

# Matching Markets under (In)complete Information\*

Lars Ehlers<sup>†</sup>

Jordi Massó<sup>‡</sup>

November 2012

---

\*L. Ehlers acknowledges financial support from the SSHRC (Canada) and the FQRSC (Québec). Support for the research of J. Massó was received through the prize “ICREA Acadèmia” for excellence in research, funded by the Generalitat de Catalunya. He also acknowledges the support of MOVE (where he is an affiliated researcher) and the Government of Catalonia, through grant SGR2009-419. His work is also supported by the Spanish Ministry of Science and Innovation, through grants ECO2008-04756 (Grupo Consolidado-C) and CONSOLIDER-INGENIO 2010 (CDS2006-00016). Part of this research was done while J. Massó was visiting the Université de Montréal and while L. Ehlers was visiting the Universitat Autònoma de Barcelona; the visits were financed by CIREQ and CREA, respectively.

<sup>†</sup>Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, Québec H3C 3J7, Canada; e-mail: [lars.ehlers@umontreal.ca](mailto:lars.ehlers@umontreal.ca)

<sup>‡</sup>Universitat Autònoma de Barcelona and Barcelona GSE. Departament d'Economia i Història Econòmica, Edifici B. Campus UAB. 08193 Cerdanyola del Vallès (Barcelona), Spain; e-mail: [jordi.massó@uab.es](mailto:jordi.massó@uab.es)

## Abstract

We introduce incomplete information to centralized many-to-one matching markets. This is important because in real life markets (i) any agent is uncertain about the other agents' true preferences and (ii) most entry-level matching is many-to-one (and not one-to-one). We show that given a common prior, a strategy profile is an ordinal Bayesian Nash equilibrium under incomplete information in a stable mechanism if and only if, for any true profile in the support of the common prior, the submitted profile is a Nash equilibrium under complete information in the direct preference revelation game induced by the stable mechanism.

*JEL Classification:* C78, D81, J44.

*Keywords:* Many-to-one Matching Market, Stability, Incomplete Information.

# 1 Introduction

Centralized many-to-one matching markets operate as follows to match the agents from two sides, the firms (colleges, hospitals, schools, etc.) and the workers (students, medical interns, children, etc.): a centralized clearinghouse collects for each participant a ranked list of potential partners and matches via a mechanism firms and workers on the basis of the submitted ranked lists. In applications many of the successful mechanisms are stable.<sup>1,2</sup> The literature has considered stability of a matching (in the sense that all agents have to be matched to acceptable partners and no unmatched pair of a firm and a worker prefer each other rather than the proposed partners) to be its main characteristic in order to survive.<sup>3</sup> This is puzzling because there exists no stable mechanism which makes truth-telling a dominant strategy for all agents (Roth, 1982). Therefore, an agent's (submitted) ranked lists of potential partners are not necessarily his true ones and the implemented matching may not be stable for the true profile. The literature has studied intensively Nash equilibria of direct preference revelation games induced by different stable mechanisms under complete information.<sup>4</sup>

We use the (ordinally) Bayesian approach in many-to-one matching markets by assuming that nature selects a preference profile according to a commonly known probability distribution on the set of profiles (a common prior).<sup>5</sup> Since matching

---

<sup>1</sup>See Roth (1984a), Roth and Peranson (1999), and Roth (2002) for a careful description and analysis of the American entry-level medical market. Roth (1991), Kesten (2005), Ünver (2005), and Ehlers (2008) describe and analyze the equivalent UK markets.

<sup>2</sup>Chen and Sönmez (2006), Ergin and Sönmez (2006), and Abdulkadiroğlu, Che, and Yosuda (2011) study the case of public schools in Boston, Abdulkadiroğlu and Sönmez (2003) studies the cases of public schools in Boston, Lee County (Florida), Minneapolis, and Seattle, and Abdulkadiroğlu, Pathak, and Roth (2005, 2009) study the case of public high schools in New York City.

<sup>3</sup>See, for instance, Roth (1984a) and Niederle and Roth (2003).

<sup>4</sup>See Dubins and Freedman (1981), Roth (1982, 1984b, 1985), Gale and Sotomayor (1985), Shin and Suh (1996), Sönmez (1997), Ma (1995, 2002), and Alcalde (1996).

<sup>5</sup>Roth (1989) is the first paper studying strategic incentives generated by stable mechanisms

markets require to report ranked lists and not their specific utility representations, we stick to the ordinal setting and assume that probability distributions are evaluated according to the first-order stochastic dominance criterion. Then, a strategy profile is an ordinal Bayesian Nash equilibrium (OBNE) if, for every von Neumann Morgenstern (vNM)-utility function of an agent’s preference ordering (his type), the submitted ranked list maximizes his expected utility in the direct preference revelation game induced by the common prior and the mechanism.<sup>6</sup> For direct preference revelation games under incomplete information induced by a stable mechanism, our main result shows a link between Nash equilibria under complete information and OBNE under incomplete information. More precisely, Theorem 1 states that, given a common prior, a strategy profile is an OBNE under incomplete information in a stable mechanism if and only if for any profile in the support of the common prior, the submitted profile is a Nash equilibrium under complete information at the true profile in the direct preference revelation game induced by the stable mechanism.

The paper is organized as follows. Section 2 describes the many-to-one matching market with responsive preferences and introduces incomplete information and the notion of ordinal Bayesian Nash equilibrium. Section 3 states our main result, Theorem 1, and its applications. Section 4 discusses variations of our main result and the Appendix contains all proofs.

---

under incomplete information. He shows that for particular vNM-utility representations of the ordinal preferences, Bayesian Nash equilibria under incomplete information may not satisfy appealing properties of Nash equilibria under complete information. Chakraborty, Citanna, and Ostrovsky (2010) study two-sided matching markets with interdependent preferences.

<sup>6</sup>This notion was introduced by d’Aspremont and Peleg (1988) who call it “ordinal Bayesian incentive-compatibility”. Majumdar and Sen (2004) use it to relax strategy-proofness in the Gibbard-Satterthwaite Theorem. Majumdar (2003), Pais (2005), and Ehlers and Massó (2007) have already used this ordinal equilibrium notion in one-to-one matching markets.

## 2 Many-To-One Matching Markets

Let  $W$  denote the set of workers,  $F$  denote the set of firms, and  $V \equiv F \cup W$  denote the set of agents. For each firm  $f$ , there is a maximum number  $q_f \geq 1$  of workers that  $f$  may hire,  $f$ 's *quota*. Let  $q = (q_f)_{f \in F}$  denote the vector of quotas. Each worker  $w$  has a strict preference ordering  $P_w$  over  $F \cup \{\emptyset\}$ , where  $\emptyset$  stands for being unmatched. Each firm  $f$  has a strict preference ordering  $P_f$  over  $W \cup \{\emptyset\}$ , where  $\emptyset$  stands for leaving a position unfilled. A *profile*  $P = (P_v)_{v \in V}$  is a list of preference orderings. Given  $S \subseteq V$ , we sometimes write  $(P_S, P_{-S})$  instead of  $P$ . Let  $\mathcal{P}_v$  be the set of all preference orderings of agent  $v$ . Let  $\mathcal{P} = \times_{v \in V} \mathcal{P}_v$  be the set of all profiles and let  $\mathcal{P}_{-v} = \times_{v' \in V \setminus \{v\}} \mathcal{P}_{v'}$ . Let  $R_v$  denote the weak preference associated with  $P_v$ . Given  $w \in W$ ,  $P_w \in \mathcal{P}_w$ , and  $v \in F \cup \{\emptyset\}$ , let  $B(v, P_w)$  denote the *weak upper contour set* of  $P_w$  at  $v$ ; *i.e.*,  $B(v, P_w) = \{v' \in F \cup \{\emptyset\} \mid v' R_w v\}$ . Let  $A(P_w)$  be the set of *acceptable* firms for  $w$  according to  $P_w$ ; *i.e.*,  $A(P_w) = \{f \in F \mid f P_w \emptyset\}$ . Given a subset  $S \subseteq F \cup \{\emptyset\}$ , let  $P_w|S$  denote the restriction of  $P_w$  to  $S$ . Similarly, given  $P_f \in \mathcal{P}_f$ ,  $v \in W \cup \{\emptyset\}$ , and  $S \subseteq W \cup \{\emptyset\}$ , we define  $B(v, P_f)$ ,  $A(P_f)$ , and  $P_f|S$ . A *many-to-one matching market* (or *college admissions problem*) is a quadruple  $(F, W, q, P)$ . Because  $F$ ,  $W$  and  $q$  remain fixed, a problem is simply a profile  $P \in \mathcal{P}$ . If  $q_f = 1$  for all  $f \in F$ ,  $(F, W, q, P)$  is called a *one-to-one matching market*.

A *matching* is a function  $\mu : V \rightarrow 2^V$  satisfying the following: (m1) for all  $w \in W$ ,  $\mu(w) \subseteq F$  and  $|\mu(w)| \leq 1$ ; (m2) for all  $f \in F$ ,  $\mu(f) \subseteq W$  and  $|\mu(f)| \leq q_f$ ; and (m3)  $\mu(w) = \{f\}$  if and only if  $w \in \mu(f)$ . We will often write  $\mu(w) = f$  instead of  $\mu(w) = \{f\}$ . If  $\mu(w) = \emptyset$ , we say that  $w$  is *unmatched* at  $\mu$ . If  $|\mu(f)| < q_f$ , we say that  $f$  has  $q_f - |\mu(f)|$  *unfilled* positions at  $\mu$ . Let  $\mathcal{M}$  denote the set of all matchings. Given  $P \in \mathcal{P}$  and  $\mu \in \mathcal{P}$ ,  $\mu$  is *stable* (at  $P$ ) if (s1) for all  $v \in V$ ,  $\mu(v) \subseteq A(P_v)$  (*individual rationality*); and (s2) there exists no pair  $(w, f) \in W \times F$  such that  $f P_w \mu(w)$  and either  $[w P_f \emptyset$  and  $|\mu(f)| < q_f]$  or  $[w P_f w'$  for some  $w' \in \mu(f)]$  (*pairwise stability*). Let  $C(P)$  denote the set of stable matchings at  $P$  (or the core of  $P$ ). A (direct) mechanism is a function  $\varphi : \mathcal{P} \rightarrow \mathcal{M}$ . A mechanism  $\varphi$  is *stable* if for all  $P \in \mathcal{P}$ ,  $\varphi[P]$  is stable at

$P$ . The most popular stable mechanisms are the *deferred-acceptance algorithms* (DA-algorithms) (Gale and Shapley, 1962): the firm-proposing DA-algorithm is denoted by  $DA_F$  and the worker-proposing DA-algorithm is denoted by  $DA_W$ .

A mechanism matches each firm  $f$  to a *set* of workers, taking into account only  $f$ 's preference ordering  $P_f$  over individual workers. To study firms' incentives, preference orderings of firms over individual workers have to be extended to preference orderings over subsets of workers. The preference extension  $P_f^*$  over  $2^W$  is *responsive* to  $P_f \in \mathcal{P}_f$  if for all  $S \in 2^W$ , all  $w \in S$ , and all  $w' \notin S$ : (r1)  $S \cup \{w'\}P_f^*S \Leftrightarrow |S| < q_f$  and  $w'P_f\emptyset$ ; and (r2)  $(S \setminus \{w\}) \cup \{w'\}P_f^*S \Leftrightarrow w'P_fw$ . Let  $R_f^*$  denote the weak preference associated with  $P_f^*$  and  $resp(P_f)$  denote the set of all responsive extensions of  $P_f$ . Moreover, given  $S \in 2^W$ , let  $B(S, P_f^*)$  be the *weak upper contour set* of  $P_f^*$  at  $S$ ; *i.e.*,  $B(S, P_f^*) = \{S' \in 2^W \mid S'R_f^*S\}$ .

Any mechanism and any true profile define a direct (ordinal) preference revelation game under complete information for which we can define the natural (ordinal) notion of Nash equilibrium. Given a mechanism  $\varphi$  and  $P, P' \in \mathcal{P}$ ,  $P'$  is a *Nash equilibrium* (NE) in the mechanism  $\varphi$  under complete information  $P$  if (n1) for all  $w \in W$ ,  $\varphi[P'](w)R_w\varphi[\hat{P}_w, P'_w](w)$  for all  $\hat{P}_w \in \mathcal{P}_w$ ; and (n2) for all  $f \in F$  and all  $P_f^* \in resp(P_f)$ ,  $\varphi[P'](f)R_f^*\varphi[\hat{P}_f, P'_f](f)$  for all  $\hat{P}_f \in \mathcal{P}_f$ . Truth-telling is a NE in  $\varphi$  under  $P$  if  $P$  is a NE in  $\varphi$  under  $P$ .

A *common prior* is a probability distribution  $\tilde{P}$  over  $\mathcal{P}$ . Given  $P \in \mathcal{P}$ , let  $\Pr\{\tilde{P} = P\}$  denote the probability that  $\tilde{P}$  assigns to  $P$ . Given  $v \in V$ , let  $\tilde{P}_v$  denote the marginal distribution of  $\tilde{P}$  over  $\mathcal{P}_v$ . Given a common prior  $\tilde{P}$  and  $P_v \in \mathcal{P}_v$ , let  $\tilde{P}_{-v}|_{P_v}$  denote the probability distribution which  $\tilde{P}$  induces over  $\mathcal{P}_{-v}$  conditional on  $P_v$ . It describes agent  $v$ 's (Bayesian) uncertainty about the preferences of the other agents, given that his preference ordering is  $P_v$ .<sup>7</sup>

A random matching  $\tilde{\eta}$  is a probability distribution over the set of matchings

---

<sup>7</sup>This formulation does not require symmetry nor independence of priors; conditional priors might be very correlated if agents use similar sources to form them (*i.e.*, rankings, grades, recommendation letters, etc.).

$\mathcal{M}$ . Given  $\mu \in \mathcal{M}$ , let  $\Pr\{\tilde{\eta} = \mu\}$  denote the probability that  $\tilde{\eta}$  assigns to  $\mu$ . Let  $\tilde{\eta}(w)$  denote the distribution which  $\tilde{\eta}$  induces over  $w$ 's set of potential partners  $F \cup \{\emptyset\}$ , and let  $\tilde{\eta}(f)$  denote the distribution which  $\tilde{\eta}$  induces over  $f$ 's set of potential partners  $2^W$ . Given two random matchings  $\tilde{\eta}$  and  $\tilde{\eta}'$ , (fo1) for  $w \in W$  and  $P_w \in \mathcal{P}_w$  we say that  $\tilde{\eta}(w)$  *first-order stochastically  $P_w$ -dominates*  $\tilde{\eta}'(w)$ , denoted by  $\tilde{\eta}(w) \succ_{P_w} \tilde{\eta}'(w)$ , if for all  $v \in F \cup \{\emptyset\}$ ,  $\sum_{v' \in F \cup \{\emptyset\}: v' R_w v} \Pr\{\tilde{\eta}(w) = v'\} \geq \sum_{v' \in F \cup \{\emptyset\}: v' R_w v} \Pr\{\tilde{\eta}'(w) = v'\}$ ; and (fo2) for  $f \in F$  and  $P_f \in \mathcal{P}_f$ ,  $\tilde{\eta}(f)$  *first-order stochastically  $P_f$ -dominates*  $\tilde{\eta}'(f)$ , denoted by  $\tilde{\eta}(f) \succ_{P_f} \tilde{\eta}'(f)$ , if for all  $P_f^* \in \text{resp}(P_f)$  and all  $S \in 2^W$ ,<sup>8</sup>  $\sum_{S' \in 2^W: S' R_f^* S} \Pr\{\tilde{\eta}(f) = S'\} \geq \sum_{S' \in 2^W: S' R_f^* S} \Pr\{\tilde{\eta}'(f) = S'\}$ .<sup>9</sup>

A mechanism  $\varphi$  and a common prior  $\tilde{P}$  define a direct (ordinal) preference revelation game under incomplete information. A strategy of agent  $v$  is a function  $s_v : \mathcal{P}_v \rightarrow \mathcal{P}_v$  specifying for each type  $P_v$  of  $v$  a list that  $v$  submits to the mechanism,  $s_v(P_v)$ . We restrict our analysis to pure strategies in the main text. The Appendix generalizes our main result to mixed strategies and random mechanisms. A *strategy profile* is a list  $s = (s_v)_{v \in V}$  of strategies specifying for each true profile  $P$  a submitted profile  $s(P)$ . Given a mechanism  $\varphi : \mathcal{P} \rightarrow \mathcal{M}$  and a common prior  $\tilde{P}$  over  $\mathcal{P}$ , a strategy profile  $s : \mathcal{P} \rightarrow \mathcal{P}$  induces a random matching  $\varphi[s(\tilde{P})]$  in the following way: for all  $\mu \in \mathcal{M}$ ,

$$\Pr\{\varphi[s(\tilde{P})] = \mu\} = \sum_{P \in \mathcal{P}: \varphi[s(P)] = \mu} \Pr\{\tilde{P} = P\}.$$

Using Bayesian updating, the relevant random matching for agent  $v$ , given his type  $P_v$  and a strategy profile  $s$ , is  $\varphi[s_v(P_v), s_{-v}(\tilde{P}_{-v}|P_v)]$  (where  $s_{-v}(\tilde{P}_{-v}|P_v)$  is the probability distribution over  $\mathcal{P}_{-v}$  which  $s_{-v}$  and  $\tilde{P}$  induce conditional on  $P_v$ ).

**Definition 4 (Ordinal Bayesian Nash Equilibrium)** Let  $\tilde{P}$  be a common prior.

<sup>8</sup>Observe that this definition requires that  $\tilde{\eta}$  first-order stochastically dominates  $\tilde{\eta}'$  according to *all* responsive extensions of  $P_f$ .

<sup>9</sup>It is well-known that (fo1) is equivalent to that for any vNM-representation of  $P_w$  the expected utility of  $\tilde{\eta}$  is greater than or equal to the expected utility of  $\tilde{\eta}'$  (and similarly for (fo2) and all vNM-representations of any responsive extension of  $P_f$ ). See for instance, Theorem 3.11 in d'Aspremont and Peleg (1988).

A strategy profile  $s$  is an *ordinal Bayesian Nash equilibrium* (OBNE) in the mechanism  $\varphi$  under incomplete information  $\tilde{P}$  if and only if for all  $v \in V$  and all  $P_v \in \mathcal{P}_v$  such that  $\Pr\{\tilde{P}_v = P_v\} > 0$ ,<sup>10</sup>

$$\varphi[s_v(P_v), s_{-v}(\tilde{P}_{-v|P_v})](v) \succ_{P_v} \varphi[P'_v, s_{-v}(\tilde{P}_{-v|P_v})](v) \quad \text{for all } P'_v \in \mathcal{P}_v. \quad (1)$$

Truth-telling is an OBNE in the mechanism  $\varphi$  under incomplete information  $\tilde{P}$  if and only if for all  $v \in V$  and all  $P_v \in \mathcal{P}_v$  such that  $\Pr\{\tilde{P}_v = P_v\} > 0$ ,

$$\varphi[P_v, \tilde{P}_{-v|P_v}](v) \succ_{P_v} \varphi[P'_v, \tilde{P}_{-v|P_v}](v) \quad \text{for all } P'_v \in \mathcal{P}_v. \quad (2)$$

In general for arbitrary mechanisms there is no connection between NE under complete information and OBNE under incomplete information. For instance, suppose that the common prior  $\tilde{P}^u$  is uniform in the sense that it puts equal probability on all preference profiles. Furthermore, suppose that the mechanism  $\varphi$  matches a worker and a firm if and only if they rank each other as their most preferred choice (and  $\varphi$  leaves all other positions unfilled and all other workers unmatched). Then it is easy to verify that truth-telling is an OBNE in the mechanism  $\varphi$  under the uniform prior  $\tilde{P}^u$ .<sup>11</sup> However, truth-telling is not always a NE in the mechanism  $\varphi$  under complete information since for some profiles, a firm may rank a worker first and a worker the firm second, and if the worker is unmatched, then she profitably manipulates by moving the firm to the first position of her submitted ranking. Our main result will show that such a disconnection between NE and OBNE is only possible for unstable mechanisms.

---

<sup>10</sup>In the definition of OBNE optimal behavior of agent  $v$  is only required for the preferences of  $v$  which arise with positive probability under  $\tilde{P}$ . If  $P_v \in \mathcal{P}_v$  is such that  $\Pr\{\tilde{P}_v = P_v\} = 0$ , then the conditional prior  $\tilde{P}_{-v|P_v}$  cannot be derived from  $\tilde{P}$ . However, we could complete the prior of  $v$  in the following way: let  $\tilde{P}_{-v|P_v}$  put probability one on a profile where all other agents submit lists which do not contain  $v$ .

<sup>11</sup>For any agent  $v$  and any  $P_v \in \mathcal{P}_v$ ,  $\tilde{P}_{-v}^u|_{P_v}$  is uniform over  $\mathcal{P}_{-v}$ . For all agents belonging to the opposite side of the market, the probability that she ranks  $v$  first is identical. Hence,  $v$  cannot do better than submitting the true preference relation.



### 3 The Main Result and Its Applications

The support of a common prior  $\tilde{P}$  is the set of profiles on which  $\tilde{P}$  puts positive probability:  $P \in \mathcal{P}$  belongs to the support of  $\tilde{P}$  if and only if  $\Pr\{\tilde{P} = P\} > 0$ .

**Theorem 1** *Let  $\tilde{P}$  be a common prior,  $s$  be a strategy profile, and  $\varphi$  be a stable mechanism. Then,  $s$  is an OBNE in the stable mechanism  $\varphi$  under incomplete information  $\tilde{P}$  if and only if for any profile  $P$  in the support of  $\tilde{P}$ ,  $s(P)$  is a Nash equilibrium in the stable mechanism  $\varphi$  under complete information  $P$ .*

Theorem 1 has several consequences and applications. One immediate consequence is that for determining whether a strategy profile is an OBNE, we only need to check whether for each realization of the common prior the submitted preference orderings constitute a Nash equilibrium under complete information. This means that the uniquely relevant information for an OBNE is the support of the common prior and no calculations of probabilities are necessary. This consequence is very important for applications because we need to check equilibrium play only for the realized (or observed) profiles. Furthermore, by Theorem 1, we can use properties of NE (under complete information) to deduce characteristics of OBNE.

Observe that, given a common belief  $\tilde{P}$ , the set of OBNE in a stable mechanism is non-empty. For instance, imagine that the workers and the firms are divided into “local” matching markets as follows: let  $(W_f)_{f \in F}$  be a partition of the set of workers (allowing  $W_f = \emptyset$  for some firms  $f$ ) where  $W_f$  denotes the set of workers belonging to the “local” market of  $f$ . In words, under the following strategy profile, if worker  $w$  belongs to the local market of firm  $f$ , then  $w$  ranks  $f$  uniquely acceptable if  $f$  is preferred to being unmatched and otherwise  $w$  ranks no firm acceptable. Any firm  $f$  ranks as acceptable (and in the true order) all workers which both belong to its local market and are acceptable according to its true preference relation. Let the strategy profile  $s$  be defined in the following way: (i) for any  $w \in W$  and any  $P_w \in \mathcal{P}_w$ ,  $A(s_w(P_w)) = \{f\}$  if  $f \in A(P_w)$  and  $w \in W_f$ , and  $A(s_w(P_w)) = \emptyset$  otherwise; and (ii)

for all  $f \in F$  and all  $P_f \in \mathcal{P}_f$ , let  $A(s_f(P_f)) = A(P_f) \cap W_f$  and  $s_f(P_f)|_{A(P_f) \cap W_f} = P_f|_{A(P_f) \cap W_f}$ . Then for any stable mechanism and any profile  $P$ ,  $s(P)$  is a NE under complete information. Hence,  $s$  is an OBNE in any stable mechanism under any common belief. In the special case where  $W_f = W$  for some firm  $f$ , firm  $f$  has a monopolistic market.

Restricting ourselves to truth-telling (where  $s(P) = P$  for all  $P \in \mathcal{P}$ ), Theorem 1 shows that truth-telling is an OBNE in a stable mechanism  $\varphi$  if and only if for any profile in the support of the common belief, truth-telling is a NE in  $\varphi$  under complete information for any profile in the support of the common prior. In other words, in many-to-one matching *a stable mechanism  $\varphi$  is ordinally Bayesian incentive compatible under the common prior  $\tilde{P}$  if and only if  $\varphi$  restricted to the support of  $\tilde{P}$  is incentive compatible*. Furthermore, it can be easily seen that  $P$  is a NE in the stable mechanism  $\varphi$  under complete information  $P$  if and only if  $P$  is a NE in any stable mechanism under complete information  $P$ . All what matters is the stability of the mechanism (and not which specific stable matching is chosen).

While the proof of Theorem 1's (If)-part is straightforward, its (Only if)-part proceeds roughly as follows. If for some profile  $P$  in the support of  $\tilde{P}$ ,  $s(P)$  does not constitute a NE, then some agent  $v$  has a profitably deviation from  $s(P)$  under complete information  $P$ . Using this deviation we then construct another profitable deviation for  $v$  from  $s(P)$  under complete information  $P$  such that agent  $v$  strictly increases the probability of the weak upper contour set (at his type  $P_v$ ) of assigned partner(s) of the deviation under  $P$ . This implies that strategy profile  $s$  cannot be an OBNE. The construction and the proof use repeatedly the following peculiarities of stable matchings in many-to-one matching markets (see Appendix A.1): (1) invariance of unmatched agents and unfilled positions: the set of unmatched agents and any firm's number of unfilled positions are the same for all stable matchings; and (2) comparative statics: starting from any many-to-one matching market and its workers-optimal matching, when new workers become available all firms weakly

prefer any matching, which is stable for the enlarged market, to the workers-optimal stable matching of the smaller market.

Below we turn to the applications of Theorem 1.

### 3.1 Application I: Structure of OBNE

By Theorem 1, a strategy profile is an OBNE if and only if the agents play a Nash equilibrium for any profile in the support of the common prior. Therefore, (a) the set of OBNE is identical for any two common priors with equal support and (b) the set of OBNE shrinks if the support of the common prior becomes larger. We state these two facts as Corollary 1 below.

**Corollary 1 (Invariance)** *Let  $s$  be a strategy profile and  $\varphi$  be a stable mechanism.*

- (a) *Let  $\tilde{P}$  and  $\tilde{P}'$  be two common priors with equal support. Then,  $s$  is an OBNE in the stable mechanism  $\varphi$  under  $\tilde{P}$  if and only if  $s$  is an OBNE in the stable mechanism  $\varphi$  under  $\tilde{P}'$ .*
- (b) *Let  $\tilde{P}$  and  $\tilde{P}'$  be two common priors such that the support of  $\tilde{P}'$  is contained in the support of  $\tilde{P}$ . If  $s$  is an OBNE in the stable mechanism  $\varphi$  under  $\tilde{P}$ , then  $s$  is an OBNE in the stable mechanism  $\varphi$  under  $\tilde{P}'$ .*

Now by (a) of Corollary 1, for stable mechanisms any OBNE is robust to perturbations of the common prior which leave its support unchanged. Therefore, any OBNE remains an equilibrium if agents have different priors with equal support, *i.e.* each agent  $v$  may have a private prior  $\tilde{P}^v$  but all private priors have identical (or common) support.<sup>12</sup> This consequence is especially important for applications since for many of them, the common prior assumption might be too strong.

---

<sup>12</sup>Then in Definition 4 of OBNE the common prior  $\tilde{P}$  is replaced for each agent  $v$  by his private prior  $\tilde{P}^v$ . Theorem 1 and its proof show that, a strategy profile  $s$  is an OBNE in a stable mechanism  $\varphi$  under private priors  $(\tilde{P}^v)_{v \in V}$  if and only if for all  $v \in V$  and any profile  $P$  in the support of  $\tilde{P}^v$ ,  $s_v(P_v)$  is a best response to  $s_{-v}(P_{-v})$  in  $\varphi$  under complete information  $P$ . In other words,

By (b) of Corollary 1, the set of OBNE with full support (*i.e.* all common priors which put positive probability on all profiles) is contained in the set of OBNE of any arbitrary common prior (or support). Therefore, any OBNE for a common prior with full support is an OBNE for any arbitrary prior. Hence, such OBNE are invariant with respect to the common prior and remain OBNE if the agents' priors are not necessarily derived from the same common prior (and the "local" markets example is an OBNE in any stable mechanism under any priors).

### 3.2 Application II: Truth-Telling under Correlated Preferences

In empirical applications the preferences of one side of the market are often perfectly correlated. For example, each firm may rank all workers according to an objective criterion such as their degree of qualifications or each college may rank all students according to their grades. Furthermore, it is common in labor economics or search theory to often assume that all workers have identical preferences over firms.<sup>13</sup> One-sided perfect correlation is an extreme case of interdependence of preferences where an agent's preference may depend on the preferences of the other agents on his side.

We say that a common prior  $\tilde{P}$  is *F-correlated* if for any profile  $P$  in the support of  $\tilde{P}$ , all firms have identical preferences.<sup>14</sup> Similarly we say that a prior  $\tilde{P}$  is *W-correlated* if for any profile  $P$  in the support of  $\tilde{P}$ , all workers have identical preferences. Theorem 1 also helps us to prove the following result under *F-correlated* or *W-correlated* priors. Theorem 1 also helps us to prove the following result under *F-correlated* or *W-correlated* priors. Each agent's strategy  $s_v$  chooses a best response to the other reported preferences for any profile belonging to the support of his private prior. If all private priors have equal support, then it follows that a strategy profile  $s$  is an OBNE with private priors (with common support) if and only if for any profile  $P$  in the common support,  $s(P)$  is a Nash equilibrium in the mechanism  $\varphi$  under complete information  $P$ .

<sup>13</sup>For instance, Shi (2002) provides a long list of papers on directed search models in labor markets where at least one side of the market is homogenous.

<sup>14</sup>Formally this means for all  $f, f' \in F$ ,  $A(P_f) = A(P_{f'})$  and  $P_f|W = P_{f'}|W$ .

$W$ -correlated common priors.

**Proposition 1** *Let  $\tilde{P}$  be a common prior.*

- (a) *If  $\tilde{P}$  is  $F$ -correlated or  $W$ -correlated, then truth-telling is an OBNE in any stable mechanism under incomplete information  $\tilde{P}$ .*
- (b) *Let  $s$  be a strategy profile such that  $s_w(P_w) = P_w$  for all  $w \in W$  and all  $P_w \in \mathcal{P}_w$ . If  $\tilde{P}$  is  $W$ -correlated and  $s$  is an OBNE in the stable mechanism  $DA_W$  under incomplete information  $\tilde{P}$ , then for all profiles  $P$  in the support of  $\tilde{P}$ ,  $DA_W[s(P)]$  is stable with respect to  $P$ . The analogous statement is true for the stable mechanism  $DA_F$ .*

Although Proposition 1 focuses on completely correlated priors, it is easy to extend it in the following direction. Suppose that each worker has a certain qualification and each firm only offers positions having the same job-specific qualification. Let all firms, which are interested in the same qualification, have identical preferences over all workers possessing this qualification for any realization in the common prior. Then the qualifications segregate the matching market and the conclusions of Proposition 1 apply. For example, each firm may represent a certain department in a hospital and they would like to fill their positions with physicians who studied the medical specialty of their department.

## 4 Variations

Recall for truth-telling to be an OBNE for a common belief, it must be that for any firm and any of its realized preference over firms, truth-telling first order stochastically dominates submitting any other ranking *for all* responsive extensions of the true ranking. It is natural to ask whether Theorem 1 breaks down when we restrict the set of responsive extensions firms may have.

First, it is easy to see that the proof of Theorem 1 remains true if firms responsive extensions are additive, i.e. where a firm has a numerical value for each worker and the value of a set of workers is the sum of the values of the hired workers.<sup>15</sup>

Second, we show that Theorem 1 depends on firms having responsive extensions which are not monotonic: given  $P_f \in \mathcal{P}_f$  and  $P_f^* \in \text{resp}(P_f)$ ,  $P_f^*$  is *monotonic* if for all  $S, S' \in 2^W$  such that  $|S'| < |S| \leq q_f$  and  $S \subseteq A(P_f)$ , we have  $SP_f^*S'$ . Let  $m\text{resp}(P_f)$  denote the set of all monotonic responsive extensions of  $P_f$ . We will call a strategy profile a *monotonic OBNE* in a mechanism under incomplete information if (fo2) holds for all monotonic responsive extensions of any firm's preference relation over individual workers. In the example below we show that truth-telling is a monotonic OBNE in  $DA_W$ , while, for some preference profile  $P$  in the support of the common belief, truth-telling is not a Nash equilibrium under complete information  $P$  in the direct preference revelation game induced by  $DA_W$ .

**Example 1** Consider a many-to-one matching market with three firms  $F = \{f_1, f_2, f_3\}$  and four workers  $W = \{w_1, w_2, w_3, w_4\}$ . Firm  $f_1$  has capacity  $q_{f_1} = 2$  and firms  $f_2$  and  $f_3$  have capacity  $q_{f_2} = q_{f_3} = 1$ . Consider the common belief  $\tilde{P}$  with  $\Pr\{\tilde{P} = P\} = p$  and  $\Pr\{\tilde{P} = \bar{P}\} = 1 - p$ , where  $p < 1/2$ , and  $P$  and  $\bar{P}$  are the following profiles:

$P_{f_1}$	$P_{f_2}$	$P_{f_3}$	$P_{w_1}$	$P_{w_2}$	$P_{w_3}$	$P_{w_4}$	$\bar{P}_{f_1}$	$\bar{P}_{f_2}$	$\bar{P}_{f_3}$	$\bar{P}_{w_1}$	$\bar{P}_{w_2}$	$\bar{P}_{w_3}$	$\bar{P}_{w_4}$
$w_1$	$w_1$	$w_3$	$f_3$	$f_2$	$f_1$	$f_1$	$w_1$	$w_2$	$w_4$	$f_1$	$f_2$	$f_1$	$f_3$
$w_2$	$w_2$	$w_1$	$f_1$	$f_1$	$f_3$	$f_2$	$w_2$						
$w_3$	$w_3$	$w_2$	$f_2$	$f_3$	$f_2$	$f_3$	$w_3$						
$w_4$	$w_4$	$w_4$					$w_4$						

Note that  $P_{f_1} = \bar{P}_{f_1}$ . It is straightforward to verify that both profiles have a singleton

<sup>15</sup>A responsive preference ordering  $P_f^*$  is additive if there exists an injective function  $g : W \rightarrow \mathbb{R} \setminus \{0\}$  such that for all  $S, S' \in 2^W$  with  $|S| \leq q_f$  and  $|S'| \leq q_f$ , we have  $SP_f^*S' \Leftrightarrow \sum_{w \in S} g(w) > \sum_{w' \in S'} g(w')$ . In footnotes we show that any responsive extension in the proof of Theorem 1 can be chosen to be additive.

core and  $C(P) = \{\mu\}$  and  $C(\bar{P}) = \{\bar{\mu}\}$ , where

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ \{w_3, w_4\} & w_2 & w_1 \end{pmatrix} \quad \text{and} \quad \bar{\mu} = \begin{pmatrix} f_1 & f_2 & f_3 \\ \{w_1, w_3\} & w_2 & w_4 \end{pmatrix}.$$

Let  $\varphi$  be a stable mechanism. Thus, by stability of  $\varphi$ ,  $\varphi[P] = \mu$  and  $\varphi[\bar{P}] = \bar{\mu}$ .

First we will show that for the profile  $P$  truth-telling is *not* a Nash equilibrium under complete information  $P$  in the direct preference revelation game induced by  $DA_W$ .

Let  $P'_{f_1} \in \mathcal{P}_{f_1}$  be such that  $P'_{f_1} : w_1 w_2 w_4 \emptyset w_3$ . Then  $C(P'_{f_1}, P_{-f_1}) = \{\mu'\}$  where

$$\mu' = \begin{pmatrix} f_1 & f_2 & f_3 \\ \{w_1, w_4\} & w_2 & w_3 \end{pmatrix}.$$

Hence, by stability of  $\varphi$ ,  $\varphi[P'_{f_1}, P_{-f_1}] = \mu'$ . Obviously, for all responsive extensions  $P^*_{f_1}$  of  $P_f$  we have  $\{w_1, w_4\} P^*_{f_1} \{w_3, w_4\}$ , which is equivalent to  $\varphi[P'_{f_1}, P_{-f_1}](f_1) P^*_{f_1} \varphi[P](f_1)$ .

Therefore, truth-telling is not a Nash equilibrium in any stable mechanism  $\varphi$  under complete information  $P$  (and profile  $P$  belongs to the support of the common belief  $\tilde{P}$ ).

On the other hand we will show that *truth-telling is a monotonic OBNE* in the stable mechanism  $DA_W$  under incomplete information  $\tilde{P}$ . Note that for all  $v \in V \setminus \{f_1\}$ , if  $v$  observes his preference relation, then  $v$  knows whether  $P$  was realized or  $\bar{P}$  was realized. Since at both of  $P$  and  $\bar{P}$  the core is a singleton and firms  $f_2$  and  $f_3$  have quota one, it follows from the proof of Theorem 1 in Ehlers and Massó (2007) that  $v$  cannot gain by deviating.

Next we consider firm  $f_1$ . All arguments except for the last one apply to any stable mechanism  $\varphi$ . Observe that  $P_{f_1} = \bar{P}_{f_1}$  and the random matching  $\varphi[P_{f_1}, \tilde{P}_{-f_1}|P_{f_1}]$  assigns to  $f_1$  the set  $\{w_3, w_4\}$  with probability  $p$  and the set  $\{w_1, w_3\}$  with probability  $1 - p$ . Let  $P^*_{f_1}$  be a monotonic responsive extension of  $P_{f_1}$  and  $P''_{f_1} \in \mathcal{P}_{f_1}$ . We show that

$$\varphi[P_{f_1}, \tilde{P}_{-f_1}|P_{f_1}](f_1) \succ_{P^*_{f_1}} \varphi[P''_{f_1}, \tilde{P}_{-f_1}|P_{f_1}](f_1). \quad (3)$$

We distinguish two cases. First, suppose that  $|\varphi[P''_{f_1}, P_{-f_1}](f_1)| = 1$  or  $|\varphi[P''_{f_1}, \bar{P}_{-f_1}](f_1)| = 1$ . Now if (3) does not hold, then by monotonicity of  $P^*_{f_1}$  and the fact that when sub-

mitting  $P_{f_1}$ ,  $f_1$  is assigned the set  $\{w_3, w_4\}$  with probability  $p$  and the set  $\{w_1, w_3\}$  with probability  $1 - p$  (where  $1 - p > 1/2$ ), we must have that  $\varphi[P_{f_1}'', P_{-f_1}](f_1)P_{f_1}^*\{w_1, w_3\}$  or  $\varphi[P_{f_1}'', \bar{P}_{-f_1}](f_1)P_{f_1}^*\{w_1, w_3\}$ . Obviously, from the definition of  $\bar{P}_{-f_1}$ , the last is impossible. Thus,  $\varphi[P_{f_1}'', P_{-f_1}](f_1)P_{f_1}^*\{w_1, w_3\}$  and, by responsiveness of  $P_{f_1}^*$ , we must have  $\varphi[P_{f_1}'', P_{-f_1}](f_1) = \{w_1, w_2\}$ . But then, without loss of generality, we would have  $DA_F[P_{f_1}'', P_{-f_1}](f_1) = \{w_1, w_2\}$  (because  $DA_F$  chooses the most preferred stable matching from the firms' point of view).<sup>16</sup> Since we have  $C(P) = \{\mu\}$  and  $DA_F[P_{f_1}, P_{-f_1}](f_1) = \{w_3, w_4\}$ , this would imply that in the corresponding one-to-one matching problem  $DA_F$  is group manipulable by the two copies of  $f_1$  (with each copy gaining strictly), a contradiction to the result of Dubins and Freedman (1981).<sup>17</sup>

Second, suppose that  $|\varphi[P_{f_1}'', P_{-f_1}](f_1)| = 2$  and  $|\varphi[P_{f_1}'', \bar{P}_{-f_1}](f_1)| = 2$ . Then by definition of  $\bar{P}_{-f_1}$  and  $|\varphi[P_{f_1}'', \bar{P}_{-f_1}](f_1)| = 2$ , we must have  $\varphi[P_{f_1}'', \bar{P}_{-f_1}](f_1) = \{w_1, w_3\}$ . Thus, by stability of  $\varphi$ ,  $\{w_1, w_3\} \subseteq A(P_{f_1}'')$ .

If for all  $\mu'' \in C(P_{f_1}'', P_{-f_1})$ ,  $\mu''(w_4) = \emptyset$ , then  $w_4 \notin A(P_{f_1}'')$  and by definition of  $P_{-f_1}$  and  $w_3 \in A(P_{f_1}'')$ ,  $DA_W[P_{f_1}'', P_{-f_1}](f_1) = \{w_3\}$ . Then  $f_1$  does not fill all its positions at the workers-optimal matching and by Roth and Sotomayor (1990),  $f_1$  is matched to the same set of workers at all stable matchings. Hence,  $\varphi[P_{f_1}'', P_{-f_1}](f_1) = \{w_3\}$  and (3) holds (because when submitting  $P_{f_1}''$ , firm  $f_1$  is matched with probability  $p$  to  $w_3$  and with probability  $1 - p$  to  $\{w_1, w_3\}$ ).

If for some  $\mu'' \in C(P_{f_1}'', P_{-f_1})$ ,  $\mu''(w_4) \neq \emptyset$ , then by definition of  $P_{-f_1}$ ,  $\mu''(w_4) = f_1$ ; otherwise the pair  $(w_2, f_2)$  would block  $\mu''$  at  $(P_{f_1}'', P_{-f_1})$  if  $\mu''(w_4) = f_2$  and the pair  $(w_1, f_3)$  would block  $\mu''$  at  $(P_{f_1}'', P_{-f_1})$  if  $\mu''(w_4) = f_3$ . Thus, by  $\{w_3, w_4\} \subseteq A(P_{f_1}'')$ ,  $\mu \in C(P_{f_1}'', P_{-f_1})$  and  $DA_W[P_{f_1}'', P_{-f_1}] = \mu$ . Hence, (3) holds for the stable mechanism  $DA_W$ .<sup>18</sup>

<sup>16</sup>If  $DA_F[P_{f_1}'', P_{-f_1}](f_1) \neq \{w_1, w_2\}$ , then choose  $P_{f_1}'''$  such that  $A(P_{f_1}''') = \{w_1, w_2\}$ . Then we obtain  $DA_F[P_{f_1}''', P_{-f_1}](f_1) = \{w_1, w_2\}$ .

<sup>17</sup>Their result says that in a one-to-one matching market no group of firms can profitably manipulate  $DA_F$  at the true profile under complete information (with strict preference holding for all firms belonging to the group).

<sup>18</sup>Note that Example 1 does not contradict Theorem 1. When considering the non-monotonic



The above example has another important implication: suppose that firms submit preference orderings over sets of workers instead of submitting preference orderings over individual workers only and the common belief is a distribution over profiles where firms' preference orderings are over sets of workers. Now if the common belief puts only positive probability on profiles where all firms' preference orderings are responsive and monotonic, then the above example shows that truth-telling can be an OBNE while not necessarily at all profiles in the support truth-telling is a NE under complete information.

This is partly due to the fact our main result is a statement for any common belief. Once we put certain conditions on the common belief, our main result continues to hold even if firms submit preference orderings over sets of workers. Without going into details, let  $\mathcal{P}_f^*$  denote the set of all responsive preference orderings of  $f$  over  $2^W$  and  $\mathcal{P}^* = (\times_{f \in F} \mathcal{P}_f^*) \times (\times_{w \in W} \mathcal{P}_w)$ . Let the common belief  $\tilde{P}^*$  on  $\mathcal{P}^*$  be such that for all  $P_{-f}^* \in \mathcal{P}_{-f}^*$  and all  $P_f, P'_f \in \mathcal{P}_f^*$  such that  $P_f|_{W \cup \{\emptyset\}} = P'_f|_{W \cup \{\emptyset\}}$ ,  $\Pr\{\tilde{P}^* = (P_f, P_{-f}^*)\} > 0 \Leftrightarrow \Pr\{\tilde{P}^* = (P'_f, P_{-f}^*)\} > 0$ . In words, if whenever the common belief puts positive probability on some profile, then for any firm and any other preference ordering which is responsive to the same ordering, the belief also puts positive probability on the profile where the firm's preference is replaced by this preference ordering (the support of the common belief does not distinguish preference orderings which are responsive to the same ordering over individual workers). The proof of Theorem 1 then shows that truth-telling is an OBNE in the stable mechanism  $\varphi$  if and only if for any profile in the support of  $\tilde{P}^*$ , truth-telling is a NE in the stable mechanism under complete information.

It would be interesting to identify other economic environments where a similar link between BNE under incomplete information and NE under complete information

---

extension  $P_{f_1}^*$  such that  $\{w_1\}P_{f_1}^*\{w_3, w_4\}$ , then firm  $w_1$  can gain by submitting the list  $\hat{P}_{f_1}$  where worker  $w_1$  is the unique acceptable worker (*i.e.*  $A(\hat{P}_{f_1}) = \{w_1\}$ ). Then we have both  $\varphi[\hat{P}_{f_1}, P_{-f_1}](f_1) = \{w_1\}$  and  $\varphi[\hat{P}_{f_1}, \bar{P}_{-f_1}](f_1) = \{w_1\}$ , which means that truth-telling is not an OBNE in any stable mechanism  $\varphi$  under incomplete information  $\tilde{P}$ .

holds. In those environments the strategic analysis under complete information is essential to undertake the corresponding analysis under incomplete information.

## References

- [1] A. Abdulkadiroğlu, Y.-K. Che, and Y. Yosuda. “Resolving Conflicting Preferences in School Choice: the Boston Mechanism Reconsidered,” *American Economic Review* **101**, 399-410 (2011).
- [2] A. Abdulkadiroğlu, P. Pathak, and A. Roth. “The New York City High School Match,” *American Economic Review, Papers and Proceedings* **95**, 364-367 (2005).
- [3] A. Abdulkadiroğlu, P. Pathak, and A. Roth. “Strategy-proofness versus Efficiency in Matching with Indifferences: Redefining the NYC High School Match,” *American Economic Review* **99**, 1954-1978 (2009).
- [4] A. Abdulkadiroğlu and T. Sönmez. “School Choice: A Mechanism Design Approach,” *American Economic Review* **93**, 729-747 (2003).
- [5] C. d’Aspremont and B. Peleg. “Ordinal Bayesian Incentive Compatible Representation of Committees,” *Social Choice and Welfare* **5**, 261-280 (1988).
- [6] J. Alcalde. “Implementation of Stable Solutions to Marriage Problems,” *Journal of Economic Theory* **69**, 240-254 (1996).
- [7] A. Chakraborty, A. Citanna, and M. Ostrovsky. “Two-Sided Matching with Interdependent Values,” *Journal of Economic Theory* **145**, 85-105 (2010).
- [8] Y. Chen and T. Sönmez. “School Choice: An Experimental Study,” *Journal of Economic Theory* **127**, 202-231 (2006).
- [9] V.P. Crawford. “Comparative Statics in Matching Markets,” *Journal of Economic Theory* **54**, 389-400 (1991).

- [10] L. Dubins and D. Freedman. “Machiavelli and the Gale-Shapley Algorithm,” *American Mathematical Monthly* **88**, 485-494 (1981).
- [11] L. Ehlers. “Truncation Strategies in Matching Markets,” *Mathematics of Operations Research* **33**, 327-335 (2008).
- [12] L. Ehlers and J. Massó. “Incomplete Information and Singleton Cores in Matching Markets,” *Journal of Economic Theory* **136**, 587-600 (2007).
- [13] H. Ergin and T. Sönmez. “Games of School Choice under the Boston Mechanism,” *Journal of Public Economics* **90**, 215-237 (2006).
- [14] D. Gale and L. Shapley. “College Admissions and the Stability of Marriage,” *American Mathematical Monthly* **69**, 9-15 (1962).
- [15] D. Gale and M. Sotomayor. “Ms. Machiavelli and the Stable Matching Problem,” *American Mathematical Monthly* **92**, 261-268 (1985).
- [16] O. Kesten. “An Advice to the Organizers of Entry-Level Labor Markets in the United Kingdom,” mimeo, University of Rochester (2005).
- [17] J. Ma. “Stable Matchings and Rematching-Proof Equilibria in a Two-Sided Matching Market,” *Journal of Economic Theory* **66**, 352-369 (1995).
- [18] J. Ma. “Stable Matchings and the Small Core in Nash Equilibrium in the College Admissions Problem,” *Review of Economic Design* **7**, 117-134 (2002).
- [19] D. Majumdar. “Ordinally Bayesian Incentive-Compatible Stable Matchings,” mimeo, CORE (2003).
- [20] D. Majumdar and A. Sen. “Ordinally Bayesian Incentive-Compatible Voting Rules,” *Econometrica* **72**, 523-540 (2004).

- [21] M. Niederle and A.E. Roth. “Unraveling Reduces Mobility in a Labor Market: Gastroenterology with and without a Centralized Match,” *Journal of Political Economy* **111**, 1342-1352 (2003).
- [22] J. Pais. *Incentives in Random Matching Markets*. Ph.D. Thesis, Universitat Autònoma de Barcelona (2005).
- [23] J. Pais. “Random Matching in the College Admissions Problem,” *Economic Theory* **35**, 99-116 (2008).
- [24] A.E. Roth. “The Economics of Matching: Stability and Incentives,” *Mathematics of Operations Research* **7**, 617-628 (1982).
- [25] A.E. Roth. “The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory,” *Journal of Political Economy* **92**, 991-1016 (1984a).
- [26] A.E. Roth. “Misrepresentation and Stability in the Marriage Problem,” *Journal of Economic Theory* **34**, 383-387 (1984b).
- [27] A.E. Roth. “The College Admissions Problem is not Equivalent to the Marriage Problem,” *Journal of Economic Theory* **36**, 277-288 (1985).
- [28] A.E. Roth. “Two-Sided Matching with Incomplete Information about Others’ Preferences,” *Games and Economic Behavior* **1**, 191-209 (1989).
- [29] A.E. Roth. “A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the U.K.,” *American Economic Review* **81**, 415-440 (1991).
- [30] A.E. Roth. “The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics,” *Econometrica* **70**, 1341-1378 (2002).

- [31] A.E. Roth and E. Peranson. “The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design,” *American Economic Review* **89**, 748-780 (1999).
- [32] A.E. Roth and M. Sotomayor. “The College Admissions Problem Revisited,” *Econometrica* **57**, 559-570 (1989).
- [33] A.E. Roth and M. Sotomayor. *Two-sided Matching: A Study in Game-Theoretic Modelling and Analysis*. Cambridge University Press, Cambridge, England. [Econometric Society Monograph] (1990).
- [34] S. Shi. “Directed Search Model of Inequality with Heterogenous Skills and Skill-Biased Technology,” *Review of Economic Studies* **69**, 467-491 (2002).
- [35] S. Shin and S.-C. Suh. “A Mechanism Implementing the Stable Rule in Marriage Problems,” *Economics Letters* **51**, 185-189 (1996).
- [36] T. Sönmez. “Games of Manipulation in Marriage Problems,” *Games and Economic Behaviour* **20**, 169-176 (1997).
- [37] U. Ünver. “On the Survival of Some Unstable Two-Sided Matching Mechanisms,” *International Journal of Game Theory* **33**, 239-254 (2005).

## APPENDIX

Before we prove Theorem 1, we recall the following properties of the core of a many-to-one matching market. These properties will be used frequently in the proof. It will be convenient to write  $(F, W, P; q)$  for any many-to-one matching market  $(F, W, q, P)$  in which  $q_f = 1$  for all  $f \in F$ .

### A.1 Properties of the Core

The core of a many-to-one matching market has a special structure. The following well-known properties will be useful in the sequel:<sup>19</sup>

(P1) For each profile  $P \in \mathcal{P}$ ,  $C(P)$  contains two stable matchings, the firms-optimal stable matching  $\mu_F$  and the workers-optimal stable matching  $\mu_W$ , with the property that for all  $\mu \in C(P)$ ,  $\mu_W(w)R_w\mu(w)R_w\mu_F(w)$  for all  $w \in W$ , and for all  $f \in F$ ,  $\mu_F(f)R_f^*\mu(f)R_f^*\mu_W(f)$  for all  $P_f^* \in \text{resp}(P_f)$ . The *deferred-acceptance algorithms* (*DA-algorithms*) (Gale and Shapley, 1962) are denoted by  $DA_F : \mathcal{P} \rightarrow \mathcal{M}$  and  $DA_W : \mathcal{P} \rightarrow \mathcal{M}$ : for all  $P \in \mathcal{P}$ ,  $DA_F[P] = \mu_F$  and  $DA_W[P] = \mu_W$ .

(P2) For each profile  $P \in \mathcal{P}$  and any responsive extensions  $P_F^* = (P_f^*)_{f \in F}$  of  $P_F = (P_f)_{f \in F}$ ,  $C(P)$  coincides with the set of group stable matchings at  $(P_W, P_F^*)$ , where group stability corresponds to the usual cooperative game theoretical notion of weak blocking.<sup>20</sup> This means that the set of group stable matchings (relative to  $P$ ) is invariant with respect to any specific responsive extensions of  $P_F$ .

(P3) For each  $P \in \mathcal{P}$ , the set of unmatched agents is the same for all stable matchings (see Roth and Sotomayor, 1990, Theorems 5.12 and 5.13): for all  $\mu, \mu' \in C(P)$ , and for all  $w \in W$  and  $f \in F$ , (i) if  $\mu(w) = \emptyset$ , then  $\mu'(w) = \emptyset$ ; (ii)  $|\mu(f)| = |\mu'(f)|$ ; and (iii) if  $|\mu(f)| < q_f$ , then  $\mu(f) = \mu'(f)$ .

<sup>19</sup>See Roth and Sotomayor (1990) for a detailed presentation of these properties.

<sup>20</sup>A matching  $\mu$  is weakly blocked by coalition  $S \subseteq V$  under  $(P_W, P_F^*)$  if there exists a matching  $\mu'$  such that (b1) for all  $v \in S$ ,  $\mu'(v) \subseteq S$ , (b2) for all  $w \in W \cap S$ ,  $\mu'(w)R_w\mu(w)$ , and (b3) for all  $f \in F \cap S$ ,  $\mu'(f)R_f^*\mu(f)$ , with strict preference holding for at least one  $v \in S$ .

(P4) Consider a one-to-one matching market  $(F, W, P; q)$  and suppose that new workers enter the market. Let  $(F, W', P'; q)$  be this new one-to-one matching market where  $W \subseteq W'$  and  $P'$  agrees with  $P$  over  $F$  and  $W$ . Let  $DA_W[P] = \mu_W$ . Then, for all  $f \in F$ ,  $\mu'(f)R'_f\mu_W(f)$  for all  $\mu' \in C(F, W', P'; q)$  (Gale and Sotomayor, 1985; Crawford, 1991).

(P5) Given  $(F, W, q, P)$ , split each firm  $f$  into  $q_f$  identical copies of itself (all having the same preference ordering  $P_f$ ) and let  $F'$  be this new set of  $\sum_{f \in F} q_f$  splitted firms. Set  $q'_{f'} = 1$  for all  $f' \in F'$  and replace  $f$  by its copies in  $F'$  (always in the same order) in each worker's preference relation  $P_w$ . Then,  $(F', W, P; q')$  is a one-to-one matching market for which we can uniquely identify its matchings with the matchings of the original many-to-one matching market  $(F, W, q, P)$ , and vice versa (Roth and Sotomayor, 1990, Lemma 5.6). Then, and using this identification, we write  $C(F, W, q, P) = C(F', W, P; q')$ .

## A.2 Proof of Theorem 1

Below we extend our result to random stable mechanisms<sup>21</sup> and mixed strategies.

Let  $\Delta(\mathcal{M})$  denote the set of all probability distributions over  $\mathcal{M}$ . A random mechanism is a function  $\tilde{\varphi} : \mathcal{P} \rightarrow \Delta(\mathcal{M})$  choosing for each profile  $P \in \mathcal{P}$  a distribution  $\tilde{\varphi}[P]$  over  $\mathcal{M}$ . The random mechanism  $\tilde{\varphi}$  is stable if for all  $P \in \mathcal{P}$ , the support of  $\tilde{\varphi}[P]$  is contained in  $C(P)$ . Given  $v \in V$ , let  $\Delta(\mathcal{P}_v)$  denote the set of all probability distributions over  $\mathcal{P}_v$ . A mixed strategy of agent  $v$  is a function  $m_v : \mathcal{P}_v \rightarrow \Delta(\mathcal{P}_v)$  specifying for each type  $P_v$  of  $v$  a distribution  $m_v(P_v)$  over  $\mathcal{P}_v$ . A (mixed) strategy profile is a list  $m = (m_v)_{v \in V}$ .

Given a random mechanism  $\tilde{\varphi}$ ,  $P \in \mathcal{P}$ , and  $m$ ,  $m(P)$  is a NE in  $\tilde{\varphi}$  under complete information  $P$  if for all  $v \in V$  and all  $P'_v \in \mathcal{P}_v$ ,  $\tilde{\varphi}[m(P)](v) \succ_{P_v} \tilde{\varphi}[P'_v, m_{-v}(P_{-v})](v)$ .<sup>22</sup>

<sup>21</sup>Pais (2008) provides a strategic analysis of random stable mechanisms under complete information.

<sup>22</sup>Note that implicitly here we use the fact that under complete information a mixed strategy is a best response (in the set of all mixed strategies) if and only if the mixed strategy is weakly better

**Definition 5 (OBNE in mixed strategies)** Let  $\tilde{P}$  be a common prior. A mixed strategy profile  $m$  is an *ordinal Bayesian Nash equilibrium (OBNE)* in the random mechanism  $\tilde{\varphi}$  under incomplete information  $\tilde{P}$  if and only if for all  $v \in V$  and all  $P_v \in \mathcal{P}_v$  such that  $\Pr\{\tilde{P}_v = P_v\} > 0$ ,

$$\tilde{\varphi}[m_v(P_v), m_{-v}(\tilde{P}_{-v}|P_v)](v) \succ_{P_v} \tilde{\varphi}[u_v, m_{-v}(\tilde{P}_{-v}|P_v)](v) \quad \text{for all } u_v \in \Delta(\mathcal{P}_v). \quad (4)$$

As usual, if  $s$  is an OBNE in pure strategies in the (deterministic) mechanism  $\varphi$  under  $\tilde{P}$ , then  $s$  is an OBNE in mixed strategies in the mechanism  $\varphi$  (where  $\varphi$  is a random mechanism putting probability one on a unique matching for each profile).

**Theorem 2** *Let  $\tilde{P}$  be a common prior,  $m$  be a mixed strategy profile, and  $\tilde{\varphi}$  be a random stable mechanism. Then,  $m$  is an OBNE in the random stable mechanism  $\tilde{\varphi}$  under incomplete information  $\tilde{P}$  if and only if for any profile  $P$  in the support of  $\tilde{P}$ ,  $m(P)$  is a Nash equilibrium in the random stable mechanism  $\tilde{\varphi}$  under complete information  $P$ .*

**Proof.** Let  $\tilde{P}$  be a common prior,  $m$  be a mixed strategy profile and  $\tilde{\varphi}$  be a random stable mechanism. For any probability distribution  $D$  we denote by  $\text{supp}(D)$  its support (and e.g.,  $\text{supp}(\tilde{P})$  is the support of  $\tilde{P}$ ).

( $\Leftarrow$ ) Suppose that for any profile  $P$  in the support of  $\tilde{P}$ ,  $m(P)$  is a Nash equilibrium in the mechanism  $\tilde{\varphi}$  under complete information  $P$ . Let  $v \in V$  and  $P_v \in \mathcal{P}_v$  be such that  $\Pr\{\tilde{P}_v = P_v\} > 0$ . By the previous fact, then we have for all  $P'_v \in \mathcal{P}_v$  and all  $P_{-v} \in \mathcal{P}_{-v}$  such that  $\Pr\{\tilde{P}_{-v}|P_v = P_{-v}\} > 0$ ,  $\tilde{\varphi}[m(P)](v) \succ_{P_v} \tilde{\varphi}[P'_v, m_{-v}(P_{-v})](v)$  for all  $P'_v \in \mathcal{P}_v$ . Hence,

$$\tilde{\varphi}[m_v(P_v), m_{-v}(\tilde{P}_{-v}|P_v)](v) \succ_{P_v} \tilde{\varphi}[P'_v, m_{-v}(\tilde{P}_{-v}|P_v)](v),$$

and for any  $u_v \in \Delta(\mathcal{P}_v)$ ,

$$\tilde{\varphi}[m_v(P_v), m_{-v}(\tilde{P}_{-v}|P_v)](v) \succ_{P_v} \tilde{\varphi}[u_v, m_{-v}(\tilde{P}_{-v}|P_v)](v).$$

---

than any pure strategy  $P'_v \in \mathcal{P}_v$ .



Hence,  $m$  is an OBNE in mixed strategies in  $\tilde{\varphi}$  under  $\tilde{P}$ , the desired conclusion.

( $\Rightarrow$ ) Let  $m$  be an OBNE in mixed strategies in the random stable mechanism  $\tilde{\varphi}$  under  $\tilde{P}$ .

First we show that for all  $P \in \mathcal{P}$  such that  $\Pr\{\tilde{P} = P\} > 0$ ,

$$\mu(v) \subseteq A(P_v) \text{ for all } v \in V \text{ and all } \mu \in \text{supp}(\tilde{\varphi}[m(P)]). \quad (5)$$

If for some  $P$  in the support of  $\tilde{P}$ , for some  $v \in V$ , and some  $\mu \in \text{supp}(\tilde{\varphi}[m(P)])$ ,  $\mu(v) \not\subseteq A(P_v)$ , then let agent  $v$  choose  $P_v$  instead of  $m_v(P_v)$ . By stability of  $\tilde{\varphi}$ , we have  $\mu'(v) \subseteq A(P_v)$  for all  $P'_{-v} \in \mathcal{P}_{-v}$  and all  $\mu' \in \text{supp}(\tilde{\varphi}[P_v, m_{-v}(P'_{-v})])$ . Let  $v \in F$  (the case  $v \in W$  is analogous and easier). We choose a responsive extension  $P_v^*$  of  $P_v$  such that for all  $W' \in 2^W$  with  $|W'| \leq q_v$ ,  $W'R_v^*\emptyset$  if and only if  $W' \subseteq A(P_v)$ .<sup>23</sup> Hence, by  $\mu(v) \not\subseteq A(P_v)$ ,  $\mu \in \text{supp}(\tilde{\varphi}[m(P)])$ , and  $\Pr\{\tilde{P}_{-v}|_{P_v} = P_{-v}\} > 0$ , it follows that

$$\Pr\{\tilde{\varphi}[P_v, m_{-v}(\tilde{P}_{-v}|_{P_v})](v) \in B(\emptyset, P_v^*)\} = 1 > \Pr\{\tilde{\varphi}[m_v(P_v), m_{-v}(\tilde{P}_{-v}|_{P_v})](v) \in B(\emptyset, P_v^*)\},$$

which means that  $m$  is not an OBNE in the random stable mechanism  $\tilde{\varphi}$  under  $\tilde{P}$ , a contradiction. Hence, (5) holds.

Second, suppose that for some  $P \in \text{supp}(\tilde{P})$ ,  $m(P)$  is not a NE in  $\tilde{\varphi}$  under  $P$ . Then, without loss of generality, there exists  $f \in F$  and  $P'_f \in \mathcal{P}_f$  such that

$$\tilde{\varphi}[m(P)](f) \not\geq_{P'_f} \tilde{\varphi}[P'_f, m_{-f}(P_{-f})](f).$$

Then there exists a responsive extension  $P_f^*$  of  $P_f$  and  $P'_{-f} \in \text{supp}(m_{-f}(P_{-f}))$  such that for some  $\mu' \in \text{supp}(\tilde{\varphi}[P'_f, P'_{-f}])$  and  $\mu \in \text{supp}(\tilde{\varphi}[m_f(P_f), P'_{-f}])$  we have

$$\mu'(f)P_f^*\mu(f). \quad (6)$$

---

<sup>23</sup>The responsive extension  $P_v^*$  of  $P_v$  can be chosen to be additive by selecting  $g : W \rightarrow \mathbb{R} \setminus \{0\}$  such that (i) for all  $w, w' \in W$ ,  $wP_v w' \Leftrightarrow g(w) > g(w')$ , (ii) for all  $w \in A(P_v)$ ,  $g(w) \in (0, 1)$ , and (iii) for all  $w \in W \setminus A(P_v)$ ,  $g(w) < -|W|$ . It is easy to see that for all  $W' \in 2^W$  with  $|W'| \leq q_v$ ,  $\sum_{w' \in W'} g(w') \geq 0 \Leftrightarrow W' \subseteq A(P_v)$ .

The case where a worker has a profitable deviation is analogous to the case where a firm with quota one has a profitable deviation.

Let  $\mu'(f) = \{w'_1, w'_2, \dots, w'_{|\mu'(f)|}\}$  where  $w'_1 P_f w'_2 P_f \dots P_f w'_{|\mu'(f)|}$  and  $\mu(f) = \{w_1, w_2, \dots, w_{|\mu(f)|}\}$  where  $w_1 P_f w_2 P_f \dots P_f w_{|\mu(f)|}$ . We now construct from  $P'_f$  another deviation  $P''_f$  and from  $\mu'(f)$  both a responsive extension  $P_f^{**}$  of  $P_f$  and a subset of workers  $W^*$ , and prove that the random matching  $\tilde{\varphi}[m_f(P_f), m_{-f}(\tilde{P}_{-f}|_{P_f})]$  does not first-order stochastically  $P_f$ -dominate the random matching  $\tilde{\varphi}[P''_f, m_{-f}(\tilde{P}_{-f}|_{P_f})]$  since  $\Pr\{\tilde{\varphi}[P''_f, m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^{**})\} > \Pr\{\tilde{\varphi}[m_f(P_f), m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^{**})\}$ . We proceed by distinguishing between two mutually exclusive cases.

**Case 1:** There exists  $k \in \{1, \dots, |\mu'(f)|\}$  such that  $w'_k P_f w_k$  and  $w_l R_f w'_l$  for all  $l \in \{1, \dots, k-1\}$ .

Note that  $w'_k \in A(P_f)$  because  $w'_k P_f w_k$  and by (5),  $w_k \in \mu(f) \subseteq A(P_f)$ . Let  $P''_f \in \mathcal{P}_f$  be such that  $A(P''_f) = B(w'_k, P_f)$  and  $P''_f|A(P''_f) = P'_f|A(P''_f)$ .

First we show that for all  $\mu'' \in \text{supp}(\tilde{\varphi}[P''_f, P'_{-f}])$ ,  $\mu''(f)$  contains at least  $k$  workers. Note that any profile implicitly specifies the set of agents of the matching problem. For the time being, below we specify both the profile and the quota of the matching problem.

Because  $\tilde{\varphi}$  is stable and  $\mu' \in \text{supp}(\tilde{\varphi}[P'_f, P'_{-f}])$ ,  $\mu' \in C(P'_f, P'_{-f}; q)$ . Let  $\mu''$  be the matching for the problem  $(F, W \setminus \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}, (k, q_{-f}), (P'_f, P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}))$  such that  $\mu''(f) = \{w'_1, \dots, w'_k\}$  and  $\mu''(f') = \mu'(f')$  for all  $f' \in F \setminus \{f\}$ . Then from  $\mu' \in C(P'_f, P'_{-f}; q)$  it follows that

$$\mu'' \in C(P'_f, P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; k, q_{-f}). \quad (7)$$

By our choice of  $P''_f$ , we have  $\mu''(f) \subseteq A(P''_f)$  and  $P''_f|A(P''_f) = P'_f|A(P''_f)$ . Hence, we also have by (7),

$$\mu'' \in C(P''_f, P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; k, q_{-f}). \quad (8)$$

Thus, by  $\mu''(f) = \{w'_1, \dots, w'_k\}$  and the fact that any firm is matched to the same number of workers under all stable matchings, firm  $f$  is matched to  $k$  workers for all

matchings belonging to  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; k, q_{-f})$ . Now if firm  $f$  is matched to fewer than  $k$  workers in some matching belonging to  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; q)$ , then this matching is also stable for the problem  $(P_f'', P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; k, q_{-f})$ , a contradiction to the previous fact. Hence,  $f$  is matched to at least  $k$  workers in any stable matching belonging to  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; q)$ . Now when considering the workers-optimal matching in this core, we may split firm  $f$  into  $q_f$  copies (all having the same preference  $P_f''$ ) and each copy of firm  $f$  weakly prefers according to  $P_f''$  any matching in  $C(P_f'', P'_{-f}; q)$  to this matching. Since at least  $k$  copies of  $f$  are matched to a worker under the workers-optimal matching in  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+1}, \dots, w'_{|\mu'(f)|}\}}; q)$ , at least  $k$  copies of  $f$  must be also matched to a worker under any stable matching in  $C(P_f'', P'_{-f}; q)$ . Therefore, by  $\text{supp}(\tilde{\varphi}[P_f'', P'_{-f}]) \subseteq C(P_f'', P'_{-f}; q)$ , for all  $\mu'' \in \text{supp}(\tilde{\varphi}[P_f'', P'_{-f}])$ ,  $\mu''(f)$  contains at least  $k$  workers.

Second we choose a responsive extension  $P_f^{**}$  of  $P_f$ . Let  $W^* \subseteq B(w'_k, P_f)$  be such that  $W^*$  consists of the  $k$  lowest ranked workers (according to  $P_f$ ) in the set  $B(w'_k, P_f)$ , *i.e.*  $|W^*| = k$  and for all  $w \in B(w'_k, P_f) \setminus W^*$  and all  $w^* \in W^*$ ,  $w P_f w^*$ . Let  $P_f^{**}$  be the responsive extension of  $P_f$  be such that for all  $W'' \in 2^W$ ,  $W'' R_f^{**} W^*$  if and only if the following three conditions hold: (i)  $W'' \subseteq A(P_f)$ , (ii)  $k \leq |W''| \leq q_f$ , and (iii) if  $W'' = \{w''_1, w''_2, \dots, w''_{|W''|}\}$  where  $w''_1 P_f \dots P_f w''_{|W''|}$  and  $W^* = \{w^*_1, \dots, w^*_k\}$  where  $w^*_1 P_f \dots P_f w^*_k$ , then  $w''_l R_f w^*_l$  for all  $l \in \{1, \dots, k\}$ .<sup>24</sup> Since for all  $\mu'' \in \text{supp}(\tilde{\varphi}[P_f'', P'_{-f}])$ ,  $\mu''(f)$  contains at least  $k$  workers and  $A(P_f'') = B(w'_k, P_f)$ , our construction implies that for all  $\mu'' \in \text{supp}(\tilde{\varphi}[P_f'', P'_{-f}])$ ,  $\mu''(f) P_f^{**} \mu(f)$ . More precisely, for Case 1 the set  $\mu(f)$  violates (iii) and our choice of  $P_f^{**}$  and  $W^*$  yields for all  $\mu'' \in \text{supp}(\tilde{\varphi}[P_f'', P'_{-f}])$ ,

$$\mu''(f) R_f^{**} W^* P_f^{**} \mu(f) \tag{9}$$

---

<sup>24</sup>The responsive extension  $P_f^{**}$  of  $P_f$  can be chosen to be additive by selecting  $g : W \rightarrow \mathbb{R} \setminus \{0\}$  such that (a) for all  $w, w' \in W$ ,  $w P_f w' \Leftrightarrow g(w) > g(w')$ , (b) for all  $w \in B(w'_k, P_f)$ ,  $g(w) \in [|\mathbb{W}|, |\mathbb{W}| + 1]$ , (c) for all  $w \in A(P_f) \setminus B(w'_k, P_f)$ ,  $g(w) \in (0, 1)$ , and (d) for all  $w \in W \setminus A(P_f)$ ,  $g(w) < -|\mathbb{W}|^2$ . It is easy to see that for all  $W'' \in 2^W$  with  $|W''| \leq q_f$ ,  $\sum_{w' \in W''} g(w') \geq \sum_{w \in W^*} g(w) \Leftrightarrow$  (i)-(iii) hold for  $W''$ .

and  $\mu \in \text{supp}(\tilde{\varphi}[m_f(P_f), P'_{-f}])$ .

The following claim will be the key to the proof. We show that for any profile, if some stable matching is weakly preferred to  $W^*$  under  $P_f^{**}$ , then *all* matchings, which are stable under the profile where  $f$ 's preference ordering is replaced by  $P_f''$ , are weakly preferred to  $W^*$  under  $P_f^{**}$ .

*Claim:* Let  $\hat{P} \in \mathcal{P}$ . If for some  $\hat{\mu} \in C(\hat{P})$ ,  $\hat{\mu}(f)R_f^{**}W^*$ , then for all  $\bar{\mu} \in C(P_f'', \hat{P}_{-f})$ ,  $\bar{\mu}(f)R_f^{**}W^*$ .

*Proof of Claim.* By  $\hat{\mu}(f)R_f^{**}W^*$  and our choice of  $P_f^{**}$ ,

$$\hat{\mu}(f) \cap B(w'_k, P_f) \text{ must contain at least } k \text{ workers.} \quad (10)$$

If for some  $\bar{\mu}' \in C(P_f'', \hat{P}_{-f})$ ,  $\bar{\mu}'(f)$  contains at least  $k$  workers, then all these workers belong to  $B(w'_k, P_f)$  because  $A(P_f'') = B(w'_k, P_f)$ . Thus, by our choice of  $P_f^{**}$  and  $W^*$ ,  $\bar{\mu}'(f)R_f^{**}W^*$ . Because by (P3),  $f$  is matched to at least  $k$  workers in any stable matching belonging to  $C(P_f'', \hat{P}_{-f})$  and  $A(P_f'') = B(w'_k, P_f)$ , it follows from our construction that  $\bar{\mu}(f)R_f^{**}W^*$  for all  $\bar{\mu} \in C(P_f'', \hat{P}_{-f})$ , the desired conclusion.

Suppose that for all  $\bar{\mu} \in C(P_f'', \hat{P}_{-f})$ ,  $\bar{\mu}(f)$  contains fewer than  $k$  workers. Let  $\hat{\mu}(f) = \{\hat{w}_1, \dots, \hat{w}_{|\hat{\mu}(f)|}\}$  where  $\hat{w}_1 P_f \cdots P_f \hat{w}_{|\hat{\mu}(f)|}$ . By (10),  $\hat{\mu}(f) \cap B(w'_k, P_f)$  contains at least  $k$  workers. Thus,  $k \leq |\hat{\mu}(f)|$ . For the time being, below we specify both the profile and the quota of the matching problem. Then we have  $\hat{\mu} \in C(\hat{P}; q)$ . Let  $\hat{\mu}'$  be the matching for the problem  $(F, W \setminus \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}(f)|}\}, (k, q_{-f}), (\hat{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}(f)|}\}}))$  such that  $\hat{\mu}'(f) = \{\hat{w}_1, \dots, \hat{w}_k\}$  and  $\hat{\mu}'(f') = \hat{\mu}(f')$  for all  $f' \in F \setminus \{f\}$ . Then, from  $\hat{\mu} \in C(\hat{P}_f, \hat{P}_{-f}; q)$  it follows that

$$\hat{\mu}' \in C(\hat{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}(f)|}\}}; k, q_{-f}). \quad (11)$$

Let  $\hat{w} \in \hat{\mu}'(f)$  be such that  $\hat{\mu}'(f) \subseteq B(\hat{w}, \hat{P}_f)$  (in other words,  $\hat{w}$  is the worker who is least preferred in  $\hat{\mu}'(f)$  according to  $\hat{P}_f$ ). Let  $\dot{P}_f \in \mathcal{P}_f$  be such that  $A(\dot{P}_f) = B(\hat{w}_k, P_f) \cap B(\hat{w}, \hat{P}_f)$  and  $\dot{P}_f|A(\dot{P}_f) = P_f''|A(\dot{P}_f)$ . Note that  $\hat{\mu}'(f) \subseteq A(\dot{P}_f)$ . Then we must have  $\hat{\mu}' \in C(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}(f)|}\}}; k, q_{-f})$ : first, note that  $\hat{\mu}'$  is individually

rational because both  $\hat{\mu}'(f) \subseteq B(\hat{w}_k, P_f)$  and  $\hat{\mu}'(f) \subseteq B(\hat{w}, \hat{P}_f)$  (by our choice of  $\hat{w}$ ); second, if there would exist a blocking pair for  $\hat{\mu}'$ , then by (11) and the fact that only firm  $f$ 's preference changed from  $\hat{P}_f$  to  $\dot{P}_f$ , firm  $f$  needs to be part of this blocking pair; third, if  $(w, f)$  blocks  $\hat{\mu}'$  under  $(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}'(f)|}\}}; k, q_{-f})$ , then  $w \notin \hat{\mu}'(f)$ ; now this implies  $w \neq \hat{w} \in \hat{\mu}'(f)$ , and  $w \in A(\dot{P}_f) = B(\hat{w}_k, P_f) \cap B(\hat{w}, \hat{P}_f)$ ; but then we must have  $w \hat{P}_f \hat{w}$  (because  $\hat{w}$  is the least preferred worker in  $\hat{\mu}'(f)$  and  $A(\dot{P}_f) \subseteq B(\hat{w}, \hat{P}_f)$ ) and  $(w, f)$  must also block  $\hat{\mu}'$  under  $(\hat{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}'(f)|}\}}; k, q_{-f})$ , a contradiction to (11).

Thus, since  $|\hat{\mu}'(f)| = k$ , firm  $f$  is matched to  $k$  workers for all matchings belonging to  $C(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\hat{\mu}'(f)|}\}}; k, q_{-f})$ . Now if firm  $f$  is matched to fewer than  $k$  workers for some  $\tilde{\mu} \in C(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\tilde{\mu}(f)|}\}}; q)$ , then  $\tilde{\mu}$  is also stable under  $(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\tilde{\mu}(f)|}\}}; k, q_{-f})$ , a contradiction to the previous fact. Hence,  $f$  is matched to at least  $k$  workers in any stable matching belonging to  $C(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\tilde{\mu}(f)|}\}}; q)$ . Now when considering the workers-optimal matching in this core, we may split firm  $f$  into  $q_f$  copies (all having the same preference  $\dot{P}_f$ ) and each copy of firm  $f$  weakly prefers according to  $\dot{P}_f$  any matching in  $C(\dot{P}_f, \hat{P}_{-f}; q)$  to this matching. Since at least  $k$  copies of  $f$  are matched to a worker under the workers-optimal matching in  $C(\dot{P}_f, \hat{P}_{-\{f\} \cup \{\hat{w}_{k+1}, \dots, \hat{w}_{|\tilde{\mu}(f)|}\}}; q)$ ,

$$\text{at least } k \text{ copies of } f \text{ are matched to a worker in any matching in } C(\dot{P}_f, \hat{P}_{-f}; q). \quad (12)$$

On the other hand, for all  $\bar{\mu} \in C(P_f'', \hat{P}_{-f})$ ,  $\bar{\mu}(f)$  contains fewer than  $k$  workers. Let  $\tilde{\mu} \in C(P_f'', \hat{P}_{-f})$ . Let  $\tilde{\mu}'$  be the matching for the problem  $(F, W \setminus (\tilde{\mu}(f) \setminus A(\dot{P}_f)), q, (P_f'', \hat{P}_{-\{f\} \cup (\tilde{\mu}(f) \setminus A(\dot{P}_f))}))$  such that  $\tilde{\mu}'(f) = \tilde{\mu}(f) \cap A(\dot{P}_f)$  and  $\tilde{\mu}'(f') = \tilde{\mu}(f')$  for all  $f' \in F \setminus \{f\}$ . Since  $\tilde{\mu} \in C(P_f'', \hat{P}_{-f}; q)$  and  $\tilde{\mu}(f)$  contains fewer than  $q_f$  workers, we must have  $\tilde{\mu}' \in C(P_f'', \hat{P}_{-\{f\} \cup (\tilde{\mu}(f) \setminus A(\dot{P}_f))}; q)$ . Thus, by  $\tilde{\mu}'(f) \subseteq A(\dot{P}_f)$  and  $\dot{P}_f|A(\hat{P}_f) = P_f''|A(\dot{P}_f)$ , we also obtain  $\tilde{\mu}' \in C(\dot{P}_f, \hat{P}_{-\{f\} \cup (\tilde{\mu}(f) \setminus A(\dot{P}_f))}; q)$ . Hence, in any matching belonging to this core firm  $f$  is matched to  $|\tilde{\mu}'(f)| = |\tilde{\mu}(f) \cap A(\dot{P}_f)|$  workers. Now when considering the workers-optimal matching in this core, we may

split each firm  $f' \in F \setminus \{f\}$  into  $q_{f'}$  copies (all having the same preference  $\hat{P}_{f'}$ ) and each copy of firm  $f'$  weakly prefers according to  $\hat{P}_{f'}$  any matching in  $C(\dot{P}_f, \hat{P}_{-f}; q)$  to this matching. Thus, in total all the copies of all firms  $f' \in F \setminus \{f\}$  receive at least the same number of workers in  $C(\dot{P}_f, \hat{P}_{-f}; q)$  as they did previously. Since exactly  $|\tilde{\mu}(f) \setminus A(\dot{P}_f)|$  new workers are available and  $f$  was matched to  $|\tilde{\mu}'(f)| = |\tilde{\mu}(f) \cap A(\dot{P}_f)|$  workers before, firm  $f$  can be matched to at most  $|\tilde{\mu}(f)|$  workers under any stable matching in  $C(\dot{P}_f, \hat{P}_{-f}; q)$ . Since  $|\tilde{\mu}(f)| < k$ , this contradicts (12) and the fact that under responsive preferences, firm  $f$  is matched to the same number of workers for any two matchings in  $C(\dot{P}_f, \hat{P}_{-f}; q)$ . Hence, for  $\bar{\mu} \in C(P_f'', \hat{P}_{-f})$ ,  $\bar{\mu}(f)$  cannot contain fewer than  $k$  workers.  $\diamond$

Because  $\tilde{\varphi}$  is stable, the Claim implies that for all  $\hat{P} \in \mathcal{P}$ ,

$$\Pr\{\tilde{\varphi}[\hat{P}_f, \hat{P}_{-f}](f) \in B(W^*, P_f^*)\} \leq \Pr\{\tilde{\varphi}[P_f'', \hat{P}_{-f}](f) \in B(W^*, P_f^*)\}.$$

Thus,

$$\Pr\{\tilde{\varphi}[\hat{P}_f, m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^*)\} \leq \Pr\{\tilde{\varphi}[P_f'', m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^*)\}.$$

By (9), there exists  $\dot{P}_f \in \text{supp}(m_f(P_f))$  such that  $\mu \in \text{supp}(\tilde{\varphi}[\dot{P}_f, P'_{-f}])$ . Thus, by  $\mu'(f)R_f^*W^*P_f^*\mu(f)$  and  $\mu' \in C(P_f'', P_{-f}')$ , (9) implies

$$\Pr\{\tilde{\varphi}[\dot{P}_f, P'_{-f}](f) \in B(W^*, P_f^*)\} < \Pr\{\tilde{\varphi}[P_f'', P'_{-f}](f) \in B(W^*, P_f^*)\}.$$

Hence, by  $m_f(P_f) \in \Delta(\mathcal{P}_f)$  and both  $\dot{P}_f \in \text{supp}(m_f(P_f))$  and  $P'_{-f} \in \text{supp}(m_{-f}(\tilde{P}_{-f}|_{P_f}))$ ,

$$\Pr\{\tilde{\varphi}[m_f(P_f), m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^*)\} < \Pr\{\tilde{\varphi}[P_f'', m_{-f}(\tilde{P}_{-f}|_{P_f})](f) \in B(W^*, P_f^*)\},$$

which means that  $m$  is not an OBNE in  $\tilde{\varphi}$  under  $\tilde{P}$ .

**Case 2:** Otherwise.

Then we have  $w_l R_f w'_l$  for all  $l \in \{1, \dots, \min\{|\mu(f)|, |\mu'(f)|\}\}$ . Let  $k = |\mu(f)|$ . If  $|\mu'(f)| \leq |\mu(f)|$ , then by responsiveness of  $P_f^*$  and  $\mu(f) \subseteq A(P_f)$ , we have  $\mu(f)R_f^*\mu'(f)$ , which contradicts (6). Hence, we must have  $|\mu'(f)| > |\mu(f)| = k$ ,  $q_f > k$ , and  $w'_{k+1} \in$

$A(P_f)$ . Let  $P_f'' \in \mathcal{P}_f$  be such that  $A(P_f'') = B(w'_{k+1}, P_f)$  and  $P_f''|A(P_f'') = P_f'|A(P_f'')$ . Since  $\mu(f) \subseteq B(w'_{k+1}, P_f) = A(P_f'')$  and  $\mu(f)$  does not fill the quota of firm  $f$ , we must have  $\mu \in C(P_f'', P'_{-f}; q)$ . Hence,

$$\text{firm } f \text{ is matched to } k \text{ workers under any matching in } C(P_f'', P'_{-f}; q). \quad (13)$$

On the other hand, let  $\mu''$  be the matching for the problem  $(F, W \setminus \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}, (k+1, q_{-f}), (P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}))$  such that  $\mu''(f) = \{w'_1, \dots, w'_{k+1}\}$  and  $\mu''(f') = \mu'(f')$  for all  $f' \in F \setminus \{f\}$ . Then from  $\mu' \in C(P'_f, P'_{-f}; q)$  it follows that  $\mu'' \in C(P'_f, P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; k+1, q_{-f})$ . Thus, by  $\mu''(f) \subseteq B(w'_{k+1}, P_f) = A(P_f'')$  and  $P_f''|A(P_f'') = P_f'|A(P_f'')$ ,  $\mu'' \in C(P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; k+1, q_{-f})$ . Now if firm  $f$  is matched to fewer than  $k+1$  workers in some matching belonging to  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; q)$ , then this matching is also stable for the problem  $(P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; k+1, q_{-f})$ , a contradiction to the previous fact. Hence,  $f$  is matched to at least  $k+1$  workers in any stable matching belonging to  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; q)$ . Now when considering the workers-optimal matching in this core, we may split firm  $f$  into  $k+1$  copies (all having the same preference  $P_f''$ ) and each copy of firm  $f$  weakly prefers according to  $P_f''$  any matching in  $C(P_f'', P'_{-f}; q)$  to this matching. Since at least  $k+1$  copies of  $f$  are matched to a worker under the workers-optimal matching in  $C(P_f'', P'_{-\{f\} \cup \{w'_{k+2}, \dots, w'_{|\mu'(f)|}\}}; q)$ , at least  $k+1$  copies of  $f$  must be also matched to a worker under any matching in  $C(P_f'', P'_{-f}; q)$ , which contradicts (13) and the fact that firm  $f$  is matched to the same number of workers under any matching in  $C(P_f'', P'_{-f}; q)$ . Hence, Case 2 cannot occur.  $\square$

Theorem 1 is a corollary of Theorem 2 by restricting Theorem 2 and its proof to pure strategies and deterministic mechanisms.

### A.3 Proof of Proposition 1

**Proposition 1** *Let  $\tilde{P}$  be a common prior.*

- (a) *If  $\tilde{P}$  is  $F$ -correlated or  $W$ -correlated, then truth-telling is an OBNE in any stable mechanism under incomplete information  $\tilde{P}$ .*

(b) Let  $s$  be a strategy profile such that  $s_w(P_w) = P_w$  for all  $w \in W$  and all  $P_w \in \mathcal{P}_w$ . If  $\tilde{P}$  is  $W$ -correlated and  $s$  is an OBNE in the stable mechanism  $DA_W$  under incomplete information  $\tilde{P}$ , then for all profiles  $P$  in the support of  $\tilde{P}$ ,  $DA_W[s(P)]$  is stable with respect to  $P$ . The analogous statement is true for the stable mechanism  $DA_F$ .

**Proof.** (a) Let  $\varphi$  be a stable mechanism and  $\tilde{P}$  be a common prior. Without loss of generality, let  $\tilde{P}$  be  $F$ -correlated. The case where  $\tilde{P}$  is  $W$ -correlated is analogous to the case where  $\tilde{P}$  is  $F$ -correlated and all firms have quota 1. Let  $P$  be in the support of  $\tilde{P}$ . Because all firms' preferences are identical at  $P$ , we have  $|C(P)| = 1$ , say  $C(P) = \{\mu\}$ . By stability of  $\varphi$ ,  $\varphi[P] = \mu$ . By Theorem 1, it suffices to show that  $P$  is a NE in the mechanism  $\varphi$  under complete information  $P$ .

Because all firms have identical preferences, say  $P_f : w_1 w_2 \cdots w_k \emptyset w_{k+1} \cdots$  for all  $f \in F$ ,  $\mu(w_1)$  is  $w_1$ 's most preferred firm (if any) under  $P_{w_1}$ . Then  $\mu(w_2)$  is  $w_2$ 's most preferred firm (if any) from  $F \setminus \{\mu(w_1)\}$  under  $P_{w_2}$ , and in general for  $i = 1, \dots, |W|$ ,  $\mu(w_i)$  is  $w_i$ 's most preferred firm (if any) from  $F \setminus \{\mu(w_1), \dots, \mu(w_{i-1})\}$  under  $P_{w_i}$ . By stability of  $\varphi$ , obviously no worker can profitably manipulate.

Let  $f \in F$ ,  $P'_f \in \mathcal{P}_f$ , and  $\mu' = \varphi[P'_f, P_{-f}]$ . Suppose that for some  $P_f^* \in \text{resp}(P_f)$  we have  $\mu'(f) P_f^* \mu(f)$ . Hence,  $\mu'(f) \neq \mu(f)$ . By stability of  $\varphi$ , without loss of generality we may suppose  $A(P'_f) = \mu'(f)$  and because any firm's set of acceptable workers is  $\{w_1, \dots, w_k\}$ ,  $A(P'_f) \subseteq A(P_f)$ . Again, by stability of  $\varphi$ ,  $\mu(f) \subseteq A(P_f)$ . First, suppose that  $|\mu'(f)| > |\mu(f)|$ . Note that  $\mu \in C(P)$  and  $\mu' \in C(P'_f, P_{-f})$ . Then, without loss of generality, we may suppose  $|\mu'(f)| = q_f$ .<sup>25</sup> Let  $P''_f \in \mathcal{P}_f$  be such that  $A(P''_f) = A(P_f)$  and  $P''_f | A(P''_f) = P'_f | A(P'_f)$ . By  $|\mu(f)| < q_f$ ,  $A(P''_f) = A(P_f)$  and (P3), we obtain  $\mu \in C(P''_f, P_{-f})$ . By  $|\mu(f)| \neq |\mu'(f)|$  and (P3), we must have  $\mu' \notin C(P''_f, P_{-f})$ . Thus,  $\mu'$  is blocked by some pair  $(w', f')$  under  $(P''_f, P_{-f})$ . By  $\mu' \in C(P'_f, P_{-f})$  and

---

<sup>25</sup>If  $|\mu'(f)| < q_f$ , then set  $q'_f = |\mu'(f)|$ . From (P3) (where we specify both the profile and the quotas),  $\mu \in C(P; q)$  and  $|\mu(f)| < q'_f$  imply  $\mu \in C(P; q'_f, q_{-f})$ , and similarly  $\mu' \in C(P'_f, P_{-f}; q)$  and  $\mu'(f) = A(P'_f)$  imply  $\mu' \in C(P'_f, P_{-f}; q'_f, q_{-f})$ .



$A(P'_f) \subseteq A(P''_f)$ , we must have  $f' = f$  and  $w' \notin \mu'(f)$ . But then by  $A(P'_f) = \mu'(f)$  and  $P''_f|A(P''_f) = P'_f|A(P''_f)$ , we must have  $wP_f w'$  for all  $w \in \mu'(f)$ . Now  $f$  must have an unfilled slot under  $\mu'$  and  $|\mu'(f)| < q_f$ , a contradiction.

Hence,  $|\mu'(f)| \leq |\mu(f)|$ . Let  $\underline{w}$  be the least  $P_f$ -preferred worker in  $\mu(f)$ , i.e.  $wR_f \underline{w}$  for all  $w \in \mu(f)$ . If for all  $w \in \mu'(f) \setminus \mu(f)$  we have  $\underline{w}P_f w$ , then by  $|\mu'(f)| \leq |\mu(f)|$  and responsiveness of  $P_f^*$ , we have  $\mu(f)R_f^* \mu'(f)$ , a contradiction. Let  $w_l \in \mu'(f) \setminus \mu(f)$  be such that  $w_l P_f \underline{w}$ . We show that there exists an index  $i(l) < l$  such that  $w_{i(l)} \in \mu(f) \setminus \mu'(f)$ . Let  $\mu(w_l) = f_l$ . Note that by  $f_l \neq f$ ,  $w_l P_f \underline{w}$  and  $\mu \in C(P)$  we have  $f_l P_{w_l} f$ . Thus, by  $\mu' \in C(P'_f, P_{-f})$ , we must have  $|\mu'(f_l)| = q_{f_l}$  and  $w P_{f_l} w_l$  for all  $w \in \mu'(f_l)$ . But then  $w_l$ 's position at firm  $f_l$  is filled with some new worker  $w'$ , i.e.  $w' \in \mu'(f_l) \setminus \mu(f_l)$  and  $w' P_{f_l} w_l$ . Now by  $\mu \in C(P)$  and  $\mu(w_l) = f_l$ ,  $\mu(w') P_{w'} f_l$ . If  $\mu(w') = f$ , then  $w'$  has an index  $i(l) < l$  such that  $w' = w_{i(l)}$  and  $w_{i(l)} \in \mu(f) \setminus \mu'(f)$ . Otherwise, let  $\mu(w') = f' \neq f$ . Then again as above from  $f' P_{w'} f_l$  we have  $|\mu'(f')| = q_{f'}$  and  $w P_{f'} w'$  for all  $w \in \mu'(f')$ , and  $w'$ 's position at firm  $f'$  is filled with some new worker  $w''$ , i.e.  $w'' \in \mu'(f') \setminus \mu(f')$  and  $w'' P_{f'} w'$ . By  $P'_{-f} = P_{-f}$  and the finiteness of  $W$  and  $F$ , in the end for  $w_l \in \mu'(f) \setminus \mu(f)$  there must exist  $w_{i(l)} \in \mu(f) \setminus \mu'(f)$  with  $i(l) < l$ . Furthermore, from the above arguments, we can choose  $i(l) \neq i(l')$  for all  $l \neq l'$  such that  $w_l, w_{l'} \in \mu'(f) \setminus \mu(f)$ . Since  $\mu(f) \subseteq A(P_f)$  and  $|\mu'(f)| \leq |\mu(f)|$ , responsiveness of  $P_f^*$  implies  $\mu(f)R_f^* \mu'(f)$ , a contradiction.

(b) Let  $P$  be in the support of  $\tilde{P}$ . Since  $s_w(P_w) = P_w$  for all  $w$  and  $\tilde{P}$  is  $W$ -correlated, we have by (a) that no worker can gain by manipulation. Furthermore, by Theorem 1,  $s(P)$  must be a NE in  $DA_W$  under  $P$ . Because  $\tilde{P}$  is  $W$ -correlated, all workers have identical preferences, say  $P_w : f_1 f_2 \cdots f_l \emptyset f_{l+1} \cdots$  for all  $w \in W$ . Suppose that  $DA_W[s(P)]$  is not stable with respect to  $P$ . Since  $s(P)$  is a NE in  $DA_W$  under  $P$ , no agent is matched to any partner under  $DA_W[s(P)]$  which is unacceptable according to its true preference relation. Suppose that some unmatched worker-firm pair  $(w, f)$  blocks  $DA_W[s(P)]$ . Then  $f \in A(P_w)$  and by  $s_w(P_w) = P_w$ ,  $f P_w DA_W[s(P)](w)$ . But then, along the  $DA_W$ -algorithm which produces  $DA_W[s(P)]$ , worker  $w$  proposed

to  $f$  before proposing to  $DA_W[s(P)](w)$  and because all workers' submitted lists are identical, at that step all unmatched workers proposed to  $f$  (and the set of unmatched workers shrinks from one step to the next one). Let  $w'$  be the least preferred worker according to  $P_f$  in  $DA_W[s(P)](f)$ . But now  $f$  profitably deviates from  $s(P)$  in  $DA_W$  by submitting a list  $P'_f$  where  $A(P'_f) = (DA_W[s(P)](f) \cup \{w\}) \setminus \{w'\}$ . When in  $DA_W[P'_f, s_{-f}(P_{-f})]$  worker  $w$  proposes to  $f$ , all unmatched workers propose to  $f$  in that step because all workers' submitted lists are identical. Firm  $f$  accepts  $(DA_W[s(P)](f) \cup \{w\}) \setminus \{w'\}$  which is strictly preferred to  $DA_W[s(P)](f)$  under any responsive extension  $P_f^*$  of  $P_f$ . Hence,  $s(P)$  is not a NE in  $DA_W$  under  $P$ , a contradiction. □