Side-Communication Yields Efficiency of Ascending Auctions: The Two-Items Case *

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Abstract

We analyze the realistic, popular format of an ascending auction with anonymous item-prices, when there are two items that are substitutes. This auction format entails increased opportunities for bidders to coordinate bids, as the bidding process is longer, and since bidders see the other bids and can respond to various signaling. This has happened in many real auctions, e.g. in the Netherlands' 3G Telecom Auction, and in the FCC auctions in the US.

While on the face of it, such a bidding behavior seems to harm economic efficiency, we show that side-communication may actually improve the social efficiency of the auction: We describe an ex-post (sub-game perfect) equilibrium, that uses limited side-communication, and is ex-post efficient. In contrast, without side-communication, we show that there is no ex-post equilibrium which is ex-post efficient in the ascending auction.

In the equilibrium strategy we suggest, bidders start by reporting their true demands at the first stages of the auction, and then perform a single demand reduction at a certain concrete point, determined using a single message exchanged between the bidders. We show that this limited collusion opportunity resolves the strategic problems of myopic bidding, and, quite surprisingly, improves the social welfare instead of harming it.

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1 Introduction

Auctions are often used to improve the social efficiency in one-sided markets, e.g. for sales of cellular licences, in electricity markets, in B2B transactions, and more. The format of these auctions is many times some variation of an ascending auction, where item prices are being iteratively raised as a response to bidders' demand reports. Although this format significantly expands the strategic choices of the bidders, compared to sealed-bid one-shot mechanisms, a complete understanding of its various strategic aspects is still missing.

One problematic aspect of the ascending auction format is the increased opportunity for bidders to coordinate bids, as the bidding process is longer, and since bidders see the other bids and can respond to various signaling. In the Netherlands' 3G Telecom Auction, for example, one bidder firm stopped bidding after receiving a letter from another bidder firm, threatening legal action for damages if they continued to bid (Klemperer, 2002), and it has been argued that a shorter auction process might have prevented this event. Cramton and Schwartz (2000) report on a conceptually-similar event in some of the FCC auctions in the US: while most bids were a multiple of 1000 US dollars, occasionally bids included single dollar quantities. These bids were (most probably) coordination signals between the bidders, to lower competition. Intuitively it may seem that this reduced competition harms economic efficiency. In fact the response of the auction organizers, in both cases, was to refine the auction rules in order to prevent such cases from reoccurring.

This paper shows that the completely opposite explanation to this phenomenon is also possible: that signaling is an important ingredient in an *efficient* ex-post equilibrium of the ascending auction. We study a setting of two items with no-complementarities and n players with quasi-linear utilities. We describe a socially efficient ex-post equilibrium of the ascending auction game that has the following structure: initially players bid in a straight-forward (myopic) way, reporting their true demands; but at some carefully-chosen point, some of the bidders artificially reduce demand, practically ignoring one of the items. This decision point relies on a one-bit signal that is communicated between bidders. This pattern is very similar to the collusion pattern observed in the two examples described above, however in our case social efficiency is not harmed by this strategic behavior, in fact the opposite is true.[1] To complete the picture, we additionally show that no ex-post equilibrium that is ex-post efficient exists, if communication is not used, even if there are only two players.[2]

A revenue-equivalence argument[3] implies that in every efficient ex-post equilibrium, players' payments must be the VCG payments. Using this, Gul and Stacchetti (2000) implicitly explain the necessity of a demand reduction at some point during the ascending auction process, if an

[1]Perhaps even more surprising is the fact that, in the equilibrium path we describe, the player with the smaller demand performs the demand reduction, following a signal from a player with a larger demand. This strengthens the connection to the realistic examples mentioned above.

[2]Gul and Stacchetti (2000) prove this when there are at least four items and at least three players.

[3]See for example Milgrom and Segal (2002) and Heydenreich, Muller, Uetz and Vohra (2009).

	$\{a\}$	$\{b\}$	$\{a,b\}$
v_1	10	4	12
v_2	10	5	14

phase	D_1	D_2	p_a	p_b
1	$\{a,b\}$	$\{a,b\}$	$0 \to 2$	$0 \to 2$
2	$\{a\}$	$\{a,b\}$	$2 \to 8$	2 (unchanged)
3	$\{a\}$ or $\{b\}$	$\{a,b\}$	$8 \to 9$	$2 \to 3$
4	$\{a\}$	$\{b\}$	9	3

Figure 1: Example. The table on the left shows example valuations for two players; the table on the right shows the price pattern for these valuations, with truthful demand reporting.

equilibrium behavior is assumed. They show that if players always report their true demands, then the final outcome is a Walrasian equilibrium. They additionally show that the minimal possible Walrasian prices can be strictly larger than VCG prices. Thus, to reach VCG prices (a necessity in every efficient ex-post equilibrium), a demand reduction at some point in the process may be required. In other words, truthful demand reporting sometimes causes excess competition that essentially contradicts ex-post efficiency in equilibrium. We pin-point a simple way to achieve the required demand reduction, via side-communication. Thus what initially seems as an undesirable collusion phenomenon may actually be a way-out of the excess competition embedded in the auction.

The example in Figure 1 helps making this argument more concrete. The table on the left gives example values for two players, and the right table describes the course of the ascending auction when players truthfully reveal their demands. As the table shows, initially (phase 1) both players demand both items and so both prices ascend. Player 1 stops demanding item b when its price crosses 2 (phase 2), as the profit from a alone becomes larger than the profit from a and b together. The price of item a continues to ascend, and when it reaches 8, player 1 becomes indifferent between items a and b. Thus a slight increase in a's price leads her to demand b, and then a slight increase in b's price leads her to again demand a, and so on and so forth (phase 3). This terminates when player 2 stops demanding a, when its price reaches 9. The price of item b at this point is 3 (phase 4).

One can observe that in this specific situation player 2 is not playing a best response, as she could lower her payment from 3 to 2 by removing item a from her demand already in the beginning of the third phase. This will terminate the process when a's price becomes 8, and b's price at that point is still 2. Of-course, player 2 cannot know this in advance. Trying to speculate might even cause her to stop demanding the incorrect item, as one can easily construct other examples where player 1 wins item b instead of a. In our equilibrium strategy, the players exchange a single message when the third phase starts, and as a result player 1 will stop demanding item b. This prevents item b from being over priced, which benefits player 2. This also ensures that player 2 does not need to make possibly incorrect speculations on which item to focus, which benefits player 1.

The full analysis, and a more detailed discussion, are given in the body of the paper. The bottom line is that, from the point of view of the players, this is a simple strategic equilibrium behavior that guarantees no ex-post regret, and from the point of view of the auctioneer, enabling some form of restricted signaling (as cheap talk) may help achieving ex-post efficiency.

We should remark at this point that there are several other possible formats of indirect mechanisms that are known to obtain ex-post efficiency in ex-post equilibrium. Ausubel (2006) describes an iterative auction with anonymous and linear prices that ascend or descend as a response to bidders' demand reports. Sincere bidding is an equilibrium strategy in this mechanism, and the equilibrium outcome is ex-post efficient. Alternatively, one can reach the Vickrey outcome by an ascending auction with non-anonymous and non-linear prices, as for example Parkes (1999) and Ausubel and Milgrom (2002) show.

Thus, our result should not be interpreted as saying that the only solution to the problematic aspects of the simple ascending auction format is to allow side-communication. Instead, from a conceptual point of view, our result suggests an alternative interpretation to the observed phenomenon of signaling in ascending auctions, with the conclusion that side-communication do not *necessarily* lead to inefficiency. This interpretation is strengthened by the actual equilibrium behavior that we find, which seems similar to what is seen in reality, in which the bidder with the larger demand signals the bidder with the smaller demand to perform a demand reduction, and the smaller bidder finds it in her best interest to follow this signal.

If the mechanism designer insists on avoiding any allowable side-communication, there is always the possibility to design a new ascending auction that internalizes the equilibrium behavior described here. This ascending auction will generate the necessary signaling and the resulting demand reduction, instead of giving the bidders the opportunity to do so. This is a standard revelation principle trick, that conceptually suggests a forth possible way to achieve efficiency: requiring players to answer queries that are slightly more detailed than simple demand reports, but strictly maintaining the other requirements of ascending-prices that are anonymous and linear (and no-side-communication). This way, our result contributes to a long-standing agenda in auction theory, of understanding the various possible indirect mechanisms that implement the VCG outcome.[4]

1.1 Related literature

A broad look on the advantages and disadvantages of ascending auctions is given by Milgrom (2000). This paper also discusses the possibility of collusion in a simple complete-information model. Ausubel and Schwartz (1999) and Ausubel and Cramton (2002) introduce and formally study the concept of demand reduction, in the context of auctions with many identical items.

Several papers study signaling and collusion in ascending auctions, focusing on the inefficiencies that such a collusion can create. Brusco and Lopomo (2002) show that when players have certain prior beliefs, an inefficient Bayesian equilibrium can be formed. Engelbrecht-Wiggans and Kahn (2005) independently identify a similar phenomenon. Albano, Germano and Lovo (2006) and Zheng (2006) show, using two different technical settings, that a clock (Japanese) auction is less prone

[4]This is important for various reasons, e.g. since these auctions better preserve privacy considerations, since they better fit settings where bidders do not have quasi-linear utilities, or since they make it harder for a dishonest auctioneer to artificially increase prices, to name just a few possible reasons.

to signaling, compared to an English auction where players decide by how much to increase their offer. These works consider a two-item setting, as we do here (excluding only Engelbrecht-Wiggans and Kahn (2005), that study a two-bidder setting). While they collectively establish the point that signaling and collusion can create inefficiencies, the other extreme of completely disallowing communication also leads to ex-post inefficient outcomes when players are strategic, as Goeree and Lien (2009) recently show. We suggest as an answer to allow some form of carefully-restricted communication.

As mentioned above, another possible way to reach the Vickrey outcome by an ascending auction (and by this ensuring its incentive-compatibility) is to allow non-anonymous and non-linear prices. The literature contains by now a wide and systematic knowledge of the possibilities and impossibilities of this approach. For example, De Vries, Schummer and Vohra (2007) give such an ascending auction when bidders are substitutes, by developing a primal-dual algorithm for the linear program formulation of Bikhchandani and Ostroy (2002). Lamy (2010) shows that the requirement that bidders are substitutes is necessary. When this condition does not hold, Mishra and Parkes (2007) suggest to add a final step that discounts prices, by this reaching the VCG outcome, and Lamy (2010) reduces the amount of information that needs to be revealed when such a price discount is used. Blumrosen and Nisan (2010) explain why the above papers need to assume both non-anonymity and non-linearity of prices, by showing various impossibility results for more restrictive price structures.

Collusion as cheap talk in sealed-bid auctions was also studied. Matthews and Postlewaite (1989) study a double-auction setting and show that pre-communication significantly expands the set of equilibria. McAfee and McMillan (1992) study the effect of side-transfers and pre-communication on the possibility of successful collusion in first-price auctions. Collusion in repeated auctions under various assumptions on information and communication is studied by Fudenberg, Levine and Maskin (1994), Athey and Bagwell (2001), and Skrzypacz and Hopenhayn (2004).

Collusion behavior in ascending auctions is studied in parallel by a long line of works in behavioral economics, as early as Isaac and Plott (1981). For example, recently, Kwasnica and Sherstyuk (2007) systematically show how collusion evolves in an ascending auction for two items, even without implicit communication, and Brown, Plott and Sullivan (2009) show that descending auctions are less prone to collusion than ascending auctions.

We have mentioned above two relatively early studies that analyze actual data from simultaneous ascending auctions. A more recent reference that represents this line of works is Bulow, Levin and Milgrom (2009).

1.2 Paper organization

Our formal setting is described in Section 2, followed by a closer look at the problematic aspects of truthful demand reporting in section 3. Section 4 describes the new proposed equilibrium and its analysis. Section 5 shows that without communication there does not exist an ex-post efficient equilibrium. Section 6 concludes. Several appendices complete the technical details.

2 The Setting

An auctioneer sells two items $\{a, b\}$ to n bidders. Each bidder i assigns a value $v_i(S)$ to any subset S of the items, where it is assumed that $v_i(a)+v_i(b) \geq v_i(ab) \geq \max(v_i(a), v_i(b))$ (the first inequality is a no-complementarities condition[5]; the second is a free-disposal assumption), and that $v_i(\emptyset) = 0$. Player i's valuation is known only to her. The player's utility when she receives a subset S and pays some price p is $v_i(S) - p$, and she acts strategically in order to maximize it.

To formally define the ascending auction we need some notation. For a price vector $p = (p_a, p_b)$ (p_x is the price of item $x \in \{a, b\}$), let $D_i(p) = \text{argmax}_{S \subseteq \{a,b\}}\{v_i(S) - p(S)\}$ be the demand of player i under prices p (where $p(S) = \sum_{x \in S} p_x$). Note that $D_i(p)$ can contain the two sets $\{a\}$ and $\{b\}$, as in the example of Figure 1 (player 1 in the third phase).

We say that there is no-over-demand at price p if items can be assigned to players so that each player i receives a subset of items $S_i \in D_i(p)$. Otherwise there is over-demand at price p. An item x is demanded by player i at price p if there exists $S_i \in D_i(p)$ such that $x \in S_i$, and $S_i \setminus \{x\} \notin D_i(p)$. For example, if $D_i(p) = \{\{a\}, \{a, b\}\}$ then only item a is demanded by player i (player i is indifferent about receiving b on top of a at prices p). An item x is over-demanded at p if there is over-demand at p, and there exist two players that demand x at p.[6]

For a set of items D we define 1_D to be a 0-1 vector such that $(1_D)_x = 1$ if and only if $x \in D$. Thus for a price vector p and some real number $\delta > 0$, $p + \delta \cdot 1_D$ denotes a price increase of δ for all items in D. Another useful notation is the marginal value of a given b, $v_i(a|b) = v_i(ab) - v_i(b)$.

We analyze a standard simultaneous ascending clock auction ("SAA") format: prices gradually ascend while players adjust demands until no item is over-demanded. Formally,

Definition 1 (The ascending auction)**.** *Initialize $p_a \leftarrow 0, p_b \leftarrow 0$, and perform:*

- *Players report their demands at price p. If there is no-over-demand, exit loop.*
- *Otherwise, let D be the set of over-demanded items, and δ^* be the infimum over $\delta > 0$ such that there exists a player i with $D_i(p) \neq D_i(p + \delta \cdot 1_D)$.*
- *Set $p \leftarrow p + \delta^* \cdot 1_D$ and repeat.*

Upon termination, each player i receives a demanded set $S_i \in D_i(p)$ and pays $p(S_i)$.

[5] With two-items, this is also equivalent to the "gross-substitutes" condition assumed in Gul and Stacchetti (2000).

[6] Gul and Stacchetti (2000) define a *minimal* set of over-demanded items, which is a more subtle definition. They need this since it might be possible that an item belongs to demand sets of two different players, but there is no-over-demand (using our terminology). E.g. when there are two players and $D_1(p) = \{\{a\}\}, D_2(p) = \{\{a\}, \{b\}\}$. For two items our simple definitions are sufficient to describe the ascending auction, and we do not need the more complicated definitions.

For two items, this auction is equivalent to the English auction of Gul and Stacchetti (2000). In particular, they show that when players bid myopically, i.e. report true demands throughout, this auction terminates in a minimal Walrasian equilibrium.[7] They also show that payments at this equilibrium are not smaller than VCG payments.[8, 9]

This ascending auction is viewed as a game of incomplete information. Thus, as usual, a strategy is a function of the player's private valuation and the history of the auction, which outputs a demand correspondence. A tuple of strategies forms an ex-post equilibrium if each strategy is best-response to the other strategies, for every tuple of players' values. This yields the strong property of no ex-post regret. In addition one may recall that an ex-post equilibrium is also a Bayesian-Nash equilibrium, for any possible prior. Since the ascending auction is an extensive-form game it is preferable to use the notion of ex-post sub-game perfect equilibrium, which in our context translates to requiring that a tuple of strategies will form an ex-post equilibrium for any initial prices p of the auction (and not just for initial prices $p = (0, 0)$).

As discussed in the Introduction, truthful demand reporting throughout does not constitute an ex-post equilibrium of this ascending auction. We will construct such an equilibrium by allowing players to exchange messages in the course of the auction. But first we need to better understand why truthful demand reporting fails.

3 Truthful Demand Reporting

In the heart of the construction of our equilibrium strategy lies a careful analysis of the course of the auction with truthful demand reporting. To better understand why truthfulness is not an equilibrium, consider again the example in Figure 1. This example includes two players, and we will show below that the strategic problem of truthful demand reporting arises only when there remain exactly two active players in the auction.[10] The course of the auction process in the example contains three phases:

1. **(2-items)** Both players demand both items. This phase continues until $p_b = v_1(b|a) = v_1(ab) - v_1(a)$.

[7] A Walrasian equilibrium is an allocation $S_1, ..., S_n$ of the items to the players (player i receives S_i), and item prices $p = (p_a, p_b)$, such that (1) $S_i \in D_i(p)$ for every player i, and (2) $\cup_i S_i = \{a, b\}$.

[8] The VCG mechanism is described e.g. in the text book by Mas-Collel, Whinston and Green (1995). In short, in VCG, each player i reports a valuation $\tilde{v}_i(\cdot)$, the chosen allocation $S_1, ..., S_n$ is the one that maximizes the sum $\sum_i \tilde{v}_i(S_i)$, and each player i pays $\sum_{j \neq i} \tilde{v}_j(S_j^{(-i)}) - \sum_{j \neq i} \tilde{v}_j(S_j)$, where the allocation $\{S^{(-i)}\}_{j \neq i}$ maximizes the term $\sum_{j \neq i} \tilde{v}_j(S_j^{(-i)})$. (Thus i's payment may be viewed as the aggregate damage she causes to the other players). It is a dominant strategy of each player to declare her true valuation in the VCG mechanism, i.e. to declare $\tilde{v}_i(\cdot) = v_i(\cdot)$.

[9] Interestingly, if the valuations are all unit-demand, or are all additive, then minimal Walrasian prices are always equal to VCG prices. However even a combination of these simple valuation formats causes Walrasian prices to be sometimes strictly higher than VCG prices.

[10] Conceptually, when three or more players are competing for two items, the competition is "real" and does not create bubble prices. When there remain two players, the example shows how such a bubble can be formed.

2. **(1-item)** Player 1 demands one item, a, and only the price of this item increases. This phase continues until the prices satisfy $v_1(a) - p_a = v_1(b) - p_b$.

3. **(jump)** Player 1 demands $\{\{a\}, \{b\}\}$. We term this step "jump" as in the practical auction version the player's demand constantly switches back and forth between $\{a\}$ and $\{b\}$, creating a jump-effect. In our formal auction this is captured by an indifference between $\{a\}$ and $\{b\}$.

Consider the case of an arbitrary number of players, and suppose that the auction process with truthful demand reporting reaches a price vector $p^0 = (p_a^0, p_b^0)$ such that two players demand $\{a, b\}$ at p^0 while all other players demand the empty set at p^0. It is a useful exercise to show that these three phases always occur in the specified order (though some of them may be empty) after p^0. To see this, assume without loss of generality that the two remaining active players are 1 and 2. Since all players truthfully reveal their demand we can easily describe the price path:

The two-items phase. Since both players demand both items at p^0, the two items were over-demanded from the beginning of the auction and we have $p_a^0 = p_b^0$. Additionally we have $v_i(a|b) \geq p_a^0$ and $v_i(b|a) \geq p_b^0$ for $i = 1, 2$. During this phase both items' prices will keep increasing, until they reach $\min_{i=1,2}(v_i(a|b), v_i(b|a))$. Assume without loss of generality that the minimal term is $v_1(b|a)$. Then when the price reaches a point p^1 with $p_a^1 = p_b^1 = v_1(b|a)$, player 1 will change her demand. If player 1 demands the empty set, the auction ends and player 2 wins both items. This can happen only if $v_1(b|a) = v_1(a|b) = v_1(a) = v_1(b) = p_a^1 = p_b^1$. Thus in this case player 2 pays $v_1(ab)$, and so if the auction terminates in the two-items phase, it reaches the VCG outcome.

The one-item phase. If the auction continues then it must be that $v_1(b|a) < v_1(a|b)$, and $D_1(p^1) = \{a\}$, while player 2 still demands both items. Thus in this phase only the price of item a is increased. Let p^2 be the next point where one of the players changes her demand. Note that $p_b^2 = p_b^1 = v_1(b|a)$. If it is player 2 who changes her demand first then since only a's price is increasing her new demand must be $D_2(p^2) = \{b\}$,[11] and $p_a^2 = v_2(a|b)$. At this point the auction ends, player 1 receives a and player 2 receives b, and they both pay their VCG prices. If on the other hand player 1 changes her demand first and now demands the empty set then $p_a^2 = v_1(a)$. In this case player 2 receives both items and pays $v_1(a) + v_1(b|a) = v_1(ab)$. Thus here as well, if the auction terminates, it reaches the VCG outcome.

The jump phase. If the auction continues then it must be that at the end of the previous phase player 1 has $v_1(a) - p_a^2 = v_1(b) - p_b^2 > 0$. Therefore in this phase player 2 still demands both items and player 1 demands $\{\{a\}, \{b\}\}$, and the prices of both items again increase together. Since $p_b^2 = v_1(b|a)$ we have that $p_a^2 = v_1(a|b) < v_1(a)$. Let p^3 be the next point of a demand change. If it is player 1 who changes her demand first then she must now demand the empty set,

[11] This follows since the valuations are gross-substitutes. One can explicitly verify this: $\{a\}$ cannot be a demanded set since previously $\{a, b\}$ was preferred over $\{a\}$ and only a's price increased; $\{b\}$ is preferred over the empty set since previously we had $p_b < v_2(b|a) \leq v_2(b)$ (as $\{a, b\}$ was demanded) and b's price did not increase.

and $v_1(a) - p_a = v_1(b) - p_b = 0$. In this case player 2 wins both items and pays $v_1(a) + v_1(b)$ which is strictly larger than her VCG price. If player 2 changes her demand first then her new demand must be a singleton item, and at this point the auction ends since player 1 is indifferent between the two items. Player 2 again pays more than her VCG price.

This case analysis brings us to the following conclusion:

Lemma 1. *Suppose all players truthfully report their demand, and there are two players that demand $\{a, b\}$ after all other players quit the auction. Then the auction terminates in the VCG outcome if and only if there was no jump phase in the auction.*

By Gul and Stacchetti (2000) we know that when all players truthfully report their demand, the ascending auction reaches minimal Walrasian prices. Thus the lemma implies (for the special case it considers) that VCG payments are equal to minimal Walrasian payments if and only if there was no jump during the ascending auction. In other words, when entering the jump phase, players can realize that continuing with truthful demand reporting will lead to an undesired outcome for at least one of the players, and hence may want to change their strategies. Even more importantly, players can be certain that the only way for the auction to end in prices higher than VCG prices is because a jump phase occurs. This turns out to be true in general, as the next claim shows.

Lemma 2. *Fix valuations $v_1, ..., v_n$, and let p^W denote the minimal Walrasian price vector for this instance. Assume that for at least one player, the Walrasian price that she pays for the bundle she receives in the efficient allocation is different than her VCG price. Then the ascending auction with truthful demand reporting, that starts at some arbitrary price vector $p < p^W$ (where the inequality is coordinate-wise), terminates in a "jump phase", in which: (1) only two players i, j have non-empty demand, (2) player j demands $\{a, b\}$, and (3) player i demands $\{\{a\}, \{b\}\}$.*

The proof of this Lemma uses more subtle arguments than above, and is given in Appendix A.

The example in Figure 1 illustrates this state of affairs: the VCG prices in this example are 9 for player 1 and 2 for player 2. One can verify that until the jump phase, prices in the auction are below VCG prices, and the jump phase causes the price of item b to exceed its VCG price. As mentioned in the Introduction, player 2 may try to avoid this price increase by avoiding the jump phase. A successful guess on which item to focus will increase her utility, but an unsuccessful guess may harm both players' utilities, as well as the resulting efficiency.

We will show that it is possible to fix this problematic aspect of the auction by performing a "demand reduction" when the jump phase starts. The player that intends to jump (i.e. demands $\{\{a\}, \{b\}\}$) focuses on demanding just one of the items. The choice which item to drop and ignore depends on a single message exchanged with the other active player. We will show that this strategy forms an ex-post efficient equilibrium.

4 The Equilibrium Strategy

We construct equilibrium strategies in which the players almost always report their true demand. The only exception is when there are only two active players, in a jump phase. In this case, the non-jumping player signals the jumping player to "lower competition", and indicates on which item to focus. The jumping player finds it in her best interest to actually follow this signal.

Definition 2 (The signaling strategy)**.** *Given prices $p = (p_a, p_b)$, the demand report of player i is:*

1. *If there are three or more active players or if there are two active players and $D_i(p)$ does not contain $\{\{a\}, \{b\}\}$ then i reports the true demand.*

2. *Otherwise (there are two active players and $\{\{a\}, \{b\}\} \in D_i(p)$), i sends a message "entering jump phase" to the other active player, j.*

   If player j answers "focus on x" ($x \in \{a, b\}$), player i reports the demand $\{x\}$ until $p_x = v_i(x)$, and then quits the auction. If player j does not send a valid answer, i continues to report the true demand.

3. *If another player, j, sends a "jump" message, then if $v_i(a) - v_i(b) \geq p_a - p_b$ player i answers "focus on b", otherwise ($v_i(a) - v_i(b) < p_a - p_b$) player i answers "focus on a". Afterward the player should report her true demand until the auction ends, regardless of what player j does.*

Player i answers "focus on b" in step 3 when $v_i(a) - v_i(b) \geq p_a - p_b$ since at this point $v_j(a) - p_a = v_j(b) - p_b$, and therefore $v_i(a) + v_j(b) \geq v_j(a) + v_i(b)$. Since we need to reach the VCG outcome, we want player j to focus on b and not on a. Since VCG's goal is aligned with the players' goals, it is not a big surprise that this choice will push the strategies towards equilibrium.

Let us examine how the course of the auction of Figure 1 changes when both players play the signaling strategy. The first two phases remain the same, and players report their true demand until the beginning of phase 3, which is a jump phase. At this point, player 1 sends a message to player 2, indicating that she intends to jump. Player 2 calculates $v_2(a) - v_2(b) = 10 - 5 < 8 - 2 = p_a - p_b$ and answers "focus on a" to player 1. Player 1 then reports the demand $\{a\}$, and player 2 continues to report $\{a, b\}$. Thus a's price is raised. At a price $p_a = 9 = v_2(a|b)$, player 2 changes her demand to $\{b\}$, and the auction terminates. Player 1 receives a and pays 9, and player 2 receives b and pays 2. This is exactly the VCG outcome.

The reader can verify that, for this specific example setup, no player can improve her utility by deviating from our strategy. For example, if player 1 will continue reporting her true demand in phase 3 she will still win item a, and for the same price. If she will send the jump signal earlier in the auction then once again her utility cannot be improved: if she will receive a she will pay the same, and if she will receive b she will pay $v_2(b|a)$ which overall is less profitable.

Another interesting aspect of our strategy is that the jumping player finds it in her best interest to actually follow the demand reduction request. In particular, she cannot gain by demanding the ignored item, despite the fact that at final prices, the ignored item may be more attractive than the item that she receives. In our example, player 1 receives item a and pays 9, for a resulting utility of 1, while item b's final price is 2, and at this price player 1 will extract a surplus of 2 from item b. Nevertheless, it is not hard to verify, for this specific example, that player 1 cannot receive item b for such an attractive price, and that receiving a for the price 9 is the best she can achieve.

This is not by accident, of-course, and in the remainder of this section we prove:

Theorem 1. *The signaling strategy is a symmetric ex-post (sub-game perfect) equilibrium of the ascending auction (for two items). This equilibrium yields the VCG outcome, hence ex-post efficiency is obtained.*

4.1 Analysis

We prove the theorem by linking the strategies to the VCG mechanism. A nice property of dominant-strategy mechanisms like VCG is that, when given types for all players besides some fixed player j, we can determine a price for j for any subset of items S. This price does not depend on j's type, and we use its properties throughout our analysis. More concretely, we begin with the following notation:

Definition 3 ("j's VCG price for S, given σ_{-j}")**.** *Fix a player j, a tuple of players' types σ_{-j} (which does not include j's type), and a subset of items S. Let v_j be a type for player j such that, in the VCG assignment for (v_j, σ_{-j}), j receives S.*[12] *Player j's VCG price for S, given σ_{-j}, is denoted by $p_j^{VCG}(S, \sigma_{-j})$, and is defined to be j's VCG payment in the instance (v_j, σ_{-j}).*

Using this notation, we link the equilibrium strategies to the VCG mechanism as follows. Suppose we could show the following two properties:

1. If all players follow the signaling strategy then the VCG outcome (allocation and payments) is reached.

2. If some player i deviates while the others follow the signaling strategy, and i wins a bundle S, her payment in this case is at least her VCG price for S, $p_j^{VCG}(S, v_{-j})$.

Then a standard revelation-principle argument shows that since VCG is incentive-compatible, the signaling strategy is a symmetric equilibrium (see below a formal proof in Lemma 3).

Thus, to show a sub-game-perfect equilibrium, it is enough to show that these two properties hold for any possible starting price of the auction. Unfortunately this is not true. If the auction

[12] A player can always ensure receiving exactly S by declaring an additive valuation with a high value for every element in S and a zero value for all other elements; since the VCG payment does not depend on v_j, it is the same for any choice of v_j for which j receives the same bundle S.

starts at a price higher than the VCG price for the given tuple of types v, clearly it cannot reach the VCG outcome. Consider the following example: two players, $v_1(a) = 11, v_1(b) = 10, v_1(ab) = 21, v_2(a) = v_2(b) = v_2(ab) = 9$. In the VCG outcome, player 1 wins both items and pays 9. If we start from prices $p_a = 8.5, p_b = 3$ we obviously cannot reach VCG prices since prices only ascend.

To overcome this problem, we show that the prices and allocation resulting from the auction are the same as the VCG outcome of another instance σ that contains the original players and two extra "dummy" players. By adding these two players we guarantee that the starting prices will not be higher than the VCG prices of σ. More formally, define a valuation $v^{(p)}$ to be $v^{(p)}(a) = p_a, v^{(p)}(b) = p_b$, and $v^{(p)}(ab) = p_a + p_b$, a tuple of types $\sigma = (v, v^{(p)}, v^{(p)})$, and modified conditions:

1. If all players $\{1, ..., n\}$ follow the signaling strategy then the VCG outcome <u>of σ</u> (allocation and payments) is reached.

2. If some player i deviates while the others follow the signaling strategy, and i wins a bundle S, her payment in this case is at least her VCG price for S in $\underline{\sigma_{-j}}$, $p_j^{VCG}(S, \sigma_{-j})$.

These conditions are slightly more subtle, as only players $\{1, ..., n\}$ participate in the ascending auction (the dummy players of-course do not participate, they are used only in the analysis), but we compare the outcome of the ascending auction to the VCG outcome for the tuple of types σ, that does include the dummy players. Thus, it is slightly more complicated to show that the two modified requirements hold, but if they do hold then the same standard revelation-principle argument shows that the signaling strategy is an equilibrium.

To get more intuition, let us examine again the example above. When the auction starts with prices $p_a = 8.5, p_b = 3$, player 1 demands both items, and player 2 demands item b. Thus b's price will increase, and when it will reach 8.5, a jump phase will occur. Player 2 will send a jump signal to player 1, who will reply to focus on b. The end outcome is that player 1 wins both items and pays 17.5. This is not the VCG payment for (v_1, v_2), but it is the VCG payment for σ, as defined above, with the two dummy players. One can verify that the second modified property holds here as well (i.e. when a player $i = 1, 2$ deviates and wins a different bundle S she pays at least $p_j^{VCG}(S, \sigma_{-j})$). This implies that any such deviation is not profitable (note again that only players 1, 2 participate in the ascending auction). More formally,

Lemma 3. *Suppose that the ascending auction starts from a price vector p, and fix a tuple of strategies $s_1(\cdot), ..., s_n(\cdot)$ such that, for any tuple of valuations $v_1(\cdot), ..., v_n(\cdot)$ and for any player j,*

1. *If every player $i \in \{1, ..., n\}$ plays strategy $s_i(v_i)$ then the outcome of the auction for every $i \in \{1, ..., n\}$ (allocation and payment) is the same as i's VCG outcome in the instance $\sigma = (v_1, ..., v_n, v^{(p)}, v^{(p)})$.*

2. *If every player $i \in \{1, ..., n\}, i \neq j$ plays strategy $s_i(v_i)$, and j plays some other strategy and receives some bundle S, then j's payment is at least $p_j^{VCG}(S, \sigma_{-j})$.*

Then s_i is best response to s_{-i} in the ascending auction that starts at price vector p, for every player i and every tuple of types $v_1(\cdot), ..., v_n(\cdot)$.

Proof. Suppose that in the VCG outcome for σ, player j receives bundle S. Then if she plays $s_j(v_j)$ her utility is $v_j(S) - p_j^{VCG}(S, \sigma_{-j})$. If at some point(s) in the auction process she deviates and as a result receives S' and pays p', her utility is $v_j(S') - p' \leq v_j(S') - p_j^{VCG}(S', \sigma_{-j})$. By the incentive compatibility of VCG, $v_j(S') - p_j^{VCG}(S', \sigma_{-j}) \leq v_j(S) - p_j^{VCG}(S, \sigma_{-j})$, and the claim follows. □

Corollary 1. *Any tuple of strategies for the ascending auction that satisfies the conditions of Lemma 3 for every starting price p forms an ex-post (sub-game-perfect) equilibrium.*

Therefore to prove Theorem 1 we prove that the signaling strategy satisfies the two properties detailed in Lemma 3. To prove the first property we first show in Appendix B:

Lemma 4. *Suppose that all players $1, ..., n$ follow the signaling strategy, starting at some price vector p. Then, if a signaling message is being exchanged at some phase, the auction ends in the VCG outcome for the instance $\sigma = (v_1, ..., v_n, v^{(p)}, v^{(p)})$.*

We use Lemma 4 to show that property 1 holds for the signaling strategy:

Lemma 5. *The signaling strategy satisfies the first property of Lemma 3.*

Proof. Clearly the starting price vector p is coordinate-wise weakly smaller than the minimal Walrasian prices for σ, $p^W(\sigma)$. We consider three cases:

Case 1: $p < p^W(\sigma)$ and, for the instance σ, there exists a player whose VCG payment is not equal to her Walrasian payment. In this case Lemma 2 implies that a jump will occur when the ascending auction starts at p and all players in σ are truthfully reporting their demand, since $p < p^W(\sigma)$. Since the two dummy players demand the empty set throughout the auction, the same jump also occurs if only players $\{1, ..., n\}$ participate. Therefore if all players in v play the signaling strategy, a signaling message will be exchanged. Lemma 4 now implies that the end outcome of the auction when players $1, ..., n$ play the signaling strategy is the VCG outcome of σ, as we need to show.

Case 2: $p < p^W(\sigma)$ and, for the instance σ and for every player in σ, her VCG payment is equal to her Walrasian payment. In this case, if truthful demand reporting leads to a jump phase when all players in σ participate then Lemma 4 implies that the end outcome is the VCG outcome for σ using the same argument as above. If on the other hand truthful demand reporting does not lead to a jump phase then the end outcome is the Walrasian outcome for σ which is in this case identical to the VCG outcome for σ. Since the course of the auction does not change if the players in v are playing the signaling strategy and the dummy players do not participate, the claim follows.

Case 3: There exists an item $x \in \{a, b\}$ such that $p_x = p_x^W(\sigma)$. In this case x's price will not be raised, which implies that the signaling strategy is identical to myopic bidding. Furthermore as before the course of the ascending auction would be identical if the dummy players were participating and bidding myopically as well, since they demand the empty set. In addition, the two dummy players have $v^{(p)}(x \mid \{a,b\} \setminus x) = p^W_x(\sigma)$. Thus all requirements of Lemma 13 in Appendix A hold, implying that the ascending auction reaches a VCG outcome for σ. □

The proof of the second property follows by a straight-forward case analysis:

Lemma 6. *Suppose that the ascending auction starts from initial prices p^0, and that all players besides j play the signaling strategy. Suppose that player j plays some strategy and wins $\{a,b\}$. Then j pays at least $p_j^{VCG}(ab, \sigma_{-j})$, where $\sigma = (v_1, ..., v_n, v^{(p^0)}, v^{(p^0)})$.*

Proof. We separate to two cases according to the value of $p_j^{VCG}(ab, \sigma_{-j})$:

Case 1: $p_j^{VCG}(ab, \sigma_{-j}) = v_i(ab)$ for some player i. If i is a dummy player the claim holds simply because the price ascent starts from p^0. Otherwise, player i follows the signaling strategy. Consider two sub-cases:

- Player i never enters a valid signaling step. In this case, a's final price is at least $v_i(a)$ and b's final price is at least $v_i(b)$, hence j's payment is at least $v_i(a) + v_i(b) \geq v_i(ab)$.

- Player i enters a signaling step at prices $p = (p_a, p_b)$. At p player i demands $\{a\}$ and $\{b\}$ but not $\{a,b\}$, thus $p_a \geq v_i(a|b)$ and $p_b \geq v_i(b|a)$. When player j instructs i to choose item x, player i demands x until its price is $v_i(x)$. Thus player j pays at least $v_i(ab)$.

Case 2: $p_j^{VCG}(ab, \sigma_{-j}) = v_i(a) + v_l(b)$ for some players i and l. Assume without loss of generality that player l drops before or at the same time as player i in the course of the auction. Then b's price is at least $v_l(b)$ (this also holds if player l is a dummy). If i is a dummy the claim again immediately follows. Otherwise, player i follows the signaling strategy. Consider two sub-cases:

- Player i never enters a valid signaling step or is instructed to focus on item a. In this case, a's price will be at least $v_i(a)$ and the claim follows.

- Player i enters the signaling step at prices $p = (p_a, p_b)$ and is instructed to focus on item b. Because i entered the signaling stage we have that $v_i(a) - p_a = v_i(b) - p_b$. Since $p_b \geq v_l(b)$ we conclude that $p_a \geq v_i(a) + v_l(b) - v_i(b)$. Note that b's final price is exactly $v_i(b)$, and the claim follows. □

A similar logic is used to prove the case where j wins a single item; the formal proof is given in appendix C below.

Lemma 7. *Suppose that the ascending auction starts from initial prices p^0, and that all players besides j play the signaling strategy. Suppose that player j plays some strategy and wins item a. Then j pays at least $p_j^{VCG}(a, \sigma_{-j})$, where $\sigma = (v_1, ..., v_n, v^{(p^0)}, v^{(p^0)})$.*

All the above shows that the signaling strategy satisfies the two required properties of Lemma 3, and Theorem 1 now follows.

5 No ex-post efficient equilibrium without side-communication

We show non-existence of efficient ex-post equilibrium even if there are only two players. The assumption on the number of players is without loss of generality since if there exist efficient equilibrium strategies for more than two players we can construct equilibrium strategies for two players by adding dummy players with zero values. Since the dummy players drop immediately, equilibrium strategies for more than two players imply equilibrium strategies for two players. Thus the non-existence of efficient equilibrium assuming only two players implies the non-existence of efficient equilibrium for any number of players.

The following Lemma is useful for the impossibility proof:

Lemma 8. *In any ex-post efficient equilibrium strategy for an ascending auction with two players, for any player i, if $D_i(p) = \{\{a, b\}\}$ then i must report her true demand.*

Proof. Assume by contradiction that there exist two valuations v_1, v_2 such that at some point p in the auction $D_1(p) = \{\{a, b\}\}$ but player 1 bids $\{a\}$. Since $D_1(p) = \{\{a, b\}\}$ then $p_x < v_1(x|\{a, b\} \setminus \{x\})$, for any $x \in \{a, b\}$. We use this fact to show that in another instance with two players and valuations $v_1, \tilde{v}_2$, player 2 can profit by deviating from her strategy.

In particular, choose $\tilde{v}_2$ such that $p_a < \tilde{v}_2(a) < v_1(a|b)$ and $p_b < \tilde{v}_2(b) < v_1(b|a)$. In the instance $v_1, \tilde{v}_2$ the efficient allocation is to give the two items to player 1. Thus, if player 2 follows the equilibrium strategy her resulting utility will be zero. Consider a different strategy in which player 2 plays as if her type is v_2 until price p, and at price p demands only $\{b\}$. Then, since player 1 does not demand b at this point, the auction ends and player 2 wins item b for a positive utility. Therefore, player 2 has a profitable deviation, a contradiction. □

As mentioned in the Introduction, payments in any ex-post efficient equilibrium outcome of the ascending auction must be equal to Clarke payments. This is well-known, see e.g. Gul and Stacchetti (2000). Let us briefly repeat the argument for completeness: Let $M(v_1, v_2)$ be a direct-revelation mechanism whose outcome is the equilibrium outcome of the ascending auction when the players types are (v_1, v_2). A player's utility in the ascending auction is maximized by bidding according to the equilibrium strategy, assuming the other player plays the equilibrium strategy as well, *for any tuple of types.* Therefore in the direct-revelation mechanism M it is a dominant-strategy to report the player's true type. Since VCG as well as the new mechanism M are both incentive-compatible in dominant strategies and ex-post efficient, their payments must always be the same, as shown in e.g. Heydenreich et al. (2009), and the claim follows.

	$\{a\}$	$\{b\}$	$\{a,b\}$
v_1	4	4	4
v_2	10	11	12

	$\{a\}$	$\{b\}$	$\{a,b\}$
v_1	4	4	4
$\tilde{v}_2$	11	10	12

Figure 2: Two example instances for Theorem 2.

Theorem 2. *There is no ex-post efficient equilibrium in the ascending auction game without side-communication.*

Proof. Assume two players, and suppose by contradiction that there exists an efficient ex-post equilibrium. Consider first the valuations that are described in the left table of Figure 5. The efficient outcome is to allocate a to player 1 and b to player 2. Clarke's prices in this case are 1 for player 1 and 0 for player 2.

Consider the course of the auction with the equilibrium strategies for these valuations. Since the auction must end in prices that are equal to Clarke's prices, the price of item b cannot increase at all. When the auction begins at prices $(0,0)$, player 2 must demand $\{a,b\}$ by Lemma 8. Player 1 cannot therefore include b in her reported demand. Player 1 also cannot demand the empty set since if she does the auction will end, and the outcome will be inefficient. Thus player 1 must demand $\{a\}$, and only a's price increases.

We conclude that a's final price is greater than zero for the left instance of Figure 5. Moreover, we conclude that player 1 must demand $\{a\}$ at prices $(0,0)$ whenever her valuation is v_1, since her demand report at this point $(0,0)$ depends only on her type.

Now consider the instance in the right table of Figure 5. By all the above, the end price of item a in this instance will be strictly larger than zero, since player 1 will demand a at the beginning (her valuation is v_1), and player 2 will demand both items at the beginning (by Lemma 8). However the Clarke price of item a in this instance is zero. Therefore the equilibrium strategies cannot be ex-post efficient. □

As shown in the previous sections, just one bit of cheap talk eliminates this impossibility, and enables the emergence of an efficient ex-post equilibrium.

6 Summary and Discussion

We study an ascending auction with anonymous and linear item prices, which is one of the most popular auction formats in realistic settings. Previous theoretical work shows mainly impossibilities for this auction format, and suggests to allow arbitrary price adjustments (increase and decrease), or alternatively non-anonymous and non-linear bundle prices, as a solution. In this paper we give a different solution, showing how an efficient ex-post equilibrium can be constructed by allowing limited side-communication between the bidders.

The implication of this result, that side-communication is a useful tool to promote efficiency, is certainly a non-conservative aspect of our paper; previous works on ascending auctions mainly highlight the negative aspects of collusion, as leading to inefficient outcomes in Bayesian settings. We wish to point out that we do not argue that side-communication and other signaling techniques cannot cause inefficiency. Indeed, with the appropriate Bayesian knowledge, inefficient *Bayesian-Nash* equilibria still exist, as was previously shown, and we obviously face an equilibrium selection problem.[13] The point is that the other extreme of disallowing communication all together (while using the common ascending auction format) may also lead to ex-post inefficiency. Our analysis suggests that *limited* side-communication may be the right balance of these two extremes, and we pin-point the form of communication that should be allowed.

We wish to additionally emphasize that, a-priori, it is not clear whether side-communication can give rise to an ex-post equilibrium behavior. For example, the most straight-forward strategy in which players "divide the loot" by communicating at zero-prices and coordinating demand to avoid competition all-together obviously does not constitute an ex-post equilibrium, since players always have the incentive to exaggerate and declare a very high valuation that will award them both items (after all they will not be required to pay for them). The fact that our (formal) model shows how communication can lead to efficient market division was initially a surprise, at least to us. In this context, it is an interesting question to characterize the inefficient ex-post equilibria (or all possible ex-post equilibria) that can be the result of other various formats of side-communication.

The in-equilibrium path of our construction has the following structure: initially bidders bid myopically, reporting their true demands. At a single specific point in the auction, bidders need to perform a demand reduction, to avoid the formation of bubble prices that myopic bidding may sometimes cause. The form of the demand reduction is decided by a single message exchanged between the bidders. We have argued, in a very preliminary way, that the real-life signaling examples described in the literature may match this pattern of our proposed strategy. It would be an interesting empirical question to conduct a more in-depth examination of our claim.

As mentioned in the Introduction, Mishra and Parkes (2007) suggest incorporating price discounts in order to achieve the VCG outcome in ascending auctions with non-anonymous bundle prices when players bid myopically. Our analysis shows that the price path of the ascending auction we consider here (with myopic bidding) contains enough information to recover the VCG outcome, *even if we have anonymous item prices*. Briefly, if a jump phase did not occur then the end result of the auction is the VCG outcome. Otherwise, the price of (one of) the item(s) that the non-jumping player receives should be lowered to its level at the beginning of the jump phase. Thus, conceptually, instead of allowing side-communication, the mechanism can be slightly changed by introducing a price discount, and the same effect will be obtained, while still having anonymous

[13] One can argue about this point that an ex-post equilibrium is more robust to information uncertainties, because of its no-regret property, and since bidders in actual auctions have a tendency to prefer winning of slightly over-priced items over losing them to a competitor.

item prices.

From a technical point of view, the most natural question regarding our work is perhaps what happens with more than two items. In this case, the problem becomes significantly more involved. In particular, our jump condition needs to be modified, such that players that are indifferent between *bundles* should decide on which bundle to focus. Moreover, this can happen several times during the price ascent. We have obtained some preliminary results for this case, but have chosen to separate the presentation of the two-items case, for the sake of conceptual clarity.

Appendices

A Proof of Lemma 2

Fix valuations $v_1, ..., v_n$. Let p^W be the minimal Walrasian prices for $v_1, ..., v_n$, and p_i^{VCG} the VCG payment of player $i = 1, ..., n$. Assume that there are T phases in the auction, and let $p(1), ..., p(T)$ be the price vectors at the end of each phase, so $p(T)$ are the final item prices, and $p(0) = p$ is the starting price vector. By Gul and Stacchetti (2000), $p(T)$ are the minimal Walrasian prices, and additionally the Walrasian payment of each player is at least her VCG payment. Suppose that for some player j, $p^W(S_j) > p_j^{VCG}$, where S_j is the bundle that player j receives in the efficient (Walrasian) outcome. We show that the ascending auction with truthful demand reporting terminates in a "jump phase", in which: (1) only two players i, j have non-empty demand, (2) player j demands $\{a, b\}$, and (3) player i demands $\{\{a\}, \{b\}\}$.

The proof is by four claims. We first show that under certain conditions (detailed in Lemma 9) the prices of both items during the ascending auction are strictly lower than their minimal Walrasian prices (in contrast to the a-priori possibility that the price of one item becomes equal to its minimal Walrasian price strictly before the price of the other item becomes equal to its minimal Walrasian price). We then use this to show that if these conditions are satisfied, the auction terminates in a jump phase (lemma 10). Finally, we prove that these conditions are indeed satisfied when $S_j = \{a, b\}$ (lemma 11), and when $|S_j| = 1$ (lemma 12). This immediately gives Lemma 2.

Lemma 9. *Fix valuations $v_1, ..., v_n$. Let (p_a^W, p_b^W) be the minimal Walrasian prices for these valuations. Suppose there exist two players i, j such that:*

1. $v_i(a|b) < p_a^W \leq v_j(a|b)$,
2. $v_i(b|a) < p_b^W \leq v_j(b|a)$,
3. $v_i(a) - p_a^W = v_i(b) - p_b^W \geq 0$, *and*
4. *for every item $x \in \{a, b\}$ and every player $l \neq i, j$, $v_l(x) < p_x^W$.*

Then in every step $t < T$ of the ascending auction, $p_a(t) < p_a^W$ and $p_b(t) < p_b^W$.

Proof. Suppose by contradiction that there exists a step t, $T > t \geq 1$ such that $p_a(t-1) < p_a^W$ and $p_b(t-1) < p_b^W$, but (without loss of generality) $p_a(t) = p_a^W$ while $p_b(t) < p_b^W$.

Since $p_a(t-1) < p_a(t)$ it must be the case that a is over-demanded in step t. Since $v_j(a|b) > p_a(t-1)$ and $v_j(b|a) > p_b(t-1)$ we have $D_j(p(t-1)) = \{a, b\}$, and since $v_i(a) - p_a^W = v_i(b) - p_b^W \geq 0$ we have $\emptyset \notin D_i(p(t-1))$. If $D_i(p(t-1)) = \{a, b\}$ then a's price in step t cannot increase beyond $v_i(a|b) < p_a^W$, contradicting $p_a(t) = p_a^W$. Thus $\{a, b\} \notin D_i(p(t-1))$. We consider two cases:

Case 1: b is also over-demanded. Thus $p_a(t) - p_a(t-1) = p_b(t) - p_b(t-1)$, and since $v_i(a) - p_a(t) = v_i(a) - p_a^W = v_i(b) - p_b^W < v_i(b) - p_b(t)$ we get that $D_i(p(t-1)) = \{b\}$. Since a is over-demanded there must be a player $l \neq i, j$ that demands it, and her demand changes at a price of at most $v_l(a) < p_a^W$. Thus we get that $p_a(t) < p_a^W$, a contradiction.

Case 2: b is not over-demanded. Thus no player except j demands b, and therefore $D_i(p(t-1)) = \{a\}$. Hence $v_i(a) - p_a(t-1) > v_i(b) - p_b(t-1)$. Since $v_i(a) - p_a(t) < v_i(b) - p_b(t-1)$ there is a price vector $p_a(t-1) < p_a^* < p_a(t)$ such that $v_i(a) - p_a^* = v_i(b) - p_b(t-1)$. At this point i's demand changes to $D_i(p(t-1)) = \{\{a\}, \{b\}\}$, and therefore both items a, b become over-demanded, implying that step t should end at price $p_a^* < p_a(t)$, a contradiction. □

Lemma 10. *Under the conditions of Lemma 9, step T of the ascending auction with truthful demand reporting is a jump step.*

Proof. First we argue that for any player $l \neq i, j$, $\emptyset \in D_l(p(T-1))$. Otherwise if $v_l(x) > p_x(T-1)$ for some player l and some item x then this player will change her demand no later than at price $v_l(x) < P_x^W$, contradicting the fact that T is the last step in the auction. Second, we note that $D_j(p(T-1)) = \{a, b\}$ since $p_a(T-1) < p_a^W \leq v_j(a|b)$ and $p_b(T-1) < p_b^W \leq v_j(b|a)$.

Third, we argue that $D_i(p(T-1)) = \{\{a\}, \{b\}\}$. Since $p_a(T-1) < p_a^W = p_a(T)$ and $p_b(T-1) < p_b^W = p_b(T)$ it must be that both items a and b are over-demanded at $p(T-1)$. If $D_i(p(T-1)) = \{a, b\}$ then player i will change her demand no later than the price $v_i(a|b) < P_a^W$, contradicting the fact that step T ends with $p_a(T) = p_a^W$. Thus it must be that $D_i(p(T-1)) = \{\{a\}, \{b\}\}$, which implies that step T is a jump step. □

Lemma 11. *If $S_j = \{a, b\}$ and $p_j^{VCG} < p^W(S_j)$ then the requirements of lemma 9 are satisfied.*

Proof. Since $D_j(p^W) = \{a, b\}$ we have $p_a^W \leq v_j(a|b)$ and $p_b^W \leq v_j(b|a)$. Since (p_a^W, p_b^W) are *minimal* Walrasian prices there must be a player i_a such that $v_{i_a}(a) = p_a^W$ and a player i_b such that $v_{i_b}(b) = p_b^W$. Suppose by contradiction that there exist such i_a, i_b, and $i_a \neq i_b$. Then an efficient allocation without player j would be to allocate a to i_a and b to i_b, since in this case (p_a^W, p_b^W) is a Walrasian equilibrium to the set of valuations v_{-j}. Thus $p_j^{VCG} = v_{i_a}(a) + v_{i_b}(b) = p^W(S_j)$, a contradiction. Thus we conclude that $i_a = i_b = i$, and $p_x^W > \max_{l \neq i,j} v_l(x)$ for any $x \in \{a, b\}$. Finally, if $v_i(a|b) = p_a^W$ or $v_i(b|a) = p_b^W$ then $p_j^{VCG} \geq v_i(ab) = p_b^W + p_a^W = p^W(S_j)$, a contradiction. This shows all the requirements of Lemma 9. □

Lemma 12. *If* $|S_j| = 1$ *and* $p_j^{VCG} < p^W(S_j)$ *then the requirements of Lemma 9 are satisfied.*

Proof. Suppose without loss of generality that $S_j = \{a\}$ and let i be the player that receives b in the efficient allocation. Since $p_j^{VCG} \geq \max_{l \neq i,j} v_l(a)$ and $p_j^{VCG} \geq v_i(a|b)$ we first have

$$p_a^W > \max_{l \neq i,j} v_l(a) \text{ and } p_a^W > v_i(a|b). \tag{1}$$

Since p^W are minimal Walrasian prices, it follows that at prices $p_\epsilon = (p_a^W - \epsilon, p_b^W)$ for some small $\epsilon > 0$ item a must be over-demanded. When $p_a^W - \epsilon > \max_{l \neq i,j} v_l(a)$ it follows that player i must demand a at p_ϵ. If $D_i(p_\epsilon) = \{a, b\}$ then we get that $p_a^W = v_i(a|b)$, a contradiction. Thus $D_i(p_\epsilon) = \{a\}$. For any $\epsilon > 0$ we now have $v_i(a) - (p_a^W - \epsilon) > v_i(b) - p_b^W$ and $v_i(a) - p_a^W \leq v_i(b) - p_b^W$, implying

$$v_i(a) - p_a^W = v_i(b) - p_b^W. \tag{2}$$

Using this equation we can also get, for any player $l \neq i, j$, $v_i(a) - v_i(b) + p_b^W = p_a^W > p_j^{VCG} \geq v_i(a) - v_i(b) + v_l(b)$. Rearranging, we have

$$p_b^W > \max_{l \neq i,j} v_l(b). \tag{3}$$

As above, in prices $(p_a^W, p_b^W - \epsilon)$ item b is over-demanded, and this is true for every $\epsilon > 0$. When $p_b^W - \epsilon > \max_{l \neq i,j} v_l(b)$, it follows that player j must demand b, hence

$$p_b^W = v_j(b|a) = p_i^{VCG}. \tag{4}$$

Combining $p_a^W > v_i(a|b)$ and $v_i(a) - p_a^W = v_i(b) - p_b^W$ gives

$$p_b^W > v_i(b|a). \tag{5}$$

Finally, we have $p_a^W = p_b^W + v_i(a) - v_i(b) = v_j(b|a) + v_i(a) - v_i(b)$ by the above equations, and $v_i(a) - v_i(b) \leq v_j(a) - v_j(b)$ since the assignment of a to j and b to i is efficient. These two together imply

$$p_a^W \leq v_j(a|b). \tag{6}$$

This shows all the requirements of Lemma 9. □

This concludes the argument for the correctness of Lemma 2. For another part in the paper it will be useful to rely on the last two claims which quite immediately imply:

Lemma 13. *Fix a tuple of valuations* σ *and let* p^W *denote the minimal Walrasian prices for* σ*. Suppose that there exists an item* $x \in \{a, b\}$ *and two players* $i_1, i_2 \in \sigma$ *such that* $v_{i_l}(x \mid \{a, b\} \setminus x) \geq p_x^W$ *(for* $l = 1, 2$*). Then the ascending auction with myopic bidding, that starts from any arbitrary price vector* $p \leq P^W$*, terminates in a VCG outcome for* σ*.*

Proof. Since players bid myopically the ascending auction terminates in prices p^W and in an efficient allocation. Suppose by contradiction that there exists a player j whose Walrasian payment $p^W(S_j)$ is strictly larger than her VCG payment. Then by Lemma 11 and Lemma 12 the requirements of Lemma 9 are satisfied. In particular, there exists *exactly* one player $j \in \sigma$ that satisfies $v_j(x \mid \{a,b\} \setminus x) \geq p^W_x$ for any item $x \in \{a,b\}$, which contradicts the assumption of the claim. □

B Proof of Lemma 4: Jump phase implies VCG outcome

Suppose that all players follow the signaling strategy and that there are T phases in the auction. Let $p(1), ..., p(T)$ be the price vectors at the end of each phase, so $p(T)$ are the final item prices and $p(0) = p$ is the starting price vector. By definition, if players follow the signaling strategy and there was signaling during the auction, then it must happen between phase $T-1$ and phase T. In other words there is signaling in the auction if and only if there are two players i, j such that $D_i(p(T-1)) = \{\{a\},\{b\}\}$, $D_j(p(T-1)) = \{a,b\}$ and for any other player $l \in \{1,...,n\}, l \neq i,j$, we have $\emptyset \in D_l(p(T-1))$. We need to show that in this case the auction ends in the VCG outcome for the instance $\sigma = (v_1, ..., v_n, v^{(p)}, v^{(p)})$. We start with a useful claim:

Lemma 14. *If $p_a(T-1) > v_i(a|b)$ and $p_b(T-1) > v_i(b|a)$ then there exists a player $l \in \sigma, l \neq i,j$ and an item $x \in \{a,b\}$ such that $v_l(x) = p_x(T-1)$.*

Proof. If $p_a(T-1) = p_a(0)$ or $p_b(T-1) = p_b(0)$ then player l is one of the dummy players $v^{(p)}$. Thus assume that $p_a(T-1) > p_a(0)$ and $p_b(T-1) > p_b(0)$.

If there is a phase $t < T$ such that $D_i(p(t)) = \{\{a\},\{b\}\}$ then, since player i initiates a jump signal just before phase T, in every phase $t, ..., T-1$ there was at least one additional player with non-empty demand, and that player for phase $T-1$ is the player l we are looking for.

Otherwise let t be the last phase for which $\{a,b\} \in D_i(p(t))$ ($t = 0$ if there was no such phase). Since $p_a(T-1) > v_i(a|b)$ and $p_b(T-1) > v_i(b|a)$ it must be that $t < T-1$. Since player i does not demand $\{\{a\},\{b\}\}$ before phase T it must be that she demands a singleton item y in every phase $t+1, ..., T-1$. Let the other item be x. We claim that $p_x(T-1) > p_x(t)$: If $t = 0$ then by assumption $p_x(T-1) > p_x(t)$, and if $t > 0$ then $p_x(t) = v_i(x|y)$ and $p_x(T-1) > v_i(x|y)$ so again $p_x(T-1) > p_x(t)$.

Since $p_x(T-1) > p_x(t)$ and $t < T-1$, there is some bidder $l \neq i,j$ such that $v_l(x) = p_x(T-1)$, this is the last bidder to demand x besides j. Thus we have found a player l and an item x that fit the requirements of the claim. □

We now prove Lemma 4. Suppose without loss of generality (as the two demands are completely symmetric with respect to the items) that $v_i(a) + v_j(b) \geq v_j(a) + v_i(b)$. Therefore after the communication player i focuses on item a, and the outcome is determined by the values of $v_i(a)$ and $v_j(a|b)$:

- If $v_i(a) < v_j(a|b)$ then player j receives $\{a, b\}$ and pays $v_i(a) + p_b(T-1)$.
- If $v_i(a) \geq v_j(a|b)$ then player i receives $\{a\}$ and pays $v_j(a|b)$ and player j receives $\{b\}$ and pays $p_b(T-1)$.

It is straight-forward to verify that the assignment is indeed efficient, in each of these two cases. We now verify that the players' prices are VCG prices. First, player i's payment in the second case is her VCG payment, since when player i is absent it is efficient to assign $\{a, b\}$ to player j (as $p(T-1)$ is a Walrasian equilibrium for σ_{-i}).

It remains to verify that j's payment equals her VCG payment. To do that we begin by considering the possible values of $p_a(T-1)$ and $p_b(T-1)$. Since $v_i(a) - p_a(T-1) = v_i(b) - p_b(T-1)$ we get that $p_a(T-1) = v_i(a|b)$ if and only if $p_b(T-1) = v_i(b|a)$, and similarly $p_a(T-1) > v_i(a|b)$ if and only if $p_b(T-1) > v_i(b|a)$. Since it cannot be that $p_a(T-1) < v_i(a|b)$ and simultaneously $p_b(T-1) < v_i(b|a)$ as this implies that i will demand both items, one of the previous two cases must hold.

- Case 1: $p_a(T-1) = v_i(a|b)$ and $p_b(T-1) = v_i(b|a)$. In this case $p(T-1)$ is a Walrasian equilibrium for σ_{-j}. Hence, when player j is absent it is efficient to assign $\{a, b\}$ to player i, implying that j's price is her VCG price.
- Case 2: $p_a(T-1) > v_i(a|b)$ and $p_b(T-1) > v_i(b|a)$. For this case Lemma 14 shows that there exists a player $l \in \sigma, l \neq i, j$ and an item $x \in \{a, b\}$ such that $v_l(x) = p_x(T-1)$. Therefore an efficient assignment without j is to allocate item x to player l and the other item to player i (since $p(T-1)$ is a Walrasian equilibrium for σ_{-j}).

  If $x = b$ then $p_b(T-1) = v_l(b)$, implying that j's price in both possible assignments is indeed her VCG price. If $x = a$ then since $v_i(a) - p_a(T-1) = v_i(b) - p_b(T-1)$ we have $p_b(T-1) = v_i(b) - v_i(a) + v_l(a)$, again implying that j's price in both possible assignments is her VCG price.

This concludes the proof of Lemma 4.

C Proof of Lemma 7: Prices are always at least VCG prices

Recall that $v_1, ..., v_n$ denote the true valuations of players $1, ..., n$, respectively. We assume that the ascending auction starts from initial prices p^0, and that all players besides j play the signaling strategy. We need to show if player j wins item a she pays at least $p_j^{VCG}(a, \sigma_{-j})$ (regardless of the strategy she follows), where $\sigma = (v_1, ..., v_n, v^{(p^0)}, v^{(p^0)})$. Let $i \neq j$ be the player that has the maximal value for b (i.e. $i = \text{argmax}_{l \neq j \in \sigma} v_l(b)$). Note that because i's value for b is at least the value of any other player for b, there is an efficient allocation without j in which i receives some item. Thus the maximal welfare for σ_{-j} can be either $v_i(ab)$, $v_i(b)+v_l(a)$, or $v_i(a)+v_l(b)$ (for some player l). We consider each case separately.

- $p_j^{VCG}(a,\sigma_{-j}) = v_i(a|b)$ (maximal welfare for σ_{-j} is $v_i(ab)$). If i is a dummy player then trivially j's price is at least $v_i(a|b) = p_a^0$. Otherwise, regardless of the course of the auction, if i demands b immediately after demanding $\{a,b\}$ (or at the beginning of the auction), then at the change point a's price is $v_i(a|b)$, implying the claim.

  On the other hand, if i demands a immediately after demanding $\{a,b\}$ or at the beginning of the auction then at the change point b's price is $v_i(b|a)$, and i will demand a, still regardless of the course of the auction, until item prices will satisfy $v_i(a) - p_a = v_i(b) - p_b$. Since we have $p_b \geq v_i(b|a)$ we get $p_a \geq v_i(a|b)$ and the claim holds.

- $p_j^{VCG}(a,\sigma_{-j}) = v_l(a)$ (maximal welfare for σ_{-j} is $v_i(b)+v_l(a)$). Consider the first price vector $p = (p_a, p_b)$ in the auction in which there remain exactly two active players (player j and some other player). If player l is not active at this point then a's price must be at least $v_l(a)$ (this includes the possibility that l is a dummy player).

  Otherwise, j and l are active, and at the end l wins item b. Thus $v_l(b) \geq p_b$. Since i was not one of the last two players to quit, we have that $p_b \geq v_i(b)$. Using the definition of i we have $v_i(b) \geq v_l(b) \geq p_b \geq v_i(b)$. Thus $v_l(b) = p_b$. Since l must demand b at some price vector $p' \geq p$ we must have at p' that $v_l(a) - p'_a = 0$. Since j pays at least p'_a the claim holds.

- $p_j^{VCG}(a,\sigma_{-j}) = v_i(a) - v_i(b) + v_l(b)$ (efficient allocation for σ_{-j} is $v_i(a)+v_l(b)$). If player i is not one of the last two players to remain, or if player i is a dummy player then a's price is at least $v_i(a) \geq v_i(a) - v_i(b) + v_l(b)$ (the inequality follows since $v_i(b) \geq v_l(b)$).

  Otherwise, players i and j are the last to remain active, and player i wins item b. In this case when player l quits b's price is at least $v_l(b)$ (this is true even if l is dummy). When player i demands b for the first time after l quitted, we must have that $v_i(b) - p_b \geq v_i(a) - p_a$ which implies $p_a \geq v_i(a) - v_i(b) + v_l(b)$, as needed.

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