities that a house affords is the utility that derives from the features of the house itself, for example, its size or how comfortable it is, independently of where the house is located. This paper will assume that houses are identical and will focus on opportunities that are specific to the area where the home is located and not its attribute as a shelter.

[2] For most of the paper, we interpret local productivity as labor-related and hence translating into labor earnings, but the model has an equivalent interpretation in terms of leisure, where productivity is understood as the ability of the agent to enjoy local amenities. See page 10 for a more detailed discussion of the consumption interpretation.

We want to obtain closed-form solutions and expressions that are comparable to standard mean-variance asset pricing models. To this end, we assume that agents have constant-absolute risk-aversion preferences with infinite elasticity of intertemporal substitution and that both city-productivity and stock-dividends stochastic shocks are normally distributed. While all investment decisions can be re-visited in every period, the location choice is irreversible. The distribution of individual characteristics across the population is left in a general form. We also do not impose any restrictions on the covariances between the stochastic processes driving stock dividends and city-specific technology shocks.[3]

Our main result is the characterization of the unique stationary linear equilibrium of the model, which is composed of three parts.

First, the optimal portfolio of every agent is characterized as a combination of two components: (a) An investment in local real estate that depends on the agent's exposure to local productivity shocks, (b) A portfolio of stocks and residential properties, with identical weights across agents. One can view point (a) as a manifestation of home bias. An agent who does not own property in the city where he lives is vulnerable to a combination of local productivity shocks and rent fluctuations (which are determined endogenously in the third part). This risk can be hedged away by an appropriate holding of local real estate. This hedging demand depends on the covariance between the agent's earnings and local equilibrium rents. Point (b) amounts to an extension of the two-fund theorem. Consider the portfolio made up of all stocks and residential properties in the economy minus the homes held for hedging purposes. Let us call this portfolio the adjusted market portfolio. The optimal portfolio characterization we obtain says that besides their local hedging investment, all agents hold a portfolio of risky assets with the same weights as the adjusted market portfolio. Note that as long as the local hedging demand is smaller than the local supply of properties, every household holds some local properties in the adjusted market portfolio.

Second, equilibrium asset prices depend on the contribution of each asset to systemic risk evaluated in the adjusted market portfolio. Our expressions for expected returns on stocks and houses are similar to CAPM with two important modifications: (a) The covariance matrix that determines prices now also includes local productivity shocks; namely the price of a stock is determined not only by how its dividend co-varies with other financial assets but also by how it relates to the earnings risk in different cities; (b) The quantity of real estate in each location that enters the systemic risk is the total supply of residential properties

[3] For most of the paper, we assume that there are no spillover effects across agents, namely the productivity of an agent depends on his location but not by who else lives in that location. In Section 4.5, we show that our characterization extends to a model with generic economies of agglomeration.

minus the quantity held by local residents for hedging purposes. Point (b) implies that the price of real estate in a location depends on the identity of people who live there to the extent that it determines the quantity of local homes that are left in the adjusted market portfolio.

Third, the location decisions of agents are determined endogenously together with the rent in every city. For every vector of city rents, we can compute the expected utility of every agent in every possible location. We can identify a set of measure zero of marginal residents (who are indifferent between two or more locations), which pins down aggregate demand for each location. There exists a unique vector of rents that equates supply and demand in every location. Local rents are determined in equilibrium by the human capital of marginal movers (their expected earnings and the covariance of their earnings with the local production technology). That is why in equilibrium rent fluctuations are determined by local productivity shocks.

It is important to understand that the first two parts of the equilibrium characterization rely on the endogenous location of agents. Our portfolio choice and asset pricing results hinge on a specific real estate dividend process that maps local productivity shocks into rents. Such process would be different if the underlying location model were different (or if the allocation of agents to cities were exogenous). Overall, the spatial allocation of agents affects real estate returns through three channels. First, the expected level of rents in a location is determined by the productivity of the marginal residents (houses are more expensive in cities where the marginal residents are more productive, "at equal covariance"). Second, the covariance of the earnings of the marginal agents in each city with the local productivity shock and the volatility these shocks determine the volatility of rents and thus the housing risk premium. Third, the quantity of homes that are not held by the locals determines the weight of such homes in the relevant market portfolio used to price all assets in the economy, including local homes.

We use our results to: (1) Explain differences in real estate returns across locations in terms of shock-insulation parameters of local residents; (2) Show that housing demand for hedging purposes is first increasing and then decreasing in age; (3) Understand when talent allocation across cities will not maximize aggregate expected production (which happens in our model because agents prefer to locate to cities where their human capital is less correlated to local productivity shocks); (4) Discuss the role of indices aggregating financial assets or real estate assets; (5) Measure the 'error' that we make if, within this model, we price stocks according to a classical beta (taking into account only the covariance with other stocks), rather than the correct beta.

The paper is organized as follows: Section 2 sets out the model. Section 3 presents the main equilibrium characterization result, through three propositions corresponding to: portfolio allocation (Proposition 1), asset pricing (Proposition 2), and location choice (Proposition 4). Section 4 uses the main result to discuss a number of related issues. Section 5 concludes. All proofs are in the Appendix.

Related Literature

This is the first portfolio choice and asset pricing model that combines the role of housing as access to a location and its role as an investment vehicle for households.

While the present model aims to stay as close as possible to classical mean-variance asset pricing models, it differs from them in an essential way. The dividends of residential properties are not given by an exogenous stochastic process, but are determined in equilibrium by the spatial allocation of agents. Namely, the decision of agents to live in a certain location affects local rents, which represents the dividend of local real estate. In turn, the decision of agents to move to a certain location depends on the analysis of expected earnings and rents as well as the risk of future earnings less rents. The statement and resolution of this fixed-point problem constitute the central methodological innovation of the present paper.

An extensive literature has explored the effect of housing consumption on households' life-cycle overall consumption and investment behavior. One of the early papers by Henderson and Ioannides (1983) considers an optimal consumption and saving problem when the household chooses whether to own or rent and a wedge arises endogenously between the cost of renting and owning. Henderson and Ioannides show that the consumption demand for homeownership distorts households' investment decisions. Goetzman (1993) and Brueckner (1997) explain how this distortion affects households' portfolio choice. Flavin and Yamashita (2002) compute mean-variance optimal portfolios for homeowners using U.S. data on housing and financial asset returns.[4] Cocco (2004) also computes optimal portfolios but in a calibrated dynamic model of households consumption and portfolio choice. Housing consumption is constrained to equal housing investment in both papers. Yao and Zhang (2004) introduce discrete tenure choice (rent or own total housing consumption) in a simi-

[4] Englund, Hwang and Quigley (2002) report similar computations for Sweden, Iacoviello and Ortalo-Magné (2003) for the UK, and LeBlanc and Lagarenne (2004) for France. Note that every one of these papers considers the stock market as a whole and so ignores the covariance between housing and specific stocks.

lar environment. They show the sensitivity of households' portfolio choice to tenure mode: owning a house leads households to reduce the proportion of equity investment in their net worth (a substitution effect). However, households give a greater weight to stocks relative to bonds in their portfolio because homeownership provides insurance against stocks and labor-income fluctuations (a diversification benefit). Altogether, these papers demonstrate that incorporating housing consumption in portfolio choice models helps reconcile theoretical predictions with cross-sectional observations. In particular, home investment seems a key factor in explaining the very limited participation of the young in equity markets. Credit constraints play a critical role in explaining the observed hump-shape in home ownership over the life-cycle.

Grossman and Laroque (1991) characterize optimal consumption and portfolio selection when households derive utility from a single durable good only and trading the durable require payment of a transaction cost. They show that CAPM holds in this environment, but CCAPM fails because consumption of housing is not a smooth function of wealth due to the transaction costs. Flavin and Nakagawa (forthcoming) expand on the Grossman and Laroque framework by assuming that households derive utility not only from housing but also from numeraire consumption. They show that when housing asset returns do not co-vary with stock returns, the CCAPM holds. In equilibrium, all households hold a single optimal portfolio of risky financial assets. Depending on their holding of housing, households vary how much of their wealth is invested in this portfolio but not its composition.

We obtain somewhat similar results with regards to portfolio choice (e.g., separation) and asset pricing (CAPM) although we build our model focusing on a completely different dimension of housing. In the existing literature, housing differs from stocks in the fact that the quantity owned or rented enters directly into the utility function. Housing in our model does not enter the utility function directly. Rather, the choice of a home determines the characteristics of the labor income process households enjoys and the stream of rents they will face. Choosing a home amounts to shorting an asset (commitment to pay the future stream of rents) and going long in another asset (the households stream of income) with the added feature that the returns to this last asset are specific to each agent, they depend on his human capital.

Another key difference between our paper and the literature cited above is that the stream of rents in each location in our model is determined endogenously, by the allocation of the households over space. The papers cited above assume that housing rents follow exogenous stochastic processes.

Piazzesi, Schneider, and Tuzel (2005) study a consumption-based asset pricing model where housing rents and prices are determined endogenously; the quantity of housing follows an exogenous stochastic process. Agents can invest in both housing and stocks. The focus of the analysis is on the composition risk related to fluctuations in the share of housing in the households' consumption baskets. The authors show that the housing share can be used to forecast excess returns of stocks – a prediction that appears to be borne out by the data. Lustig and Van Nieuwerburgh (2007) propose a mechanism whereby the amount of housing wealth in the economy affect the ability of households to insure idiosyncratic income risk and thus shifts the market price of risky assets, housing included. In Lustig and Van Nieuwerburgh (2005) the authors present empirical evidence of the relevance of the ratio of housing wealth to human wealth for returns of stocks. In Lustig and Van Nieuwerburgh (2006), they extend their framework to consider risk sharing across regions. Empirical evidence indicates that the amount of housing wealth in each region affects the sensitivity of local consumption to local income. We share with Piazzesi et al. (2005) and Lustig and Van Nieuwerburgh (2005, 2006, 2007) the same focus on the equilibrium properties of housing rents and the risk premia. Lustig and Van Nieuwerburgh's (2006) is particularly close to us because there are several locations. However, our paper differs because the location choice is endogenous and it leads to asset pricing based on marginal residents.

Our approach to the modeling of housing as an enabling asset follows from the tradition of urban economics. Our location choice model follows the standard multi-cities framework of Rosen (1979) and Roback (1982) where residential properties provide access to the local labor market and locations are differentiated by potential surplus. As in Rosen and Roback and the many more recent papers that build on this framework (e.g., Gyourko and Tracy, 1991, Kahn, 1995, Glaeser and Gyourko, 2005), we assume households face a unit housing consumption requirement and derive utility from consumption of numeraire only.

Because we are concerned with portfolio choice in a dynamic environment, we assume households are risk averse. Risk aversion in the face of stochastic streams of income and rent provides a motivation for ownership of local residential properties – homeownership – in our model. This approach builds on the work of Ortalo-Magné and Rady (2002), Sinai and Souleles (2005), Hilber (2005), Davidoff (2006) and others who provide evidence of the relevance of such motivation for housing investment.

It is beyond the scope of this paper to review the vast literature concerned with the determinants of housing prices. Typically in this literature, the equilibrium discount factor for housing is the risk free rate due to assumptions of risk neutrality of consumers or producers (e.g., Davis and Heathcote, 2003, Ortalo-Magné and Rady, 2006, Van Nieuwerburgh and Weill, 2007, Kiyotaki, Michaelides and Nikolov, 2007).

DeMarzo, Kaniel and Kremer (2004) consider an economy with multiple communities and local goods as well as a global good. In this dynamic setting, some agents (the laborers) are endowed with human capital which will be used to produce local goods in future periods, but they are currently subject to borrowing constraints. Other agents (the investors) own shares in firms that produce the global good. This simple set-up yields a number of powerful results. Investors care about their relative wealth in the community because they bid for scarce local goods. This generates an externality in portfolio choice, which leads to the potential presence of multiple equilibria (in the stable equilibria, investors display a strong home bias). Moreover, if there is a behavioral bias, the presence of this externality amplifies the bias through the portfolio decisions of rational investors. Like DeMarzo et al. (2004), our paper studies the properties of portfolio choice and asset pricing under uncertainty in the presence of community effects. However, our model differs from theirs in a number of important dimensions: (i) There are no credit constraints; (ii) Our local good does not produce utility directly but it enables agents to realize their human capital potential; (iii) Our spatial allocation is endogenous.[5]

2 Model

2.1 Geography and population

Consider an overlapping generation economy where a mass 1 of agents is born in every period. Each agent in the t-cohort is born at the beginning of period t, lives for S periods, and dies at the beginning of period $t+S$. Hence, at every time t, there is a mass S of agents alive in the economy.

There are L cities, denoted with index $l = 1, ..., L$ and a countryside denoted with index $l = 0$. City l has an exogenously given mass of houses. For convenience let n^l be the mass of houses per cohort that will be active on the housing market so that total supply of housing in city l equals $S \times n^l$. We assume that housing supply is scarce in cities:

$$\sum_{l=1}^{L} n^l < 1;$$

[5] Our results on home bias are also related to the international finance literature on the home bias puzzle (Stockman and Dellas, 1989). However, we differ in our focus on real estate and in that location choice is endogenous.

but it is abundant once the countryside is included:[6]

$$\sum_{l=0}^{L} n^{l} > 1.$$

Each house accommodates exactly one agent.

2.2 Production

The income of a person who lives in the countryside is (normalized to) zero. Productivity in city l follows the process

$$y_{t}^{l} = y_{t-1}^{l} + \tau_{t}^{l}$$

where τ_{t}^{l} is a random variable, independently and identically distributed across time. We discuss the covariance of these shocks below in the Random Shock Structure section.

At birth, each agent draws:

- a vector of city-specific endowment surplus, $\boldsymbol{\varepsilon} = [\varepsilon^{l}]^{l=1,\ldots,L}$, with $\varepsilon^{l} \in (-\infty, \infty)$.
- a matrix of city and age specific insulation parameters: $\boldsymbol{\rho} = [\rho_{s}^{l}]_{s=0,\ldots,S.}^{l=1,\ldots,L}$, with $\rho_{s}^{l} \in [0,1]$. Assume $\rho_{0}^{l} = 0$ for all l.

The parameters $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ are i.i.d across generations. Their joint distribution within a generation is left in a general form $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, with the only requirement that it should be continuous and have full support.

An agent's income equals his product. At time $t+s$, the income of an agent living in city l, born at time t, with parameters $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ is

$$y_{t,t+s}^{l}\left(\varepsilon^{l}, \boldsymbol{\rho}^{l}\right) = y_{t-1}^{l} + \varepsilon^{l} + \sum_{m=0}^{s}\left(1-\rho_{m}^{l}\right)\tau_{t+m}^{l},$$

for $s = 0, \ldots, S-1$. Hence, the income of each agent can be decomposed into a permanent part, which captures the initial productivity of the agent in his location and a time-dependent part, which is determined by the local productivity shocks in the city and that agent's sensitivity to his city's shocks. We call ε^{l} the *city-agent effect* and ρ_{s}^{l} the *shock insulation effect.* We represent below the income earned by an agent born at time t, living

[6] Alternatively, the "countryside" can be interpreted as homelessness, with the assumption that employers refuse to hire homeless people.

in city l, for each of the first three years of his life.

$$\begin{aligned}
y_{t,t}^{l} &= \overbrace{\varepsilon^{l}}^{\text{city-agent effect}} + \overbrace{y_{t}^{l}}^{\text{city-cohort effect}} ; \\
y_{t,t+1}^{l} &= \overbrace{\varepsilon^{l}}^{\text{city-agent effect}} + \overbrace{y_{t}^{l}}^{\text{city-cohort effect}} + \overbrace{\left(1-\rho_{1}^{l}\right)\tau_{t+1}^{l}}^{\text{year 1 catch-up}}; \\
y_{t,t+2}^{l} &= \overbrace{\varepsilon^{l}}^{\text{city-agent effect}} + \overbrace{y_{t}^{l}}^{\text{city-cohort effect}} + \overbrace{\left(1-\rho_{1}^{l}\right)\tau_{t+1}^{l}}^{\text{year 1 catch-up}} + \overbrace{\left(1-\rho_{2}^{l}\right)\tau_{t+2}^{l}}^{\text{year 2 catch-up}}.
\end{aligned}$$

Similar formulations determine the agent's earnings until he reaches age $S-1$.[7] At age S, we assume the agent does not earn anything. It is mathematically convenient to set $\rho_S = 0$ for all agents even if it is irrelevant to the agents' earnings.

The city-agent effect is a standard object in multi-city models with heterogeneous agents. Depending on their human capital, agents face different earning opportunities in different locations.

The shock-insulation effect captures two economic phenomena. First, agents may be exposed to a technological cohort-specific effect (documented by Goldin and Katz, 1998). The human capital of certain people, especially the young, may be more flexible. When a technological innovation appears, the income of certain agents will be more affected than the income of others. Second, certain agents – like senior workers and public sector workers – may be part of an implicit labor insurance agreement. Their wage is more insulated from productivity shocks.

It is reasonable – but not strictly necessary for the analysis – to assume that the insulation parameter, for a shock that occurs at a given age, is increasing in the age of the agent: $\rho_{s+1}^{l} > \rho_{s}^{l}$. Of course the two extreme cases are full insulation $(\rho_{s}^{l} = 1)$ and full exposure $(\rho_{s}^{l} = 0)$.[8]

For concreteness, we interpret $y_{t,t+s}^{l}$ as monetary income, but there exists an alternative interpretation in terms of non-monetary benefits that is equivalent from a mathematical standpoint. The term $y_{t,t+s}^{l}$ is now viewed as a money-equivalent of the utility afforded by the amenities present in location l. In turn the utility can be decomposed into an agent-city

[7] The structure of ε and ρ could be much more complex than the one we have here and still be amenable to analysis in the present mean-variance set-up. For instance, we could imagine that the city-agent effect is not constant over the life of the agent but it follows a random walk. Also, we could assume that the extent to which a shock that occurs at age s affect future incomes depends on the age of the agent.

[8] We find it natural to restrict ρ_{s}^{l} to be between zero and one, but our mathematical analysis is valid even if $\rho_{s}^{l} > 1$ (the agent's productivity is negatively correlated to local shocks) and $\rho_{s}^{l} < 0$ (the agent is over-exposed to local shocks).

effect (taste for that particular location) and a shock component (perhaps an environmental or a social risk) multiplied by the agent's sensitivity to that type of shock. Of course, the model can also be interpreted as a mix of monetary and non-monetary benefits.

At birth, every agent chooses in what city (or the countryside) to live. He cannot move afterwards. If an agent lives and thus produces in city l, he must rent exactly one unit of housing.[9]

2.3 Housing market

The market rent in city l at time t is denoted with r_t^l and will be determined in equilibrium. The housing market is frictionless. There are no transaction costs associated with renting, buying or selling property. In particular there is no difference between living in a owned or a rented house.

Agents may invest in divisible shares of any city's housing stock and revise their decision at every period. Let $a_{t,t+s}^l$ denote the amount of housing of city l owned by an agent born at time t of age s.

The market price of a unit of housing in city l at time t is p_t^l. The agent revises his housing investment at the beginning of every period. For accounting purposes, imagine that the agent liquidates all his housing assets and then buys the desired amount in each period. At the beginning of period $t+s$, the agent acquires $a_{t,t+s}^l$ units in city l at total cost $a_{t,t+s}^l p_{t+s}^l$. During period t, the agent collects rent on his housing investment for a total of $a_{t,t+s}^l r_{t+s}^l$. At the beginning of the next period, the agent liquidates the housing investment and receives $a_{t,t+s}^l p_{t+s+1}^l$. We denote $\boldsymbol{a}_{t,t+s}$ the vector of the agent's housing investments, $\boldsymbol{a}_{t,t+s} = \left[a_{t,t+s}^l\right]_{l=1,...,L}$.

2.4 Stock Market

Besides housing, there is another class of securities, which we call stocks. These are claims on productive assets, which – as in regular asset pricing models – produce an exogenous (but stochastic) stream of income. There are Sz^k units of type-k asset, with $k \in \{1, ..., K\}$ and $z^k > 0$. A unit of stock k produces dividend d_t^k at time t. The dividend follows the stochastic process:

$$d_t^k = d_{t-1}^k + \nu_t^k$$

[9] The assumption that people cannot move is essential for tractability. Without it, the model cannot be analyzed within the normal-CARA framework. However, the assumption is not necessary for qualitative results about the role of housing as a hedge. Ortalo-Magné and Prat (2006) develop a (much simpler) model where people can move freely.

where $\boldsymbol{\nu}$ is i.i.d. across time (and the probability distribution will be discussed below).

As for housing, every agent can buy units of every stock and revise his portfolio allocation in every period. The market price of stock k at a given point in time is q_t^k. At the beginning of period $t+s$, the agent acquires $b_{t,t+s}^k$ units of stock k at total cost $b_{t,t+s}^k q_{t+s}^k$. During period $t+s$, the agent receives dividend on his investment in k for a total of $b_{t,t+s}^k d_{t+s}^k$. At the beginning of the next period, the agent liquidates the stock investment and receives $b_{t,t+s}^k q_{t+s+1}^k$. We denote $\boldsymbol{b}_{t,t+s}$ the vector of the agent's stock investments, $\boldsymbol{b}_{t,t+s} = \left[b_{t,t+s}^k\right]_{k=1,\ldots,K}$.

2.5 Distribution of Random Shocks

There are two sources of exogenous shocks in our economy: a vector $\boldsymbol{\tau}$ of local productivity shocks and a vector $\boldsymbol{\nu}$ of capital productivity shocks. The shocks are independently and identically distributed over time, according to a normal distribution with mean 0 and covariance matrix $\boldsymbol{\Sigma}$: $(\boldsymbol{\tau}_t, \boldsymbol{\nu}_t) \sim N(0, \boldsymbol{\Sigma})$.

It is important that we allow for correlation between local shocks and dividends. A certain industry may be more affected by shocks in a certain market and viceversa. We also allow for correlation of shocks across cities.

2.6 Consumption and Savings

As the goal of this paper is to arrive at a mean-variance closed-form expression for asset prices, we assume that agents derive CARA utility $-\exp(-\gamma w)$ from wealth at the end of their life, w, where γ is the standard risk-aversion parameter.

Agents face no credit constraints and can borrow and lend freely at discount rate $\beta \in (0,1)$. For simplicity, we assume that agents are born with no wealth (this does not affect their decisions, given that they have CARA preferences).

2.7 Non-Negativity Constraints

Asset pricing models with normally distributed shocks suffer from a well-known technical problem. As the value of the dividends can become negative, agents may find themselves in situations where they would want to dispose of assets they own. If they could, the distribution of asset values would no longer be normal and the model would not be tractable. Hence, all models in this class assume, implicitly or explicitly, that agents cannot dispose of assets. Typically, this assumption is unrealistic because in practice both agents and firms are protected by limited liability. Instead, in the model stocks can have negative prices, and their owners must pay to get rid of them.

Our CARA-normal set-up inherits this non-negativity problem. In particular, the productivity in a city could become negative and house prices there may be negative.[10]

The usual response to this criticism, which applies here as well, is that the unconstrained model should be viewed as an approximation of the model with non-negativity constraints, as long as the starting values are sufficiently far from zero.

2.8 Timing

To re-cap, the order of moves, for an agent born at time t is as follows:

1. At birth, the agent chooses in which location l he will spend the rest of his life.
2. At the beginning of each period $t+0, ..., t+S$, the agent learns the values of the random shocks for that period, $\boldsymbol{\nu}_{t+s}$ and $\boldsymbol{\tau}_{t+s}$.
3. For $s = 0, ..., S-1$, at the beginning of period $t+s$ the agent revises his housing and stock investments ($\boldsymbol{a}_{t,t+s}$ and $\boldsymbol{b}_{t,t+s}$).
4. At $t+s$, the agent also pays rent r^l_{t+s} for one unit of housing in the location where he lives. He collects dividends and rents on the assets that he owns.
5. At the end of his life, at time $t+S$, the agent liquidates his investments ($\boldsymbol{a}_{t,t+S-1}$ and $\boldsymbol{b}_{t,t+S-1}$) and consumes the wealth that he has accumulated.[11]

3 Analysis

An equilibrium is an allocation of households across cities, a vector of optimal portfolio holdings of housing and stocks for each agent, housing rents and prices for each city and stock prices such that: (i) The location choice and portfolio holdings solve the agents' problem; (ii) The housing markets in each city clear; (iii) The stock markets clear.

A *stationary equilibrium* is an equilibrium where the mass of agents of a generation t who live in a given city l is the same across generations.[12]

[10] We assume that homeowners have an obligation to rent their property (they pay a large fine if it IS vacant).

[11] The agent does not work or pay rent in the last period of his life ($t+S$). He consumes his wealth in the beginning of the period and he dies.

[12] A non-stationary equilibrium would have the following structure. As agents cannot move after they locate to city l, the stock of rented accomodation used by the t-cohort will not become available until

Define a *linear equilibrium* as an equilibrium where stock prices, rents, and house prices can be expressed respectively as:

$$q_t^k = \frac{1}{1-\beta} d_t^k - \bar{q}^k; \tag{1}$$

$$r_t^l = y_t^l + \bar{r}^l; \tag{2}$$

$$p_t^l = \frac{1}{1-\beta} r_t^l - \bar{p}^l; \tag{3}$$

where $\bar{\boldsymbol{q}} = \left[\bar{q}^k\right]^{k=1,\ldots,K}$, $\bar{\boldsymbol{p}} = \left[\bar{p}^l\right]^{l=1,\ldots,L}$ are price discounts and $\bar{\boldsymbol{r}} = \left[\bar{r}^l\right]^{l=1,\ldots,L}$ is a rent premium to be determined in equilibrium. The rent is equal to local productivity plus a local constant. House and stock prices are equal to the discounted value of a perpetuity that pays the current rent or dividend minus an asset-specific discount.

Price discounts can also be interpreted as expected returns of zero-cost portfolios.[13] Throughout the analysis we describe $\bar{p}^l$ and $\bar{q}^k$ as price discounts or expected returns, depending on the context.

Our strategy for finding equilibria is as follows. We start by conjecturing that we are in a stationary linear equilibrium. We postulate a feasible allocation of agents to cities and we solve the portfolio problem of a generic agent living in a given city. As it turns out, solving this agent problem is enough to characterize analytically stock prices and house prices up to a vector of city-specific constants. With this information, we compute the expected utility of every agent conditional on city choice. We determine aggregate location demand given any price vector by comparing expected utilities across cities. Finally, by equalizing the number of homes in each location to the number of agents in each location, we obtain the vector of city-specific constants that closes the model. As it turns out, this vector exists and is unique, proving that that our initial conjecture on linear prices was correct.[14]

members of the t-cohort die at then end of $t+S$. Hence, if the t-cohort is, say, over-represented, then the $t+S+1$-cohort will be equally over-represented. The non-stationary equilibria are characterized by cycles of length $S+1$.

[13] For instance, the expected return of a zero-cost one-unit portfolio invested in housing in city l (evaluated in today's dollars) is

$$\begin{aligned} & E\left[\beta p_{t+1}^l - \left(p_t^l - r_t^l\right)\right] \\ = \; & \frac{\beta}{1-\beta} r_t^l - \beta \bar{p}^l - \frac{\beta}{1-\beta} r_t^l + \bar{p}^l \\ = \; & (1-\beta)\,\bar{p}^l \end{aligned}$$

[14] It is tempting to consider the two first parts of the analysis, portfolio choice and asset pricing, in isolation. But they are only valid if the third part is present too. If one assumed a different location model or an exogenous allocation of agents to cities, the three price processes in (1), (2), and (3) would be different and

As agents have CARA preferences. their lifetime utility can be decomposed into:

$$E\left[u_t^l\right] = E\left[w_t^l\right] - \gamma V\left[w_t^l\right]$$

The following proposition re-writes the two components of the agents utility and uses them to compute his optimal portfolio choice and his expected utility (in what follows we focus on one agent and we drop the argument representing the agent-specific characteristics: $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$.

Proposition 1 (Portfolio Allocation) *Suppose that prices and rents are given by equations (1), (2), and (3), with given $\bar{r}$'s, $\bar{q}$'s and $\bar{p}$'s. Consider any allocation of agents to cities. Consider an agent born at period t characterized by a vector $\boldsymbol{\varepsilon}$ and a matrix $\boldsymbol{\rho}$. If this agent lives in l and chooses investment profiles $[\boldsymbol{a}_{t,t+s}, \boldsymbol{b}_{t,t+s}]_{s=0,\ldots,S-1}$, the expectation and the variance of his end-of-life wealth can be written respectively as:*

$$\begin{aligned} E\left[w_t\right] &= \frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s\left(\varepsilon^l - \bar{r}^l + (1-\beta)\left(\left(1-\beta^{S-s}\right)\rho_{s+1}^l\bar{p}^l + \sum_{j=1}^{L}\tilde{a}_{t,t+s}^j\bar{p}^j + \sum_{k=1}^{K}b_{t,t+s}^k\bar{q}^k\right)\right) \\ Var\left[w_t\right] &= \frac{1}{\beta^S}\frac{\beta^2}{(1-\beta)^2}\sum_{s=0}^{S-1}\beta^{2s}Var\left[\sum_{j=1}^{L}\tilde{a}_{t,t+s}^j\tau_{t+s+1}^j + \sum_{k=1}^{K}b_{t,t+s}^k\nu_{t+s+1}^k\right] \end{aligned}$$

where

$$\tilde{a}_{t,t+s}^j = \begin{cases} a_{t,t+s}^l - \left(1-\beta^{S-s}\right)\rho_{s+1}^l & \text{if } j = l \\ a_{t,t+s}^j & \text{otherwise} \end{cases}.$$

The agent's optimal investment profile is given by

$$\begin{bmatrix} \tilde{\boldsymbol{a}}_{t,t+s} \\ \boldsymbol{b}_{t,t+s} \end{bmatrix} = \frac{(1-\beta)^3}{2\gamma\beta^{s+2}}\beta^S\boldsymbol{\Sigma}^{-1}\begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix},$$

for $s = 0, ..., S-1$, and his expected log-utility is

$$U^l = \frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s\left(\varepsilon^l - \bar{r}^l + (1-\beta)\left(1-\beta^{S-s}\right)\rho_{s+1}^l\bar{p}^l\right) + S\frac{(1-\beta)^4}{4\gamma\beta^2}\begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix}'\boldsymbol{\Sigma}^{-1}\begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix}$$

Proposition 1 says that the optimal portfolio of any agent can be decomposed into:

- Investment in a mutual fund that contains all stocks and houses in all cities, with weights $(\tilde{\boldsymbol{a}}, \boldsymbol{b})$. The mutual fund is the same for all agents. All agents within a cohort

Propositions 1 and 2 would no longer hold. For instance, if agents could move between cities during their lifetime, it is not clear that the rent the price in city l would depend only on productivity in city l.

We see this as both a weakness and a strength of spatial asset pricing. On the one hand, one cannot have a meaningful discussion about real estate prices in multiple locations without a spatial model in the background. On the other hand, this opens the door to a wealth of testable implications encompassing spatial and financial variables.

buy the same amount of mutual fund shares (but older agents buy more shares, purely because of the discount rate β). Given a vector of expected returns (which for now is still exogenous), the weights $(\tilde{\boldsymbol{a}}, \boldsymbol{b})$ that the mutual fund puts on various stocks and real estate assets are given by a standard CAPM allocation. The portfolio puts more weight on an asset if its returns are less correlated to other assets and they have a higher expected value.

- Demand for real estate in the city where the agent lives, driven by a desire to hedge shocks to disposable income due to rent fluctuations. As the price of a house is linear in the rent, a house in a certain city is a perfect hedge against rent fluctuations in that city. The hedging demand is given by $\left(1-\beta^{S-s}\right)\rho^l_{s+1}$. Hence, it depends on how well the agent is insulated from local productivity shocks at time t. The hedging demand varies across agents and across time for a given agent (the cross-sectional and life-cycle implications of this are explored in detail in the Discussion section). However, the hedging demand does not depend on the expected return of real estate in that city (if a city has a high return, that will be reflected in the mutual fund share only).[15]

Now that we have solved the portfolio allocation problem for any given vector of premia, we can find the equilibrium expected returns. Denote any (measurable) allocation of agents to cities with the indicator function $I^l_{\boldsymbol{\varepsilon},\boldsymbol{\rho}}$, which takes value 1 if agents with personal characteristics $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$ locate to city l, and zero otherwise (such that $\sum_{l=0}^{L} I^l_{\boldsymbol{\varepsilon},\boldsymbol{\rho}} = 1$ for all $\boldsymbol{\varepsilon}$ and $\boldsymbol{\rho}$).

Proposition 2 (Asset Pricing) *Suppose that rents are given by equations (2), with given $\bar{r}$'s. Consider any allocation of agents to cities. Then, prices are given by equations (1) and (3) with discounts:*

$$\begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix} = 2\gamma S \frac{\beta}{(1-\beta)^2\left(1-\beta^S\right)}\boldsymbol{\Sigma}\begin{bmatrix} \boldsymbol{n}-\boldsymbol{R} \\ \boldsymbol{z} \end{bmatrix}$$

where $\boldsymbol{R} = \left[R^1, ..., R^l, ..., R^L\right]'$ *and*

$$R^l = \frac{1}{S}\sum_{s=0}^{S-1}\left(1-\beta^{S-s}\right)\int_{\boldsymbol{\varepsilon}}\int_{\boldsymbol{\rho}} I^l_{\boldsymbol{\varepsilon},\boldsymbol{\rho}}\rho^l_{s+1}\phi\left(\boldsymbol{\varepsilon},\boldsymbol{\rho}\right)d\boldsymbol{\varepsilon}d\boldsymbol{\rho}.$$

[15] Davis and Willen (2000) obtain a related result (Proposition 1 in their paper) on the decomposition of the optimal portfolio of agents who face labor risk into a speculative component and a hedging component.

Houses and stocks are priced based on their contribution to systemic risk according to a classical CAPM formula. Proposition 2 finds the correct definition of systemic risk for this model. The weights of stocks in the market portfolio correspond to the quantity of stocks available, as in the regular CAPM. However, the weights of real estate are reduced by the total hedging demand. Namely, the weight of houses in city l is equal to the mass of homes n^l minus the integral of the hedging demand by residents of l: R^l.

To explore the pricing expressions in Proposition 2 further, define the *adjusted market portfolio* M as a portfolio allocation that includes

$$\frac{n^l - R^l}{Q} \text{ units of housing in city } l \text{ for every city } l$$

$$\frac{z^k}{Q} \text{ units of stock } k \text{ for every stock } k$$

with $Q = \sum_{l=1}^{L} \left(n^l - R^l\right) + \sum_{k=1}^{K} z^k$. The mutual fund that all agents buy contains the adjusted market portfolio.

Denote the expectation and the variance of the adjusted market portfolio, respectively, with $\bar{p}^M$ and $Var\left(M\right)$. Define $Cov\left(l, M\right)$ as the covariance between the return of real estate in city l and the return of M. For every stock k define $Cov\left(k, M\right)$ similarly. Then:

Corollary 3 *The expected return of real estate in city l is given by*

$$\bar{p}^l = \frac{Cov\left(l, M\right)}{Var\left(M\right)} \bar{p}^M,$$

and the expected return of stock k is

$$\bar{p}^k = \frac{Cov\left(k, M\right)}{Var\left(M\right)} \bar{p}^M.$$

The expression in the Corollary is akin to the classical CAPM pricing formula where $\frac{Cov(l,M)}{Var(M)}$ is a beta-factor for housing in city l. The main innovation in our setting lies in the identification of the adjusted market portfolio, for which this formula is true.[16]

To close the analysis, we still have to prove that our price conjectures are correct, to determine the vector of rent premia $\bar{\boldsymbol{r}}$, and to find the vector of hedging demands $\boldsymbol{R}$. All this depends on the location of agents, which we study next.

For an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, the log-utility of locating in city l is given by U in Proposition 1, where now $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ are defined in terms of primitives through

[16] For instance, if one defined the market portfolio without the $-R$ correction, such beta representation would not be valid.

Proposition 2. For every $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, let

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) \equiv \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l \sum_{s=1}^{S}\left(1-\beta^{S-s+1}\right)\rho_s^l.$$

Also let

$$\bar{U} = S\frac{(1-\beta)^4}{4\gamma\beta^2}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}'\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}$$

Then, we can write the utility of locating in city l as[17]

$$U^l = \frac{1-\beta^S}{1-\beta}\left(\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l\right) + \bar{U}.$$

Namely, the agent's utility can be decomposed into a component that is common to all agents (and depends on investment in the mutual fund) and an agent-specific component that depends on the city-agent effect ε^l and the shock-insulation vector $\boldsymbol{\rho}^l$ that the agent faces is he chooses to locate in city l..

A given agent locates in city l if and only if $U^l = \max_m U^m$. For every L-vector $\hat{\boldsymbol{r}}$, we can write the aggregate demand for location l as

$$\nu^l(\hat{\boldsymbol{r}}) = \int_{(\boldsymbol{\varepsilon},\boldsymbol{\rho}):\bar{u}^l(\boldsymbol{\varepsilon},\boldsymbol{\rho})-\hat{r}^l=\max_m(\bar{u}^m(\boldsymbol{\varepsilon},\boldsymbol{\rho})-\hat{r}^m)} \phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})\, d(\boldsymbol{\varepsilon}, \boldsymbol{\rho}).$$

We obtain:

Proposition 4 (Location Choice) *There is a unique linear stationary equilibrium. In it, an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^l = \max_{j=0,\ldots L}\left(\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \bar{r}^j\right)$$

and $\bar{\boldsymbol{r}}$ is the unique value of the vector $\hat{\boldsymbol{r}}$ such that $\nu^l(\hat{\boldsymbol{r}}) = n^l$ in all cities.

The equilibrium rent in city l is

$$r_t^l = y_t^j + \bar{r}^l.$$

Proposition 4 characterizes rents in terms of the decisions of marginal residents – agents who in equilibrium are indifferent between living in one location or another. Despite the fact that the payoff of an agent in a given city is determined by $S+1$ parameters (ε^l plus

[17] To see this, note that:

$$\frac{1-\beta}{1-\beta^S}\sum_{s=0}^{S-1}\beta^s\left(\varepsilon^l + (1-\beta)\left(1-\beta^{S-s}\right)\rho_{s+1}^l\bar{p}^l\right) = \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l\sum_{s=1}^{S}\left(1-\beta^{S-s+1}\right)\rho_s^l$$

the vector $\boldsymbol{\rho}^l$), the expected utility U^l of the agent in that city can be condensed into a simple expression containing $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$. For any possible vector of rents $\hat{\boldsymbol{r}}$, the demand function $\nu^l(\hat{\boldsymbol{r}})$ establishes how many agents will live in each location. The assumption that the distribution of individual characteristics $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ has full support guarantees that the demand function is continuous and hence the existence of a solution $\bar{\boldsymbol{r}}$ in the system $\boldsymbol{\nu}(\bar{\boldsymbol{r}}) = \boldsymbol{n}$. One can also prove that $\boldsymbol{\nu}(\hat{\boldsymbol{r}})$ is monotonic in $\hat{\boldsymbol{r}}$ and hence the solution is unique.

The equilibrium rent vector identifies a set of measure zero of agents who are indifferent among all cities and the countryside. These "super-marginal" residents have personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ such that $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) = \bar{r}^l$ for every l (note that in the countryside $\bar{u}^0(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) = \bar{r}^0 = 0$). As in many spatial economics models, the location choice of other agents can be understood in terms of relative utility comparisons with super-marginal agents.

A key feature of our location equilibrium is that the geographical distribution of individual characteristics is time-invariant. The identity of super-marginal agents is the same in every cohort. This guarantees that local productivity shocks are fully incorporated into local rents according to our simple linear expression $r_t^l = y_t^j + \bar{r}^l$.

3.1 Example

While we obtained closed-form solutions for portfolio decisions and asset premia, Proposition 4 does not express rents in closed form. This is natural as the probability distribution over individual characteristics, $\phi(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, is left in a general form. By making appropriate assumptions over personal characteristics and geography, one can arrive at closed-form expressions for all variables, as the following example illustrates.

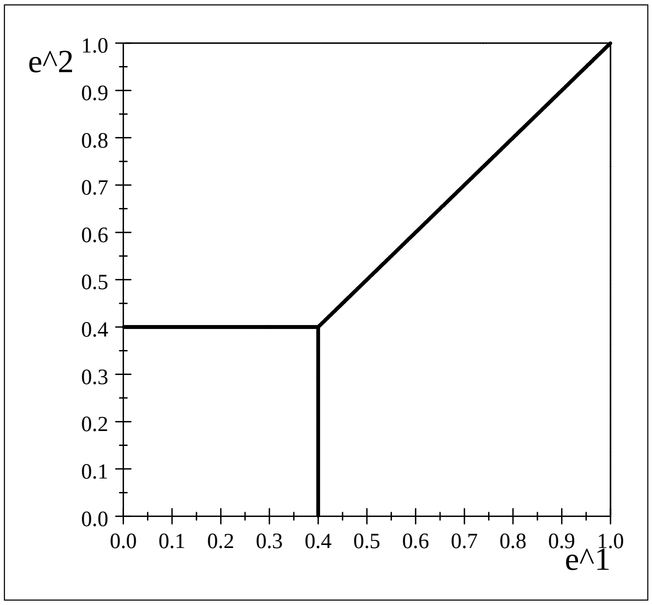
Assume that:

- Agents in each cohort draw city-specific endowments $\boldsymbol{\varepsilon}$ from a uniform distribution defined over $[0,1]^L$;
- At each age, all agents face the same city-specific insulation parameter $\left[\tilde{\rho}_s^l\right]_{s=0,\ldots,S}^{l=1,\ldots,L}$;
- All cities have same size: $n^l = \frac{1}{L}N$ for every l, with $N \in (0,1)$.

Proposition 5 *An agent with human capital $\boldsymbol{\varepsilon}$ locates in city l if: (i) $\varepsilon^l = \max_m \varepsilon^m$; and (ii) $\varepsilon^l \geq (1-N)^{\frac{1}{L}}$. The equilibrium rent in city l is*

$$r^l = (1-N)^{\frac{1}{N}} + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l \sum_{s=1}^{S}\left(1-\beta^{S-s+1}\right)\tilde{\rho}_s^l$$

In the special case with two cities only ($L = 2$), the equilibrium allocation is depicted in the plot below . The agents who locate in the countryside are those with a low ε^1 and a low ε^2 (the bottom right square region) locate in the countryside. Those who locate in city 1 have $\varepsilon^1 \geq (1-N)^{\frac{1}{2}}$ and $\varepsilon^1 \geq \varepsilon^2$ (bottom right trapezoid). Those who locate in city 2 have $\varepsilon^2 \geq (1-N)^{\frac{1}{2}}$ and $\varepsilon^2 \geq \varepsilon^1$.



4 Discussion

Our spatial asset pricing model yields a rich set of implications, which we explore in the present section. We begin by discussing cross-sectional and life-cycle implications. We then turn to talent allocation across cities. We explore the pricing of portfolios of stocks and portfolios of real estate. We conclude with a short discussion of how the model can be extended to include economies of agglomeration.

4.1 Returns on Housing across Cities

Our model yields predictions on cross-sectional differences in real estate returns (Proposition 2 and Corollary 3). To get a qualitative feel for those predictions, consider a simple benchmark: assume that shocks across cities are uncorrelated and suppose there are no stocks. Let $Var\left(\tau^l\right) = \sigma_l^2$. Proposition 2 yields

$$\bar{p}^l = 2\gamma \frac{\beta}{(1-\beta)^2\left(1-\beta^S\right)} \sigma_l^2 \left(n^l - R^l\right)$$

The expected return in a city is an increasing function of the variance of shocks in that city and of the outstanding real estate stock $n^l - R^l$. In turn, the latter is a decreasing function of the average shock insulation parameter (R^l) in that city. The value of R^l is determined in equilibrium.

Consider a location that specializes in an industry and thus with low shock-insulation parameters: All residents, whether old or young, are affected by industry productivity shocks in the same way. The residents have a low demand for housing for hedging purposes. The city's homeownership rate is low, and so are prices. On the contrary, a city centered around an industry with high shock-insulation parameters – perhaps a high-tech industry where the old struggle to catch up with innovation or a highly protected sector, where older worker face implicit insurance – will display a high hedging demand for housing, high homeownership rates, and high prices at "equal rents."

4.2 Home Ownership over the Life Cycle

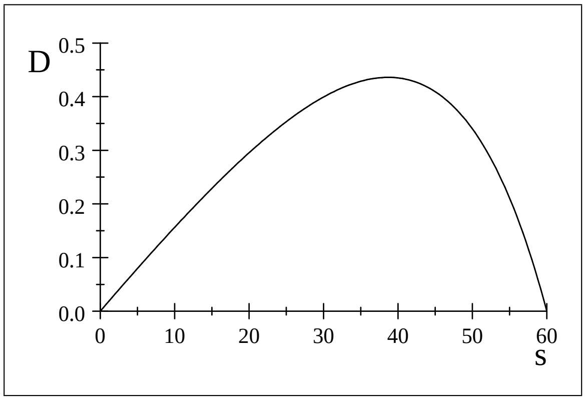
The model also yields intertemporal predictions on home-ownership rates. We know from Proposition 1 that housing demand for hedging purposes depends on the shock-insulation parameter, which in turn varies with age. The hedging demand by someone at age s anticipating a shock-insulation parameter the following period of ρ_{s+1}^l is

$$D_s^l = \left(1 - \beta^{S-s}\right) \rho_{s+1}^l.$$

If one assumes that the shock-insulation parameter can be written as $\rho_s^l = k\frac{s-1}{S-1}$, with $k \in (0,1)$ (implying $\rho_1^l = 0$ and ρ_s^l linearly increasing in age), we have

Proposition 6 *Local home ownership has an inverted U-shape over i's lifetime. For every agent i in city l, there exists an age $\hat{s}$ such that local homeownership is increasing until $\hat{s}$ and decreasing afterwards.*

For instance, if $\beta = 0.95$, $S = 60$, and $k = 1$, the hedging demand over the life-cycle is plotted in the graph below.



This result offers another explanation – alternative to credit constraints – for why homeownership rates should be lower for younger people. When young, households do not need much insurance against rent shocks because their earnings provides such insurance. As they get older earnings provide less insurance, their hedging demand for homeownership increases. Against this force is the fact that as agent gets older, the number of remaining periods of life decreases, reducing the demand for insurance; this last point was made by Sinai and Souleles (2005) who also provide evidence of its empirical relevance.

4.3 Talent Allocation

Does our market equilibrium have the potential to create productive inefficiency?

Let us begin by defining and characterizing productive efficiency in this context. Let the economy's total product at time t be

$$Y_t = \sum_{l=1}^{L} \int_{(\boldsymbol{\varepsilon},\boldsymbol{\rho}):\bar{u}^l(\boldsymbol{\varepsilon},\boldsymbol{\rho})-\bar{r}^l=\max_m(\bar{u}^m(\boldsymbol{\varepsilon},\boldsymbol{\rho})-\bar{r}^m)} y^l_{t,t+s}\left(\boldsymbol{\varepsilon}^l,\boldsymbol{\rho}^l\right)\phi\left(\boldsymbol{\varepsilon},\boldsymbol{\rho}\right)d\left(\boldsymbol{\varepsilon},\boldsymbol{\rho}\right)$$

Suppose a planner wishes to maximize the expected discounted sum of future total products

$$Y = \sum_{s=0}^{\infty} \beta^s E\left[Y_{t+s}\right]$$

Proposition 7 *The allocation of agents to cities that maximizes Y depends only on $\boldsymbol{\varepsilon}$ not on $\boldsymbol{\rho}$: An agent with $\boldsymbol{\varepsilon}$ locates in city l if $\varepsilon^l - \bar{\varepsilon}^l = \max_m \varepsilon^m - \bar{\varepsilon}^m$, where $\bar{\boldsymbol{\varepsilon}}$ is the unique vector that guarantees that the mass of agents in every city equals housing supply.*

Proposition 8 *Exactly one of the following statements is true:*

(i) For all cities, $\bar{p}^l = 0$;

(ii) The linear stationary equilibrium does not maximize Y.

The previous proposition says that productive efficiency is reached if and only the expected return on real estate is zero in every city. In that case, insurance against the rent risk is available a cost zero (if the return is positive insurance carries a negative price). Agents base their location decisions exclusively on $\boldsymbol{\varepsilon}$. Expected returns on real estate are zero when: (i) The covariance matrix Σ is such that there is no systemic risk; (ii) The local productivity shocks are uncorrelated and the number of cities goes to infinity (there is still systemic risk coming from stocks); (ii) All agents have $\rho_s^l = 0$ in all cities at all ages. Outside these restrictive conditions, the distribution of $\boldsymbol{\rho}$ matters in location choices and the equilibrium allocation does not maximize expected product. Of course, productive inefficiency does not imply overall inefficiency. Our market equilibrium is constrained-efficient given the insurance options available in the model.

To reinforce the point of this proposition, we fully solve a restricted version of our model in closed-form. For ease of exposition, we let $S = 2$, and restrict the stock market to a single stock. We assume agents enjoy a constant insulation parameter ρ over life. Each cohort is equally divided in two agent-types: type 0 agents have no insulation ($\rho = 0$), type 1 agents have full insulation, $\rho = 1$. The distribution of agent-city match parameter is independent of agent type, ε, uniform over the unit interval. An agent (ε, ρ) locates in the city if and only if

$$\frac{1}{\beta^2}\sum_{s=0}^{1}\beta^s\left(\varepsilon - \bar{r} + (1-\beta)\left(1-\beta^{2-s}\right)\rho\bar{p}\right) \geq 0$$

The marginal city dwellers of type 0, $\hat{\varepsilon}^0$, and type 1, $\hat{\varepsilon}^1$ satisfy

$$\begin{cases} \hat{\varepsilon}^0 = \bar{r} \\ \hat{\varepsilon}^1 = \bar{r} - \frac{(1-\beta)\bar{p}\left(1+\beta-2\beta^2\right)}{1+\beta} \end{cases}.$$

The market clearing condition on the spatial market is $\frac{\left(1-\hat{\varepsilon}^1+1-\hat{\varepsilon}^0\right)}{2} = n$, which yields a solution for the rent premium as a function of the housing price discount

$$\bar{r} = 1 - n + \frac{(1-\beta)\left(1+\beta-2\beta^2\right)}{2(1+\beta)}\bar{p}.$$

The asset market clearing conditions are

$$\begin{bmatrix} 2n \\ 2z \end{bmatrix} = \left(1+\frac{1}{\beta}\right)\frac{(1-\beta)^3}{2\gamma}\boldsymbol{\Sigma}^{-1}\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} + \begin{bmatrix} \left(1-\beta^2\right)\left(n-\frac{1}{2}+\frac{\bar{r}}{2}\right) \\ 0 \end{bmatrix}.$$

Let $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_h^2 & \sigma_{hs} \\ \sigma_{hs} & \sigma_s^2 \end{bmatrix}$. Replacing $\bar{r}$ with the equation above and rearranging yields a solution for the stock price discount, $\bar{q}$, as a function of $\bar{p}$, and a solution for $\bar{p}$, hence a full characterization of the equilibrium

$$\bar{q} = \frac{2\gamma}{(1-\beta)^3\left(1+\frac{1}{\beta}\right)}\left(\sigma_{hs}\left(2n-\left(1-\beta^2\right)\left(\frac{n}{2}+\frac{(1-\beta)\left(1+\beta-2\beta^2\right)}{4(1+\beta)}\bar{p}\right)\right)+2z\sigma_s^2\right),$$

$$\bar{p} = \left(2n\sigma_h^2-\left(1-\beta^2\right)\frac{n}{2}\sigma_h^2+2z\sigma_{hs}\right)/\left(\left(1+\frac{1}{\beta}\right)\frac{(1-\beta)^3}{2\gamma}+\frac{(1-\beta)^2\left(1+\beta-2\beta^2\right)}{4}\sigma_h^2\right).$$

With numerical values $\beta = n = z = -\sigma_{hs} = .5$ and $\sigma_h = \sigma_s = \gamma = 1$, the equilibrium solution is $\hat{\varepsilon}^0 = \bar{r} = 0.71$, $\hat{\varepsilon}^1 = 0.3$, $\bar{p} = 1.25$, $\bar{q} = 3.375$. Maximizing output would have required $\hat{\varepsilon}^0 = \hat{\varepsilon}^1 = .5$.

4.4 Housing and Stock Indices

As in CAPM one can price any portfolio with respect to the market. In this model, the relevant market is defined by the adjusted market portfolio M, discussed in Corollary 3.

In particular, one can price a housing-only index with weights $\frac{\boldsymbol{n}-\boldsymbol{R}}{\boldsymbol{1}\cdot[\boldsymbol{n}-\boldsymbol{R}]}$ (we call it H) and a stock-only index with weights $\frac{\boldsymbol{z}}{\boldsymbol{1}\cdot\boldsymbol{z}}$ (called S). We have:

$$\bar{p}^H = \frac{Cov(H,M)}{Var(M)}\bar{p}^M$$
$$\bar{p}^S = \frac{Cov(S,M)}{Var(M)}\bar{p}^M$$

Note that H can be interpreted as an index tracking the market portfolio of REITs: it is the housing demand vector that is the same for all agents. It includes all houses that are not owned by local residents for hedging purposes. The following result is immediate (by putting together the two return expressions above):

Corollary 9 *The relative returns of the housing index and the stock index are given by*

$$\bar{p}^H = \frac{Cov(H,M)}{Cov(S,M)}\bar{p}^S$$

The corollary implies that, ceteris paribus, the difference between real estate returns and stock returns is related to home-ownership rates. The higher the fraction of residential property owned by local residents, the lower the returns on real estate.

Our model can also be used for predictions on stock returns. Often, the return of a stock is computed according to a CAPM formula that takes into account stocks only. Namely, the return of stock k is assumed to be

$$\tilde{q}^k = \frac{Cov(k,S)}{Var(S)}\bar{p}^S.$$

In our setting, this expression is of course incorrect, because it does not take into account the presence of housing. The correct expression is $\bar{q}^k = \frac{Cov(k,M)}{Var(M)} \bar{p}^M$. The ratio between the wrong expression and the correct one is

$$\frac{\tilde{q}^k}{\bar{q}^k} = \frac{Cov\,(k,S)}{Cov\,(k,M)} \frac{Var\,(S)}{Var\,(M)} \frac{\bar{p}^S}{\bar{p}^M}$$

If one assumes that dividend shocks are more volatile than the whole economy which includes productivity shocks ($Var\,(S) > Var\,(M)$) and stock k is more correlated with the stock index than with the whole economy ($Cov\,(k,S) > Cov\,(k,M)$), then we must conclude that the ratio between the two expressions is greater than one, namely the beta's predicted by the stock-only CAPM are systematically higher than the beta's predicted by our model.

4.5 Economies of Agglomeration

In the core of the paper we assumed that there are no production externalities (or amenity externalities, if one embraces the amenity interpretation of our model). Our set-up can be easily extended to incorporate externalities. Most results still hold, except possibly uniqueness.

Assume that the income of an agent if he locates in l is now given by

$$y^l_{t,t+s}\left(\varepsilon^l, \boldsymbol{\rho}^l\right) = y^l_{t-1} + \varepsilon^l\left(\boldsymbol{E}^l\right) + \sum_{m=0}^{s}\left(1-\rho^l_m\right)\tau^l_{t+m},$$

where $\boldsymbol{E}^l$ is the collection of ε^l of other agents living in city l.

It is easy to see that Propositions 1 and 2 hold as stated. Proposition 4 can be re-stated as follows. For every $\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{E}^l\right)$, let

$$\bar{u}^l\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{E}^l\right) = \varepsilon^l\left(\boldsymbol{E}^l\right) + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l \sum_{s=1}^{S}\left(1-\beta^{S-s+1}\right)\rho^l_s$$

As before, an agent locates in city l if and only if $U^l = \max_m U^m$.

An allocation of agents to cities is described by $\boldsymbol{E} = \left(\boldsymbol{E}^1, ..., \boldsymbol{E}^L\right)$. Hold $\boldsymbol{E}$ constant. For every L-vector $\hat{r}$, the aggregate demand for location l is

$$\nu^l\left(\hat{r}, \boldsymbol{E}\right) = \int_{(\boldsymbol{\varepsilon},\boldsymbol{\rho}):\bar{u}^l\left(\boldsymbol{\varepsilon},\boldsymbol{\rho},\boldsymbol{E}^l\right)-\hat{r}^l=\max_m\left(\bar{u}^m(\boldsymbol{\varepsilon},\boldsymbol{\rho},\boldsymbol{E}^m)-\hat{r}^m\right)} \phi\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}\right) d\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}\right)$$

Proposition 10 *An allocation $\boldsymbol{E}$ is part of a linear stationary equilibrium if and only if: (i) for all $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$, an agent with personal characteristics $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l if and only if*

$$\bar{u}^l\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{E}^l\right) - \bar{r}^l = \max_m\left(\bar{u}^m\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{E}^m\right) - \bar{r}^m\right)$$

and (ii) $\bar{r}$ is the unique value of the vector $\hat{r}$ such that $\nu^l\left(\hat{r}, \boldsymbol{E}\right) = n^l$ in all cities.

Thus, the equilibrium characterization part of Proposition 4 is still valid. What is missing is existence and uniqueness, which will depend on the properties of the functions $\varepsilon^l(\cdot)$. While it would not be difficult to find conditions on $\varepsilon^l(\cdot)$ that ensure existence, multiplicity of equilibrium is an intrinsic feature of models with economies of agglomeration. Our model does not help predict which equilibrium will arise, but it describes portfolio allocation and asset pricing in each equilibrium.

5 Conclusion

Our model is just a first step towards a theory of spatial asset pricing. The goal of the present paper was to obtain a tractable framework that stayed as close as possible to CAPM. Despite its simplicity, the model generates a wealth of testable implications that span financial variables as well as spatial ones.

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Appendix

Proof of Proposition 1

The cash flow at period $t+s$ for agent born at t, living in city l is

$$\begin{aligned} v_{t,t+s} &= y^l_{t,t+s} - r^l_{t+s} \\ & -\sum_j \left(\left(p^j_{t+s} - r^j_{t+s}\right) a^j_{t,t+s} - p^j_{t+s} a^j_{t,t+s-1}\right) \\ & -\sum_k \left(\left(q^k_{t+s} - d^k_{t+s}\right) b^k_{t,t+s} - q^k_{t+s} b^k_{t,t+s-1}\right) \end{aligned}$$

for $s = 0, ..., S-1$ and

$$v_{t,t+S} = \sum_j p^j_{t+S} a^j_{t,t+S-1} + \sum_k q^k_{t+S} b^k_{t,t+S-1}.$$

The end-of-life wealth of an agent born in t (evaluated at the beginning of his life) is:

$$w_t = \frac{1}{\beta^S} \sum_{s=0}^{S} \beta^s v_{t,t+s}$$

Plug in the income process and the linear prices:

$$\begin{aligned} w_t &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(y^l_{t-1} + \varepsilon^l + \sum_{m=0}^{s} \left(1-\rho^l_m\right) \tau^l_{t+m} - y^l_{t-1} - \sum_{m=0}^{s} \tau^l_{t+m} - \bar{r}^l \right) \\ & + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a^j_{t,t+s} \left(\frac{\beta}{1-\beta} \tau^j_{t+s+1} + (1-\beta)\, \bar{p}^j \right) \\ & + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b^j_{t,t+s} \left(\frac{\beta}{1-\beta} \nu^j_{t+s+1} + (1-\beta)\, \bar{q}^k \right) \\ &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l - \bar{r}^l\right) - \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \sum_{m=0}^{s} \rho^l_m \tau^l_{t+m} \\ & + \frac{1}{\beta^S} \sum_j \sum_{s=0}^{S-1} \beta^s a^j_{t,t+s} \left(\frac{\beta}{1-\beta} \tau^j_{t+s+1} + (1-\beta)\, \bar{p}^j \right) \\ & + \frac{1}{\beta^S} \sum_k \sum_{s=0}^{S-1} \beta^s b^j_{t,t+s} \left(\frac{\beta}{1-\beta} \nu^j_{t+s+1} + (1-\beta)\, \bar{q}^k \right) \end{aligned}$$

because $\rho_0^l = 0$. Note that

$$\begin{aligned}
\sum_{s=0}^{S-1} \beta^s \sum_{m=0}^{s} \rho_m^l \tau_{t+m}^l &= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \sum_{m=s}^{S-1} \beta^m = \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \beta^s \sum_{m=0}^{S-1-s} \beta^m \\
&= \sum_{s=0}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s \left(1-\beta^{S-s+1}\right)}{1-\beta} \\
&= \sum_{s=1}^{S-1} \rho_s^l \tau_{t+s}^l \frac{\beta^s \left(1-\beta^{S-s+1}\right)}{1-\beta} \text{because } \rho_0 = 0 \\
&= \sum_{s=0}^{S-2} \beta^s \left(1-\beta^{S-s}\right) \rho_{s+1}^l \frac{\beta}{1-\beta} \tau_{t+s+1}^l \\
&= \sum_{s=0}^{S-1} \beta^s \left(1-\beta^{S-s}\right) \rho_{s+1}^l \frac{\beta}{1-\beta} \tau_{t+s+1}^l \text{ because } \rho_S \equiv 0
\end{aligned}$$

Then,

$$\begin{aligned}
w_t &= \sum_{s=0}^{S-1} \beta^s \left(\left(\varepsilon^l - \bar{r}^l\right) - \left(1-\beta^{S-s}\right) \rho_{s+1}^l \frac{\beta}{1-\beta} \tau_{t+s+1}^l \right) \\
&\quad + \sum_{s=0}^{S-1} \beta^s \sum_j a_{t,t+s}^j \left(\frac{\beta}{1-\beta} \tau_{t+s+1}^j + (1-\beta) \bar{p}^j \right) + \sum_{s=0}^{S-1} \beta^s \sum_k b_{t,t+s}^k \left(\frac{\beta}{1-\beta} \nu_{t+s+1}^k + (1-\beta) \bar{q}^k \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
E\left[w_t\right] &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l - \bar{r}^l + (1-\beta) \left(\left(1-\beta^{S-s}\right) \rho_{s+1}^l \bar{p}^l + \sum_{j=1}^{L} \tilde{a}_{t,t+s}^j \bar{p}^j + \sum_{k=1}^{K} b_{t,t+s}^k \bar{q}^k \right) \right) \\
Var\left[w_t\right] &= \frac{1}{\beta^{2S}} \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2s} Var\left[\sum_{j=1}^{L} \tilde{a}_{t,t+s}^j \tau_{t+s+1}^j + \sum_{k=1}^{K} b_{t,t+s}^k \nu_{t+s+1}^k \right]
\end{aligned}$$

where $\tilde{a}_{t,t+s}^l = a_{t,t+s}^l - \left(1-\beta^{S-s}\right) \rho_{s+1}^l$ and $\tilde{a}_{t,t+s}^j = a_{t,t+s}^j$ for all $j \neq l$. In a matrix form, this is re-written as

$$\begin{aligned}
E\left[w_t\right] &= \frac{1}{\beta^S} \sum_{s=0}^{S-1} \beta^s \left(\varepsilon^l - \bar{r}^l + (1-\beta) \left(\left(1-\beta^{S-s}\right) \rho_{s+1}^l \bar{p}^l + \begin{bmatrix} \tilde{\boldsymbol{a}}_{t,t+s} \\ \boldsymbol{b}_{t,t+s} \end{bmatrix}' \begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix} \right) \right) \\
Var\left[w_t\right] &= \frac{1}{\beta^{2S}} \frac{\beta^2}{(1-\beta)^2} \sum_{s=0}^{S-1} \beta^{2s} \begin{bmatrix} \tilde{\boldsymbol{a}}_{t,t+s} \\ \boldsymbol{b}_{t,t+s} \end{bmatrix}' \boldsymbol{\Sigma} \begin{bmatrix} \tilde{\boldsymbol{a}}_{t,t+s} \\ \boldsymbol{b}_{t,t+s} \end{bmatrix}
\end{aligned}$$

The first-order conditions yield

$$\begin{bmatrix} \tilde{\boldsymbol{a}}_{t,t+s} \\ \boldsymbol{b}_{t,t+s} \end{bmatrix} = \frac{(1-\beta)^3}{2\gamma\beta^{s+2}} \beta^S \boldsymbol{\Sigma}^{-1} \begin{bmatrix} \bar{\boldsymbol{p}} \\ \bar{\boldsymbol{q}} \end{bmatrix},$$

Plug back into the utility function:

$$\begin{aligned}
U &= \frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s\left(\varepsilon^l-\bar{r}^l\right)+\frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s\left(1-\beta^{S-s}\right)\rho_{s+1}^l(1-\beta)\bar{p}^l \\
&+\frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s(1-\beta)\left(\frac{(1-\beta)^3}{2\gamma\beta^{s+2}}\beta^S\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}'\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}\right) \\
&-\gamma\frac{1}{\beta^{2S}}\frac{\beta^2}{(1-\beta)^2}\sum_{s=0}^{S-1}\beta^{2s}\frac{(1-\beta)^3}{2\gamma\beta^{s+2}}\frac{(1-\beta)^3}{2\gamma\beta^{s+2}}\beta^{2S}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix} \\
&= \frac{1}{\beta^S}\sum_{s=0}^{S-1}\beta^s\left(\varepsilon^l-\bar{r}^l+(1-\beta)\left(1-\beta^{S-s}\right)\rho_{s+1}^l\bar{p}^l\right)+S\frac{(1-\beta)^4}{4\gamma\beta^2}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}'\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}
\end{aligned}$$

■

Proof of Proposition 2

The demands for assets excluding the hedging motive can be written as

$$\begin{bmatrix}\tilde{\boldsymbol{a}}_{t-s,t}\\ \boldsymbol{b}_{t-s,t}\end{bmatrix}=\frac{(1-\beta)^3}{2\gamma\beta^{s+2}}\beta^S\boldsymbol{\Sigma}^{-1}\begin{bmatrix}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{bmatrix}$$

for $s=0,...,S-1$. Since all agents have the same portfolio and there is a measure one of agents in each cohort, the aaggregate portfolio demand for assets (excluding the hedging motive), is

$$\sum_{s=0}^{S-1}\begin{bmatrix}\tilde{\boldsymbol{a}}_{t-s,t}\\ \boldsymbol{b}_{t-s,t}\end{bmatrix}=\left(1+\frac{1}{\beta}+\frac{1}{\beta^2}+...+\frac{1}{\beta^{S-1}}\right)\begin{bmatrix}\tilde{\boldsymbol{a}}_{t,t}\\ \boldsymbol{b}_{t,t}\end{bmatrix}=\frac{1-\beta^S}{(1-\beta)\beta^{S-1}}\begin{bmatrix}\tilde{\boldsymbol{a}}_{t,t}\\ \boldsymbol{b}_{t,t}\end{bmatrix}.$$

The housing demand in city l by people with age s due to the hedging motive is

$$\left(1-\beta^{S-s}\right)\int_{\boldsymbol{\varepsilon}}\int_{\boldsymbol{\rho}}I_{\boldsymbol{\varepsilon},\boldsymbol{\rho}}^l\rho_{s+1}^l\phi(\boldsymbol{\varepsilon},\boldsymbol{\rho})\,d\boldsymbol{\varepsilon}d\boldsymbol{\rho}.$$

It is then easy to see that the total housing demand in city l due to the hedging motive is SR^l, where R^l is defined as in the statement of the proposition.

The supply of houses minus the hedging demand in every city is $S(\boldsymbol{n}-\boldsymbol{R})$. The housing market clearing condition is therefore

$$\frac{1-\beta^S}{(1-\beta)\beta^{S-1}}\tilde{\boldsymbol{a}}_{t,t}=S(\boldsymbol{n}-\boldsymbol{R}).$$

Hence

$$\tilde{\boldsymbol{a}}_{t,t}=S\frac{(1-\beta)\beta^{S-1}}{\left(1-\beta^S\right)}(\boldsymbol{n}-\boldsymbol{R})$$

and by analogy

$$\boldsymbol{b}_{t,t}=S\frac{(1-\beta)\beta^{S-1}}{\left(1-\beta^S\right)}\boldsymbol{z}.$$

Plugging in the demand function yields a solution to the housing and stock risk premia yields

$$S\frac{(1-\beta)\,\beta^{S-1}}{\left(1-\beta^{S}\right)}\left[\begin{array}{c}(\boldsymbol{n}-\boldsymbol{R})\\ \boldsymbol{z}\end{array}\right]=\frac{(1-\beta)^{3}}{2\gamma\beta^{2}}\beta^{S}\boldsymbol{\Sigma}^{-1}\left[\begin{array}{c}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{array}\right]$$

$$2\gamma S\frac{\beta}{(1-\beta)^{2}\left(1-\beta^{S}\right)}\boldsymbol{\Sigma}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]=\left[\begin{array}{c}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{array}\right].$$

■

Proof of Corollary 3

Note that

$$\begin{aligned}Cov\,(l,M) &= \frac{1}{Q}\left(\sum_{m=1}^{L}(n^{m}-R^{m})\,Cov\left(\tau^{l},\tau^{m}\right)+\sum_{k=1}^{K}z^{k}Cov\left(\tau^{l},\nu^{k}\right)\right)\\ Var\,(M) &= \frac{1}{Q^{2}}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]^{\prime}\boldsymbol{\Sigma}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]\end{aligned}$$

The expected return of a zero-cost market portfolio containing one unit of M is given by

$$\begin{aligned}\bar{p}^{M} &= \frac{\sum_{l=1}^{L}\left(n^{l}-R^{l}\right)\bar{p}^{l}+\sum_{k=1}^{K}z^{k}\bar{q}^{k}}{\sum_{l=1}^{L}\left(n^{l}-R^{l}\right)+\sum_{k=1}^{K}z^{k}}=\frac{1}{Q}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]^{\prime}\left[\begin{array}{c}\bar{\boldsymbol{p}}\\ \bar{\boldsymbol{q}}\end{array}\right]\\ &= 2\gamma\frac{\beta}{(1-\beta)^{2}\left(1-\beta^{S}\right)}S\frac{1}{Q}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]^{\prime}\boldsymbol{\Sigma}\left[\begin{array}{c}\boldsymbol{n}-\boldsymbol{R}\\ \boldsymbol{z}\end{array}\right]=2\gamma\frac{\beta}{(1-\beta)^{2}\left(1-\beta^{S}\right)}SQVar\,(M)\,.\end{aligned}$$

Similarly

$$\begin{aligned}\bar{p}^{l} &= 2\gamma\frac{\beta}{(1-\beta)^{2}\left(1-\beta^{S}\right)}S\left(\sum_{m=1}^{L}(n^{m}-R^{m})\,Cov\left(\tau^{l},\tau^{m}\right)+\sum_{k=1}^{K}z^{k}Cov\left(\tau^{l},\nu^{k}\right)\right)\\ &= 2\gamma\frac{\beta}{(1-\beta)^{2}\left(1-\beta^{S}\right)}SQCov\,(l,M)\end{aligned}$$

Hence, we can write

$$\frac{\bar{p}^{l}}{\bar{p}^{M}}=\frac{Cov\,(l,M)}{Var\,(M)}.$$

The proof for k follows similar lines and is omitted. ■

Proof of Proposition 4

It is immediate to see that a solution to $\boldsymbol{\nu}\left(\hat{\boldsymbol{r}}\right)=\boldsymbol{n}$ constitutes a linear stationary equilibrium: no agent wants to change location, by definition $r_{t}^{l}=y_{t}^{j}+\bar{r}^{l}$, and the conditions for Propositions 1 and 2 are satisfied.

To prove existence, note that $\nu^{l}\left(\hat{\boldsymbol{r}}\right)$ is continuous in $\hat{r}$ and that $\lim_{\hat{r}^{l}\to-\infty}\nu^{l}\left(\hat{r}\right)=1$ and $\lim_{\hat{r}^{l}\to\infty}\nu^{l}\left(\hat{r}\right)=0$.

To prove uniqueness, suppose that the system $\boldsymbol{\nu}(\hat{\boldsymbol{r}}) = \boldsymbol{n}$ has two distinct solutions $\bar{\boldsymbol{r}}$ and $\bar{\boldsymbol{r}}'$. Assume without loss of generality that there exists a non-empty set of cities $\tilde{L}$ for which $(\bar{r}')^l < \bar{r}^l$. The set of agents who locate in a city in $\tilde{L}$ is given by

$$\left\{(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) : \max_{l \in \tilde{L}} \left(\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^l\right) \geq \max_{j \notin \tilde{L}} \left(\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^j\right)\right\}$$

Note however, that this set must become strictly larger when $\bar{r}$ is replaced by $\bar{r}'$, because all elements $\bar{u}^l(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^l$ on one side become strictly larger and all elements $\bar{u}^j(\boldsymbol{\varepsilon}, \boldsymbol{\rho}) - \hat{r}^j$ on the other side do not become larger. Hence, more agents will want to locate in cities in $\tilde{L}$, but this is impossible as the mass of agents who locate in $\tilde{L}$ must sum up to $\sum_{l \in \tilde{L}} n^l$ in both solutions. ■

Proof of Proposition 5

In the limit,

$$\bar{u}^l = \varepsilon^l + \frac{1-\beta}{1-\beta^S}\left(1 - (S+1)\beta^S + S\beta^{S+1}\right)\bar{p}^l \tilde{\rho}^l_{s+1}.$$

As $\tilde{\rho}^l_{s+1}$ are the same for all agents and the $\tilde{\varepsilon}$'s are uniformly distributed, we write

$$\nu^l(\hat{r}) = \int_{\tilde{\boldsymbol{\varepsilon}} : \tilde{\varepsilon}^l - \hat{r}^l = \max_m (\tilde{\varepsilon}^m - \hat{r}^m)} d\tilde{\boldsymbol{\varepsilon}}$$

This problem is symmetric in l. Hence, the unique solution to $\nu^l(\hat{r}) = \frac{1}{L}N$ for $l = 1, ..., L$ must be symmetric in l, namely $\bar{r}^l = \bar{r}$. This means that the mass of agents who locates in the city is $\nu^0(\hat{r}) = \bar{r}^L$. This implies $\bar{r} = (1-N)^{\frac{1}{N}}$. The equilibrium rent is given by

$$r^l = \bar{r} + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l \sum_{s=1}^{S}\left(1 - \beta^{S-s+1}\right)\tilde{\rho}^l_s.$$

■

Proof of Proposition 6

Given the assumed relationship between ρ^l_s and s,

$$D^l_s = \left(1 - \beta^{S-s}\right) k \frac{s}{S-1}.$$

It is easy to see that $\lim_{s \to 0^+} D^l_s = 0$ and $D^l_S = 0$. Next note that

$$\frac{d}{ds}D^l_s = \left(1 - \beta^{S-s}\right) k \frac{1}{S-1} + \log\beta \cdot \beta^{S-s} k \frac{s}{S-1}$$

and

$$\frac{d^2}{ds^2}D^l_s = 2\log\beta \cdot \beta^{S-s} k \frac{1}{S-1} - (\log\beta)^2 \beta^{S-s} k \frac{s}{S-1} < 0$$

■

Proof of Proposition 7

Consider any allocation of agents to cities. Suppose an agent with $\left(\varepsilon^l, \varepsilon^m\right)$ is allocated to city l and another agent with $\left(\left(\varepsilon^l\right)', (\varepsilon^m)'\right)$ is allocated to m. Swapping agents does not increase total expected production if and only if

$$\varepsilon^l - \left(\varepsilon^l\right)' \geq \varepsilon^m - (\varepsilon^m)'$$

If this holds true for every agent, one can find a unique vector $\bar{\boldsymbol{\varepsilon}}$ such that the condition in the statement is satisfied.

Next, we ask under what conditions market equilibrium leads to productive efficiency. ■

Proof of Proposition 8

According to proposition 4, in a linear stationary equilibrium agents are assigned to cities according to

$$\bar{u}^l\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}\right) = \varepsilon^l + \frac{(1-\beta)^2}{1-\beta^S}\bar{p}^l \sum_{s=1}^{S}\left(1-\beta^{S-s+1}\right)\rho_s^l$$

suppose that an agent with a certain $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ locates in city l. His next preferred city is m, and the utility difference between the two cities is given by

$$D = \bar{u}^l\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}\right) - \bar{u}^m\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}\right),$$

where D is sufficiently low. Consider another agent with $(\boldsymbol{\varepsilon}', \boldsymbol{\rho}')$ which is identical to $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ except that $(\varepsilon')^l = \varepsilon^l + \delta$ and $\sum_{s=1}^{S}\left(1-\beta^{S-s-1}\right)(\rho')_s^l = \sum_{s=1}^{S}\left(1-\beta^{S-s-1}\right)\rho_s^l - \alpha$. Given a positive $\bar{p}^l$, it is always possible to find α and δ such that $\bar{u}^l\left(\boldsymbol{\varepsilon}', \boldsymbol{\rho}'\right) < \bar{u}^m\left(\boldsymbol{\varepsilon}', \boldsymbol{\rho}'\right)$. By the assumption that ϕ has full support, agents with $(\boldsymbol{\varepsilon}, \boldsymbol{\rho})$ and $(\boldsymbol{\varepsilon}', \boldsymbol{\rho}')$ exist. The sum of expected outputs of the two agents would be higher if the agents switched cities. ■

Proof of Proposition 10

The first part is immediate. If $\boldsymbol{E}$ is an allocation and prices are linear, then every agent is using $\bar{u}^l\left(\boldsymbol{\varepsilon}, \boldsymbol{\rho}, \boldsymbol{E}^l\right) - \bar{r}^l$ as a criterion to locate and rents must equate demand and supply. The argument for the uniqueness of $\bar{r}^l$ (given $\boldsymbol{E}$) is unchanged from the proof of Proposition 4. ■