Asynchronous Revision Games*

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Abstract

We consider situations where players prepare their actions before playing a game. Prepared actions are mutually observable, and opportunities to revise prepared actions arrive stochastically, by independent Poisson processes (one for each player). We show that the optimal trigger strategy equilibrium path can be characterized by an optimal control problem. Using this result, we examine what happens if one player has a higher arrival rate than the other.

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1 Introduction

In social or economic problems, agents often need to prepare their actions before they interact. They may obtain some information about what others are preparing and revise their prepared actions accordingly. To analyze such situations in a stylized model, Kamada and Kandori (2011; "KK" hereafter) introduced *revision games*. In a revision game, players play a "component game" only once at the prespecified deadline, before which they obtain stochastic opportunities to revise their actions. Prepared actions are assumed to be mutually observable. KK characterized the optimal trigger strategy equilibrium and showed that a certain level of cooperation is sustainable in a revision game. In other words, cooperation is possible even though a game is played only once, if (i) players prepare and revise their actions and (ii) players mutually monitor their prepared actions. KK showed that the optimal trigger strategy equilibrium is described by a simple differential equation, and the paper provided various economic applications. One key assumption in KK was that revisions are synchronous—there is one Poisson process according to which revision opportunities arrive, and at each opportunity all players revise simultaneously. This assumption enables them to obtain a clear-cut analysis, but in real-life applications players' revision opportunities are often not necessarily synchronized.

In this paper we extend the analysis of KK to the situation in which players' revision opportunities are asynchronous. Assuming that payoff is additively separable $\pi_i(a_i, a_{-i}) = b(a_{-i}) - c(a_i)$, we are able to characterize the optimal trigger strategy equilibrium by a differential equation. We first consider the case with homogeneous arrival rates, where the arrival rate of each player i is $\lambda_i = \lambda$, and show that the optimal trigger strategy equilibrium has exactly the same path of actions as in the synchronous model with arrival rate λ. We go on to consider the case with heterogeneous arrival rates. Unlike the case of homogeneous arrival rate, the characterization of the optimal trigger strategy is no longer self-evident, and we obtain the full characterization by means of an optimal control problem. Using this result, we examine what happens if one player has a higher arrival rate than the other. In the optimal trigger strategy equilibrium, the player with the higher arrival rate exerts more effort than the other player, when the deadline is not so close. Near the deadline, however, the effort levels of players are reversed: the player with the lower arrival rate exerts more effort.

Calcagno and Lovo (2010) and Kamada and Sugaya (2010) also considered asynchronous revision games for a class of games that is different from the one we analyze in the present paper. They focused on a certain class of games with finitely many actions, which includes coordination game and battle of the sexes. They showed that, in asynchronous revision games, a unique Nash equilibrium of the component game is selected, even if the component game has multiple equilibria. A crucial difference is that we consider games with continuous actions. With continuous actions, players have more freedom to fine-tune their actions near the deadline, and this degree of freedom creates multiple equilibria (and hence the possibility of cooperation) in the revision game.

2 Model

Consider a normal form game with two players $i = 1, 2$. Player i's action and payoff are denoted by $a_i \in A_i$ and $\pi_i(a_i, a_{-i})$, respectively. (Throughout the paper we denote the opponent of player i by $-i$.) This game is played at time 0, but players have to prepare their actions in advance, and they also have some stochastic opportunities to revise their prepared actions. Hence, technically the game under consideration is a dynamic game with preparation and revisions of actions, where the normal-form game π is played at the end of the dynamic game (time 0). To distinguish the entire dynamic game and its component π, the former is referred to as a *revision game* and π is referred to as the *component game*.

Time is continuous, $-t \in [-T, 0]$ with $T > 0$. At time $-T$, each player i chooses an action from A_i simultaneously. In time interval $(-T, 0]$, revision opportunities for player i arrive according to a Poisson process with arrival rate $\lambda_i > 0$ defined over the time interval $(-T, 0]$. At each arrival, i chooses an action from A_i. There is no cost of revision. At period 0, the payoffs $\pi(a') = (\pi_1(a'_1), \pi_2(a'_2))$ materialize, where a'_i is i's finally-revised action.

We assume that players observe all the past events in the revision game, and analyze subgame perfect equilibria. We assume that player i observes when revision opportunities arrived to player j, so that i can see if j has actually followed the equilibrium action path $x_j(t)$.

Unless otherwise noted, we assume throughout this paper that *the payoff function is additively separable with respect to each player's action.* Specifically, we consider payoff functions of the following form: For each $i = 1, 2$,

$$\pi_i(a_i, a_{-i}) = b(a_{-i}) - c(a_i),$$

where $a_i \in A_i = A = [0, \infty)$ and let a^* be the maximizer of $b(a) - c(a)$.[1] We also assume $b(0) = c(0) = 0$, and both b and c are continuous and strictly increasing (at this point, they may not be differentiable; we will assume differentiability in the full analysis in Section 6). Notice that there is a unique Nash equilibrium, $(a_1, a_2) = (0, 0)$.

In general, player i's revision plan depends not only on the timing of revision but also on the opponent's action that is fixed at the time of revision (hence a revision plan is represented by a function $x_i(t, a_{-i})$, where a_{-i} is the fixed action of the opponent at revision time $-t$). If the payoff is separable across players' actions, as we will show below, we can effectively ignore the dependence of action path with respect to the opponent's action, in the sense to be made precise in what follows. However, if the payoff function is not additively separable with respect to each player's action (as in the Cournot duopoly game), the dependence of revision plans on the opponent's action cannot be ignored, hence the analysis would be much more complicated than given in what follows. For example, it is not necessarily an optimal deviation to play the best response against the opponent's current action. We will demonstrate that even in the case of separable payoff functions, many complications and subtlety arise. Although the full analyses on non-separable payoff functions are beyond the scope of this paper, in the discussion section of this paper we will show that, even without separability, we can characterize the equilibrium payoff in the limit as the relative arrival rates diverge.

The characterization of the optimal equilibrium path is complicated, and the full analysis is provided in Setion 6. It will turn out in that section that when the deadline is close, the optimal equilibrium path follows the following *binding trigger strategy equilibrium path*:

Definition 1 The **binding trigger strategy equilibrium path** is a path given by $(x_1(t), x_2(t))_{t \in [0,T]}$ such that the following properties hold:

- If player i obtains a revision opportunity at time $-t$ and there has been no deviation in the past, then she prepares $x_i(t)$.
- If there has been a deviation in the past, she prepares action 0.

[1] We note that the case with $A = [0, \overline{a}]$ for some $\overline{a} < \infty$ may be more realistic, but the characterization of the optimal path would be a little bit messier, because the optimal path may overshoot a^* and may hit the maximum action $\overline{a}$. For the moment, we consider a cleaner case $\overline{a} = \infty$. Later we may consider the case with $\overline{a} < \infty$.

- The incentive compatibility conditions for both players bind for all time $-t \in [-T, 0]$.

The next section describes the basic incentive compatibility constraint. Then we proceed from easy to difficult. We start by Section 4 from the analysis of the case of homogeneous arrival rates, in which we find that the binding trigger strategy equilibrium path characterizes the actual optimal path just as in the case of synchronous revision that KK analyzed. Then in Section 5 we consider the binding trigger strategy equilibrium path. Full analysis of the optimal path is given in Section 6.

3 The Incentive Compatibility Constraint

Specifically, fixing the opponent's action a_j, player i's payoff from cooperation path at time $-t$ in the biding trigger strategy is

$$e^{-\lambda_j t} b(a_j) + \int_0^t b(x_j(\tau))\lambda_j e^{-\lambda_j \tau} d\tau - \left(e^{-\lambda_i t} c(x_i(t)) + \int_0^t c(x_i(\tau))\lambda_i e^{-\lambda_i \tau} d\tau \right). \quad (1)$$

On the other hand, i's payoff from defection is

$$e^{-\lambda_j t} b(a_j).$$

Hence the incentive compatibility condition for player i is:

$$e^{-\lambda_i t} c(x_i(t)) \leq \int_0^t \left(b(x_j(\tau))\lambda_j e^{-\lambda_j \tau} - c(x_i(\tau))\lambda_i e^{-\lambda_i \tau} \right) d\tau. \quad (2)$$

Notice that this condition does not depend on a_j, the fixed action of the opponent. This is the sense in which we said "we can effectively ignore the dependence of action path with respect to the opponent's action." The intuition for this is simple: Whether or not player i cooperates at time $-t$, the only case where the opponent's fixed action matters in either case is when the opponent j will not have any further opportunity in the future. This happens with the same probability in the two cases, and by separability what player i is preparing does not affect the payoff from j's fixed action, $b(a_j)$.

4 Homogeneous Arrival Rates

In this subsection, we consider the case in which two players' arrival rates are identical, i.e. $\lambda_1 = \lambda_2$. The case of heterogeneous arrival rates are discussed in the next subsection.

In the case of homogeneous arrival rates, there is a simple characterization of the optimal symmetric trigger strategy equilibrium. To see this, substitute $\lambda_1 = \lambda_2 = \lambda$ in the incentive compatibility condition (2), and note that the resulting condition is precisely identical to the incentive compatibility condition that we provided in KK. This gives us the following proposition:

Proposition 1 *The optimal trigger strategy equilibrium in KK also constitute the optimal trigger strategy equilibrium in the case of asynchronous revisions with equal arrival rates* $\lambda_i = \lambda_j = \lambda$.

That is, the results from KK apply in the case of homogeneous arrival rates.

When arrival rates are heterogeneous, however, the simple characterization in the above proposition no longer applies. We need to work with two distinct incentive constraints (for the two players) simultaneously, which complicates the analysis. We consider such a case in the next subsection.

5 Heterogeneous Arrival Rates (The Binding Trigger Strategy Equilibrium Path)

In this subsection we consider the case of heterogeneous arrival rates. Without loss of generality, assume $\lambda_1 < \lambda_2$.

Rearrange the incentive compatibility condition (2) to get

$$B_i(t) := e^{-\lambda_i t} c(x_i(t)) + \int_0^t c(x_i(\tau))\lambda_i e^{-\lambda_i \tau} d\tau \leq \int_0^t b(x_j(\tau))\lambda_j e^{-\lambda_j \tau} d\tau =: P_i(t). \tag{3}$$

This inequality has the following interpretation: $B_i(t)$ is the amount that player i can save by optimally deviating from the path $x(\cdot)$ at time $-t$, which is equal to the expected cost that i needs to pay on the path $x(\cdot)$. That is, $B_i(t)$ is the benefit of deviation. On the other hand, $P_i(t)$ is the penalty associated with deviation at $-t$. Since the opponent j follows the path $x(\cdot)$ at time $-t$, the loss is incurred only when there is another chance in the time interval $(-t, 0]$, which is why there is only one term in the right hand side. Overall, the inequality is saying that the benefit from deviation should be no larger than the penalty associated with it.

Our main question in this subsection is which player exerts more effort in the binding trigger strategy path. There are virtually two effects of player i's having a higher arrival rates than the opponent j. First, i is unlikely to be punished in the future upon deviation because j's arrival rate is low, so i has a larger incentive to deviate, which suggests i's action needs to be low. On the other hand, the benefit from i's deviation is low because i could have revision opportunities in the future many times near the deadline, so the expected amount that i can save by deviating is small anyway, which suggests i's action needs to be high. As we will see in what follows, these explanations are only a part of the story, and different effects are more relevant than others at different time points in the revision game. This results in the *reversal* of amounts of efforts that players exert at some time point $-t$. Notice that when two players' homogeneous arrival rates are increased by the same amount, Proposition 1 in the previous subsection and the "Arrival Rate Invariance" in KK imply that two players' actions are still the same. This means that the two effects (as well as the ones that we have not explained here but will do so in what follows) offset to each other. The complication arises when the arrival rates are changed by different amounts.

The plan of this subsection is as follows. First we show that the binding trigger strategy path is strictly increasing if there is a nontrivial cooperation at some time point $-t$, which is needed to prove the subsequent results. Then we consider two cases, $t \simeq 0$ and t large, to see which player exerts more effort in the binding trigger strategy path. These two parts imply that the relative amounts of the effort must be reversed at some time $-t$. We provide a numerical example in which this reversal takes place in the optimal trigger strategy path. The final part considers a bit different question, in which we ask whether having an infinitely low arrival rates relative to the opponent guarantees the Stackelberg outcome. To ease the load of argument, let us assume that the binding trigger strategy path is continuous.[2]

[2]This will follow from the same argument as in the main section.

5.1 Strict Increasingness of the Paths

Suppose that the binding trigger strategy path x_i is not strictly increasing. Then there must exist s and t such that $s < t$ and $x_i(s) = \max_{\tau \in [0,t]} x_i(\tau)$.[3] We compare the benefits and penalties from deviation at s and t.

First, compare the benefits:

$$B_i(t)-B_i(s) \leq e^{-\lambda_i t}c(x_i(t))-e^{-\lambda_i s}c(x_i(s))+\int_s^t c(x_i(s))\lambda_i e^{-\lambda_i \tau}d\tau = e^{-\lambda_i t}(c(x_i(t))-c(x_i(s)) \leq 0,$$

where the equality holds only when $x_i(s) = x_i(t)$. Second, compare the penalties:

$$P_i(t) - P_i(s) = \int_s^t b(x_j(\tau))\lambda_j e^{-\lambda_j \tau} d\tau \geq 0,$$

where the equality holds only when $x_j(\tau) = 0$ for almost all $\tau \in [s,t]$. But these two mean that, by the definition of s, $x_i(\tau) = 0$ for all $\tau \in [0,t]$. Hence, a nontrivial binding trigger strategy path must be strictly increasing on $[0,t]$. If there is no upper bound of t such that there exists s such that $x_i(s) = \max_{\tau \in [0,t]} x_i(\tau)$, this proves that a nontrivial binding trigger strategy path must be strictly increasing on $[0,\infty)$. If there is an upper bound, then it means that the path is strictly increasing on (t,∞), so again the proof is done. We summarize this point in the following proposition:

Proposition 2 *In the binding trigger strategy path with $x_j(t) > 0$ for some j and t, x_i is increasing for each $i = 1,2$.*

In what follows we consider the case where there exist binding and optimal trigger strategy paths such that $x_j(t) > 0$ for some j and t. Now we compare the incentives faced by two players in two cases: (i) the case when the deadline is very close ($t \simeq 0$) and (ii) the case when the deadline is very far away (t very large).

5.2 Case (i): $t \simeq 0$

First, consider case (i). In this case, $B_i(t)$ and $P_i(t)$ in the incentive compatibility condition (3) are approximately zero because x_i is close to the Nash action 0. We first show that it cannot be the case that $x_1(s) \leq x_2(s)$ for all$s \in [0,t]$ when $t > 0$ is close to zero. To see

[3]The continuity of the path x_i ensures the existence of the maximum.

this, suppose for the contrary that $x_1(s) \leq x_2(s)$ for all $s \in [0, t]$ when $t > 0$ is close to zero.

By the binding incentive constraints $B_i(t) = P_i(t)$, $i = 1, 2$, we have

$$\frac{e^{\lambda_1 s} B_1(s)}{e^{\lambda_2 s} B_2(s)} = \frac{e^{\lambda_1 s} P_1(s)}{e^{\lambda_2 s} P_2(s)} \quad \text{for all } s \in [0, t]. \tag{4}$$

By the fact that $x_1(\cdot)$ is increasing and our premise $x_1(s) \leq x_2(s)$ for all $s \in [0, t]$, we have $c(x_1(\tau)) < c(x_1(s)) \leq c(x_2(s))$ for all $\tau \leq s$. Therefore, the second term in $B_1(s)$ is bounded above by $\int_0^t c(x_2(s))\lambda_1 e^{-\lambda_1 \tau} d\tau$. Hence we have

$$\begin{aligned} \frac{e^{\lambda_1 s} B_1(s)}{e^{\lambda_2 s} B_2(s)} &< \frac{c(x_1(s)) + e^{\lambda_1 s} \int_0^t c(x_2(s))\lambda_1 e^{-\lambda_1 \tau} d\tau}{c(x_2(s))} \\ &= \frac{c(x_1(s))}{c(x_2(s))} + e^{\lambda_1 s}\left(1 - e^{\lambda_1 s}\right) \end{aligned}$$

Since $\frac{c(x_1(s))}{c(x_2(s))} \leq 1$ and $e^{\lambda_1 s}\left(1 - e^{\lambda_1 s}\right) \to 0$ as $s \to 0$, we obtain

$$\lim_{s \to 0} \frac{e^{\lambda_1 s} B_1(s)}{e^{\lambda_2 s} B_2(s)} \leq 1.$$

In contrast, we have

$$\begin{aligned} \frac{e^{\lambda_1 s} P_1(s)}{e^{\lambda_2 s} P_2(s)} &> \frac{\lambda_2 e^{(\lambda_1 - \lambda_2)s} \int_0^s b(x_2(\tau)) d\tau}{\lambda_1 \int_0^s b(x_1(\tau)) d\tau} \\ &\geq \frac{\lambda_2 e^{(\lambda_1 - \lambda_2)s} \int_0^s b(x_2(\tau)) d\tau}{\lambda_1 \int_0^s b(x_2(\tau)) d\tau} = \frac{\lambda_2 e^{(\lambda_1 - \lambda_2)s}}{\lambda_1}, \end{aligned}$$

and therefore

$$\lim_{s \to 0} \frac{e^{\lambda_1 s} P_1(s)}{e^{\lambda_2 s} P_2(s)} \geq \frac{\lambda_2}{\lambda_1} > 1.$$

Thus we have obtained

$$\lim_{s \to 0} \frac{e^{\lambda_1 s} B_1(s)}{e^{\lambda_2 s} B_2(s)} < \lim_{s \to 0} \frac{e^{\lambda_1 s} P_1(s)}{e^{\lambda_2 s} P_2(s)},$$

which contradicts (4). Hence, for any sufficiently small t, there is always some $s \leq t$ such that $x_2(s) < x_1(s)$.

The intuition for this result is simple. First, i's benefit from deviation at time $-t$ is determined mostly by what she saves at $-t$, as there is almost no revision chances in the future. This amount is independent of the arrival rate, and is increasing in the cost, and hence in the action. On the other hand, the penalty associated with deviation at time $-t$ pertains to the future events in nature, hence must depend on the arrival rates. Since there is very little time left until the deadline, the probability that there will be multiple revision opportunities in the future is negligible compared to the probability that there will be a single revision opportunity. This means that the relative likelihood that the punishment is triggered is determined by the ratio of the arrival rates, and the magnitude conditional on being punished is determined by the benefit from the opponent's future cooperation, which is increasing in the opponent's future action. Overall, if 2's action is higher than 1's from time $-t$ on, then 2's benefit from deviation is higher than 1's because 2 has much more to save than 1 does, while 2's penalty is lower than 1's because 2 expects fewer chances to be punished in the future and the magnitude of the penalty conditional on being punished is smaller. But this means that if 1's incentive compatibility condition is binding then 2's cannot bind.

The conclusion up to this point implies that it is either that there is $\bar{t} > 0$ such that for all $t \in (0, \bar{t})$, $x_2(t) < x_1(t)$ holds, or that there is an infinite sequence of times $\{t_k\}_{k=0}^{\infty}$ such that $t_k \to 0$, $x_1(t_k) = x_2(t_k)$, and there exists $\epsilon > 0$ such that $x_1(t) < x_2(t)$ for all $t \in (t_k, t_k + \epsilon)$ for all k. An analogous argument as the one provided here shows that the latter cannot hold. We provide the proof for this in Appendix D.

To summarize, we obtain the following proposition:

Proposition 3 *There exists $\bar{t} > 0$ such that for all $t \in (0, \bar{t})$, $x_2(t) < x_1(t)$ where $(x_1(t), x_2(t))_{t \in [0,T]}$ is the optimal trigger strategy equilibrium path.*

5.3 Case (ii): t large

Next, we consider case (ii), i.e. the case when the deadline is far away. We show that the inequality in Proposition 3 must be reversed at some point in time.

To see this, suppose that $x_2(\tau) \leq x_1(\tau)$ for all $\tau \in [0, t]$. We show that for sufficiently large t, the incentive compatibility condition for one player must be unable to hold with equality at time $-t$.

Consider the limits of $B_i(t)$ and $P_i(t)$ as $t \to \infty$. For $B_i(t)$, in the limit we are left with the second term. Since $x_2(\tau) \leq x_1(\tau)$ for all $\tau \in [0, t]$ and that the exponential distribution with parameter λ_1 first order stochastically dominates that with parameter λ_2 ($> \lambda_1$), for sufficiently large t we must have $B_1(t) > B_2(t)$. On the other hand, for the penalty term, by exactly the same argument we must have $P_1(t) < P_2(t)$ for sufficiently large t. Together, we cannot have $B_1(t) = P_1(t)$ and $B_2(t) = P_2(t)$ simultaneously.

The intuition for this is again simple. If a player cheats when there is much time to reach the deadline, punishment will be triggered almost certainly. Since $\lambda_1 < \lambda_2$, player 1 expects less revision chances near the deadline so the amount that she can save is larger than the case when she has a higher arrival rate, and if $x_2 \leq x_1$ in the future then the amount that she can save conditional on having a revision opportunity is no smaller than what 2 would be able to save. On the other hand, the penalty associated with 1's deviation is determined by the expected benefit that 2 brings to her, and it is smaller than what 1 brings to 2 because $x_2 \leq x_1$ in the future and 2 has more opportunities near the deadline than 1 does. Overall, player 1 expects a higher benefit and a lower penalty of deviation than player 2, so if 2's incentive compatibility condition is binding then 1's cannot bind. We summarize this finding in the following proposition:

Proposition 4 *In the biding trigger strategy equilibrium path $(x_1(t), x_2(t))_{t\in[0,\infty)}$, it cannot be that $x_1(t) > x_2(t)$ for all $t \in (0, \infty)$.*

5.4 Reversal of the Optimal Trigger Strategy Paths: Good Exchange Game

Proposition 4 suggests that in the binding trigger strategy path, *the sizes of x_1 and x_2 are reversed at some time $-t$.* Later we will show that the optimal path follows the binding path until one player's action hits a^*. This means that if the point of reversal is at an action below a^*, then the reversal occurs in the optimal trigger strategy path as well. Numerical computation reveals that this can indeed happen: In the good exchange game example ($\pi_i(x_i, x_j) = b(x_j) - c(x_i) = x_j - x_i^2$) with arrival rates $\lambda_1 = 1$ and $\lambda_2 = 5$, the reversal occurs before the path reaches the optimal action $a^* = \frac{1}{2}$. Figure 1 depicts the optimal trigger strategy path. Precisely, the following properties are true:

1. Near $t = 0$, $x_1(t) > x_2(t)$, as we have shown above.

2. When t is larger than some threshold value, however, the inequality is reversed: $x_1(t) < x_2(t)$.

3. $x_1(t)$ is concave and $x_2(t)$ is convex in the binding trigger strategy equilibrium path.

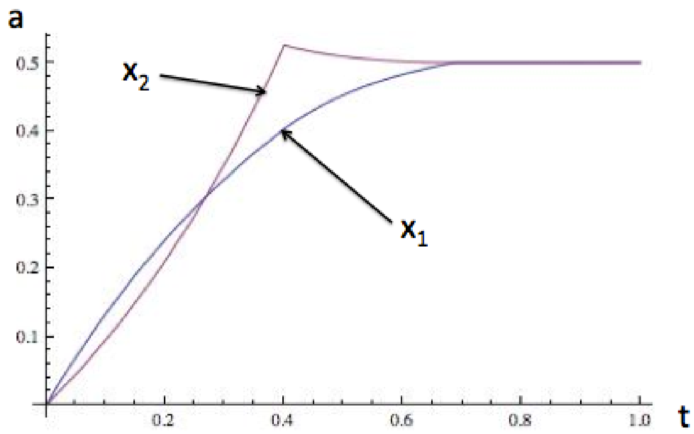

Figure 1: **The optimal path for the good exchange game in asynchronous revision game:** $\lambda_1 = 1$, $\lambda_2 = 5$.

It is possible to prove Item 3 for $\pi_i = b(x_j) - c(x_i) = x_j - x_i^2$, but we do not know if this is generally true. Since the proof is a bit complicated, it is omitted.

6 Full Analysis of the Optimal Path

6.1 The Definition of The Optimal Path

We now assume differentiability and further regularity conditions and analyze the property of the optimal path in depth. That is, we assume $b' > 0$, $b" < 0$, $c" > 0$, $c'(0) = 0$, $c'(a) > 0$ for $a > 0$. At the unique Nash equilibrium $(a_1, a_2) = (0, 0)$, the first and second order conditions of payoff maximization $-c' = 0$ and $-c'' < 0$ are satisfied. We consider the case $\lambda_1 \leq \lambda_2$.

A trigger strategy equilibrium is characterized by the equilibrium path (x_1, x_2), where $x_i : [0, T] \to A, i = 1, 2$. Recall that trigger strategy requires that player i should revise his action to $x_i(t)$, when he has a revision opportunity at time $-t$. If any player deviates from the equilibrium revision path (x_1, x_2), player i chooses the Nash action 0 in all revision opportunities. The payoff to player i at (x_1, x_2) is given by

$$\begin{aligned} V_i(x_1, x_2) &= b(x_{-i}(T))e^{-\lambda_{-i}T} - c(x_i(T))e^{-\lambda_i T} \\ &\quad + \int_0^T \left(b(x_{-i}(t))\lambda_{-i}e^{-\lambda_{-i}t} - c(x_i(s))\lambda_i e^{-\lambda_i t} \right) dt. \end{aligned}$$

To derive the optimality conditions below, we restrict our attention to the following paths.

Definition 2 $X^{PC} := \{(x_1, x_2)|\ x_i : [0, T] \to A \text{ is piecewise continuous for } i = 1, 2\}$.

The *optimal (trigger strategy equilibrium) path* (x_1, x_2) is defined to be the one that maximizes the sum of revision game payoffs subject to the (trigger strategy) incentive constraints:

Problem 1:

$$\max_{(x_1,x_2)\in X^{PC}} \sum_{i=1}^{2} \left[\left(b(x_i(T)) - c(x_i(T))\right) e^{-\lambda_i T} + \int_0^T \left(b(x_i(t)) - c(x_i(t))\right) \lambda_i e^{-\lambda_i t} dt \right] \tag{5}$$

$$\text{s.t. } \forall i \forall t \quad \int_0^t b(x_{-i}(s))\lambda_{-i}e^{-\lambda_{-i}s}ds \geq c(x_i(t))e^{-\lambda_i t} + \int_0^t c(x_i(s))\lambda_i e^{-\lambda_i s}ds.$$

The objective function is equal to the sum of revision game payoffs V_1+V_2, after rearranging terms. The left hand side of the constraint is the benefit lost when player i deviates at time $-t$, while the right hand side represents the cost saved by the deviation.

6.2 Necessary and Sufficient Conditions

First we rewrite the optimization problem (Problem 1) in the conventional form of optimal control. First, note that we can consider

$$c_i(t) := c(x_i(t))$$

as the control variable. Accordingly, define

$$B(c) := b(c^{-1}(c)),$$

and note that, for $c > 0$,

$$B' = \frac{b'}{c'} > 0 \text{ and}$$

$$B'' = \frac{b''}{c'} - \frac{b'c''}{(c')^2} < 0,$$

As we assume that $b'(0) > 0$, $b"(0) < 0$ and $c"(0) > 0$ are all finite, $c'(0) = 0$ implies

$$B'(0) = \infty \text{ and } B''(0) = -\infty.$$

Note that B is a strictly increasing concave function. With those redefinition of variables, the objective function is concave and the set of variables that satisfy the incentive constraints become convex (see (9) below). Define the state variable by

$$k_i(t) := \int_0^t B(c_{-i}(s))\lambda_{-i}e^{-\lambda_{-i}s}ds - \int_0^t c_i(s)\lambda_i e^{-\lambda_i s}ds. \tag{6}$$

Note that, with this definition, the incentive constraint is simply expressed as $k_i(t) \geq c(x_i(t))e^{-\lambda_i t}$. Next, we replace the first term in the objective function (5) by a function of the terminal state variable $k_i(T)$.

Definition 3 $W_i(k_i(T))$ is the optimal value associated with

$$\max_{c_i(T)} \left(B(c_i(T)) - c_i(T)\right) e^{-\lambda_i T} \tag{7}$$

$$\text{s.t. } k_i(T) \geq c_i(T)e^{-\lambda_i T}.$$

Recall that $c(a^*)$ is the value of c that maximizes $B(c) - c$. When $c(a^*)$ is not feasible (i.e., $k_i(T) < c_i(a^*)e^{-\lambda_i T}$), the above constraint is binding and the optimal $c_i(T)$ is equal to $k_i(T)e^{\lambda_i T}$. Hence we have

$$W_i(k) = \begin{cases} B(ke^{\lambda_i T})e^{-\lambda_i T} - k & \text{if } k < c_i(a^*)e^{-\lambda_i T} \\ \\ \left(B(c(a^*)) - c(a^*)\right) e^{-\lambda_i T} & \text{otherwise} \end{cases} \tag{8}$$

Note that $W_i(\cdot)$ is a concave function.

Now define

$$C^{PC} := \{(c_1, c_2) |\ c_i : [0, T] \to [0, c(\overline{a})] \text{ is piecewise continuous for } i = 1, 2\}$$

Then, our optimization problem (Problem 1) can be expressed as an optimal control problem:

Problem 2:

$$\max_{(c_1,c_2)\in C^{PC}} \sum_{i=1}^{2} \left[\int_0^T \left(B(c_i(t)) - c_i(t) \right) \lambda_i e^{-\lambda_i t} ds + W_i(k_i(T)) \right]$$

$$\text{s.t.} \quad \forall i \forall t \quad k_i(t) \geq c_i(t) e^{-\lambda_i t} \tag{9}$$

$$\dot{k}_i(t) = B(c_{-i}(t)) \lambda_{-i} e^{-\lambda_{-i} t} - c_i(t) \lambda_i e^{-\lambda_i t} \tag{10}$$

$$k_i(0) = 0 \tag{11}$$

Since we consider piecewise continuous c_i, (10) is required for almost all t (i.e., outside a measure zero set where c_i jumps). The incentive constraint (9) is a "mixed inequality constraint" on a flow (control) variable ($c_i(t)$) and a stock (state) variable ($k_i(t)$). Conditions for optimality in such problems are found in, for example, Sethi and Thompson (2000, Chapter 3).

Let $\mu_i(t)$ and be the Lagrange multiplier associated with the law of motion of state variable (10), and define Hamiltonian

$$\begin{aligned} & H(t) \\ : & = \sum_{i=1}^{2} \left[\underbrace{\left(B(c_i(t)) - c_i(t) \right) \lambda_i e^{-\lambda_i t}}_{\text{The integrand of the objective function}} + \mu_i(t) (\underbrace{B(c_{-i}(t)) \lambda_{-i} e^{-\lambda_{-i} t} - c_i(t) \lambda_i e^{-\lambda_i t}}_{\text{The right hand side of (10)}}) \right] \\ = & \sum_{i=1}^{2} \left[(1 + \mu_{-i}(t)) B(c_i(t)) - (1 + \mu_i(t)) c_i(t) \right] \lambda_i e^{-\lambda_i t}. \end{aligned}$$

Optimality requires that $H(t)$ is to be maximized for each t (with respect to the control variables $c_i(t)$, $i = 1, 2$) subject to the incentive constraint (9). Hence we consider Lagrangian

$$L(t) := H(t) + \sum_{i=1}^{2} \gamma_i(t) \left(k_i(t) - c_i(t) e^{-\lambda_i t} \right),$$

where $\gamma_i(t)$ is the Lagrange multiplier associated with the incentive constraint (9).

Optimality conditions are

$$\mu_i(T) = W_i'(k_i^*(T)) \text{ (transversality condition)},$$

the Kuhn-Tucker conditions for the constrained maximization of the Hamiltonian

$$\frac{\partial L(t)}{\partial c_i(t)} = 0,$$

$$\gamma_i(t) \geq 0 \text{ and } \gamma_i(t)\left(k_i(t) - c_i(t)e^{-\lambda_i t}\right) = 0,$$

and

$$\frac{\partial L(t)}{\partial k_i(t)} + \dot{\mu}_i(t) = 0,$$

together with the original constraints (9)-(11). Those conditions are necessary for optimality, and they are also sufficient when

1. the integrand of the objective function ($\sum_{i=1}^{2}(B(c_i(t)) - c_i(t))\lambda_i e^{-\lambda_i t}$) is concave in $c_i(t), k_i(t)$, $i = 1, 2$,
2. terminal value function ($\sum_{i=1}^{2} W_i(k_i(T))$) is concave in $k_i(T)$, $i = 1, 2$, and
3. the set of $(c_i(t), k_i(t))$ that satisfies the mixed inequality constraint (9) is convex.

Those requirements 1-3 are satisfied in our model. Hence we obtained the following characterization of optimal path.

Proposition 5 *Path (c_1^*, c_2^*) is the optimal solution to Problem 2 (and hence (x_1^*, x_2^*) defined by $(c(x_1^*), c(x_2^*)) = (c_1^*, c_2^*)$ is the optimal path that solves Problem 1), if and only if the following conditions hold. There exit a continuous, piecewise continuously differentiable function $\gamma_i(t)$, a piecewise continuous function $\mu_i(t)$, and the state variable (k_1^*, k_2^*) determined by*

$$\dot{k}_i^*(t) = B(c_{-i}^*(t))\lambda_{-i}e^{-\lambda_{-i}t} - c_i^*(t)\lambda_i e^{-\lambda_i t}, \tag{12}$$

$$k^*(0) = 0 \tag{13}$$

that satisfy the following conditions for $i = 1, 2$.

$$\forall t \quad k_i^*(t) \geq c_i^*(t)e^{-\lambda_i t}, \tag{14}$$

$$\mu_i(T) = \begin{cases} B'(c_i^*(T)) - 1 & \textit{if } k_i^*(T) < c(a^*)e^{-\lambda_i T} \\ \\ 0 & \textit{otherwise} \end{cases}, \tag{15}$$

$$c_i^*(T) = \begin{cases} k_i^*(T)e^{\lambda_i T} & \textit{if } k_i^*(T) < c(a^*)e^{-\lambda_i T} \\ \\ c(a^*) & \textit{otherwise} \end{cases}, \tag{16}$$

and for almost all t,

$$\lambda_i \left[(1 + \mu_{-i}(t))B'(c_i^*(t)) - (1 + \mu_i(t))\right] - \gamma_i(t) = 0, \tag{17}$$

$$\gamma_i(t) \geq 0 \textit{ and } \gamma_i(t)\left(k_i^*(t) - c_i^*(t)e^{-\lambda_i t}\right) = 0, \tag{18}$$

$$\dot{\mu}_i(t) = -\gamma_i(t). \tag{19}$$

Note that (17) and (19) correspond to $\frac{\partial L(t)}{\partial c_i(t)} = 0$ and $\frac{\partial L(t)}{\partial k_i(t)} + \dot{\mu}_i(t) = 0$ respectively. Also note that (15) corresponds to the transversality condition $\mu_i(T) = W_i'(k_i^*(T))$, because W_i' can be calculated by (8) as

$$W_i'(k_i^*(T)) = \begin{cases} B'(\underbrace{c_i(T)}_{k_i^*(T)e^{\lambda_i T}}) - 1 & \text{if } k_i^*(T) < c(a^*)e^{-\lambda_i T} \\ \\ 0 & \text{otherwise} \end{cases}.$$

Let us now provide intuitive interpretation of the optimality conditions. The key conditions are (17) and (18). They show that the optimal path maximizes, at each point t,

$$(1 + \mu_{-i}(t))B(c_i(t)) - (1 + \mu_i(t))c_i(t) \tag{20}$$

$$\text{subject to } k_i(t) \geq c_i(t)e^{-\lambda_i t}. \tag{21}$$

Contrast this to the simple maximization of $\pi_1(t) + \pi_2(t)$, which requires

$$B(c_i(t)) - c_i(t)$$

is to be maximized, subject to the same incentive constraint (21). In addition to the direct benefit $B(c_i(t))$, increasing $B(c_i(t))$ is accompanied by additional benefit of relaxing the other player's incentive constraints for in $(t,T]$. This indirect effect is captured by $\mu_{-i}(t)$ in the correct objective function (20). Similarly, the direct cost $c_i(t)$ is accompanied by additional cost of tightening the player's own incentive constraint in $(t,T]$, and this effect is captured by $\mu_i(t)$ in (20). A subtle but important remark is in order for $t=T$. This remark plays an important role in characterizing the optimal path.

Remark T: The Hamiltonian maximization condition for $t=T$,

$$\max_{c_i}(1+\mu_{-i}(T))B(c_i)-(1+\mu_i(T))c_i \tag{22}$$

$$\text{s.t. } k_i(t)\geq c_i e^{-\lambda_i t}$$

is different from the condition to determine $c_i(T)$ (= the optimal control at T):

$$\max_{c_i} B(c_i)-c_i \tag{23}$$

$$\text{s.t. } k_i(t)\geq c_i e^{-\lambda_i t}.$$

The former program (22) determines $\lim_{t\uparrow T} c_i(t)$, while the latter (23) determines the actual $c_i(T)$. Later we show that $\lim_{t\uparrow T} c_i(t)$ is actually different from $c_i(T)$ in some cases (see Propositions 7, 8 and 9). Since the Hamiltonian maximization condition is required for *almost all* t, the solution to (22) can be different from the solution to (23), and this is not a contradiction. In such a case, the Hamiltonian maximization condition turns out to be satisfied for all $t\in[0,T)$ but not on a measure-zero set $\{T\}$ (hence the solution to (22) is $\lim_{t\uparrow T} c_i(t)\neq c_i(T)$). Nonetheless, we calculate the solution to (22) in such a case, because this turns out to be useful in computing the optimal path for $t\in[0,T)$. To summarize, we should interpret the program (22) as the optimization condition for t slightly smaller than T (more precisely, (22) determines $\lim_{t\uparrow T} c_i(t)$, not (necessarily) $c_i(T)$).

Now recall that the indirect incentive effects captured by $\mu_i(t)$ arise in the interval $(t,T]$, and this interval shrinks when t increases. Hence, $\mu_i(t)$, $i=1,2$, should be decreasing. This is captured by (19) and (18), which shows $\dot{\mu}_i(t)=-\gamma_i(t)\leq 0$. Note that, because $\gamma_i(t)$

is the Lagrange multiplier associated with the constraint (21), it measures the marginal benefit of relaxing the incentive constraint at t, consistent with our intuitive explanation. For $t \approx T$, the only major indirect effects is to change the incentive constraint of the initial action choice at T, and this is represented by the transversality condition (15).

6.3 The Optimal Path

In this section, we derive the optimal path from the optimality conditions in Proposition 5. Note that the differential equations for the control variables $c_i(t)$, $i = 1, 2$ are derived by two of the optimality conditions, (12) and (14), *if the latter (the incentive constraint) is binding.* By differentiating $k_i(t) = c_i(t)e^{-\lambda_i t}$, we have $\dot{k}_i(t) = \dot{c}_i(t)e^{-\lambda_i t} - \lambda_i c_i(t)e^{-\lambda_i t}$. From this and (12), we can derive the differential equation for our control variable, when (14) is binding:

$$\dot{c}_i(t) = \lambda_{-i} B(c_{-i}(t)) e^{(\lambda_i - \lambda_{-i})t}. \tag{24}$$

Recall that our control variable is $c_i = c(x_i)$ and $B(c(x_i)) = b(x_i)$, where x_i represents an action in the component game. Hence the above equation can be transformed into our differential equation of the binding path of action

$$\dot{x}_i(t) = \lambda_{-i} \frac{b(x_{-i}(t))}{c'(x_i(t))} e^{(\lambda_i - \lambda_{-i})t}. \tag{25}$$

We can use this result to show that, in the model with heterogeneous arrival rates $\lambda_1 \neq \lambda_2$, the non-trivial path (x_1^0, x_2^0) with binding incentive constraints is optimal, when T is small so that x_t^0 is yet to hit the optimal action a^*.

Note: We need to show the existence of the non-trivial solution. The same argument as in Revision Games (the finite time condition) would do.

Note that this is not completely obvious. Recall that we are examining the case where $a^N < a^*$. In the synchronous case, choosing maximum possible symmetric action $x(t)$ at each t both improves total payoff and relaxes the incentive constraints of $s > t$. In the asynchronous case, however, choosing maximum possible action $x_i(t)$ at each t improves total payoff but it makes the incentive constraints of $s > t$ more stringent (because when i deviates at $s > t$, i can save more cost). Therefore, it might be more profitable to take a smaller action at t, because in that way player i can supply more effort at some $s > t$. Proposition above actually show that this is not the case. The very rough intuition is that the marginal productivity of player i's effort (to improve the total payoff) is higher at t than at s ($s > t$), because i's action is smaller at t. Hence the aforementioned manipulation does not pay.

Proposition 6 *Let (c_1^0, c_2^0) be the non-trivial path with binding incentive constraints and suppose that T is small so that $c_i^0(T) < c(a^*)$ for $i = 1, 2$ Then, (c_1^0, c_2^0) is optimal.*

Proof. Let $\mu_i(t)$, $i = 1, 2$ be the solution to the system of differential equations

$$\dot{\mu}_i(t) = -\lambda_i \left[(1 + \mu_{-i}(t)) B'(c_i^0(t)) - (1 + \mu_i(t)) \right], i = 1, 2 \tag{26}$$

with boundary conditions

$$\begin{aligned} \mu_i(T) &= B'(k_i^*(T) e^{\lambda_i T}) - 1 \\ &= B'(c_i^0(T)) - 1 > 0 \quad i = 1, 2, \end{aligned} \tag{27}$$

where the second equality and the last inequality follows from $c_i^0(T) < c(a^*)$ (recall $B'(c(a^*)) = 1$), under which the terminal incentive constraint $k_i^*(T) \geq c_i(T) e^{-\lambda_i T}$ is binding. Let us also define

$$\gamma_i(t) := \lambda_i \left[(1 + \mu_{-i}(t)) B'(c_i^0(t)) - (1 + \mu_i(t)) \right] \tag{28}$$

Our task is to show that $\gamma_i(t) \geq 0$. If this is shown, (c^0, γ, μ) satisfies all the conditions in Proposition 5 and therefore optimal.

First, consider $t = T$. By (27), we have

$$\begin{aligned} &(1 + \mu_{-i}(T)) B'(c_i^0(T)) - (1 + \mu_i(T)) \\ &> B'(c_i^0(T)) - (1 + \mu_i(T)) = 0. \end{aligned}$$

The inequality comes from $\mu_{-i}(T) > 0$ and the equality is implied by $\mu_i(T) = B'(c_i^0(T)) - 1$. Hence $\gamma_i(T) > 0$.

Consider

$$Q := \{t \in [0,T] |\ \forall s \geq t \ \ \gamma_i(s) > 0,\ i = 1,2\}.$$

Since $T \in Q$, Q is not empty and therefore we can define

$$t' := \inf Q.$$

Since $\gamma_i(T) > 0$, $i = 1,2$, by continuity of γ_i, we have

$$t' < T. \tag{29}$$

We suppose $t' > 0$ and find a contradiction. We first claim that there is i such that

$$\gamma_i(t') = 0. \tag{30}$$

If $\gamma_i(t') > 0$ for both $i = 1,2$, by continuity of $\gamma_i(t)$, we must have $\inf Q < t'$, a contradiction.

Now we show that

$$\gamma_i'(t') < 0. \tag{31}$$

Note that, $\gamma_i(t') = 0$ implies $\dot{\mu}_i(t') = -\gamma_i(t') = 0$. Hence, by differentiating (28), we obtain

$$\gamma_i'(t') = \dot{\mu}_{-i}(t')B'(c_i^0(t')) + (1 + \mu_{-i}(t'))B''(c_i^0(t'))\dot{c}_i^0(t').$$

Now we evaluate the each term in this expression as follows.

(i) Since $\dot{\mu}_{-i}(t) = -\gamma_{-i}(t) < 0$ for all $t \in Q$, by taking limit $t \to t'$, we obtain $\dot{\mu}_{-i}(t^1) \leq 0$.

(ii) Since $\dot{\mu}_{-i}(t) = -\gamma_{-i}(t) < 0$ for all $t \in Q$ and $\dot{\mu}_{-i}(T) > 0$, we obtain $\mu_{-i}(t') > 0$.

Those facts, together with $B' > 0$, $B'' < 0$ and $\dot{c}_i^0(t^1) > 0$, show (31).

When (29), (30) and (31) are satisfied, however, there must be $t \in Q$ which is slightly larger than t' such that $\gamma_i(t) < 0$, which contradicts the definition of Q.

Hence $\inf Q$ must be equal to zero, and this implies $\gamma_i(t) > 0$ for all t and all i. ■

With this result, we are now ready to characterize fully the optimal path when T is large. Consider again (c_1^0, c_2^0), the non-trivial path with binding incentive constraints. Note that (c_1^0, c_2^0) is the non-trivial solution (i.e., the solution satisfying $c_i(t) \neq c(a^N) = 0$ for $t > 0$, $i = 1,2$) to (24) with boundary condition $c_i(0) = 0$, $i = 1,2$. By (24), both $c_1^0(t)$ and $c_2^0(t)$ are strictly increasing, and we now show that at least one of them hits the optimal level $c(a^*)$ at some finite time. The reason is the following. Consider player i who has the largest arrival rate ($\lambda_i \geq \lambda_{-i}$) and fix any $c^0_{-i}(t') > 0$. Then, for $t \geq t'$, $\dot{c}^0_i(t)$ is strictly positive and bounded away from 0:

$$\dot{c}^0_i(t) = \lambda_{-i} B(c^0_{-i}(t)) e^{(\lambda_i - \lambda_{-i})t} > \lambda_{-i} B(c^0_{-i}(t')) > 0.$$

Hence $c^0_i(t)$ must hit $c(a^*)$ at a finite time.

The above argument guarantees that there is player j whose binding path $c^0_j(t)$ hits the optimal level $c(a^*)$ first:

Definition 4 Let j be the player whose binding path hits the optimal level first, and denote the hitting time by t^0: $c^0_j(t^0) = c(a^*)$ and $c^0_{-j}(t^0) \leq c(a^*)$.

Since $c^0_{-j}(t)$ is strictly increasing, the above condition $c^0_{-j}(t^0) \leq c(a^*)$ guarantees that $c^0_{-j}(t)$ hits $c(a^*)$ after or at t^0 (if it ever does).

If both players' binding paths hit the optimal simultaneously ($c^0_{-j}(t^0) = c(a^*)$), then Proposition 6 implies that the binding path is optimal. More precisely, we have the following very simple characterization.

Corollary 1 *Suppose the binding paths $c^0_1(t)$ and $c^0_2(t)$ hit the optimal level $c(a^*)$ at the same time t^0. Then, the following holds. If $T \leq t^0$, the binding path $(c^0_1(t), c^0_2(t))$ for all $t \in [0, T]$ is optimal. If $T > t^0$,*

$$c_i(t) = \begin{cases} c^0_i(t) & \text{for } t \in [0, t^0] \\ c(a^*) & \text{for } t \in (t^0, T] \end{cases}$$

is optimal.

In particular, the Corollary above provides the optimal in the case of symmetric arrival rates $\lambda_1 = \lambda_2$, as we have shown in the companion paper Kamada and Kandori (2011). In what follows, we consider the remaining case and therefore assume:

Assumption A6.3: When player j's binding path hits the optimal at t^0, the other player's binding path is below the optimal level: $c^0_j(t^0) = c(a^*)$ and $c^0_{-j}(t^0) < c(a^*)$.

We are going to show that the incentive constraints of both players continue to bind even after t^0. This means that the optimal action overshoots the optimal level a^*.

Proposition 7 *Under A6.3, there exists $t^1 \in (t^0, \infty]$ such that the following statements are true if and only if $t^0 \leq T \leq t^1$: The incentive constraints of both players are binding for all $t \in [0, T)$, and the optimal path is given by*

$$(c_j(t), c_{-j}(t)) = \begin{cases} (c^0_j(t), c^0_{-j}(t)) & \text{if } t \in [0, T) \\ \\ (c(a^*), c^0_{-j}(T)) & \text{if } t = T \end{cases} \tag{32}$$

Furthermore, t^1 is the unique solution to

$$B'(c^0_{-j}(t))B'(c^0_j(t)) - 1 = 0 \tag{33}$$

if the solution exists, and otherwise $t^1 = \infty$.

Remarks: Since $T > t^0$ implies $c^0_j(T) > c(a^*)$, this Proposition shows two interesting features of the optimal path, when $t^1 \geq T > t^0$. First, the action of player j (whose incentive is not binding near T) "overshoots" the optimal level $c_j(t) > c(a^*)$, for t close to T. Second, there is discontinuity in player j's optimal action at $t = T$; $\lim_{t \to T} c_j(t) = c^0_j(T) > c_j(T) = c(a^*)$. The intuition is as follows. At T, player j simply takes optimal action a^* to maximize the total payoff. At $t = T - \varepsilon$, however, player j exerts higher effort than the optimal ($c^0_j(t) > c(a^*)$) to improve the other player's action in $(t - \varepsilon, T]$.

Proof. Step 1: First, we show that (32) is optimal only if $t^0 \leq T \leq t^1$. If $c_j(T) = c(a^*)$ is optimal, the incentive constraint at T is slack for player j (i.e., $c(a^*)e^{-\lambda_j T} \leq k_i(T)$). Since $c^0_j(T)$ satisfies the binding incentive constraint $c^0_j(T)e^{-\lambda_j T} = k_i(T)$, we must have

$$c^0_j(T) \geq c(a^*).$$

Recall that $c^0_j(t^0) = c(a^*)$ and $c^0_j(t)$ is strictly increasing. Hence the above inequality is equivalent to

$$t^0 \leq T. \tag{34}$$

In summary, (32) is optimal only if (34) is satisfied.

If $(c_j(T), c_{-j}(T)) = (c(a^*), c^0_{-j}(T))$ is optimal, then by the transversality condition (15), we must have

$$\begin{cases} \mu_j(T) := 0 \\ \mu_{-j}(T) := B'(c^0_{-j}(T)) - 1. \end{cases} \tag{35}$$

With this definition, consider the Hamiltonian maximization problem at T with respect to j's action

$$\text{(PT)} \quad \max_{c_j} \left(1 + \mu_{-j}(T)\right) B(c_j) - (1 + \mu_j(T))c_j$$
$$\text{s.t. } c_j e^{-\lambda_j T} \leq k_j(T).$$

Our candidate path (32) is optimal only if the solution to this program (PT) is $c^0_j(T)$. This follows from the fact that (PT) determines $\lim_{t \to T} c_j(t)$ (recall the Remark T). More precisely, the argument goes as follows. Note that $c^0_j(T)$ satisfies the binding incentive constraint $c^0_j(T)e^{-\lambda_j} = k_j(T)$. If $c^0_j(T)$ is not the solution to (PT), then we have an interior solution c^*_j with $c^*_j e^{-\lambda_j T} < k_j(T)$. Then, by continuity, the optimization program

$$\text{(Pt)} \quad \max_{c_j} \left(1 + \mu_{-j}(t)\right) B(c_j) - (1 + \mu_j(t))c_j$$
$$\text{s.t. } c_j e^{-\lambda_j t} \leq k_j(t).$$

also has an interior solution satisfying $c_j(t)e^{-\lambda_j T} < k_j(t)$ for some interval $(\widehat{t}, T)$. This means that the binding path $c^0_j(t)$, which satisfies $c^0_j(t)e^{-\lambda_j T} = k_j(t)$ for all t, is not the solution to (Pt) on $(\widehat{t}, T)$, and the necessary conditions for optimal path (17) and (18) (which are also the necessary conditions for the solution to (Pt)) cannot be satisfied for almost all t by $c^0_j(t)$.

Hence, the necessary condition for our candidate path (32) to be optimal is that $c^0_j(T)$ is the solution to (PT). Since $c^0_j(T)$ satisfies the constraint of (PT) with equality, $c^0_j(T)$ is the solution to (PT) if and only if

$$\frac{d}{dc_j}\left[\left(1 + \mu_{-j}(T)\right) B(c_j) - (1 + \mu_j(T))c_j\right]\Big|_{c_j = c^0_j(T)}$$
$$= B'(c^0_{-j}(T))B'(c^0_j(T)) - 1 \geq 0. \tag{36}$$

Let us now examine the properties of

$$\varphi(t) := B'(c^0_{-j}(t))B'(c^0_j(t)) - 1.$$

Recall that t^0 is the time where $c^0_j(t^0) = c(a^*)$ and our maintained assumption is $c^0_{-j}(t^0) < c(a^*)$ (see A6.3). The latter implies $B'(c^0_{-j}(t^0)) - 1 > 0$ and therefore we have $\varphi(t^0) > 0$ because

$$\begin{aligned} & B'(c^0_{-j}(t^0))B'(c^0_j(t^0)) - 1 \\ > \; & B'(c^0_j(t^0)) - 1 = B'(c(a^*)) - 1 = 0. \end{aligned}$$

Since B' is strictly decreasing and $c^0_i(t)$, $i = 1, 2$ are strictly increasing, $\varphi(t)$ is strictly decreasing. Hence either there is a unique finite $t^1 > t^0$ such that $\varphi(t^1) = 0$ holds ($\Leftrightarrow$(33)), or $\varphi(t) > 0$ for all t (in which case $t^1 = \infty$). In either case, $\varphi(t) \geq 0$ if and only if $t \leq t^1$, and therefore (36), the Hamiltonian maximization condition at T for j's action, is satisfied if and only if

$$T \leq t^1. \tag{37}$$

Hence, we have obtained two necessary conditions (34) and (37). Therefore, a necessary condition for (32) to be optimal is

$$t^0 \leq T \leq t^1.$$

Step 2: We now show that (32) is optimal if $t^0 \leq T \leq t^1$. We are going to check all the optimality conditions in Proposition 5 are satisfied.

[1] Terminal conditions (16) and (15): In Step 1, we have shown that $c^0_j(T) \geq c(a^*)$ when $t^0 \leq T$. Since $c^0_j(T)$ satisfies the incentive constraint, $c_j(T) = c(a^*)$ also satisfies the incentive constraint. This implies (i) $c_j(T) = c(a^*)$ satisfies optimality condition (16) for $i = j$ and (ii) if we define

$$\mu_j(T) := 0, \tag{38}$$

it satisfies the transversality condition (15) for $i = j$.

Next we show that $c_{-j}(T) = c^0_{-j}(T)$ satisfies optimality condition (16). This condition holds if $c^0_{-j}(T) \leq c(a^*)$, or equivalently,

$$B'(c^0_{-j}(T)) - 1 > 0 \tag{39}$$

(because $B'(c(a^*)) - 1 = 0$ and $B'' < 0$). Recall that $c_j^0(t^0) < c_j^0(T)$ (because $t^0 < T$) and therefore $B'(c_j^0(t^0)) > B'(c_j^0(T))$. Furthermore, $B'(c_j^0(t^0)) = 1$ because $c_j^0(t^0) = c(a^*)$, and therefore we have

$$1 > B'(c_j^0(T)).$$

Now recall that Step 1 shows that

$$\varphi(T) = B'(c_{-j}^0(T))B'(c_j^0(T)) - 1 \geq 0$$

for any $T \leq t^1$. The two inequalities above imply (39), and therefore $c_{-j}^0(T) \leq c(a^*)$. This implies that (i) $c_{-j}(T) = c_{-j}^0(T)$ satisfies optimality condition (16) for $i = -j$ and (ii) if we define

$$\mu_{-j}(T) := B'(c_{-j}^0(T)) - 1 > 0, \tag{40}$$

it satisfies the transversality condition (15) for $i = -j$.

[2] The Hamiltonian maximization conditions (17) and (18) for T: In Step 1, we have shown that the solution to

$$\text{(PT)} \quad \max_{c_j} (1 + \mu_{-j}(T)) B(c_j) - (1 + \mu_j(T))c_j$$
$$\text{s.t. } c_j e^{-\lambda_j T} \leq k_j(T).$$

is equal to $c_j^0(T)$, when $T \leq t^1$. The optimality conditions (17) and (18) for this problem is satisfied if we define

$$\gamma_j(T) := \lambda_j \left[(1 + \mu_{-j}(T)) B'(c_j^0(T)) - (1 + \mu_j(T)) \right].$$

Note that

$$\gamma_j(T) \geq 0, \tag{41}$$

because Step 1 shows $(1 + \mu_{-j}(T)) B'(c_j^0(T)) - (1 + \mu_j(T)) = B'(c_{-j}^0(T))B'(c_j^0(T)) - 1 = \varphi(T) \geq 0$ if $T \leq t^1$.

Next, consider the Hamiltonian maximization condition at T for $-j$'s action:

$$\max_{c_{-j}} (1 + \mu_j(T)) B(c_{-j}) - (1 + \mu_{-j}(T))c_{-j}$$
$$\text{s.t. } c_{-j} e^{-\lambda_{-j}} \leq k_{-j}(T).$$

By our definitions (38) and (40), the objective function is $B(c_{-j}) - B'(c^0_{-j}(T))c_{-j}$. Its unconstrained maximizer is $c^0_{-j}(T)$, because it satisfies the first order condition $B'(c_{-j}) - B'(c^0_{-j}(T)) = 0$. Hence, if we define

$$\gamma_{-j}(T) := \lambda_{-j}\left[(1+\mu_j(T))\, B'(c^0_{-j}(T)) - (1+\mu_{-j}(T))\right] = 0, \tag{42}$$

the optimality conditions (17) and (18) at T for $-j$ are satisfied.

[3] **The remaining conditions (17), (18) and (19) for $t \in [0, T)$:** To satisfy the remaining conditions of optimality, let $\mu_i(t)$, $i = 1, 2$ be the solution to the system of differential equations

$$\dot{\mu}_i(t) = -\lambda_i\left[(1+\mu_{-i}(t))B'(c^0_i(t)) - (1+\mu_i(t))\right], i = 1, 2 \tag{43}$$

with boundary conditions (38) and (40). Let us also define

$$\gamma_i(t) := \lambda_i\left[(1+\mu_{-i}(t))B'(c^0_i(t)) - (1+\mu_i(t))\right]$$

If we show that $\gamma_i(t) \geq 0$ for all t, all the remaining conditions are satisfied. Note that

$$\gamma_i'(T) = \dot{\mu}_{-i}(T)B'(c^0_i(T)) + (1+\mu_{-i}(T))B''(c^0_i(T))\dot{c}^0_i(T) - \dot{\mu}_i(T).$$

Also note that, for any $\iota = 1, 2$,

(i) $\dot{\mu}_\iota(T) = -\gamma_\iota(T) \leq 0$ (by (41) and (42)),

(ii) $\mu_\iota(T) \geq 0$ (by (38) and (40), and

(iii) $B' > 0$, $B'' < 0$ and $\dot{c}^0_\iota(T) > 0$.

Hence, we conclude that, for any i,

$$\gamma_i(T) = 0 \Rightarrow \gamma_i'(T) < 0.$$

This and $\gamma_i(T) \geq 0$ for any i implies that

$$Q := \{t \in [0, T] |\ \forall s \in [t, T)\ \ \gamma_i(s) > 0,\, i = 1, 2\}.$$

contains some interval $(\widetilde{t}, T)$. Then, by the same argument as in the proof of Proposition 6, we can show that $\inf Q = 0$. Therefore, we obtained $\gamma_i(t) > 0$ for all $t \in [0, T)$ and any

i. Hence we conclude $\gamma_i(t) \geq 0$ for all t and all i. ■

Remark: The proof shows that

$$\dot{\mu}_j(T) = -\gamma_j(T) \begin{cases} < 0 & \text{if } T < t^1 \\ = 0 & \text{if } T = t^1 \end{cases},$$

$$\dot{\mu}_{-j}(T) = -\gamma_{-j}(T) = 0, \text{ and}$$

$$\dot{\mu}_i(T) = -\gamma_i(t) < 0 \text{ for all } i \text{ and all } t \in [0, T).$$

Lastly, we determine the optimal path for $T > t^1$. When $T > t^1$, the incentive constraint for player j becomes non-binding for some $t < T$. More precisely, the Hamiltonian maximization condition at t for j's action,

$$\max_{c_j} (1 + \mu_{-j}(t)) B(c_j) - (1 + \mu_j(t))c_j$$
$$\text{s.t. } c_j e^{-\lambda_j} \leq k_j(t).$$

has an interior solution $c_j(t)$ with $c_j(t)e^{-\lambda_j} < k_j(T)$, for some $t < T$. As we verify below, $\mu_{-j}(t) = 0$ while player j's incentive constraint is non-binding, and therefore the first order condition for the optimization problem given above is

$$(1 + \mu_{-j}(t))B'(c_j(t)) - 1 = 0. \tag{44}$$

From this condition, we derive the differential equation for non-binding action $c_j(t)$ as follows (its optimality is going to be rigorously verified in Proposition 8 below). With an abuse of notation define $\mu_{-j}(c_j(t))$ to satisfy the above equality:

$$\mu_{-j}(c_j(t))) := \frac{1}{B'(c_j(t))} - 1. \tag{45}$$

Note that, under (45)

$$\dot{\mu}_{-j} = -\frac{B''(c_j(t))}{[B'(c_j(t))]^2}\dot{c}_j. \tag{46}$$

For $c_j(t)$ to be optimal, it must satisfy one of the optimality conditions $\dot{\mu}_{-j}(t) = -\lambda_{-j}[(1+\underbrace{\mu_j(t)}_{0})B'(c_{-j}(t)) - (1+\mu_{-j}(t))]$. Under (46), this is satisfied if

$$\dot{c}_j(t) = \lambda_{-j}\frac{B'(c_j(t))}{B''(c_j(t))}\left(B'(c_j(t))B'(c_{-j}(t)) - 1\right) \ . \tag{47}$$

In contrast, it turns out that while player j's action follows the differential equation above, player $-j$'s action is given by the binding incentive constraint and therefore satisfies the differential equation (24) for the binding path. In what follows, we show that a part of the optimal path when $T > t^1$ is given by "one-sided biding path" defined as:

Definition 5 A **one-sided binding path** $(c_j^{\#}, c_{-j}^{\#})$ is a solution to the system of differential equations

$$\begin{cases} \dot{c}_j(t) = \lambda_{-j}\frac{B'(c_j(t))}{B''(c_j(t))}\left(B'(c_j(t))B'(c_{-j}(t)) - 1\right) \\ \\ \dot{c_{-j}}(t) = \lambda_i B(c_j(t))e^{(\lambda_{-j}-\lambda_j)t} \end{cases} \tag{48}$$

Remark: Since $B'(0) = \infty$, (48) is not well-defined when $c_j = 0$ or $c_{-j} = 0$. Therefore a solution $(c_j^{\#}, c_{-j}^{\#})$ to (48) is defined to be a pair of *strictly* positive functions, defined over some time interval $(\underline{t}, \overline{t})$ ($\underline{t}$ and $\overline{t}$ can be $-\infty$ and ∞ respectively). In other words, for $i = 1, 2$, $c_i^{\#}(\cdot) > 0$ over its domain $(\underline{t}, \overline{t})$.

First, we consider $T \in (t^1, t^2]$, where t^2 is the time when the binding path of player $-j$ hits the optimal level:

Definition 6 Let t^2 be the time when $c_{-j}^0(t^2) = c(a^*)$. If $c_{-j}^0(t^2)$ never reaches $c(a^*)$, let $t^2 = \infty$.

The following lemma confirms that t^2 is in fact larger than the threshold t^1 defined in Proposition 7:

Lemma 1 $t^1 < t^2$.

Proof. t^1 satisfies $B'(c_{-j}^0(t^1))B'(c_j^0(t^1)) - 1 = 0$. We show that $t^1 \geq t^2$ leads to a contradiction. If $t^1 \geq t^2$, then $c_{-j}^0(t^2) = c(a^*)$, $B'' < 0$, $\dot{c}_{-j}^0 > 0$ and $B'(c(a^*)) = 1$imply

$B'(c_j^0(t^1)) - 1 > 0$. Since Proposition 7 asserts $t^1 > t^0$ (t^0 is the time when $c_j^0 = c(a^*)$), $\dot{c}_j^0 > 0$ and $B'' < 0$ imply $B'(c(a^*)) - 1 > 0$, which contradicts $B'(c(a^*)) - 1$. ■

Now we are ready to characterize the optimal path for $T \in (t^1, t^2]$.

Proposition 8 *When $T \in (t^1, t^2]$, the optimal path exists and it is given by the following conditions:*

$$c_j(T) = c(a^*), \tag{49}$$

$$c_j^T := \lim_{t \to T} c_j(t) > c(a^*), \tag{50}$$

$$c_{-j}(T) < c(a^*), \tag{51}$$

$$\exists s \quad (c_j(t), c_{-j}(t)) = \begin{cases} (c_j^0(t), c_{-j}^0(t)) \text{ if } t \in [0, s] \\ \\ (c_j^{\#}(t), c_{-j}^{\#}(t)) \text{ if } t \in [s, T) \\ \\ (c(a^*), c_{-j}^{\#}(T)) \text{ if } t = T \end{cases}, \tag{52}$$

where $(c_j^{\#}(t), c_{-j}^{\#}(t))$ is the solution to (48) with boundary condition $(c_j^{\#}(T), c_{-j}^{\#}(T)) = (c_j^T, c_{-j}(T))$. Furthermore,

$$\dot{c}_j^{\#}(T) = 0 \text{ and } \dot{c}_j^{\#}(t) < 0 \text{ for } t \in [s, T) \tag{53}$$

and s, c_j^T, and $c_{-j}(T)$ are determined by

$$B'(c_{-j}(T))B'(c_j^T) - 1 = 0, \tag{54}$$

and

$$\begin{cases} c_j^0(s) = c_j^{\#}(s) \\ c_{-j}^0(s) = c_{-j}^{\#}(s) \end{cases}. \tag{55}$$

Remarks: The first two conditions (49) and (50) show that $c_j(t)$ overshoots the optimal level $c(a^*)$ and is discontinuous at T. In contrast, (51) implies that $c_{-j}(t)$ cannot reach optimal. Condition (52) says that it is optimal to follow the binding path $(c_j^0(t), c_{-j}^0(t))$ until some time s, and then to follow a one-sided binding path $(c_j^{\#}(t), c_{-j}^{\#}(t))$ after that. Condition (54) corresponds to the first order condition for the Hamiltonian maximization at T with respect to player j's action. The last condition (55) requires that the two

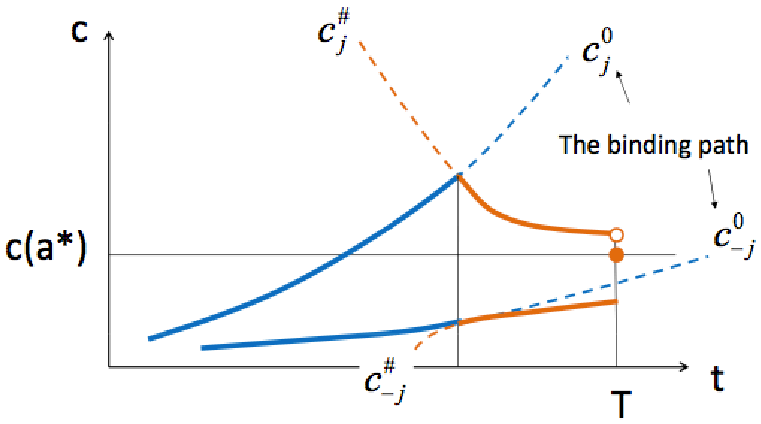

Figure t'

The heavy curves represent the optimal path when $T^0 < T < t'$.

paths, $(c_j^0(t), c_{-j}^0(t))$ and $(c_j^{\#}(t), c_{-j}^{\#}(t))$, should be pasted continuously at s. Because player $-j$'s action follows the same differential equation $\dot{c}_{-j}(t) = \lambda_i B(c_j(t))e^{(\lambda_{-j}-\lambda_j)t}$ both on $(c_j^0(t), c_{-j}^0(t))$ and $(c_j^{\#}(t), c_{-j}^{\#}(t))$, the first condition in (55) implies $\dot{c}_{-j}^0(s) = \dot{c}_{-j}^{\#}(s)$. Namely, player $-j$'s path is *pasted smoothly* at s. In contrast, $c_j^{\#}(t)$ is decreasing and $c_j^0(t)$ is strictly decreasing. Hence, player j's path is not pasted smoothly and *has a kink at* s. **Figure t'** depicts the typical shape of the optimal path.

Proof. For a parameter $v > 0$, let $(c_j^v(t), c_{-j}^v(t))$ be the solution to (48) with boundary condition

$$(c_j^v(T), c_{-j}^v(T)) = (u, v),$$

where u is determined by v by

$$B'(u)B'(v) - 1 = 0 \tag{56a}$$

to satisfy (54). The path $(c_j^{\#}(t), c_{-j}^{\#}(t))$ in the proposition is equal to $(c_j^v(t), c_{-j}^v(t))$ for the right choice of parameter v. The proof proceeds in three steps. In Step 1, we show $\dot{c}_j^v(t) \le 0$ for any $v > 0$. This shows (53) and also helps us to prove Steps 2 and 3. In Step 2, we show that, with the right choice of parameter v, $(c_j^{\#}(t), c_{-j}^{\#}(t)) := (c_j^v(t), c_{-j}^v(t))$ is pasted to $(c_j^0(t), c_{-j}^0(t))$ at some time s (condition (55) holds). In Step 3, we check that all optimality conditions are satisfied.

Step 1: Recall the Remark about the system of differential equations (48) to notice that, by definition, $c_j^v(t)$ and $c_{-j}^v(t)$ are *strictly positive* functions defined over some time interval. We denote the time interval (the set of t over which $c_j^v(t)$ and $c_{-j}^v(t)$ are defined) by D^v. Since

$$\dot{c^v}_j(t) = \lambda_{-j} \frac{B'(c_j^v(t))}{B''(c_j^v(t))} \left(B'(c_j^v(t)) B'(c_{-j}^v(t)) - 1 \right),$$

we have

$$\dot{c^v}_j(T) = 0.$$

because (56a) implies

$$B'(c_j^v(T)) B'(c_{-j}^v(T)) - 1 = 0. \tag{57}$$

Next we show

$$\dot{c}_j^v(t) < 0 \text{ for all } t \in D^v \cap [0, T). \tag{58}$$

and for all $v > 0$. By $B' > 0$ and $B'' < 0$, (58) is equivalent to

$$\zeta(t) := B'(c_j^v(t)) B'(c_{-j}^v(t)) - 1 > 0 \text{ for all } t \in D^v \cap [0, T). \tag{59}$$

The argument to show $\zeta(t) > 0$ is similar to the proof of Proposition 6. Consider

$$Q := \{ t \in D^v \cap [0, T) | \ \forall s \in D^v \cap [t, T) \ \ \zeta(s) > 0 \}.$$

First, we claim

$$\zeta(t) = 0 \Rightarrow \zeta'(t) < 0. \tag{60}$$

This is shown as follows. Since $\dot{c^v}_j(t) = \lambda_{-j} \frac{B'(c_j^v(t))}{B''(c_j^v(t))} \zeta(t)$, $\zeta(t) = 0$ implies $\dot{c^v}_j(t) = 0$. Therefore, by differentiating $\zeta(t) = B'(c_j^v(t)) B'(c_{-j}^v(t)) - 1$, we obtain

$$\zeta'(t) = B''(c_j^v(t)) B'(c_{-j}^v(t)) \dot{c}_{-j}^v(t) < 0,$$

because $\dot{c}_{-j}^v = \lambda_i B(c_j^v(t)) e^{(\lambda_{-j} - \lambda_j)t} > 0$.

Since $\zeta(T) = 0$ (by (57)), (60) implies $t \in Q$ for t slightly smaller than T. Hence, Q is non-empty and therefore we can define

$$\inf Q =: t' < T.$$

We suppose $t' > \inf D^v$ and find a contradiction. We first claim

$$\zeta(t') = 0. \tag{61}$$

If $\zeta(t') > 0$, by continuity of $\zeta(t)$ and our premise $t' > \inf D^v$, we must have $\inf Q < t'$, a contradiction. Second, by (61) and (60), there must be $t \in Q$ that is slightly larger than t' such that $\zeta(t) < 0$, which contradicts the definition of Q. Hence, $t' > \inf D^v$ is impossible and therefore

$$t' = \inf D^v$$

(by definition, t' cannot be strictly below $\inf D^v$). This shows (59) and therefore (58) holds.

Step 2: We now show that, with the right choice of parameter v, $(c_j^{\#}(t), c_{-j}^{\#}(t)) := (c_j^v(t), c_{-j}^v(t))$ coincides with $(c_j^0(t), c_{-j}^0(t))$ at some time. To this end, we first define the time $t = t(v)$ at which $c_{-j}^v(t)$ coincides with $c_{-j}^0(t)$. Our task is to show that, with the right choice of v, $c_j^v(t)$ also touches $c_j^0(t)$ at the same time $t = t(v)$. Consider

$$f(v,t) := c_{-j}^0(t) - c_{-j}^v(t)$$

and and note that $t(v)$ satisfies $f(v, t(v)) = 0$. For some values of v, there may be more than one $t(v)$ to satisfy $f(v, t(v)) = 0$, or there may be no such $t(v)$. Hence, to define $t(v)$ rigorously, we apply the implicit function theorem. To simplify notation let us define

$$w := c_{-j}^0(T).$$

Note that $f(w,T) = 0$, because $c_{-j}^w(T) = w = c_{-j}^0(T)$. Note that

$$\begin{aligned}\frac{\partial f}{\partial t}(w,T) &= \dot{c}_{-j}^0(T) - \dot{c}_{-j}^w(T)\\ &= \lambda_i \left(B(c_j^0(T)) - B(c_j^w(T))\right) e^{(\lambda_{-j}-\lambda_j)T}\\ &> 0.\end{aligned} \tag{62}$$

The last inequality is shown as follows. Note that

$$\varphi(t) := B'(c_{-j}^0(t))B'(c_j^0(t)) - 1$$

is a strictly decreasing function and t^1 is defined by $\varphi(t^1) = 0$ (see Proposition 7). Hence our premise $t^1 \leq T$ implies

$$B'(c^0_{-j}(T))B'(c^0_j(T)) - 1 < 0.$$

In contrast, by definition, (56a) should be satisfied for $u = c^w_{-j}(T)$ and $v = w = c^0_{-j}(T)$, and therefore we have

$$B'(c^w_{-j}(T))B'(c^0_j(T)) - 1 = 0.$$

Hence $c^w_{-j}(T) < c^0_{-j}(T)$ (because $B' > 0$ and $B'' < 0$). Since B is increasing, we obtained (62).

Hence, $\frac{\partial f}{\partial t}(w, T) \neq 0$ and also note that $f(\cdot, \cdot)$ is continuous. Hence, by the implicit function theorem, we can define a function $t(v)$ that satisfies $f(v, t(v)) = 0$ ($\Leftrightarrow c^0_{-j}(t(v)) = c^v_{-j}(t(v))$) and $t(w) = T$, defined on a neighborhood of $w = c^0_{-j}(T)$. We gradually decrease v below $w = c^0_{-j}(T)$ in this neighborhood and show that eventually the paths of the other player j, $c^0_j(t)$ and $c^v_j(t)$, cross at $t(v)$.

To show that, let R be the set of $v \in [0, w]$ such that, for all $v' \in [v, w]$

(i) the implicit function $t(\cdot)$ is defined at v', and

(ii) $\frac{\partial f}{\partial t}(v', t(v')) > 0$.

Since $\frac{\partial f}{\partial t} = \dot{c}^0_{-j} - \dot{c}^v_{-j} = \lambda_i \left(B(c^0_j) - B(c^v_j) \right) e^{(\lambda_{-j} - \lambda_j)T}$, the second requirement (ii) is equivalent to

(ii') $c^0_j(t(v')) > c^{v'}_j(t(v'))$.

Next, we show that $c^0_{-j}(t^1) > 0$ is a lower bund of R. Recall that t^1 is defined by

$$B'(c^0_{-j}(t^1))B'(c^0_j(t^1)) - 1 = 0.$$

Hence, if we choose $v = c^0_{-j}(t^1)$, then by (56a) $u = c^0_j(t^1)$, and therefore we obtain

$$c^v_j(T) = v = c^0_j(t^1) \tag{63}$$

and

$$c^v_{-j}(T) = u = c^0_{-j}(t^1). \tag{64}$$

Note that $\dot{c}^v_{-j} = \lambda_i B(c^v_j) e^{(\lambda_{-j} - \lambda_j)T} > 0$, because c^v_j is defined to be strictly positive (see the Remark after (48)). Furthermore, $c^0_{-j}(t)$ is also strictly increasing. Those facts, together with $c^v_{-j}(T) = c^0_{-j}(t^1) > 0$ ((64)) and $t^1 < T$ imply that either

(a) $c^v_{-j}(t)$ and $c^0_{-j}(t)$ never cross for $t < T$ (in which case, $t(v)$ is not defined), or

(b) they cross at $t(v) < t^1$.

If (a) is the case, $v = c^0_{-j}(t^1)$ is a lower bound of R, since condition (i) is violated. Hence, we consider case (b), and show that (ii') is violated.

In case (b), Since $c^v_j(t)$ is non-increasing by Step 1 and $c^0_j(t)$ is strictly increasing, (63) and $t^1 < T$ imply that

$$c^0_j(t) < c^v_j(t) \text{ if } t < t^1.$$

Since $t(v) < t^1$ in case (b), condition (ii') is violated. Therefore, conditions (i) and (ii') cannot be simultaneously satisfied for $v' = v = c^0_{-j}(t^1)$, and this implies that $c^0_{-j}(t^1) > 0$ is a lower bound of R.

The above argument shows that $\underline{v} := \inf R > 0$ exists. Since $t(v)$ is a continuous function, we can define

$$t(\underline{v}) := \lim_{v \downarrow \underline{v}} t(v)$$

and

$$f(\underline{v}, t(\underline{v})) = c^0_{-j}(t(\underline{v})) - c^{\underline{v}}_{-j}(t(\underline{v})) = 0$$

is satisfied. By (ii'), $c^0_j(t(v)) > c^v_j(t(v))$ for all $v \in R$, and therefore

$$c^0_j(t(\underline{v})) \geq c^{\underline{v}}_j(t(\underline{v})).$$

If $c^0_j(t(\underline{v})) > c^{\underline{v}}_j(t(\underline{v}))$, we have $\frac{\partial f}{\partial t}(\underline{v}, t(\underline{v})) > 0$ (because (ii) and (ii') are equivalent). Then, because $\frac{\partial f}{\partial t}(\underline{v}, t(\underline{v})) \neq 0$, we can apply the implicit function theorem and $t(\cdot)$ can be extended to a neighborhood of $\underline{v}$, where $c^0_j(t(v)) > c^v_j(t(v))$ hold by continuity. In particular, we have $v < \underline{v}$ such that (i) and (ii') are satisfied for all $v' \in [v, w]$. This means $v \in R$, which contradicts $v < \underline{v} = \inf R$. Hence we conclude

$$c^0_j(t(\underline{v})) = c^{\underline{v}}_j(t(\underline{v})),$$

and therefore at $s := t(\underline{v})$ the pasting condition

$$\begin{cases} c^0_j(s) = c^{\#}_j(s) \\ c^0_{-j}(s) = c^{\#}_{-j}(s) \end{cases}$$

is satisfied if we define $(c_j^{\#}(t), c_{-j}^{\#}(t)) := (c_j^{\underline{v}}(t), c_{-j}^{\underline{v}}(t))$.

Lastly, let us verify that $(c_j^{\#}(t), c_{-j}^{\#}(t))$ thus defined satisfies all the conditions in the Proposition. First, $(c_j^{\#}(t), c_{-j}^{\#}(t))$ is the solution to (48) with boundary condition $(c_j^{\#}(T), c_{-j}^{\#}(T)) = (\underline{u}, \underline{v})$, where $\underline{u}$ is defined by

$$B'(\underline{u})B'(\underline{v}) - 1 = 0.$$

Hence, if we define $(c_j^T, c_{-j}(T)) := (\underline{u}, \underline{v})$, it satisfies (54). Next we show (51). Recall that $c_{-j}^0(t^2) = c(a^*)$ and $T \leq t^2$ imply $c_{-j}^0(T) \leq c(a^*)$. Since $c_{-j}^0(T) = w \in R$ and $c_{-j}^0(T) \neq \inf R$, we obtain

$$c_{-j}(T) = \underline{v} = \inf R < c(a^*),$$

and therefore (51) is satisfied. Since $B'(c(a^*)) = 1$ and B' is decreasing,

$$B'(c_j^T)B'(c_{-j}(T)) - 1 = 0$$

and $c_{-j}(T) < c(a^*)$ imply

$$0 > B'(c_j^T) - 1.$$

This in turn implies $c_j^T > c(a^*)$ (condition (50)). Finally, the monotonicity (53) holds because of Step 1.

Step 3: We are now ready to verify that all the optimality conditions in Proposition 5 are satisfied. Define the path $(c_j(t), c_{-j}(t))$ to satisfy (52) with $s = t(\underline{v})$ as defined in Step 2. Also define

$$\mu_j(t) := 0 \text{ for } t \in [s, T],$$

and for $t \in [s, T]$, and define $\mu_{-j}(t)$ be for $t \in [s, T]$, by

$$\mu_{-j}(T) = B'(c_{-j}(T)) - 1.$$

and

$$\begin{aligned} &(1 + \mu_{-j}(t))B'(c_j(t)) - (1 + \mu_j(t)) \\ &= (1 + \mu_{-j}(t))B'(c_j(t)) = 0. \end{aligned} \tag{65}$$

Define also $m := \mu_{-j}(s)$. Next we let $(\mu_j(t), \mu_{-j}(t))$ for $t \in [0, s)$ to be the solution to the system of differential equations

$$\dot{\mu}_i(t) = -\lambda_i \left[(1+\mu_{-i}(t))B'(c_i(t)) - (1+\mu_i(t))\right], i = 1, 2$$

with boundary condition

$$\begin{cases} \mu_j(s) = 0 \\ \mu_{-j}(s) = m \end{cases}.$$

Next define

$$\gamma_i(t) := \lambda_i \left[(1+\mu_{-i}(t))B'(c_i(t)) - (1+\mu_i(t))\right], \ i = 1, 2.$$

Finally define $k_i(t)$ by $k_i(0) = 0$ and

$$\dot{k}_i(t) = B(c_{-i}(t))\lambda_{-i}e^{-\lambda_{-i}t} - c_i(t)\lambda_i e^{-\lambda_i t}, \ i = 1, 2.$$

Now we check (c, μ, γ, k) thus defined satisfies the conditions in Proposition 5. First, we check the incentive constraint $k_i(t) \geq c_i(t)e^{-\lambda_i t}$. By construction, $c_{-j}(t)$ for all t and $c_j(t)$ for $t \in [0, s]$ satisfy the incentive constraint with equality. For $t \in (s, T]$, we show that $c_j(t)$ satisfies the incentive constraint. Recall that

$$c_j(t) = \begin{cases} c_j^{\#}(t) \text{ for } t \in (s, T] \\ \\ c(a^*) \text{ for } t = T \end{cases}$$

and also that $c_j^{\#}(t) = c_j^T > c(a^*)$ (condition (50)). Hence, to show that the $c_j(t)$ satisfies the incentive constraint for all $t \in (s, T]$, it is sufficient to show that $c_j^{\#}(t)$ satisfies the incentive constraint for all $t \in (s, T]$.

Let us now define

$$h(t) := k_j(t)e^{\lambda_j t}.$$

The incentive constraint is expressed as $h(t) \geq c_j(t)$ and note that $h(t) = c_j^0(t)$ for $t \in [0, s]$. Since $c_j(t)$ is strictly increasing, we have

$$\begin{aligned} 0 &< \dot{c}_j^0(s) = \lim_{\Delta \downarrow 0} \frac{h(s) - h(s-\Delta)}{\Delta} \\ &= B(c_{-j}(s))\lambda_{-j}e^{(\lambda_j - \lambda_{-j})s} + \lambda_j(h(s) - c_j(s)). \end{aligned}$$

Define P be the set of $t \in [s,T]$ such that, for all $\tau \in [s,t]$

[1] $B(c^{\#}_{-j}(\tau))\lambda_{-j}e^{(\lambda_j-\lambda_{-j})\tau} + \lambda_j(h(\tau) - c^{\#}_j(\tau)) > 0$ and

[2] $h(\tau) - c^{\#}_j(\tau) \geq 0$.

Since $s \in P$, P is non-empty and we can define $\sup P \leq T$. Now we show that $\sup P = T$ (and this implies the incentive constraint of $c_j(t)$ for $t \in (s,T]$, because of [2]). To show this, we assume $t^+ := \sup P < T$ and derive a contradiction. Note that $h'(t) = B(c^{\#}_{-j}(t))\lambda_{-j}e^{(\lambda_j-\lambda_{-j})t} + \lambda_j(h(t) - c^{\#}_j(t))$ and condition [2] implies $h(t^+) - c^{\#}_j(t^+) \geq 0$. Hence $h'(t^+) > 0$, because $c_{-j}(t^+) > 0$. Since $c_j(t)$ is decreasing for $t \in [s,T)$, conditions [1] and [2] must also hold for all $\tau \in [s, t^+ + \varepsilon]$, for sufficiently small $\varepsilon > 0$. This contradicts $t^+ = \sup P < T$, and therefore we must have $\sup P = T$. This implies (by condition [2]) the incentive constraint of $c^{\#}_j(t)$ for $t \in (s,T]$. Hence we have shown that all incentive constraints are satisfied.

Since the incentive constraint is satisfied at T, it is easy to check that terminal conditions (15) and (16) hold. Conditions (17) is satisfied by our definition of γ. Given that (17) is satisfied, the remaining conditions to be checked boil down to

$$\dot{\mu_i}(t) \leq 0 \text{ and } \dot{\mu_i}(t)\left(k_i(t) - c_i(t)e^{-\lambda_i t}\right) = 0, \tag{66}$$

(an alternative expression of (18)) and,

$$\dot{\mu_i}(t) = -\lambda_i\left[(1+\mu_{-i}(t))B'(c_i(t)) - (1+\mu_i(t))\right], \tag{67}$$

(an alternative expression of (19)). First, consider player $i = j$ for $t \in (s,T]$. Since $\mu_j(t)$ is defined to be zero for $t \in (s,T]$, condition (66) is satisfied. Condition (67) also holds because $\mu_{-j}(t)$ for $t \in (s,T]$ is defined to satisfy $(1+\mu_{-i}(t))B'(c_i(t)) - 1 = 0$.

Second, consider player $i = -j$ for $t \in (s,T]$. Because $(1+\mu_{-i}(t))B'(c_i(t)) - 1 = 0$ for all $t \in (s,T]$, by differentiating the both sides we obtain

$$\dot{\mu_{-j}}(t)B'(c_j(t)) + (1+\mu_{-j}(t))B''(c_j(t))\dot{c_j}(t) = 0. \tag{68}$$

This equation, together with $\dot{c_j} = \lambda_{-j}\frac{B'(c_j)}{B''(c_j)}\left(B'(c_j)B'(c_{-j}) - 1\right)$ and $(1+\mu_{-i})B'(c_i) - 1 = 0$, implies that condition (67) is satisfied for $i = -j$. Equation (68) also implies that $\dot{\mu_{-j}}(t)$ and $\dot{c_j}(t)$ have the same sign, and $\dot{c_j}(t) \leq 0$ (by Step 1) shows $\dot{\mu_{-j}}(t) \leq 0$. The second condition in (66) is satisfied for $i = -j$, because by the construction of c_{-j}, the incentive constraint is always binding and therefore $k_{-j}(t) - c_{-j}(t)e^{-\lambda_{-j}t} = 0$. Hence we have shown

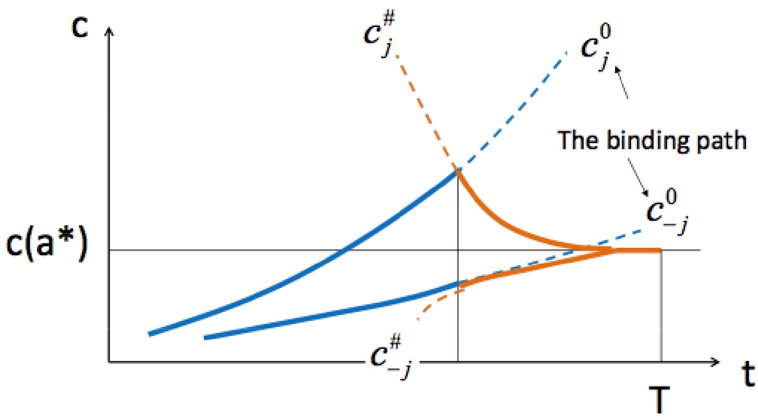

Figure T large

The heavy curves represent the optimal path in Case 1 when t' ≤ T.

that conditions (66) and (67) are satisfied for all players for $t \in (s, T]$.

Finally, we show (66) and (67) for $t \in [0, s]$. Condition (67) is satisfied by definition. The second condition in (66), $\dot{\mu}_i(t)\left(k_i(t) - c_i(t)e^{-\lambda_i t}\right) = 0$ is satisfied because the incentive constraint for all players are binding for $t \in [0, s]$ and therefore $k_i(t) - c_i(t)e^{-\lambda_i t} = 0$. The remaining condition $\dot{\mu}_i(t) \leq 0$ is shown by the same argument as in the proof of Proposition 6. ■

Finally, we consider the case $T > t^2$. Recall that t^2 is the smallest t when the binding paths of both players, $c_i^0(t)$, $i = 1, 2$, are more than or equal to the optimal level $c(a^*)$. The optimal path for $T > t$ is given by the following proposition. The graph of the optimal path (for Case 1 in the proposition) is given in **Figure T large**.

Proposition 9 *When $T > t^2$, the optimal path $(c_j(t), c_{-j}(t))$ exists and one of the following holds.*

Case 1:

$$\exists \tau^* \leq T \; \forall i \quad c_i(t) = c(a^*) \text{ for } t \in [\tau^*, T] \tag{69}$$

$$\exists s \quad (c_j(t), c_{-j}(t)) = \begin{cases} (c_j^0(t), c_{-j}^0(t)) \text{ if } t \in [0, s] \\ \\ (c_j^{\#}(t), c_{-j}^{\#}(t)) \text{ if } t \in [s, \tau^*] \end{cases}, \tag{70}$$

where $(c_j^{\#}(t), c_{-j}^{\#}(t))$ *is the solution to (48) with boundary condition* $c_i^{\#}(\tau^*) = c(a^*)$, $i = 1, 2$. *Furthermore,*

$$\dot{c}_j^{\#}(\tau^*) = 0 \text{ and } \dot{c}_j^{\#}(t) < 0 \text{ for } t \in [s, \tau^*) \tag{71}$$

and s *and* τ^* *are determined by*

$$\begin{cases} c_j^0(s) = c_j^{\#}(s) \\ c_{-j}^0(s) = c_{-j}^{\#}(s) \end{cases}. \tag{72}$$

Case 2: *The optimal path satisfies all the conditions in Proposition 8.*

Proof. For a parameter $\tau \geq t^2$, let $(c_{j,\tau}(t), c_{-j,\tau}(t))$ be the solution to (48) with boundary condition

$$\forall i \quad c_i(\tau) = c(a^*).$$

By the same argument as in Step 1 in the proof of Proposition 8, we have

$$\dot{c}_{-j,\tau}(t) \leq 0$$

for all $\tau \geq t^2$ and all t for which $c_{-j,\tau}(t)$ is defined.

By definition, $c_{-j,t^2}(t^2) = c(a^*)$ and therefore

$$c_{-j}^0(t^2) - c_{-j,t^2}(t^2) = 0.$$

Now define

$$F(\tau, t) := c_{-j}^0(t) - c_{-j,\tau}(t)$$

and note that $F(t^2, t^2) = 0$. Also note that $F(\cdot, \cdot)$ is continuous. In addition, we have

$$\frac{\partial F}{\partial t}(t^2, t^2) = \dot{c}_{-j}^0(t^2) - \dot{c}_{-j,t^2}(t^2)$$

$$= \lambda_i \left(B(c_j^0(t^2)) - B(c_{j,t^2}(t^2)) \right) e^{(\lambda_{-j} - \lambda_j)t^2} \tag{73}$$

$$> 0.$$

The above inequality is implied by

$$\begin{aligned} c_{j,t^2}(t^2) &= c(a^*) \quad \text{(by definition)} \\ &= c_j^0(t^0) \quad \text{(by definition)} \\ &< c_j^0(t^2) \quad \text{(by } t^0 < t^2 \text{ and } \dot{c}_j^0 > 0). \end{aligned}$$

Hence all the conditions of the implicit function theorem are satisfied, and there is the unique implicit function $s(\tau)$ defined by

$$t^2 = s(t^2) \text{ and } f(\tau, s(\tau)) = 0,$$

in a neighborhood of t^2. Let H be the set of $\tau \in [t^2, \infty)$ such that, for all $\tau' \in [t^2, \tau]$,

(a) $s(\tau')$ is defined, and

(b) $\frac{\partial F}{\partial t}(\tau', s(\tau')) > 0$.

By (73), condition (b) is equivalent to

(b') $c_j^0(s(\tau')) > c_{j,\tau'}(s(\tau'))$.

Since $t^2 \in H$, H is non-empty. We have the following two cases.

Case 1: $\tau^* := \sup H \leq T$. In this case, by the same line of argument as in Step 2 of the proof of Proposition 8 shows that $c_j^0(s(\tau^*)) = c_{j,\tau^*}(s(\tau^*))$. Hence the pasting condition (72) in Case 1 is satisfied for $\tau^* := \sup H$, and we can check the path described in Case 1 is optimal by the same line of argument as in Step 3 of the proof of Proposition 8.

Case 2: $\sup H \leq T$ does not hold. Then, both conditions (a) and (b') are satisfied for $\tau' = T$. Recall $(c_j^v(t), c_{-j}^v(t))$ defined in the proof of Proposition 8: $(c_j^v(t), c_{-j}^v(t))$ is the solution to (48) with boundary condition

$$(c_j^v(T), c_{-j}^v(T)) = (u, v),$$

where u is determined by v by

$$B'(u)B'(v) - 1 = 0.$$

Note that $v = c(a^*)$ implies $u = c(a^*)$, because $B'(c(a^*)) = 1$. Also recall

$$f(v,t) := c^0_{-j}(t) - c^v_{-j}(t)$$

introduced in the proof of Proposition 8. Note that, by definition,

$$\begin{aligned} f(c(a^*), s(T)) &= c^0_{-j}(s(T)) - c^{c(a^*)}_{-j}(s(T)) \\ &= c^0_{-j}(s(T)) - c_{-j,T}(s(T)) = 0. \end{aligned}$$

Also by our premise that condition (b') holds for $\tau' = T$, we have

$$\begin{aligned} c^0_j(s(T)) &> c_{j,T}(s(T)) \\ &\Leftrightarrow c^0_j(s(T)) > c^{c(a^*)}_{-j}(s(T)) \end{aligned}$$

and therefore

$$\frac{\partial f}{\partial t}(c(a^*), s(T)) > 0,$$

because $\frac{\partial f}{\partial t} = \dot{c^0_{-j}} - \dot{c^v_{-j}} = \lambda_i \left(B(c^0_j) - B(c^v_j) \right) e^{(\lambda_{-j} - \lambda_j)T}$. Hence we have $f(c(a^*), s(T)) = 0$ and $\frac{\partial f}{\partial t}(c(a^*), s(T)) \neq 0$, and therefore by the implicit function theorem, there exists function $t(v)$ defined on a neighborhood of $c(a^*)$ such that

$$f(v, t(v)) = c^0_{-j}(t(v)) - c^v_{-j}(t(v)) = 0, \text{ and}$$

$$t(c(a^*)) = s(T).$$

Now let us define R' to be the set of $v \in [0, c(a^*)]$ such that, for all $v' \in [v, c(a^*)]$

(i) the implicit function $t(\cdot)$ is defined at v', and

(ii) $\frac{\partial f}{\partial t}(v', t(v')) > 0$.

By the same argument as in Step 2 of the proof of Proposition 8, we can show that $\underline{v} := \inf R' > 0$ exists and $(c^{\#}_j(t), c^{\#}_{-j}(t)) := (c^v_j(t), c^v_{-j}(t))$ coincides with $(c^0_j(t), c^0_{-j}(t))$ at $s := t(\underline{v})$. The optimality of the path described in Case 2 is shown by the same argument as in the proof of Proposition 8. ■

7 Discussion: Commitment Power of a Low Arrival Rate

Since $\lambda_1 < \lambda_2$, It is natural to conjecture that player 1 has a greater ability to commit to an action that induces player 2 to play an action favorable to player 1. We deal with this issue by considering the limit that the arrival rates become extreme. The following proposition holds for any component game, whether additively separable or not.

Proposition 10 *Fix any component game that satisfies Assumptions A2 and A3 in KK with action space* $A = [\underline{a}, \bar{a}]$*, where the payoff functions can either be additively separable or not. For any* $\epsilon > 0$*, there exists* T *large enough and* $\delta > 0$ *such that for all* (λ_1, λ_2) *such that* $\frac{\lambda_1}{\lambda_2} < \delta$*, player 1's expected payoff in any subgame perfect equilibrium of the revision game is at least* $\max_x \pi(x, BR(x)) - \epsilon$.[4]

That is, given a fixed length of the revision phase, player 1 becomes a "Stackelberg leader" if she has a very small chance to revise her action compared to the opponent. This is intuitive: If player 1's arrival rate is very small compared to player 2's, then there is a time $\bar{t}$ such that 1 expects almost no chances to further revise her action in future while 2 expects future opportunities with probability close to 1. By continuity (A2) and the assumption of unique best reply (A3), given 1's Stackelberg action a^S, player 2's action at an opportunity after $\bar{t}$ is close to the best reply to a^S. This means that by taking a^S before $\bar{t}$ (which is one possible deviation from any equilibrium), player 1 can ensure a payoff close to the Stackelberg payoff. The formal proof is relegated to Appendix D.

References

[1] Calcagno, R. and S. Lovo (2010) "Preopening and Equilibrium Selection," mimeo.

[2] Kamada, Y, and M. Kandori (2011), "Revision Games", mimeo.

[3] Kamada, Y. and T. Sugaya (2010) "Asynchronous Revision Games with Deadline: Unique Equilibrium in Coordination Games," mimeo.

[4] Sethi, S. and G. Thompson (2000), *Optimal Control Theory (2nd Edition)*, Springer.

[4]For simplicity we state the result for symmetric payoff functions, but it is straightforward to see that the proof applies also for asymmetric cases.

Appendix

Here we provide the remaining proofs.

Proof of Proposition 3

Given the proof provided in the main text, it is sufficient to prove that it cannot be the case that there is an infinite sequence of times $\{t_k\}_{k=0}^{\infty}$ such that $t_k \to 0$, $x_1(t_k) = x_2(t_k)$, and there exists $\epsilon > 0$ such that $x_1(t) < x_2(t)$ for all $t \in (t_k, t_k + \epsilon)$ for all k. Below we prove this claim.

Suppose that there exists such a sequence. We will derive a contradiction.

First, compare the benefits:

$$\frac{B_1(t_k+\epsilon)-B_1(t_k)}{B_2(t_k+\epsilon)-B_2(t_k)} = \frac{e^{-\lambda_1(t+\epsilon)}c(x_1(t_k+\epsilon)) - e^{-\lambda_1 t_k}c(x_1(t_k)) + \int_{t_k}^{t_k+\epsilon} c(x_1(\tau))\lambda_1 e^{-\lambda_1\tau}d\tau}{e^{-\lambda_2(t+\epsilon)}c(x_2(t_k+\epsilon)) - e^{-\lambda_2 t_k}c(x_2(t_k)) + \int_{t_k}^{t_k+\epsilon} c(x_2(\tau))\lambda_2 e^{-\lambda_2\tau}d\tau}.$$

Notice that since the third term in the numerator is less than $c(x_1(t_k+\epsilon))\lambda_1\epsilon e^{-\lambda_1 t_k}$, the numerator is smaller than

$$e^{-\lambda_1 t_k}(c(x_1(t_k+\epsilon)) - c(x_1(t_k))) + c(x_1(t_k+\epsilon))e^{-\lambda_1 t_k}(\lambda_1\epsilon - (1-e^{-\lambda_1\epsilon})).$$

Observe that for any fixed $t_k > 0$, the second term becomes negligible compared to the first term as $\epsilon \to 0$. In the same manner, we can bound the denominator from below by

$$e^{-\lambda_2(t_k+\epsilon)}(c(x_2(t_k+\epsilon)) - c(x_2(t_k))) + c(x_2(t_k))e^{-\lambda_2 t_k}(\lambda_2\epsilon - (1-e^{-\lambda_2\epsilon})),$$

where the second term becomes negligible compared to the first term as $\epsilon \to 0$ for any fixed $t_k > 0$. Hence the ratio of the numerator to the denominator in the limit as $\epsilon \to 0$ is determined by the comparison of the first terms, i.e.

$$\limsup_{k\to\infty}\limsup_{\epsilon\to 0}\frac{B_1(t_k+\epsilon)-B_1(t_k)}{B_2(t_k+\epsilon)-B_2(t_k)} = \limsup_{k\to\infty}\limsup_{\epsilon\to 0}\frac{e^{-\lambda_1 t_k}(c(x_1(t_k+\epsilon)) - c(x_1(t_k))}{e^{-\lambda_2(t_k+\epsilon)}(c(x_2(t_k+\epsilon)) - c(x_2(t_k)))}.$$

But $e^{-\lambda_1 t_k}/e^{-\lambda_2(t_k+\epsilon)} \to 1$ and $x_1(t_k) = x_2(t_k)$ for all k and $x_1(\tau) < x_2(\tau)$ for all $\tau \in (t_k, t_k+\epsilon)$, we have that the above limit is no more than 1.

Second, compare the penalties:

$$\frac{P_1(t) - P_1(s)}{P_2(t) - P_2(s)} = \frac{\int_s^t b(x_2(\tau))\lambda_2 e^{-\lambda_2 \tau} d\tau}{\int_s^t b(x_1(\tau))\lambda_1 e^{-\lambda_1 \tau} d\tau} > \frac{\lambda_2 e^{-\lambda_2 t} \int_s^t b(x_2(\tau))d\tau}{\lambda_1 e^{-\lambda_1 s} \int_s^t b(x_1(\tau))d\tau}.$$

Again, by $\lambda_1 < \lambda_2$ and the assumption that $x_1(\tau) < x_2(\tau)$ for all $\tau \in (s, t)$, the ratio converges to a number strictly greater than 1 as ϵ tends to 0.

However, since on the optimal trigger strategy path the incentive compatibility condition (2) has to hold with equality for small enough time, $\frac{B_1(t)-B_1(s)}{B_2(t)-B_2(s)}$ must be equal to $\frac{P_1(t)-P_1(s)}{P_2(t)-P_2(s)}$ for small enough $\epsilon > 0$. Contradiction.

Proof of Proposition 10

For any $\gamma > 0$, there exists $\delta > 0$ such that for all (λ_1, λ_2) such that $\frac{\lambda_1}{\lambda_2} < \delta$, there exists $\bar{t}$ such that $\lambda_1 \cdot \bar{t} < \gamma$, and $\frac{1}{\gamma} < \lambda_2 \bar{t}$.

Suppose that at $-t \in (-\bar{t}, 0]$, player 1 plays a^S and player 2 obtains a revision opportunity. Since player 2 in equilibrium must get no less than what she would get by consistently taking a best reply to a^S at time $-t$ onwards no matter what the history is, 2's expected payoff must be no less than

$$e^{-\lambda_1 t}\pi(BR(a^S), a^S) + (1 - e^{-\lambda_1 t})\underline{\pi} \geq e^{-\gamma}\pi(BR(a^S), a^S) + (1 - e^{-\gamma})\underline{\pi}.$$

For any $\epsilon' > 0$, there exists $\gamma > 0$ sufficiently small such that the right hand side (hence the left hand side) is no less than $\pi(BR(a^S), a^S) - \epsilon'$. Then, by A2 and A3, player 2's action at any time $-t$ must lie in some neighborhood of $BR(a^S)$, $[BR(a^S) - \xi, BR(a^S) + \xi]$, where $\gamma \to 0$ implies $\epsilon' \to 0$, which in turn implies $\xi \to 0$.

Now we consider the minimum possible payoff of player 1 by playing $a^S := \arg\max_a \pi(a, BR(a))$ at all time $-t \in [-T, 0]$. Since any subgame perfect equilibrium is a Nash equilibrium, player 1 must obtain a payoff no less than the payoff that he would receive by playing this strategy.

The conclusion so far implies that player 1's expected payoff by playing a^S at all time is no less than

$$(1 - e^{-\lambda_2 \bar{t}}) \min_{a'^S) - \xi, BR(a^S) + \xi]} \pi(a^S, a'^{-\lambda_2 \bar{t}}\underline{\pi} \geq (1 - e^{-\frac{1}{\gamma}}) \min_{a'^S) - \xi, BR(a^S) + \xi]} \pi(a^S, a'^{-\frac{1}{\gamma}}\underline{\pi}$$

Since $\xi \to 0$ as $\gamma \to 0$, the right hand side converges to $\pi(a^S, BR(a^S))$ by A2 as $\gamma \to 0$.

This completes the proof.