# Mechanism Design With Budget Constraints and a Continuum of Agents*

Michael Richter
*Yeshiva University*

July 8, 2013

**Abstract**

This paper finds welfare- and revenue-maximizing mechanisms for assigning a divisible good to a population of budget-constrained agents where agents' have independently distributed private valuations and budgets. Both of these optimal mechanisms feature a linear price for the good. The welfare-maximizing mechanism additionally has a uniform lump sum transfer to all agents and a higher linear price than the revenue-maximizing mechanism. This transfer increases welfare because it relaxes the key difficulty in the aforementioned setting: agents with high valuations cannot purchase an efficient amount of the good because of their budget constraints. The welfare-maximizing result can therefore be interpreted as a version of the second welfare theorem. I show that both optimal mechanisms can be implemented using dominant strategies. In addition, I consider extensions where I relax the independence condition and introduce linear production.

*I would like to thank John Asker, Yeon-Koo Che, Maciej Kotowski, Richard P McLean, Martí Mestieri, Leonardo Pesjachowicz, Nikita Roketskiy, Vasiliki Skreta, Ennio Stachetti, Sergio Vicente, Kemal Yildiz, seminar participants in NYU and especially, my advisers, Debraj Ray and Ariel Rubinstein for helpful comments and suggestions. Any errors are the author's alone.

# 1 Introduction

Suppose that a principal has a finite supply of a divisible good and wishes to distribute it to a large population of agents in a utilitarian welfare-maximizing fashion. Agents have private individual budget constraints and valuations for the good. If agents were not budget constrained, the principal could calculate the market unit price for the good and sell it at that price. In my model, this would be approximately the highest valuation. This maximizes welfare because high value agents self-select by purchasing the good and agents with low valuations do not. However, when agents are budget constrained, this mechanism no longer works because agents who value the good highly are unable to purchase an efficient quantity of the good at the market price.

Motivated by the above problem, I address the following question, "What is the optimal mechanism for allocating a good to a population with private valuations and budget constraints?" Throughout, optimality will refer to either utilitarian welfare or revenue maximization and I solve for the optimal mechanisms for each of these two objectives.

To consider a simple example, suppose that there are two agents with unit valuations $v_1$, $v_2$ and a common budget constraint $w$ such that $w < v_1 < 2w < v_2$. To maximize welfare, the planner would like to sell the good to agent 2, but he runs into the problem that agent 1 would like to participate as well. Since the good is divisible, one possible mechanism is for the principal to sell one half of the good to each agent for the price $w/2$. Although this is incentive compatible, the principal can in fact achieve the first-best by giving each agent $w$ and selling the good at a price of $2w$. Both agents can afford this price because of the transfer, but only agent 2 will want to purchase the good at this price. Moreover, this mechanism produces a balanced budget as each agent receives $w$ and the high valuation agent pays $2w$. This outcome could be alternatively implemented by giving each agent half a unit of the good, and then allowing agent 2 to buy agent 1's allocation for the price $w$. This achieves the first-best outcome as above.

While the above example differs from the setting that I consider, it illustrates that transfers to agents' can be used to weaken budget constraints and improve welfare. To further clarify the setting considered in this paper, agents' utility is linear in quantity of the good and money. Agents' valuations and budget constraints are independently distributed and private. The principal has a finite supply of the good and must satisfy a weak balanced budget constraint. Without the balanced budget constraint, the principal could uniformly distribute large amounts of money to achieve the first-best welfare-maximization outcome. The balanced budget condition does not constrain the revenue-maximization mechanism because the principal's objective already involves obtaining positive revenue.

In both the welfare and revenue-maximizing settings, the optimal mechanisms that I find feature a linear price $p$ for the good. In line with the above example, the utilitarian welfare maximizing mechanism additionally features a uniform lump-sum transfer from the principal to the agents, whereas the revenue-maximizing mechanism does not. With regards to implementation, as in the example, instead of cash transfers, the principal can alternatively make in-kind transfers of the good and then allow resale. In this way, one sees that the principal satisfies his weak balanced budget constraint because transfers to the agents are internally recouped in the mechanism by the sale of the good. Importantly, these cash or in-kind transfers to the agents are uniform because valuations and budgets are unobservable, so high-value or low-wealth agents cannot be targeted to receive higher transfers.

The principal makes transfers to agents in the welfare maximization setting to relax their budget constraints. However, it is unclear if this is a good strategy for the principal in the revenue maximization setting. It is clear that he does not wish to transfer all of his revenue away, as in the welfare-maximization setting, but it may be that in the revenue case, there is some intermediate level of transfers to agents that will relax their budget constraints and therefore lead to additional revenue. However, when I find the revenue-maximizing mechanism, I find that transfers are not beneficial. This is because monetary transfers to the agents increases gross revenue, but by a smaller amount than what is transferred, so that the net revenue change is negative. In independent work, Pai and Vohra (2011) also find that transfers from the principal are not profitable for a revenue-maximizing seller in an auction setting with finite buyers, each of whom has a finite set of possible valuations and budgets.

Economic situations where a planner wishes to distribute a divisible good to budget-constrained agents abound. For the case of welfare-maximization, such settings may be: the privatization of a government-owned enterprise or the provision of healthcare or education in a government regulated system. In each of the above settings, budget constraints can be significant and can stand in the way of an efficient allocation of resources. As discussed in the example above, an implementation of the welfare maximizing mechanism is the in-kind uniform disbursal of the good and then allowing resale. This implementation resembles the privatization of government-owned industries in the Czech Republic and Russia where vouchers were distributed and then voucher auctions were conducted.

For the case of revenue maximization or a joint objective, one may think of: the privatization of companies, the sale of government land, or the sale of Treasury bonds. In particular, the method of selling Treasury bonds that is now used (since 2002) will closely resemble a dominant strategy implementation of the optimal mechanism found in this paper. There have been other interesting theoretical investigations of the sale of Treasury bonds. Ausubel (2004) studies a different setting where agents have decreasing marginal utilities and no budget constraints and finds a different mechanism to be optimal. Additionally, notice that even if a planner wishes to maximize societal welfare from some larger planning problem, when selling an individual good/company, there may be a preference to maximize revenue in selling the current object so that the revenue generated from that sale can be applied to other welfare-improving endeavors.

To obtain my result, I need to make two assumptions on agents' valuations. For a fixed level of wealth, agents' payments are tied down through their allocations and incentive compatibility constraints. The first assumption that I make is that agents' valuations satisfy the decreasing hazard ratio property. This property stipulates that agents' virtual valuations are increasing in their underlying valuations. Since the principal's revenue is determined by agents' virtual valuations, as in Myerson (1981), one then has that allocating the good to higher valuation agents is not just welfare improving, but revenue improving as well.

The other assumption that I make is new and stipulates that the density function of agents' valuations is decreasing. This assumption increases the feasibility of allocating the object to higher valuation agents, in the sense that doing so decreases top types' payments. Since the agents with the highest valuations make the largest payments, if their payments decrease when high types get higher allocations, then such an allocation is feasible.

I make no assumptions on the source of an agent's budget constraint, but I now provide a few reasons for how they may arise. One is due to agents' limited wealth. However, this is not the only possible reason. Alternatively, agents may be liquidity constrained due to imperfect capital markets. In addition, it may be the case that agents cannot borrow against the good itself because it is a final consumption good or yields uncertain returns.

The addition of budget constraints has been studied widely in the literature. For example, Che and Gale (1998) showed that the standard revenue equivalence results of Myerson (1981) do not hold when agents have budget constraints. Specifically, they find that in a standard auction setting, the first price auction outperforms the second price auction. This is because agents hedge their bids in a first-price setting and therefore budget constraints are less binding.

When budget constraints are individual and unobservable, agents' types become two-dimensional. The multi-dimensional mechanism design literature is large and features two common approaches. One is to assume that each of the agent's two characteristics can take on one of two different values and the problem becomes reduces to finite parameter programming (see Armstrong and Rochet (1999)). The other is to show that there is an equivalent one-dimensional formulation and solve that problem (for example, see Armstrong (1996), Jehiel et al. (1996), and Che and Gale (2000)).

The approach that I take is much closer to the second approach above. I begin by solving the mechanism design problem when all agents have a single budget level. This is a one-dimensional problem, because agents differ only in their valuations. In a setting with a finite set of buyers and a single uniform budget constraint setting, Laffont and Robert (1996) and Maskin (2000) have solved for the revenue maximizing mechanism and the utilitarian welfare maximizing mechanisms, respectively.

After solving the one-dimensional problem, I then use the optimal mechanisms found therein to solve the mechanism design problem across budget levels. However, there is one technical point that needs to be addressed. In the above one-dimensional problem, there are no incentive compatibility constraints across different budget levels because they are derived using only one budget level. This approach leads to a difficulty, namely that the overall mechanism fails to satisfy incentive compatibility constraints across different budget levels. This is because agents may desire to lie about their wealth. However, I show that the optimal mechanism across budget levels is a linear pricing mechanism, and in such mechanisms there is no advantage to lying downwards about one's own budget. Agents cannot lie upwards about their budgets because such lies are unaffordable.

Now, I discuss how the optimal mechanisms found here differ from others found in the budget-constrained mechanism design literature. In this strand of literature, there are three papers that closely relate to mine. The first is Che and Gale (2000) who consider revenue maximization with one buyer. In contrast to the optimality of the linear pricing function that I find, they find a convex pricing function in terms of probabilities to be revenue maximizing. This difference is driven by the conditions in the current paper being aggregate instead of individual which prevents the mechanism design problem from directly scaling from one wealth level to another. Secondly, Pai and Vohra (2011) considers utilitarian welfare maximization or revenue maximization with a finite number of agents. In their setting, the optimal mechanisms found consist of a different value-cutoff for every wealth-type. All agents with valuations at the cutoff or above pool together and receive the same allocation while agents below these cutoffs receive smaller allocations. The difference in their results from mine are largely driven because of the finite discrete setting with a finite set of agents.

Finally, the most related paper is **?**. They consider an indivisible good setting with unit demand and a unit measure of agents. They examine three different mechanisms and compare them using utilitarian welfare as a criterion. One of the feasible mechanisms that they consider closely resembles my welfare-maximizing mechanism except that agents receive the good with final probabilities $0, S$, or 1 where $S$ is the aggregate supply of the good. This mechanism is feasible but not optimal in their setting nor mine and resembles the welfare maximizing mechanism found here. The difference is that in my model, agents will receive the good with intermediate quantities as well. In addition, the mechanism that I find specifies quantities instead of probabilities because I consider allocating a divisible good with certainty rather than a probabilistically allocating an indivisible good. In addition, they solve for the welfare maximizing mechanism in a two valuation, two budget setting. It features cash subsidies and a convex price for probabilities of receiving the good.

Another key difference between this paper and the other related ones is that it states the welfare-maximizing mechanism can be implemented via uniform lump-sum transfers and a linear price. In this sense, it can be interpreted as a version of the Second Welfare Theorem because the market plus transfers succeeds in implementing the constrained utilitarian efficient outcome whereas this is not the case in settings analyzed in the above related papers.

Given this insight, I show that both the welfare-maximizing and revenue-maximizing mechanisms found, can be dominant strategy implemented. The welfare-maximizing mechanism is implementable via a uniform transfer of the good to all agents, and then allowing resale. This resembles the privatization schemes implemented in Eastern Europe. The revenue-maximizing mechanism can be implemented through the posting of decreasing demand schedules, solving for the market clearing price, and implementing sales at this price. This resembles the auction mechanism that has been used to sell Treasury bonds since 2002.

An additional advantage of the simple structure of the optimal mechanisms that I find, is that analytical extensions and comparative statics can be easily performed. Specifically, I consider a relaxation of the independence assumption of valuations and budgets, and solve for the revenue-maximizing mechanism in that setting. It is typically nonlinear. In addition, I consider extensions to the cases of linear production and indivisible goods.

The structure of the paper is as follows: In section 2, I introduce the framework, and the essential assumptions. Section 3 contains the necessary lemmas and the main result where I find the optimality of a linear pricing function for achieving utilitarian efficiency or revenue maximization. In Section 4, I show that the proposed mechanism is dominant-strategy implementable. In section 5, I consider extensions for indivisible goods, production, and a weakening of the independence condition. In section 6, I conclude. Finally, in the appendix, I include all proofs and an example of how the theorem fails when the assumptions I make do not hold.

# 2 Framework

## 2.1 Setup

In the model studied here, there is a single good to be distributed by a planner with finite supply $S$. There is a unit measure of agents, who are defined by two attributes, wealth, $w$, and value, $v$. Agents are risk-neutral with linear utility in the quantity of the good and money. An agent's per-unit value for the good is determined by her value type, $v$. If an agent has quantity of the good $x$ and money $m$, then her utility is $U(v, x, m) = xv + m$. An agent's consumption of money is nonnegative, this is her budget constraint.

Agents' attributes are distributed according to two independent distributions, $F$ for values, and $G$ for budgets, each with continuously differentiable densities on a bounded non-negative support. This assumption stipulates that knowing an agent's budget or value type reveals no information about the agent's other attribute. So, there is no incentive to favor high or low budget agents from a purely correlation point of view.

For example, under the Homestead Act of 1862, independence of valuations and budgets seems reasonable because it is unclear whether wealthier agents should have higher or lower valuations for the land being settled. A similar argument could be made for the provision of healthcare. However, in some cases, such as the distribution of food or housing allowances, it could be argued that agents' budgets and values are negatively correlated. In other cases, such as those discussed in Esteban and Ray (2006), agents are firms and the good being distributed are production licenses. There, wealthy firms may have a higher valuation to produce because those firms are more efficient, and that is how they originally became wealthy. In the next two sections, I will focus on the case of independent distributions of agents' types and in section 5, I will relax this assumption in a positive correlation direction.

**Note:** Notice that the distributions $F, G$ define the aggregate makeup of the population. The model that I formulate here is one with no uncertainty and private types. All integrals in this paper represent an aggregation over all agents rather than taking an expectation with respect to some underlying uncertainty.

I consider two different mechanism design problems. In the first, a principal (perhaps the government or another public institution) wishes to maximize the welfare of the agents. Therefore, the principal wishes to assign the good to the agents with the highest valuations for the good. The mechanism design problem is to find assignment and transfer rules $x, t : V \times W \to \mathbb{R}$ that maximize the total utilitarian welfare of society.

In the other problem, the goal of the principal (perhaps the government or a corporation) is to find the revenue-maximizing incentive compatible assignment and transfer rules. For both problems: $x$, $t$ are taken to be measurable functions. Also, notice that $x$ and $t$ are deterministic allocation and transfer rules. This is without loss of generality in my setting because of linear utility and the specific incentive compatibility constraints I impose.

Now, I briefly discuss other mechanism design formulations that appear in the mechanism design with budget-constraints literature and then present the welfare-maximization problem that I solve. Previously, Che and Gale (2000) have examined revenue maximization in a single seller, single buyer setting with discrete types. Concurrently with my paper, Pai and Vohra (2011) examine welfare and revenue maximization in a single seller, multiple buyer setting with discrete types. Also, Che, Gale and Kim (2011) examine welfare maximization in a single seller, unit measure of buyers setting with discrete types.

There are a couple of novelties in my formulation of the efficiency / revenue-maximization problems. The first is that, I am working in a continuum setting with a continuum of types, so the objective function and all constraints are modified accordingly. Another is that, in the other papers in this literature, $x$ represents a probability of receiving an indivisible good as opposed to a quantity of the good as it does in my model. Finally, in Che and Gale (2000) and Pai and Vohra (2011), because there is one good being sold, at most one agent may receive it, so $x \leq 1$.

The stipulation that $x \leq 1$ is also made in Che, Gale and Kim (2011), where it is assumed that agents demand exactly one unit of the good. Such a model can be phrased as agents having linear utility in the good from $x = 0$ to $x = 1$ and zero additional utility in $x$ thereafter. In my model, agents demand as much of the good as affordable

I will now define below the *welfare-maximization* problem. The welfare-maximization criterion I will use is that of utilitarian efficiency. For an argument of why utilitarian welfare is a reasonable criterion, see Vickrey (1945). Notice that I have already used the revelation principle in formulating the problem below as a direct mechanism.

**Definition:** Welfare-Maximization Problem

Maximize

$$\mathcal{W}(x,t) := \int_W \int_V x(v,w) v f(v) dv g(w) dw \tag{1}$$

s.t.

$$\int_W \int_V t(v,w) f(v) dv g(w) dw \geq 0 \tag{BB}$$

$$\int_W \int_V x(v,w) f(v) dv g(w) dw \leq S \tag{LS}$$

$$0 \leq x(v,w) \qquad \forall v, w \tag{NN}$$

$$t(v,w) \leq w \qquad \forall v, w \tag{BC}$$

$$\mathbb{1}_{\{w' \leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v, w, v', w' \tag{IC}$$

$$vx(v,w) - t(v,w) \geq 0 \qquad \forall v, w \tag{IR}$$

The above conditions are respectively: (BB) budget balance, (LS) a limited supply condition, (NN) a non-negativity condition, (BC) budget constraints, (IC) incentive compatibility, and (IR) individual rationality.

The budget balance condition above states that the planner cannot inject money into the system. If he were able to, then he could distribute near infinite amounts and relieve every agents' budget constraint.

The non-negativity condition states that only the planner supplies the good, i.e. agents cannot be allocated a negative quantity of the good.

Before discussing the budget constraint (BC) and incentive compatibility conditions (IC), I will define the *revenue-maximization* problem.

**Definition:** Revenue-Maximization Problem

Replace the welfare objective function (1) to be maximized with the following revenue function to be maximized:

$$\mathcal{R}(x,t) := \int_W \int_V t(v,w) f(v) dv g(w) dw \tag{1'}$$

The revenue-maximization problem has a different objective function, but retains all the constraints of the welfare-maximization problem.

**Notation:** Throughout the paper, I refer to "welfare maximizing" or "welfare optimal". By "welfare maximizing", I intend to refer to the welfare maximization problem, and by "welfare optimal", I intend to refer to the solution to the welfare maximization problem.

## 2.2 Budget Constraints and Incentive Compatibility

The budget constraint condition (BC) states that agents cannot be asked to make a transfer $t(v,w)$ strictly larger than their wealth $w$. This condition is what separates the problem studied here from an unconstrained budget setting.

The budget constraint (BC) is the same as that found in Che and Gale (2000) and Pai and Vohra (2011), but differs from Che, Gale and Kim (2011). The difference is that the latter paper uses: $t(v,w) \leq wx(v,w)$. In the previous three models, $x(v,w)$ is the probability of receiving the good. So, if agents could only make payments in the event of receiving the good, then the budget constraint found in Che, Gale and Kim (2011) could be though of as *ex-post* budget constraint, whereas the budget constraint $t(v,w) \leq w$ could be considered as an *ex-ante* budget constraint. However, in the setting that I consider, $x(v,w)$ represents quantities of a divisible good instead of probabilities for an indivisible good, so in this setting, the mechanism is deterministic and there is no ex-ante/ex-post distinction.

The incentive compatibility condition that I impose above states that wealthy agents *can* imitate poorer agents, but poor agents *cannot* imitate wealthier agents. This type of condition has been explained in the literature before as being appropriate in a setting where agents can post bonds and therefore, agents can not lie upwards about their wealth type because they are unable to post larger bonds. The alternate incentive compatibility condition typically considered in the literature is:

$$\mathbb{1}_{\{t(v',w') \leq w\}}(vx(v',w') - t(v',w')) \leq vx(v,w) - t(v,w) \qquad \forall v,w,v',w' \tag{IC'}$$

This condition is less restrictive on agents and hence more restrictive on the class of admissible mechanisms because it says that agents can imitate any type whose transfers they can afford. As it turns out, the optimal mechanisms that I find will satisfy this stronger incentive compatibility condition as well, and therefore will be optimal for either formulation.

**Note:** Notice that the incentive compatibility conditions (IC) or (IC') have both been defined in terms of a simultaneous deviation in declaration of wealth and value. These incentive compatibility constraints could instead be expressed in terms of one-dimensional deviations as follows:

$$x(v', w) - t(v'w) \leq vx(v, w) - t(v, w) \qquad \forall v, v' \qquad \text{(Value-IC)}$$
$$\mathbb{1}_{w' \leq w}(vx(v, w') - t(v, w')) \leq vx(v, w) - t(v, w) \qquad \forall w, w' \qquad \text{(Wealth-IC)}$$

The above two one-dimensional ICs imply the two-dimensional IC for the following reason. If type $(v, w)$ does not want to pretend to be $(v, w')$ and type $(v, w')$ does not want to pretend to be $(v', w')$, then type $(v, w)$ does not want to pretend to be $(v', w')$ because $(v, w)$ and $(v, w')$ have the same preferences over outcomes. In other words, the only determinant of an agent's preferences is his value type, his wealth type only determines feasibility. If an agent's wealth also affected his preferences, then the two one-dimensional ICs may not imply the single two-dimensional IC constraint. As for notation, if a mechanism only satisfies the value-IC constraint or the wealth-IC constraint, then I will call it *value-admissible* or *wealth-admissible*. If a mechanism satisfies both constraints, I will refer to it as *admissible*. Unless otherwise stated, admissibility always refers to (IC) and not (IC').

**Note:** Referring back to the formulation of the welfare objective function $\mathcal{W}$, notice that $\mathcal{W}$ is defined without regard to aggregate transfers. The budget balance constraint requires that the planner cannot introduce money into the system, but there could be a positive net transfer of money from the agents to the planner thereby reducing the agents' utility. However, for any admissible mechanism, this money could then be redisbursed back to the agents in the form of a uniform lump sum transfer without affecting any of the given constraints.[1] So, any solution to the problem where transfers are included into the welfare function will correspond to a

[1] If I worked with condition (IC') instead of (IC), the uniform disbursal of money might make a mechanism non-admissible because then additional deviations would be affordable. Fortunately, this would affect the analysis of the problem and not the conclusions because the optimal mechanisms that I find satisfy (IC') as well.

solution of the above formulation and vice-versa. Also, as it turns out, the optimal mechanism will not have any net transfers from the agents because it is not in the principal's interest to take money out of the system and thereby make the budget constraints harder.

### 2.3 Assumptions

The following two assumptions will be imposed for the duration of the paper. These two assumptions will permit the subsequent analysis that I carry out and will lead to a simple class of optimal mechanisms. Both assumptions are on the distribution of value types among the agents:

**Assumption 1:** $\frac{1-F(v)}{f(v)}$ is weakly decreasing.

**Assumption 2:** $f(v)$ is weakly decreasing.

Assumption 1 above is the decreasing hazard ratio assumption. This assumption implies that allocating more of the good to higher value types from lower value types will lead to an increase in revenue. This is because an agent's virtual valuation is increasing in their own value, as in Myerson (1981).

This assumption is important because it ties together agents valuations and their virtual valuations. Specifically, if the above assumption holds, then a revenue-maximizing principal wants to get the good to the agents with the highest valuations, because doing so also increases the revenue generated by a given mechanism.[2]

Assumption 2 states that higher valuations are weakly less likely then lower valuations. This assumption is satisfied by the uniform distribution, the exponential distribution, the left truncation of a normal distribution (recall that negative valuations are not possible), and any convex combination of the above (as long as they have the same initial value $\underline{v}$).

Analogues of Assumptions 1 and 2 for a discrete setting are employed in Pai and Vohra (2011). Unlike Pai and Vohra (2011) and Che, Gale and Kim (2011), there will be no further assumptions on the budget distribution outside of requiring a continuous distribution of budgets. Through the proof, one can see that the budget distribution could even be taken to be more general, but with an increase in tediousness resulting from needing to deal with mixed distributions.

[2] Assumption 1 is fairly standard. However, it is actually the case that all the theorems (outside of Theorem 9, Part 2) hold in this paper if one instead assumes the weaker regularity assumption of Myerson (1981), specifically that $v - \frac{1-F(v)}{f(v)}$ is weakly increasing. For that theorem, the stronger assumption on values is only needed for the given equations to uniquely pin down the revenue-maximizing parameters.

# 3 The Main Result

## 3.1 A Preview

In this subsection, I give a brief preview of the main result and a numerical example. My main result shows that the optimal mechanisms, from either a welfare or revenue point of view are linear mechanisms with uniform transfers. So, I first define such mechanisms:

**Definition:** A *linear mechanism with uniform transfers* is characterized by two parameters, $(p, T)$. Both of these parameters are taken to be non-negative. In these linear mechanisms, agents will receive a uniform lump sum transfer $T$ and can purchase as much of the good as they wish at the unit price $p$. Agents with "low" valuations, specifically with $v < p$ will therefore purchase none of the good and just receive the transfer $T$. On the other hand, agents with "high" valuations, specifically $v > p$ will have induced wealth $w + T$ and therefore will purchase $\frac{w+T}{p}$ quantity of the good.

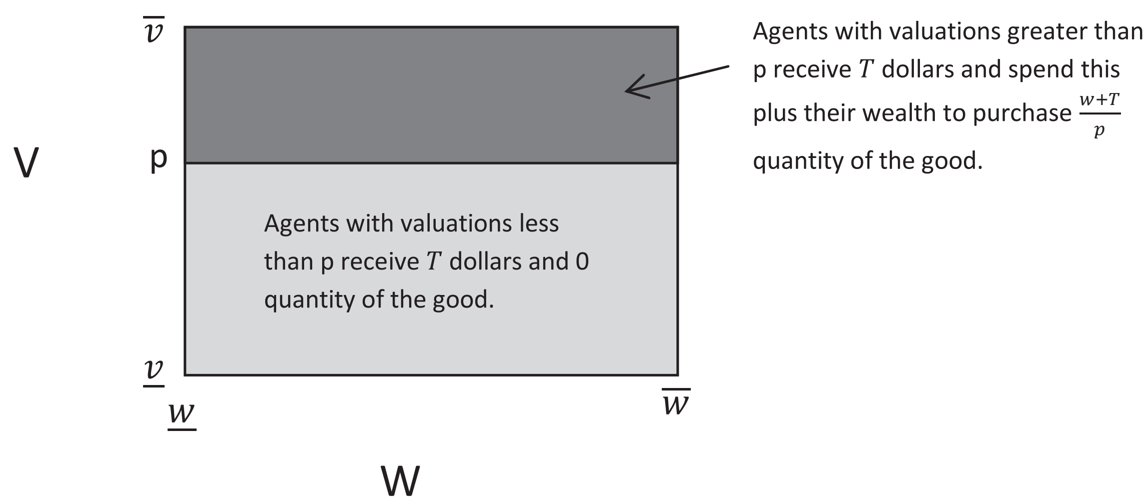


Figure 1: A Linear Mechanism with Uniform Transfers

The above mechanism is incentive compatible because all agents are choosing their optimal bundles out of the options that they can afford.

Now, I will present my main theorem. The next subsection will include three lemmas necessary for its proof.

**Main Theorem:** *Under Assumptions 1 & 2, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with uniform transfers, $(p_W, T_W)$. The welfare-optimal per-unit price and transfers are such that $T_W = S \cdot p_W$ and $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$.*

2. *The revenue-optimal mechanism is a linear mechanism with uniform transfers, $(p_R, T_R)$. The revenue-optimal per-unit price and transfers are such that $T_R = 0$ and $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$.*

3. *$p_W > p_R$*

**Notation:** I write $\mathbb{E}[w]$ above to simplify notation. This expectation denotes the average wealth of an agent in the society. Since agents' wealth and valuation distributions are independent, the expectation is $\mathbb{E}[w] = \int\limits_{\underline{w}}^{\bar{w}} wg(w)dw$.

Notice that the above theorem states that linear mechanisms with uniform lump-sum transfers are welfare- and revenue-optimal in the class of all admissible mechanisms, not just the class of linear mechanisms.

**A Numerical Example:** Consider the situation of a planner who has a unit supply of the good to be distributed, i.e. $S = 1$. In addition, suppose that agents' valuations and budgets are both uniformly distributed on $[0, 1]$, i.e. $F = G = U(0, 1)$.

**Welfare Maximization:** According to the calculations given in the theorem above, the welfare maximizing price is defined so that $(1 - p_W)\frac{1}{2} = p_W^2$. The unique solution to this equation is $p_W = \frac{1}{2}$. In addition, the transfers that agents receive are equal to $T = \frac{1}{2}$ as well. So, in the welfare-maximizing mechanism, each agent receives a cash transfer of $1/2$ and the market price for the good is $1/2$. Agents self-select into two regimes based upon their valuations. Those with valuations below $1/2$ just receive the transfer. Agents with valuations above $1/2$ receive the transfer and use it along with their wealth to purchase the good. Specifically, agents with original wealth 0 will now have wealth $1/2$ and will therefore be able to purchase one unit at the unit-price of $1/2$. On the other hand, agents with original wealth 1 will have wealth $3/2$ after the uniform lump-sum transfer and therefore be able to purchase 3 units at the unit-price of $1/2$.

**Revenue Maximization:** As for the revenue-maximizing mechanism, there are no uniform lump-sum transfers to the agents. The market clearing price, given in the theorem is $(1 - p_R)\,\frac{1}{2} = p_R$ and therefore is uniquely determined as $p_R = \frac{1}{3}$. As noted in the theorem, this market-clearing price is lower than in the welfare-maximizing setting because agents are receiving no transfers. Since agents in the revenue-maximizing setting are "poorer", then they have a lower aggregate demand function and a lower market-clearing price. Agents with valuations below 1/3 purchase none of the good. Agents with valuations above 1/3 purchase as much of the good as they can afford. Therefore, the amount purchased ranges from agents with wealth 0 will purchase 0 units and agents with wealth 1 who will purchase 3 units.

Notice that the maximum amount purchased in the revenue-maximizing setting happens to accord with the maximum amount purchased in the welfare-maximizing setting. This needn't be the case in general. In addition, under the two mechanisms, agents fall under three possible comparisons.

1. Agents with valuations less than 1/3 purchase none of the good good in either mechanism, but receive a uniform lump-sum transfer of 1/2 in the welfare-maximizing mechanism. Therefore, these agents are clearly better off in the welfare setting.

2. Agents with valuations between 1/3 and 1/2 purchase the good in the revenue-maximizing setting and receive lump-sum transfers in the welfare-maximizing setting. The agent whose utility increases the most in the revenue-maximizing setting is that of the agents with $v = 1/2$ and $w = 1$. These agents purchase three units for one dollar in the revenue-maximizing setting, which yields the utility 3/2. On the other hand, these agents in the welfare-maximizing setting, have one dollar and receive another 1/2 dollar in the uniform lump-sum transfer. So, the best off agent in the revenue-maximizing setting is indifferent between the welfare-maximizing mechanism and the revenue-maximizing mechanism and all other agents in this valuation range are strictly worse off in the revenue-maximizing mechanism.

3. As for agents with valuations $v \geq 1/2$, these agents purchase as much as they can afford in either mechanism. However, one notices that the amount the agents can purchase in the revenue maximizing mechanism $\frac{w}{1/3} = 3w$ is weakly less than the amount agents purchase in the welfare-maximizing mechanism $\frac{w+1/2}{1/2} = 2(w + 1/2) = 2w + 1$, so agents are weakly worse off in the revenue-maximizing mechanism. In fact, the inequality is strict except for those agents with $v \in [1/2, 1]$ and $w = 1$.

Therefore, one notices that in this example, the welfare-maximizing mechanism is a **Pareto improvement** of the revenue-maximizing mechanism. However, this is not a feature that needs to hold generally. More specifically, if one added a zero measure of agents with $w = 2$ and values uniformly distributed on $[0, 1]$, then agents with $w = 2$ and $v = 1$ are worse off in the welfare-maximizing mechanism. This is because they value the good highly, but can only purchase $\frac{2+1/2}{1/2} = 5$ units in the welfare-maximizing mechanism and can purchase $\frac{2}{1/3} = 6$ units in the revenue-maximizing mechanism. So, while the welfare-maximizing mechanism improves utilitarian welfare when compared to the revenue-maximizing mechanism, it may or may not be Pareto-improving as well.

Also, I would like to note here that the result does not necessarily hold if Assumptions 1 and 2 are not satisfied. Specifically, please refer to the Appendix for a counterexample.

### 3.2 Argument Outline

The following is a brief outline of the argument used in the paper:

**Step 1:** I show that any mechanism is simultaneously welfare and revenue dominated by a mechanism where every agent receives a take-it-or-leave-it offer based upon his wealth type. I will call such mechanisms, *take-it-or-leave-it mechanisms.*

**Step 2:** I prove that any take-it-or-leave-it mechanism is welfare and revenue dominated by one that charges a linear price.

**Step 3:** I show that one can consider an equivalent linear mechanism with *uniform lump-sum transfers.*

**Step 4:** I find the welfare- and revenue-optimal linear mechanisms with uniform transfers.

There are a couple points in the above outline that could be potentially troublesome that I now discuss further. First, *take-it-or-leave-it mechanisms* are value admissible, but may not be wealth-admissible. In fact, the only such wealth-admissible mechanisms are mechanisms where the wealthy agents are being favored. However, this does not pose a problem because in Step 2 above, I show that the optimal take-it-or-leave-it mechanism is a linear pricing mechanism, and hence admissible.

Therefore, while I temporarily focus on value-admissible mechanisms which may not be wealth-admissible, the optimal such mechanism is admissible with respect to both the value-ICs and the wealth-ICs.

The next problem that comes up is: while this linear mechanism is admissible with respect to (IC), it is not necessarily admissible with respect to (IC') because wealthier agents may be receiving larger transfers than poor agents. In step 3, I show that there is an equivalent (in terms of welfare, revenue, and unit prices) admissible linear mechanism that features a uniform transfer to all agents. Therefore I will have shown that for every admissible mechanism there is an admissible linear mechanism with uniform transfers that simultaneously welfare and revenue dominates it with respect to either IC condition.

Then, all that is left for the theorem is to find the welfare- and revenue-maximizing linear mechanisms with uniform transfers. This is accomplished in the proof of the theorem and their characterizations are given in the statement of the theorem. This is the *only* point where the welfare-optimal and revenue-optimal mechanisms will differ.

### 3.3 Three Lemmas

In the first lemma, I show that if all agents have the same known budget $w$, then any admissible mechanism is simultaneously welfare and revenue dominated by a take-it-or-leave-it offer $(P, Q)$. I use the notation $P$ here because this is an aggregate price, not a per-unit price.The per-unit price will be the threshold $\hat{v}$ defined in the next lemma. Moreover, the optimal take-it-or-leave-it offer has the feature that the price $P$ equals the known wealth of the agents. In these take-it-or-leave-it mechanisms, an agent is able to exchange his entire budget $w$ for a supply of the good $\hat{x}(w)$.

I show this dominance via a weight-shifting argument as outlined in Figure 3.3 where two allocation functions are drawn. The solid dark allocation function consists of a take-it-or-leave-it offer. All agents with valuations below $\hat{v}$ receive none of the good and all agents with valuations above $\hat{v}$ receive the same amount. The light dotted allocation function in the middle is another admissible allocation function. The weight-shifting argument relies upon finding a $\hat{v}$ and shifting the allocation function from the light dotted allocation function to the darker solid one. The choice of $\hat{v}$ is uniquely determined because the shift that takes place needs to preserve the same supply. So, the good is being shifted from the region to the left of $\hat{v}$ to the region to the right of $\hat{v}$ under the above transformation.

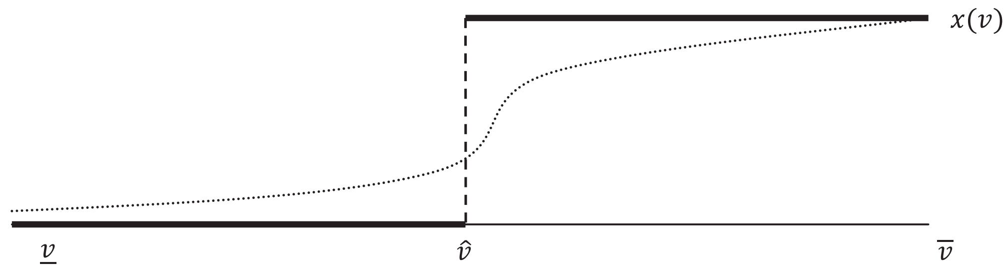


Figure 2: Agents below $\hat{v}$ no longer receive the good whereas all agents above $\hat{v}$ all receive the same share at type $\bar{v}$.

**Note:** In the dark one-step allocation function above, it may be the case that the implied transfer that the highest type pays, $t(\bar{v})$ is strictly less than $w$. In this case, $\hat{v}$ can be increased as well as the transfer $Q$ so that the indifference of type $\hat{v}$ is maintained and welfare/revenue will both be improved. This step is also performed in the following lemma.

**Lemma 1** *Under Assumptions 1 & 2, for a fixed level of wealth $w$, an admissible*[3] *allocation rule $x(v, w)$, and a transfer rule $t(v, w)$, there is a unique admissible welfare-optimal take-it-or-leave-it offer $\hat{x}$ with a transfer $\hat{t}$ that maintains the same reserve utility $U(\underline{v}, w)$ and supply $\hat{S} = \int_{\underline{v}}^{\bar{v}} x(v, w) f(v) dv$. Moreover, this take-it-or-leave-it offer has the following properties:*

1. *This offer is welfare-improving*
2. *This offer is revenue-improving*
3. *It is the case that $\hat{t} = w$*

The above lemma says that any admissible mechanism is dominated by one where an agent receives a take-it-or-leave-it offer with price equal to his wealth. Therefore, one can focus on the class of take-it-or-leave-it mechanisms when searching for the optimal mechanism. Recall that the take-it-or-leave-it mechanisms are

[3]The fact that we are starting off with an admissible pair of rules $x$, $t$ is fundamental for the proof here.

value-admissible but not wealth-admissible. I will show that the optimal take-it-or-leave-it mechanism is admissible and is equivalent to a mechanism that satisfies (IC') as well.

While the formal proof of the above lemma is regulated to the appendix, the general idea is as follows. I show that shifting the allocation of the good from lower value agents to higher valuation agents is feasible in the sense that no agent is asked to pay more than their wealth. This is because agents were previously respecting their budget constraints in the admissible mechanism and the highest value agent's payments weakly decrease after the shift. Since this agent purchases weakly more than all other agents, all agents are asked an affordable transfer. Shifting the good from lower value types to higher value types clearly improves welfare. The revenue improvement is due to Assumption 1 which states that agents' virtual valuations are increasing in their underlying value, so the good is being shifted to agents with higher virtual valuations as well.

Now, I formally define a take-it-or-leave-it mechanism. Notice that the definition is given in terms of cutoffs $\hat{v}(w)$ instead of quantities $\hat{x}(w)$. Either definition is equivalent via the indifference equation $\hat{x}\hat{v} = U(\underline{v}, w) + w$. These cutoffs serve as *both* the per-unit price that agents with budget $w$ pay and the threshold for agents who buy the good.

**Definition:** Take-it-or-leave-it Mechanisms
The class $\mathcal{T}$ of *take-it-or-leave-it mechanisms* are all $(x, t)$ such that $x : V \times W \to \mathbb{R}_+,\ t : V \times W \to \mathbb{R}$ and they satisfy the following conditions:

$$\exists \hat{v} : W \to [\underline{v}, \bar{v}] \tag{2}$$

$$\int_W \int_V t(v, w) f(v) dv g(w) dw \geq 0 \tag{3}$$

$$\int_W \int_V x(v, w) f(v) dv g(w) dw \leq S \tag{4}$$

$$\forall v, w,\ v \geq \hat{v}(w) \Rightarrow x(v, w) = \frac{w + U(\underline{v}, w)}{v},\ t(v, w) = w \tag{5}$$

$$\forall v, w,\ v < \hat{v}(w) \Rightarrow x(v, w) = 0, t(v, w) = -U(\underline{v}, w) \leq 0 \tag{6}$$

First, I would like to note that in the above definition of take-it-or-leave-it mechanisms, attention is being restricted to mechanisms where the take-it-or-leave-it transfer is equal to an agent's wealth. It is sufficient to consider such mechanisms because in Lemma 1, we have shown that any admissible mechanism is simultaneously welfare and revenue dominated by such a mechanism. Take-it-or-leave-it mechanisms are a special subclass of all value-admissible mechanisms. They are not necessarily wealth-admissible because agents may wish to lie downwards about their wealth to achieve a cheaper per-unit price $\hat{v}(w)$.

In the definition above, one can think of $\hat{v}(w)$ as being the per-unit price that an agent of wealth $w$'s take-it-or-leave-it offer has. This per-unit price then serves as a cutoff where agents with valuations higher than the per-unit price will fully expend their budgets buying the good and agents with valuations lower than the per-unit price will not purchase the good and receive a transfer. Therefore, agents IC conditions are built into equations 5 and 6 and it is assumed that agents cannot lie about their wealth. In addition, in the above setting, notice that $U(\underline{v}, w) = -t(\underline{v}, w)$ and therefore equation 6 is also the individual rationality condition stating that agents who do not receive the good cannot be expected to pay anything.

The next lemma shows that any take-it-or-leave-it mechanism is welfare and revenue dominated by a linear mechanism, in other words, one where the cutoff values $\hat{v}(w)$ are constant.

**Lemma 2** *Under Assumptions 1 & 2 the welfare-optimal mechanism and the revenue-optimal mechanisms are both linear pricing systems.*

**Proof:** See Appendix.

The two above lemmas show that any admissible mechanism is simultaneously welfare and revenue dominated by a linear pricing mechanism. These mechanisms are admissible for (IC), but not necessarily for (IC'). This is because agents below the cutoff price/threshold are receiving cash transfers that may be increasing in their wealth. The next lemma shows that for any such linear mechanism, there is a linear mechanism with lump-sum transfers that generates the same welfare and revenue. This mechanism therefore satisfies (IC') as well.

**Lemma 3** *Any linear pricing mechanism that satisfies (BB), (LS), (NN), (BC), (IR), and (IC) is simultaneously welfare and revenue equivalent to a linear mechanism with uniform lump-sum transfers (which therefore satisfies condition (IC') as well).*

### 3.4 The Main Theorem

Now, all that remains is to find the optimal linear mechanism with uniform transfers for the cases of welfare and revenue-maximization. The theorem shows that the welfare optimum is characterized by the supply and budget balance conditions while the revenue-maximizing mechanism is characterized by the supply condition and $T = 0$. I restate the theorem below and the proof is found in the appendix.

**Main Theorem:** *Under Assumptions 1 & 2, the following three statements hold:*

1. *A welfare-optimal mechanism is a linear mechanism with uniform transfers, $(p_W, T_W)$. The welfare-optimal per-unit price and transfers are such that $T_W = S \cdot p_W$ and $(1 - F(p_W))\mathbb{E}[w] = S \cdot p_W \cdot F(p_W)$.*

2. *The revenue-optimal mechanism is a linear mechanism with uniform transfers, $(p_R, T_R)$. The revenue-optimal per-unit price and transfers are such that $T_R = 0$ and $(1 - F(p_R))\mathbb{E}[w] = S \cdot p_R$.*

3. *$p_W > p_R$*

The above theorem shows that the optimal constrained mechanism from an efficiency point of view is one where the mechanism designer charges a uniform price to every agent. Agents with high valuations will purchase as much as they can afford.

In fact, the welfare-maximizing mechanism is the linear mechanism with the *highest* feasible per-unit price and the revenue-maximizing mechanism is the linear mechanism with the lowest feasible per-unit price. This is because there is an inverse relationship between transfers to the agents and the market-clearing price. The welfare-maximizing principal makes as large transfers as possible to the agents to relax their budget constraints, and this increases the market-clearing price. This is why the budget balance constraint binds in the welfare-maximizing mechanism. On the other hand, the revenue-maximizing principal makes no transfers to the agents and this results in the least market-clearing price possible. As a consequence, the per-unit price is higher for the welfare-maximizing mechanism than for the revenue-maximizing mechanism.

# 4 Dominant Strategy Implementability

## 4.1 Implementation via Dominant Strategies

Turning to implementation, one may note that the above mechanism does not specify what happens if a mass of agents misreports their types. A simple solution is to say that in this event the government chooses to just not give the good to anybody. This heavy handed solution introduces many undesirable equilibria.

Fortunately, there is another dominant strategy way to implement the above optimal mechanism. Notice that one may always implement the welfare-maximizing mechanism via a uniform lump sum transfer. This suggests that an alternative way to implement the above mechanism is to transfer a uniform amount $S$ of the good to every agent and then allow agents to buy/sell the good among themselves. This in fact is the case and leads to a dominant strategy allocation rule.

**Note:** This implementation closely resembles one of the mechanisms found in Che, Gale and Kim (2011) called Random Allocation with Resale. A key difference here is that they have indivisible goods which are resold, and so agents receive the good with final probability 0, $S$, or 1 based upon their type. In my mechanism, intermediate allocation possibilities will be realized.

In addition, implementation of the revenue optimal mechanism is achievable via dominant strategies. A dominant strategy implementation would be to ask each agent to post a demand schedule. The planner then finds the maximal market-clearing price and clears the market at this price.

**Theorem 4** *Under Assumptions 1, 2, both the welfare-maximizing and the revenue-maximizing mechanisms are implementable through dominant strategies.*

From the above result, one can see that both the welfare-maximizing and revenue-maximizing mechanisms are dominant strategy implementable. This implies that these mechanisms are robust in the sense that each agent is weakly better off telling the truth regardless of all other agent's declarations. This theorem speaks to the robustness and ease of implementability of the welfare- and revenue-maximizing mechanisms. Moreover, I would like to point out that these mechanisms are dominant strategy implementable even when the assumptions on types do not hold. These assumptions are necessary for the optimality arguments, but not for the implementation arguments.

## 5 Extensions

### 5.1 Nonlinear Pricing Schemes

In this section, I analyze a setting where the revenue-maximizing mechanism may be nonlinear. Specifically, I replace the assumption of the independence of wealths and values by the following assumption. As mentioned in the introduction, there may be settings where higher wealth agents (firms) have higher valuations for the good (production limits).

**Assumption 3:** $\frac{1-F(v,w)}{f(v,w)}$ is weakly decreasing in w.

Under the above assumption, I obtain the following theorem.

**Theorem 5** *Under Assumptions 1-3, the revenue-maximizing mechanism is a take-it-or-leave-it mechanism with zero transfers. The cutoffs $\hat{v}(w)$ are weakly decreasing in $w$ and if Assumption 3 holds strictly, then the cutoffs $\hat{v}(w)$ and $\hat{v}(w) > \underline{v}$, then the cutoffs are strictly decreasing.*[4]

The proof is relegated to the appendix, but I will discuss the intuition here. As before, it is sufficient to restrict one's attention to take-it-or-leave-it mechanisms. First, notice that it is not in the seller's interest to make any transfers to the agents because for that wealth level, he could just offer a smaller supply of the good at the same linear price. This is revenue-improving because the seller is still taking in the same revenue, selling a smaller supply, and saving on the transfers to the agents.

The other step of the proof is to show that the optimal cutoff levels $\hat{v}(w)$ are decreasing in wealth. If this is the case, and there are no transfers to the agents, then this take-it-or-leave-it mechanism is admissible. The intuition here is that this mechanism favors the wealthy. Both sets of IC constraints are satisfied because wealthy types would not like to imitate poorer types (after all, they are favored) and poor types would like to imitate wealthy types when the wealthy agents get a cheaper per-unit price, but they are unable to do so.

In addition, the above take-it-or-leave-it mechanism has an interesting interpretation. A decreasing cutoff function corresponds to a concave pricing function. In

[4]In fact, a weaker condition would imply strict decreasingness, specifically that $\forall v$, $\frac{1-F(v,\underline{w})}{f(v,\underline{w})} > \frac{1-F(v,\bar{w})}{f(v,\bar{w})}$. Instead of requiring strict decreasingness everywhere, this condition requires at least one point of strict decreasingness along every horizontal value strip.

other words, as the quantity purchases increases, the per-unit price is decreasing. So, we have found a situation where agents have fully linear utility, yet quantity discounts are profit-maximizing as a way to favor the wealthy agents (who have higher virtual valuations). This would not be the case in a linear utility model without budget constraints.

### 5.2 Unit Demand and Probability Shares

Until now, this paper has been wholly focused on optimal mechanisms in the case of divisible goods and budget constraints. One may then wonder about the case of indivisible goods. Notice that if agents are risk-neutral and have fully linear utility, then $x$ can be interpreted as probability shares of obtaining the good and the above analysis carries through. For example, $x = 1.3$ could be interpreted as receiving one unit of the good with certainty and an additional unit of the good with probability 30%.

Another question would then be: "What if agents have unit demand?" I do not provide a full characterization in this case, but I would like to note that the mechanism previously provided is valid in the case that no agent receives a quantity of the good greater than 1. This is because I have imposed no unit demand restriction, and if the mechanism previously found is optimal in a wider class of mechanisms, then is it in optimal in a smaller class as well.

**Corollary 6** *If $x(\bar{v}, \bar{w}) \leq 1$, then the outlined optimal mechanism is optimal for the unit demand problem as well.*

1. *For welfare-optimality, the necessary and sufficient condition is* $S \leq \left(1 - F\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \left(\frac{w(1-S)}{\bar{w}} + S\right) g(w)dw.$
2. *For revenue-optimality, the necessary and sufficient condition is* $S \leq (1 - F(\bar{w})) \int_{\underline{w}}^{\bar{w}} \frac{w}{\bar{w}} g(w)dw.$

Notice that quantities sold in the welfare-maximizing mechanism are higher than quantities sold in the revenue-maximizing mechanism, so one should intuitively expect that condition 2 above should be easier to satisfy than condition 1. This is indeed the case.[5]

[5] First, notice that the RHS is bounded above by 1, therefore $S$ must be less than 1 for either condition to be satisfied. This is indeed the case because $(1 - F(\bar{w})) \int_{\underline{w}}^{\bar{w}} \frac{w}{\bar{w}} g(w)dw \leq \left(1 - F(\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \frac{w(1-S)+wS}{\bar{w}} g(w)dw$
$\leq \left(1 - F(\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \frac{w(1-S)+\bar{w}S}{\bar{w}} g(w)dw = \left(1 - F\left(\frac{\bar{w}}{1-S}\right)\right) \int_{\underline{w}}^{\bar{w}} \left(\frac{w(1-S)}{\bar{w}} + S\right) g(w)dw.$

### 5.3 Production

The fact that the optimal mechanisms found in this paper are simple makes the extension of the given model to other settings fairly tractable. In this subsection, I analyze the situation where the principal is able to produce the good at a linear cost $c \in [\underline{v}, \bar{v})$ instead of taking supply to be fixed. Almost all of the setting remains the same except that $S$ is now a choice variable and the budget balance condition changes. Specifically, (BB) is replaced with:

$$\int_W \int_V t(v,w) f(v) dv g(w) dw - cS \geq 0 \tag{BB'}$$

Notice that since $S$ is now a choice variable, a mechanism is now the tuple $(x, t, S)$. However, applying the analysis from Section 3, one can think of a mechanism as being chosen from a two-stage process. Specifically, the planner chooses the production level $S$ and then the allocation/transfer rules $x, t$. From this point of view, which is without loss of generality, one can see that Lemmas 1-3 still apply, assuming that the mechanism $(x, t, S)$ is admissible in the first place. This leads us to the following theorem.

**Theorem 7** *Under Assumptions 1 & 2 and linear production costs, the following four statements hold:*

1. *The welfare-optimal mechanism is a linear mechanism with uniform transfers* $(p_W, T_W)$ *and supply* $S_W$. *The welfare-optimal price, transfers, and supply are such that* $(1 - F(p_W)) \int_W w g(w) dw = S_W p_W$, $p_W = c$, *and* $T_W = 0$..
2. *The revenue-optimal mechanism is a linear mechanism with uniform transfers* $(p_R, T_R)$ *and supply* $S_R$. *The revenue-optimal price, transfers, and supply are such that* $(1 - F(p_R)) \int_W w g(w) dw = S_R p_R$, $\frac{1-F(p_R)}{f(p_R)} c = p_R (p_R - c)$, *and* $T_R = 0$.
3. $S_W > S_R$
4. $p_W < p_R$

There are a few important points about the above theorem that I would like to note. The first interesting point is that there are no transfers from the principal to the agents in the linear pricing model. This is not surprising in the revenue-maximization mechanism, but is a bit surprising in the welfare-optimal mechanism. Basically, this can be taken as a statement that, in a linear pricing model it is a better use for the principal to spend any excess money manufacturing the good, then distributing it via uniform lump-sum transfers.

Since the welfare-maximizing principal is no longer making transfers to the agents, the original rationale of why prices are higher in the welfare-maximizing mechanism goes away. If we recall the main theorem, there the welfare-maximizing principal made uniform lump-sum transfers and allowed the market to clear. So, the welfare-maximizing linear price was *higher* than the revenue-maximizing linear price because of these transfers. When there is linear production, the welfare-maximizing principal will manufacture more of the good then the revenue-maximizing principal and make no transfers. This will then lead to the opposite conclusion about the relevant prices, specifically that the market price under the revenue-maximizing principal is *lower* than the market price under the welfare-maximizing principal.

# 6 Conclusion

In this paper, I have solved a pair of optimal mechanism design problems in a setting with a finite supply of a single divisible good and a continuum of agents. The main purpose of this paper is to consider a setting where agents may be budget constrained. This constraint prevents an implementation where the planner can sell all of the good to agents with near maximal valuation.

Instead, I discover the optimal mechanisms take the form of linear prices with uniform lump-sum transfers. These lump-sum transfers alleviate agents' budget constraints in the case of welfare-maximization and are equal to zero in the case of revenue-maximization. Agents use these transfers along with their wealth to purchase the good from the planner if they are above the linear price. In the case of welfare maximization, this theorem can be interpreted as a version of the second welfare theorem in the presence of budget constraints.

As noted above, in the case of revenue-maximization, the planner offers no transfers to agents. This means that the planner's gains in revenue from selling additional units from the good are more than outweighed by the costs of these transfers. This is in line with Pai and Vohra (2011) who obtain this finding in a finite setting with discrete values and budgets.

A key difference between this paper and that of Pai and Vohra (2011) and Che, Gale and Kim (2011) is that the optimal mechanisms in the current paper feature linear prices. The optimal mechanisms are more complicated in their settings due to the difference in settings. Specifically, their settings feature a discrete set of types, unit demands, and indivisible goods. In addition, there is a difference in the budget constraint found in this paper and Che, Gale and Kim (2011) that contributes to the difference in optimal mechanisms.

After these optimality results, I showed that the above mechanisms are implementable via dominant strategies. The dominant strategy implementations are quite different in the two different settings due to the two different optimization goals. The dominant strategy implementations use the fact that the setting is one where is a unit measure of agents and therefore agents individual type declarations do not have an effect on the overall economy. The dominant strategy implementation of the welfare-maximizing mechanism bears a strong resemblance to the random allocation with resale mechanism proposed by Che, Gale and Kim (2011), but my optimal mechanism is distinct in a significant way to their optimal mechanism. Specifically, in their mechanism all agents receive the good with final probability 0, $S$, or 1, whereas in my model, intermediate allocations are realized as well.

Additionally, I prove that when agents' valuation hazard rates are weakly decreasing in wealth types, then the revenue-maximizing mechanism may feature nonlinear prices. It is nonlinear in the sense that it is implementable via a concave pricing rule, and therefore offers a justification of quantity discounts, even when agents have linear utility.

In addition, I considered the case where agents have unit demand. I provide necessary and sufficient conditions when the mechanism found in this paper continues to be welfare and/or revenue optimal in that change of setting. These conditions may be thought of as being such that the supply of the good is not too large.

Finally, I considered an extensions to linear production. In this case, I find that the principal makes no transfers to the agents. I also find that the per-unit prices are lower in the welfare-maximizing mechanism than the revenue-maximizing mechanism in the case of linear production. This is the opposite of the finite supply case where there *are* transfers and where the welfare-maximizing mechanism has a higher per-unit price than the revenue-maximizing mechanism.

To conclude, I think that it is interesting to study mechanism design with budget constraints because budget constraints are prevalent and generally have a real impact on the structure of the optimal mechanism. In addition, this is a topic of study where there are plenty of additional questions of merit as well.

## References


**Armstrong, Mark**, "Multiproduct Nonlinear Pricing," *Econometrica*, 1996, *64* (1), pp. 51–75.

— **and Jean-Charles Rochet**, "Multi-dimensional screening:: A user's guide," *European Economic Review*, 1999, *43* (4-6), 959 – 979.

**Ausubel, Lawrence M.**, "An Efficient Ascending-Bid Auction for Multiple Objects," *American Economic Review*, 2004, *94* (5), 1452–1475.

**Border, Kim C.**, "Implementation of Reduced Form Auctions: A Geometric Approach," *Econometrica*, 1991, *59* (4), 1175–1187.

**Che, Yeon-Koo and Ian Gale**, "Standard Auctions with Financially Constrained Bidders," *Review of Economic Studies*, January 1998, *65* (1), 1–21.

— **and** — , "The Optimal Mechanism for Selling to a Budget-Constrained Buyer," *Journal of Economic Theory*, June 2000, *92* (2), 198–233.

— **,** — **, and Jimwoo Kim**, "Assigning Resources to Budget-Constrained Agents," *Review of Economic Studies, Forthcoming*, 2011.

**Cremer, Helmuth, Pierre Pestieau, and Jean-Charles Rochet**, "Direct Versus Indirect Taxation: The Design of the Tax Structure Revisited," *International Economic Review*, 2001, *42* (3), 781–800.

**Esteban, Joan and Debraj Ray**, "Inequality, Lobbying, and Resource Allocation," *The American Economic Review*, 2006, *96* (1), pp. 257–279.

**Jehiel, Philippe, Benny Moldovanu, and Ennio Stacchetti**, "How (Not) to Sell Nuclear Weapons," *The American Economic Review*, 1996, *86* (4), pp. 814–829.

**Kojima, Naoki**, "The Pricing Mechanism to the Buyer with a Budget Constraint and an Indirect Nechanism," *Working Paper*, 2004.

**Kotowski, Maciej**, "First-Price Auctions with Budget Constraints," *Working Paper*, 2010.

**Laffont, Jean-Jacques and Jacques Robert**, "Optimal auction with financially constrained buyers," *Economics Letters*, 1996, *52* (2), 181 – 186.

**Maskin, Eric S.**, "Auctions, development, and privatization: Efficient auctions with liquidity-constrained buyers," *European Economic Review*, 2000, *44* (4-6), 667 – 681.

**McAfee, R.Preston and John McMillan**, "Multidimensional incentive compatibility and mechanism design," *Journal of Economic Theory*, 1988, *46* (2), 335 – 354.

**Mestieri, Martí**, "Wealth Distribution and Human Capital: How Borrowing Constraints Shape Educational Systems," *Working Paper.*

**Milgrom, Paul and Ilya Segal**, "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, March 2002, *70* (2), 583–601.

**Monteiro, Paulo K. and Frank H. Page Jr.**, "Optimal selling mechanisms for multiproduct monopolists: incentive compatibility in the presence of budget constraints," *Journal of Mathematical Economics*, 1998, *30* (4), 473 – 502.

**Myerson, Roger B.**, "Optimal Auction Design," *Mathematics of Operations Research*, February 1981, *6* (1), pp. 58–73.

**Pai, Mallesh and Rakesh Vohra**, "Optimal Auctions with Financially Constrained Bidders," *Working Paper*, 2011.

**Pigou, A.**, *The Economics of Welfare* number v. 4. In 'The Economics of Welfare.', London: MacMillan and Co., 1920.

**Rogerson, Richard**, "Indivisible labor, lotteries and equilibrium," *Journal of Monetary Economics*, January 1988, *21* (1), 3–16.

**Tirole, Jean**, *The Theory of Industrial Organization*, Vol. 1 of *MIT Press Books*, The MIT Press, 1988.

**Vickrey, William**, "Measuring Marginal Utility by Reactions to Risk," *Econometrica*, 1945, *13* (4), pp. 319–333.

**Wilson, Robert**, *Nonlinear Pricing*, USA: Oxford University Press, 1993.

# Appendix

**Proof of Lemma 1:**

Without loss of generality, consider the case where $x(\underline{v}, w) = 0$. If this is not the case, then $x$ can be changed so that it is so and only $t(\underline{v}, w)$ is decreased by $x(\underline{v}, w)\underline{v}$. This preserves $U(\underline{v}, w)$ the utility of an agent with value $\underline{v}$, so he still does not wish to deviate after this change. Now, an agent with value $v > \underline{v}$ is receiving utility $U(v) \geq U(\underline{v})$ and would receive $U(\underline{v})$ if he lied to be the lowest type. Therefore, she also has no incentive to deviate to pretend to be $\underline{v}$ and incentive compatibility is preserved.

For simplicity, denote $\hat{x} := x(\bar{v}, w)$. There is some $\hat{v}$ such that $(1 - F(\hat{v}))\hat{x} = \hat{S}$. In addition, let $\hat{t} = \hat{x}\hat{v} - t(\underline{v}, w)$. Now, consider the alternate transfer/allocation mechanism where

$$\hat{x}(v, w) = \begin{cases} 0 & v < \hat{v} \\ \hat{x} & v \geq \hat{v} \end{cases} \qquad \hat{t}(v, w) = \begin{cases} t(\underline{v}, w) & v < \hat{v} \\ \hat{t} & v \geq \hat{v} \end{cases}$$

It needs to be shown that this alternate mechanism is feasible, improves welfare, and improves revenue for the planner. Notice that $x$ is weakly increasing, so $\hat{x}$ is an improvement over $x$ because some of the good that was going to lower-valuation types under the allocation function $x$ is being shifted to higher-valuation agents under the allocation function $\hat{x}$.

As for feasibility, it needs to be shown that $\hat{t} < w$. From the value-IC condition, transfers are given as: $t(v, w) = t(\underline{v}, w) + x(v, w)v - \int_{\underline{v}}^{v} x(z, w)dz$.

Then $t(\bar{v}, w) = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)dz = t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} x(z, w)f(z)\frac{1}{f(z)}dz$ $\geq t(\underline{v}, w) + x(\bar{v}, w)\bar{v} - \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)\frac{1}{f(z)}dz = \hat{t}(\bar{v}, w)$. In the previous calculation, notice that the inequality follows because $\frac{1}{f(z)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z, w)f(z)dz = \int_{\underline{v}}^{\bar{v}} \hat{x}(z, w)f(z)dz$, and because $\hat{x}(v, w) - x(v, w)$ is a weakly negative, then weakly positive, function of $v$.

The above argument is important, because it yields the following inequalities: $w \geq t(\bar{v}, w) \geq \hat{t}(\bar{v}, w)$ and therefore $\hat{t}$ is feasible.

Finally, its needs to be shown that

$$\mathcal{R}(\hat{t}) := \int_{\underline{v}}^{\bar{v}} \hat{t}(v, w)f(v)dv \geq \int_{\underline{v}}^{\bar{v}} t(v, w)f(v)dv$$

Using the transfer equation from above, $\mathcal{R}(t) = \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v,w)f(v)dv$. As above, we can see that $\int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) \hat{x}(v,w)f(v)dv > \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1-F(v)}{f(v)}\right) x(v,w)f(v)dv$ because $\frac{1-F(v)}{f(v)}$ is decreasing which implies $v - \frac{1-F(v)}{f(v)}$ is increasing, $\int_{\underline{v}}^{\bar{v}} x(z,w)f(z)dz$ $= \int_{\underline{v}}^{\bar{v}} \hat{x}(z,w)f(z)dz$, and because $\hat{x}(v,w) - x(v,w)$ is a negative, then positive, function of $v$. Therefore $\mathcal{R}(\hat{t}) \geq \mathcal{R}(t)$.

Therefore the optimal transfer/allocation mechanism for the planner is a take-or-leave-it offer. If it is not the case that $t(\bar{v}, w) = w$, then increase $\hat{x}$ by $\epsilon$. As before, define $\hat{v}$ s.t. $(1 - F(\hat{v}))\hat{x} = \hat{S}$ and $\hat{t} := \hat{x}\hat{v} - t(\underline{v}, w)$. Notice that $\hat{v}$ is the threshold type who accepts the take-it-or-leave-it offer, i.e. this is the amount being spent per unit. Since this has just increased, we know that revenue has increased. Moreover, we know that welfare has increased because there is now a higher threshold type. Finally, we can find an $\epsilon$ small enough s.t. $\hat{t}$ is still less than $w$, hence we still have feasibility. ■

**Proof of Lemma 2:**

Notice that for a take-it-or-leave-it mechanism, the important parameters are $x(w)$, $U(w)$, $v(w)$, and $S(w)$. These variables are respectively, the quantity of the take-it-or-leave-it offer, the reserve utility offered, the threshold value (which is also the per-unit price), and the aggregate supply that types with wealth $w$ receive.

Focusing on one wealth strip and suppressing the function arguments, one obtains the following set of defining equations.

$$S = (1 - F(v))x \tag{7}$$

$$xv = U + w \tag{8}$$

$$\mathcal{W} = \int_{v}^{\bar{v}} zf(z)dzx \tag{9}$$

$$\mathcal{R} = (1 - F(v))(U + w) - U \tag{10}$$

I will show that $\frac{d\mathcal{W}}{dS}$ and $\frac{d\mathcal{R}}{dS}$ only depend on $w$ via $v$. Then, I will show that $\mathcal{W}$ and $\mathcal{R}$ are in fact concave in $S$. This implies that if $v(w) < v(w')$, then there is a joint welfare and revenue improvement by removing some supply from agents with wealth $w$ and giving it to agents with wealth $w'$.

Replacing $x$ in the equations above and taking derivatives, one has

$$\frac{\partial S}{\partial v} = \frac{-vf(v) - (1 - F(v))}{v^2}(U + w) \tag{11}$$
$$\frac{\partial \mathcal{W}}{\partial v} = \frac{-v^2 f(v) - \int_v^{\bar{v}} zf(z)dz}{v^2}(U + w) \tag{12}$$
$$\frac{\partial \mathcal{R}}{\partial v} = -f(v)(U + w) \tag{13}$$

Therefore,

$$\frac{d\mathcal{W}}{dS} = \frac{v^2 f(v) + \int_v^{\bar{v}} zf(z)dz}{vf(v) + (1 - F(v))} \tag{14}$$
$$\frac{d\mathcal{R}}{dS} = \frac{v^2 f(v)}{vf(v) + (1 - F(v))} \tag{15}$$

Therefore, both of the above derivatives are independent of agents actual wealth levels, all that is relevant is the critical value level $v$. This suggests that a uniform $v$ is optimal and this will in fact be proven once it is shown that the second derivatives are negative. However, notice that $\frac{dS}{dv} < 0$, so it will in fact suffice to show that $\frac{\partial}{\partial v}\frac{d\mathcal{W}}{dS} > 0$ and the same for $\mathcal{R}$.

The second derivatives are

$$\frac{\partial}{\partial v}\frac{d\mathcal{W}}{dS} = \frac{vf(v)(vf(v) + (1 - F(v))) + vf'(v)(v(1 - F(v)) - \int_v^{\bar{v}} zf(z)dz)}{(vf(v) + (1 - F(v)))^2} \tag{16}$$
$$\frac{\partial}{\partial v}\frac{d\mathcal{R}}{dS} = \frac{2vf(v)(vf(v) + 1 - F(v)) + v^2 f'(v)(1 - F(v))}{(vf(v) + (1 - F(v)))^2} \tag{17}$$

The first term of the numerator of the welfare equation is positive. The second term is positive as well because $f'(z) \le 0$ and $v(1 - F(v)) - \int_v^{\bar{v}} zf(z)dz \le v(1 - F(v)) - \int_v^{\bar{v}} vf(z)dz = 0$.

As for the revenue equation, notice that $\frac{1-F(v)}{f(v)}$ decreasing and this implies that $f'(v)(1 - F(v)) \ge -f(v)^2$. Therefore, the numerator of the revenue equation is bounded below by $v^2 f(v)^2 + 2vf(v)(1 - F(v)) \ge 0$. ■

**Proof of Lemma 3:**

From the previous argument, one sees that the optimal mechanism is a linear-pricing system $p$. However, if non-uniform transfers are being made to the agents with lowest valuations $\underline{v}$, then this mechanism is not admissible with respect to (IC'). This is because, agents who do not wish to purchase the good will wish to misreport their types in order to secure a more favorable transfer. Formally, admissibility with respect to (IC') will fail if for $w \neq w'$, it is the case that $t(w, v) \neq t(w', v)$.

This problem is easily solved by distributing a uniform transfer equal to $\int_{\underline{w}}^{\bar{w}} t(w, \underline{v})g(w)dw$ to agents who do not purchase the good. This may change the allocation for many agents (not just the ones who do not purchase the good), but notice that this change is welfare and revenue equivalent.[6]

Formally, the quantity demanded at the price p does not change:

$$\begin{aligned}
&(1 - F(p)) \int_{\underline{w}}^{\bar{w}} \frac{w + t(\underline{v}, w)}{p} g(w)dw \\
=&\frac{1 - F(p)}{p} \left( \int_{\underline{w}}^{\bar{w}} wg(w)dw + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, w)g(w)dw \right) \\
=&\frac{1 - F(p)}{f(p)} \int_{\underline{w}}^{\bar{w}} \left( w + \int_{\underline{w}}^{\bar{w}} t(\underline{v}, z)g(z)dz \right) g(w)dw
\end{aligned}$$

where the first line is the amount demanded with non-uniform transfers and the last line is the quantity demanded with uniform transfers. This implies that the market clearing price $p$ does not change. Since the quantity supplied does not change, and the market clearing price does not change, neither does welfare or revenue. This is because for any $v$ above $p$, the same quantity is being bought and the same transfers are being made for that horizontal slice of agents. ■

**Proof of Main Theorem:**

Suppose that the linear mechanism with uniform transfers $(p, T)$ does not supply the entire supply $S$ of the good. Then the price $p$ can be slightly reduced and this increases both the welfare of the agents (because every agent is receiving weakly more) and increases revenue (because a larger set of agents is paying their entire wealth). Therefore, for either optimal mechanism, it must be that the supply constraint binds: $(1 - F(p))\frac{\mathbb{E}[w]+T}{p} = S$.

[6] Recall that agents were using their wealth $w$ and transfer $t(w, \underline{v})$ to buy the good at price $p$, so agents who were receiving favorable transfers could purchase more of the good. Smoothing the transfers therefore affects purchasers because their effective budget changes.

Welfare-Maximization:

For the case of welfare-maximization, suppose that there is some leftover revenue. Then the planner could increase the transfers $T$ slightly and increase the price $p$ slightly and improve the overall utilitarian welfare. Therefore, the welfare-maximizing mechanism needs to satisfy the budget balance constraint, specifically: $(1 - F(p))\mathbb{E}[w] = F(p)T$.

Solving for $p$ and $T$ yields the conditions provided in the theorem.

Revenue-Maximization:

For the case of revenue-maximization, suppose that $T > 0$. Then, an alternate mechanism where $p$ is unchanged, and $T$ is changed to 0 generates strictly more welfare. Therefore, the revenue-maximizing mechanism $(p, T)$ is such that $T = 0$ and the supply constraint above binds.

Substituting in for $T$ and multiplying the supply equation by $p$ yields the conditions provided in the theorem. ■

**Proof of Theorem 5:**

Notice that the cutoff functions are determined by Equation 15, restated here in the case of non-independence of values and budgets.

$$\frac{d\mathcal{R}}{dS} = \frac{v^2 f(v,w)}{vf(v,w) + (1 - F(v,w))}$$

Now, because values and budgets are not-independent, there may be differential returns to transferring supply from some wealth levels to other wealth levels. It needs to be checked that these returns are increasing in wealth. Technically, it is necessary and sufficient that the above condition is increasing in wealth. Dividing the top and bottom of the equation through by $f(v,w)$ yields:

$$\frac{d\mathcal{R}}{dS} = \frac{v^2}{v + \frac{1-F(v,w)}{f(v,w)}} \tag{18}$$

By Assumption 3, I have that the last term of the denominator is weakly decreasing in $w$, and hence, the whole expression is weakly increasing in $w$. In addition, the derivative is always positive. Therefore, the revenue-maximizing mechanism is where all $S$ of the good is being sold, and where $\frac{d\mathcal{R}}{dS}$ is constant for every $w$. This is precisely what is desired, and completes the proof. ■

**Proof of Theorem 4:**

1). Allow every agent to report their type $\theta(v, w)$. Find the market clearing price $p$. This price may not be unique, in which case, take $p$ to be the maximal market clearing price. Market clearing works by everyone who declares a value $v > p$ receives $x(v, w) = w/p$ quantity of the good in exchange for a transfer $-px(v, w)$. Agents who declare a value $v < p$ give up their $S$ quantity of the good in exchange for a transfer of $pS$. Finally, agents who declare $v = p$ of which there may be a non-zero measure randomly receive or sell a proportionate share of the total leftover aggregate supply/demand so that markets clear. If a measure 0 set of agents report $v$ such that $v = p$, then to tie break, we say that all of these agents fully purchase the good.

Now, suppose that an agent made a report so that $(v, w) \neq \theta(v, w)$. If an agent overreports their value, it is either the case that they receive the same allocation as truthful revelation, or they receive the good at a price $p > v$. In either event, they weakly prefer to report their true type $v$. The same goes for underreports of values where agents may accidentally sell the good when they do not wish to.

As for misreports of wealth, an agent cannot overreport his wealth. He can underreport his wealth, but this strictly hurts him when he is a buyer and offers no benefits when he is a seller. Therefore, it is weakly dominant for every agent to report their wealth truthfully.

2). For the revenue-maximizing mechanism, it is dominant strategy for an agent to demand $w/p$ for all prices $p < v$ and to demand 0 for all prices $p > 0$. When the price is $p$ itself, an agent's demands are immaterial, so for tiebreaking, let him demand $w/p$ in this situation as well.

The planner looks at all reports, take the maximal market-clearing price and clears the market at this price. If agents report so that the any price $p' > p$ has insufficient demand and price $p$ has over-demand, then agents who demand at the price $p$ are only partially filled.

An agent has two types of misreports available. He can demand less when $p < v$ or more when $p > v$. The first type of misreport is undesirable because it means that the agent may purchase less of the good then he wishes. The second type of misreport is undesirable because it means that the agent may purchase the good when he does not wish to. Agents cannot demand more for $p < v$ because that would exceed their budget constraints and cannot demand less when $p > v$ because their demand is already 0 and negative demands are not possible. $\blacksquare$

**Proof of Theorem 7:**

From the Main Theorem, it is known that the optimal mechanism after a supply level $S$ is chosen is a linear pricing mechanism with uniform transfers $(p, T)$.

Welfare-Maximization:

As in the main theorem, the Budget Balance and Supply conditions can be assumed to hold with equality. Otherwise, either money or good can be uniformly distributed and this weakly improves welfare. Therefore, setting the supply demanded equal to the overall supply and letting transfers be maximal yields:

$$(1 - F(p))\frac{\int_W wg(w)dw + T}{p} = S \tag{19}$$

$$Sp - Sc = T \tag{20}$$

which in turn implies

$$(1 - F(p))\left(\int_W wg(w)dw + Sp - Sc\right) = Sp$$

and therefore

$$\frac{(1 - F(p))\int_W wg(w)dw}{F(p)p + (1 - F(p))c} = S \tag{21}$$

The Welfare Equation is:

$$\mathcal{W} = S\frac{\int_p^{\bar{v}} vf(v)dv}{1 - F(p)}$$

Substituting into the Welfare Equation yields:

$$\mathcal{W} = \frac{\int_p^{\bar{v}} vf(v)dv}{F(p)p + (1 - F(p))c}\int_W wg(w)dw$$

Now, the numerator is decreasing in $p$ and the denominator is increasing in $p$ (since it's derivative is $f(p)(p - c) + F(p)$), so overall welfare is decreasing in $p$. Similarly, if one looks at Equation (21), there too the numerator is decreasing in

$p$ and the denominator is increasing in $p$, so supply is inversely related to price $p$. Therefore, if welfare is decreasing in $p$, one wants the smallest price $p$ possible, which in turn implies the largest supply $S$ possible due to their inverse relationship. This occurs when transfers are equal to 0 in Equation (20).

So, at the welfare optimum, it is the case that $T = 0$, $p = c$ and $(1-F(p))\left(\int_W wg(w)dw\right) = Sp$.

Revenue-Maximization:

As in the main theorem, the Supply condition can be assumed to hold with equality, and transfers can be taken to be zero. This yields:

$$(1 - F(p))\frac{\int_W wg(w)dw}{p} = S \tag{22}$$

$$0 = T \tag{23}$$

The Revenue Equation is:

$$\mathcal{R} = Sp - Sc$$

Substituting into the Revenue Equation yields:

$$\mathcal{R} = \frac{(p-c)(1-F(p))}{p}\int_W wg(w)dw$$

Maximizing with respect to $p$ requires:

$$[(1 - F(p)) - (p - c)f(p)]\,p + (p - c)(1 - F(p)) = 0$$

Rearranging gives:

$$p(p - c) = \frac{c(1 - F(p))}{f(p)}$$

Restricting attention to $p > c$, the left hand side (LHS) is increasing in $p$ and the right hand side (RHS) is decreasing in $p$. Furthermore the LHS > RHS when $p = \bar{v}$ and the LHS < RHS when $p = c$. Moreover, one knows that the First Order

Condition is sufficient because $\mathcal{R} = 0$ when $p = c$ or $\bar{v}$ and $\mathcal{R} > 0$ for intermediate $p \in (c, \bar{v})$. ■

### A Nonexample

In this subsection, I provide an example where the welfare maximizing mechanism is not a linear mechanism. This will be a "counterexample" to the consequence of my theorem in the sense that my theorem points to the optimality of linear mechanisms. Of course, it is not a "counterexample" to the theorem as I consider a situation where the antecedents of the theorem do not hold. Specifically, the assumptions on agents' valuations that I make in the body of the paper will not hold in the following example.

Recall that, by a linear mechanism, I mean that there is a per-unit price $p$ and to purchase a quantity of the good $x$, the price is $px$. In addition, my main result of the optimality of linear mechanisms applies to a either a single uniform budget constraint or individual unobservable budget constraints.

Therefore, for simplicity, I will show a "counterexample" in the single uniform budget setting where a linear mechanism is not optimal. In this case, a linear mechanism has the additional feature that it is a take-it-or-leave-it mechanism. Specifically, any linear mechanism has a threshold value. Above this threshold, every agent receives the same quantity for the same transfer, and below this threshold, no agent purchases the good.

I prove the non-optimality of a take-it-or-leave-it mechanism in this setting by demonstrating a mechanism that dominates it.

Consider the following value distribution:

$$f(x) = \begin{cases} 1/12 & \text{if } 1 \leq x \leq 2 \\ 1/3 & \text{if } 2 < x \leq 3 \\ 7/12 & \text{if } 3 < x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

The above distribution has three steps from 1 to 4. Suppose that $S = 11/12$ and $w = 2$ and consider a mechanism that sells one unit of the good at the price 2. This is a linear price mechanism for a single wealth level and will generate welfare:[7]

[7] The welfare calculation is performed by looking at each of the populations and multiplying quantity (=1) times density (= 1/3 or 7/12) times average value (= 5/2 or 7/2).

$$\mathcal{W}_1 = \frac{1}{3} \cdot \frac{5}{2} + \frac{7}{12} \cdot \frac{7}{2} = \frac{23}{12} = 1.9333$$

On the other hand, consider a mechanism where the planner sells the good according to:

$$x(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 32/29 & \text{if } 3 < x \leq 4 \end{cases} \qquad t(v) = \begin{cases} 19/29 & \text{if } 1 \leq x \leq 3 \\ 2 & \text{if } 3 < x \leq 4 \end{cases}$$

The above I call a two-step mechanism since there are two different possible allocations and the above mechanism generates welfare

$$\mathcal{W}_2 = \frac{19}{29} \cdot \frac{1}{12} \cdot \frac{3}{2} + \frac{19}{29} \cdot \frac{1}{3} \cdot \frac{5}{2} + \frac{32}{29} \cdot \frac{7}{12} \cdot \frac{7}{2} = 2.881$$

The two-step mechanism generates less revenue and more welfare than the one-step mechanism. The additional revenue from the one-step mechanism can be redistributed to the population, so that its welfare can be improved by finding a higher linear price at which trade can take place. Doing so would then make the welfare comparison between the two above mechanisms unclear. However, such a redistributional improvement can be prevented by adding a large population of agents with valuations between 0 and 1. These agents will then absorb most of the cash distributions. I show how this is done in the subsequent paragraphs.

Define a measure $g$ as:

$$g(x) = \begin{cases} m & \text{if } 0 \leq x < 1 \\ f(x) & \text{otherwise} \end{cases}$$

The above is a measure and not a distribution because the integral of $g$ is equal to $m+1$ and not 1. Now, I will compare two mechanisms based upon the previously defined ones. The first is the one-step mechanism as defined before, with all of the money taken in, redistributed to the population so that a higher per-unit price can be established. This is the welfare-maximizing one-step mechanism in the class of all one-step mechanisms. The other is the two-step mechanism from before with no cash distributions. Therefore, it is clear that $0 = \mathcal{R}_1(m) < \mathcal{R}_2(m)$.

The important point is that as $m \to \infty$ the additional welfare value of the extra revenue that the one-step function generates converges to 0. Specifically, consider if all money that is received is redistributed, then, the one-step mechanism implies a cutoff price $p$ where $(1 - F(p))w = Sp\frac{F(p)+m}{1+m}$. Moreover, recall that $(1 - F(2))w = S \cdot 2$. So, as $m \to \infty$, it is the case that $\frac{F(p)+m}{1+m} \to 1$ and therefore the threshold price $p \to 2$. This implies $\mathcal{W}_1(m) \to \mathcal{W}_1 < \mathcal{W}_2 = \mathcal{W}_2(m)$.

So, for large enough $m$, it is the case that $\mathcal{W}_1(m) < \mathcal{W}_2(m)$. While $g$ is not a distribution, one can consider a rescaling that is, specifically, let $h(x) = \frac{g(x)}{m+1}$ and supply be $\frac{S}{m+1}$. Then the above allocation mechanisms are still applicable, but generate welfare $\frac{\mathcal{W}_1(m)}{m+1}$ and $\frac{\mathcal{W}_2(m)}{m+1}$ respectively. Therefore, it is the case that the 2-step mechanism generates strictly higher revenue and welfare then the 1-step mechanism for the distribution $h$ and supply $\frac{S}{m+1}$. $\blacksquare$