# Endogenous Network Dynamics

Frank H. Page, Jr.[1]
Department of Economics
Indiana University
Bloomington, IN 47405
fpage@indiana.edu

Myrna H. Wooders[2]
Department of Economics
Vanderbilt University
Nashville, TN 37235
m.wooders@vanderbilt.edu

October 23, 2009[3]

[1]Also, Centre d'Economie de la Sorbonne, Universite Paris 1, Pantheon-Sorbonne.
[2]Also, Department of Economics, University of Warwick, Coventry CV4 7AL, UK.
[3]This paper was begun while Page and Wooders were visiting CES-CERMSEM at the University of Paris 1 in June 2006. The authors thank CES-CERMSEM and Paris 1 for their hospitality. The authors thank Matt Jackson, J.-J. Herings, Luca Merlino, Jacomo Corbo, and the participants in various conferences in 2009 for many insightful comments: CTN Workshop (Maastricht, January, 2009), PET09 (Galway, June), SAET09 (Ischia, July), the Sorbonne Workshop on Economic Theory (Paris, June, 2009), and the DIMES International Conference (Paris, June, 2009). ENDPW31L.tex. URLs: http://mypage.iu.edu/~fpage/, http://www.myrnawooders.com.

**Abstract**

We model the structure and strategy of social interactions prevailing at any point in time as a directed network and we address the following open question in the theory of social and economic network formation: given the rules of network and coalition formation, the preferences of individuals over networks, the strategic behavior of coalitions in forming networks, and the trembles of nature, what network and coalitional dynamics are likely to emergence and persist. Our main contributions are (i) to formulate the problem of network and coalition formation as a dynamic, stochastic game, (ii) to show that this game possesses a pure stationary equilibrium (in pure network and coalition formation strategies), (iii) to show that, together with the trembles of nature, this stationary equilibrium determines an equilibrium Markov process of network and coalition formation, and (iv) to show that this endogenous process possesses a *finite*, nonempty set of ergodic measures, and generates a *finite*, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction. Moreover, we extend to the setting of endogenous Markov dynamics the notions of path dominance core (Page-Wooders, 2009) and pairwise stability (Jackson-Wolinsky, 1996), and we show that in order for any network-coalition pair to emerge and persist, it is necessary that the pair reside in one of finitely many basins of attraction. As an example, we consider a dynamic game of club network formation and we show that, even if multiple club memberships are allowed, the only club networks which emerge and persist are those where players are members of a single club. The results we obtain here for endogenous network dynamics and stochastic basins of attraction are the dynamic analogs of our earlier results on endogenous network formation and strategic basins of attraction in static, abstract games of network formation (Page and Wooders, 2009), and build on the seminal contributions of Jackson and Watts (2002), Konishi and Ray (2003), and Dutta, Ghosal, and Ray (2005).

KEYWORDS: endogenous network dynamics, dynamic stochastic games of network formation, equilibrium Markov process of network formation, basins of attraction, Harris decomposition, ergodic probability measures, dynamic path dominance core, dynamic pairwise stability.

JEL Classifications: A14, C71, C72

# 1 Introduction

## 1.1 Overview

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the "with whom") and the strategy ("the how") of interactions change. Our objectives here are to model the structure *and* strategy of interactions prevailing at any point in time as a directed network and to address the following open question in the theory of social and economic network formation: given the rules of network formation, the preferences of individuals over networks, the strategic behavior of coalitions in forming networks, and the trembles of nature, what *network and coalitional dynamics* are likely to emergence and persist. Thus, we propose to study the emergence of endogenous network and coalitional dynamics from strategic behavior and the randomness in nature.

Our main contributions are (i) to formulate the problem of network formation as a dynamic, stochastic game, (ii) to show that this game possesses an equilibrium in pure stationary network and coalition formation strategies, (iii) to show that, together with the trembles of nature, these equilibrium strategies determine an equilibrium Markov process of network and coalition formation which respects the rules of network formation and the preferences of individuals, and (iv) to show that, although uncountably many networks may form, this equilibrium Markov process generates a *finite*, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction, and possesses a *finite*, nonempty set of ergodic measures.

In our prior work on static abstract games of network formation (Page and Wooders, 2009a), we have shown that, given the rules of network formation and the preferences of individuals, these games possess *strategic basins of attraction* and these contain all networks that are likely to emerge and persist as the game unfolds. Moreover, we have shown that when any one of these strategic basins contains only one network, then the game possesses a network (i.e., the single network contained in the singleton basin) that is stable against all coalitional network deviation strategies - and thus the game has a nonempty *path dominance core*. Finally, we have shown in Page-Wooders (2009a) that depending on how we specialize the rules of network formation and the dominance relation over networks, any network contained in the path dominance core is pairwise stable (Jackson-Wolinsky, 1996), strongly stable (Jackson-van den Nouweland, 2005), Nash (Bala-Goyal, 2000), or consistent (Chwe, 1994).

We show here that there are many parallels between the static abstract game formulation and our prior results for static games and the results we obtain here for our Markov dynamic game formulation. This is suggested already by the seminal paper by Jackson and Watts (2002) on the evolution of networks. Jackson and Watts present a basic theory (and to our knowledge the first theory) of stochastic dynamic network formation over a finite set of linking networks governed by a Markov chain generated by the myopic strategic behavior of players (following the Jackson-Wolinsky rules of network formation) and the trembles of nature. Their model builds on the earlier, nonstochastic model of dynamic network formation due to Watts (2001) - as far as we know, the first models of network dynamics (see also Skyrms and Pemantle, 2000)). By considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain, they show that any pairwise stable network is necessarily contained in the support of an invariant measure - that is, in the support of a probability measure that places all its support on sets of networks likely to form in the long run. We show here that similar conclusions can be reached for directed networks with many arc types governed by arbitrary network formation rules.

In a general Markov game setting, with farsighted players, what precisely does it mean for a network to be pairwise stable - or stable in any sense? For example, if the state space of networks is large, then the endogenous Markov process of network formation is likely to have many invariant measures - and in fact many ergodic probability measures (i.e., measures that place all their probability mass on a single absorbing set). Which absorbing set contains networks stable in the sense of pairwise stability, or strong stability, or Nash stability? These are some of the questions we answer here in our study of endogenous network dynamics.

We conjecture that in any reasonable dynamic, stochastic model of network formation the endogenously determined Markov process of network and coalition formation will possess ergodic probability measures and generate basins of attraction. We show here that in fact the endogenous Markov process possesses only finitely many ergodic measures and basins of attraction. This endogenous finiteness property of equilibrium has serious implications for empirical work on networks. In particular, since nature does not afford the empirical observer multiple observations across states but rather only multiple observations across time, the fact that only finitely many long run equilibrium sets are possible and more importantly, the fact that on these sets (i.e., on these basins of attraction) state averages are equal to time averages gives meaning and significance to time series observations which seek to infer the long run equilibrium network. Moreover, to the extent that networks can truly represent various social and economic interactions, our understanding of how and why the network formation process moves toward or away from any particular basin can potentially shed new light on the persistence or transience of many social and economic conditions. For example, how and why does a particular path of entrepreneurial and scientific interactions carry an economy beyond a tipping point and onto a path of economic growth driven by a particular industry - and why might it fail to do so? How and why does a particular path of product line-nonlinear pricing schedule configurations lead a strategically competitive industry to become more concentrated - or fade? These are some of the applied questions which hopefully can be addressed using a model of endogenous network dynamics.

In order to illustrate some our fundamental ideas concerning dynamic stochastic games of network formation and stability of endogenous network dynamics, we consider a dynamic stochastic game of club network formation. We show that, in general, these games possess an equilibrium in pure stationary bang-bang strategies. This conclusion has the interesting implication that, even if multiple club memberships are allowed, the only club networks which emerge and persist under the endogenous network dynamic are those where players are members of a single club.

## 1.2 Endogenous Network Dynamics

Our approach to endogenous dynamics is motivated by the observation that the stochastic process governing network and coalition formation through time is determined not only by nature's randomness (or nature's trembles) through time - as envisioned in random graph theoretic approaches - *but also* by the strategic behavior of individuals and coalitions through time in attempting to influence the networks and coalitions that emerge under the prevailing rules of network formation and the trembles of nature. Thus, here we will develop a theory of endogenous network and coalitional dynamics that brings together elements of random graph theory and game theory in a dynamic stochastic game model of network and coalition formation. While dynamic stochastic games have been used elsewhere in economics (see, for example, Amir (1991, 1996), Amir and Lambson (2003), and Chakrabarti (1999, 2008), Duffie, Geanakoplos, Mas-Colell, and McLennan (1994), Mertens and Parthasarathy (1987, 1991), Nowak (2003, 2007)), their application to the analysis of the evolution of social and economic networks is relatively new.

Our plan of analysis has two parts. In part (1) we will construct our dynamic game model of network *and* coalition formation, and then show that this game has an equilibrium in pure stationary strategies. Our model has six primitives consisting of the following: (i) a feasible set of directed networks representing all possible configurations of social or economic interactions, (ii) a feasible set of coalitions allowed to form under the rules of network formation for the purpose of proposing alternative networks, (iii) a state space consisting of feasible network-coalition pairs, (iv) a set of players and player constraint correspondences specifying for each player and in each state the set of feasible alternative networks that a player can propose under the rules of network formation as a member of the current or status quo coalition - and as a nonmember, (v) a set of player discount rates and payoff functions defined on the graph of players' product constraint correspondence, and (vi) a stochastic law of motion. This stochastic law of motion represents nature and specifies the probability with which each possible new status quo network-coalition (i.e., new state) might emerge as a function of the status quo network-coalition pair (i.e., the current state) and the profile of player-proposed new status quo networks (i.e., the current action profile). Using these primitives, we will construct a discounted stochastic game model of network formation, and then show that this game possesses an equilibrium in pure stationary network proposal strategies.

Finally, in part (1) we will show that, together with the stochastic law of motion, these stationary strategies determine an equilibrium Markov process of network and coalition formation. More importantly, we will be able to conclude via classical results due to Blackwell (1965) (also, Himmelberg, Parthasarathy, and vanVleck (1976)), Nowak and Raghavan (1992), and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994)) that these equilibrium pure stationary strategies are optimal against player defections to *any other history-dependent* network proposal strategies - thus showing that our decision to focus on pure stationary strategies is well-founded.

In part (2), we will analyze the stability properties of the endogenous Markov process of network and coalition formation. In particular, using methods of stability analysis essentially due to Nummelin (1984) and Meyn and Tweedie (1993) - and based on the profound work of Doeblin (1937, 1940) - we will show that the equilibrium Markov process of network and coalition formation possesses ergodic probability measures and generates basins of attraction. We will then study in some detail the number and structure of these basins of attraction as well as the structure of set of invariant probability measures. More importantly, we will show that, in a state space with uncountably many networks, the equilibrium process possesses only *finitely many* ergodic measures and basins of attraction. Also, in part (2) we will introduce the notions of dynamic stability and consistency and using these notions extend the definitions of path dominance core and pairwise stability to the dynamic Markov setting developed here. We will then show that networks that are stable with respect to these notions of stability must necessarily reside in the basins of attraction generated by the endogenous network dynamic. To illustrate, we show that in the case of dynamic club network formation, under the endogenous network dynamic generated by the equilibrium pure stationary bang-bang strategies, all dynamically consistent networks are single-membership networks. Thus, we will show that dynamic club formation naturally leads to an endogenous bang-bang network dynamic.

### 1.3 Related Literature

To our knowledge, the first paper to study endogenous dynamics in a related model is the paper by Konishi and Ray (2003) on dynamic coalition formation. The primitives of their model consist of (i) a finite set of outcomes (possibly a finite set of networks), (ii) a set of coalitional constraint correspondences specifying for each coalition and each status quo outcome, the set of new outcomes a coalition might bring about if allowed to do so, and (iii) a discount rate and set of player payoff functions defined on the set of all outcomes. Konishi and Ray show that their model possesses an equilibrium process of coalition formation, that is, a stochastic law of motion governing movement from one outcome to another such that (a) if a move from one outcome to another takes place with positive probability, then for some coalition this move makes sense in that no coalition member is made worse off by the move and no further move makes all coalition members better off, and (b) if for a given outcome there is another outcome making all members of some coalition better off and no further outcome makes this coalition even better off, then a move to another outcome takes place with probability 1 (i.e., the probability of standing still at the given outcome is zero). The notion of a player being better off is reckoned in terms of a player's valuation function implied by the maximization of the expected discounted stream of payoffs with respect to the stochastic law of motion. Stated loosely, then, Konishi and Ray show that for their model there is a law of motion which generates coalitionally Pareto improving moves from one outcome to another (i.e., in our case it would be from one network to another).

Our model differs from the model of Konishi and Ray in several respects. First, in our model movements from one network (outcome) to another are largely determined by the strategic behavior of individuals within feasible coalitions. In Konishi and Ray, coalitions are passive and strategic behavior plays no part in determining the movement from one outcome to another. They simply show that there model is consistent with there being a law of motion which moves the outcome along in a coalitionally Pareto improving way. In this sense - i.e., in the sense that movement is nonstrategic - their model is more closely related to random graph theoretic models of network dynamics. In our model, equilibrium strategic behavior, together with natures trembles, are central to determining equilibrium network dynamics.

Second, whereas Konishi and Ray, for technical reasons, restrict attention to a finite set of outcomes (in our model, a finite set of networks), we allow for uncountably many networks - this to allow for consideration of networks with a large number of nodes or networks with uncountably many arc types. This generalization is more than a technical nicety. In order to capture the myriad and potentially complex nature of interactions between players (say for example in a stock market or in a contracting game with multiple principals and multiple agents) we must allow there to be uncountably many possible types of interactions. In our model the set of potential interactions are represented by a set of arc types with each arc type (or arc label) representing a particular type of interaction (or connection) between nodes in a directed network. Thus, because we allow for uncountably many arc types in describing the possibly finite number of interactions between nodes, in our model there are uncountably many possible networks (or outcomes in the language of Konishi and Ray). Moreover, in order to model large networks (i.e., networks with many nodes), in our model we can allow there to be infinitely many nodes - although here we focus exclusively on the finite nodes case. Third, while Konishi and Ray restrict attention at the outset to Markov laws of motion, we will show that our strategically determined equilibrium Markov process of network and coalition formation is robust against all possible alternative dynamics induced by history-dependent types of strategic behavior. Thus, at least for the class of Konishi-Ray types of models, we will show that Markov laws of motion are stable and robust with respect to other forms of history-dependent laws of motion.[1]

Finally, whereas Konishi and Ray focus on the existence of an equilibrium process of coalition formation, here we will not only establish the existence of a strategically determined equilibrium process of network and coalition formation, but also we will show that this process possesses a nonempty set of ergodic measures and generates basins of attraction.

Dutta, Ghosal, and Ray (2005) extend the Konishi-Ray type model to consider a particular form of strategic behavior (i.e., strategic behavior governed by a particular set of network formation rules) in a dynamic game of network formation over a finite set of undirected linking networks (rather than directed networks). They show that their model has a Nash equilibrium and identify conditions under which efficiency can

[1] By a Markov law of motion we mean a stochastic law of motion where probabilistic movements from one outcome or network to another depend only on the current outcome rather than on some history of outcomes.

be sustained in equilibrium - thus, continuing in a dynamic setting the seminal work of Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) on equilibrium and efficiency. Here our focus is on equilibrium and stability rather than equilibrium and efficiency and our analysis is carried out in a dynamic, stochastic game model of network and coalition formation, admitting all forms of network formation rules, over an uncountable set of directed networks. While Dutta, Ghosal, and Ray restrict attention to Markov strategies and show that there is an equilibrium in this class of network formation strategies, here, we show that there is an equilibrium in the class of all pure stationary network and coalition proposal strategies and that this type of equilibrium is optimal relative to the class of *all* history-dependent, probabilistic network formation strategies. Moreover, as mentioned above, we show that in general, the resulting equilibrium Markov network and coalitional dynamics possess ergodic measures and generate network and coalitional basin of attraction.

We view the starting point of our research to be the pioneering work of Jackson and Watts (2002) already discussed briefly above. Our model of endogenous network and coalitional dynamics extends their work on stochastic network dynamics in several respects. First, in our model players behave farsightedly in attempting to influence the path of network and coalition formation - farsighted in the sense of dynamic programing (e.g., Dutta, Ghosal, and Ray (2005))[2]. Moreover, in our model the game is played over a (possibly) uncountable collection of directed networks under general rules of network formation which include not only the Jackson-Wolinsky rules, but also other more complex rules. In our model the law of motion is such that the trembles of nature are Markovian rather than i.i.d. as in Jackson and Watt, and are functions of the current state and the current profile of network and coalition proposals by players. Extending the notion of pairwise stability to a dynamic setting, one of the benchmarks for our research is to show that in a Markov model of network and coalition formation, if a network is dynamically pairwise stable, then in order to persist, it must be contained in one of finitely many basins of attraction, and therefore, contained in the support of an ergodic probability measure.

[2] See Chwe (1994), Page, Wooders, and Kamat (2005), and Page and Wooders (2005) for notions of farsighted behavior in static, abstract games.

# 2 Primitives

## 2.1 The Space of Directed Networks

### 2.1.1 Definition

We begin by giving the formal definition of a directed network. Let $N$ be a finite set of nodes with typical elements denoted by $i$ and $j$ equipped with the discrete metric $d_N$, and let $A$ be a metric space of arcs with typical element denoted by $a$ equipped with metric $d_A$.[3] Arcs represent types or attributes of connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.

**Definition 1** (*Directed Networks*)

*Given node set $N$ and arc set $A$, a directed network, $G$, is a nonempty, closed subset of $A \times (N \times N)$. The collection of all directed networks is denoted by $P_f(A \times (N \times N))$.*

A directed network $G \in P_f(A \times (N \times N))$ consists of a closed set of ordered pairs of the form $(a, (i, j))$ where $a$ is an arc type and $(i, j)$ is an ordered pair of nodes. We will refer to any ordered pair $(a, (i, j)) \in G$ as a *connection* in network $G$. Thus, a network $G$ is a nonempty closed set of connections specifying how the nodes in $N$ are connected by the arcs in $A$. In a directed network the node order matters. In particular, $(a, (i, j)) \in G$ means that nodes $i$ and $j$ are connected by a type $a$ arc *from* node $i$ *to* node $j$.

Under our definition of a directed network, we allow an arc to go from a given node back to that given node (i.e., *loops* are allowed).[4] Also, under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow a particular arc $a$ to go *from* a node $i$ *to* a node $j$ multiple times.

The following notation is useful in describing networks. Given directed network $G \in P_f(A \times (N \times N))$, let

$$\left. \begin{array}{c} G(a) := \{(i, j) \in N \times N : (a, (i, j)) \in G\}, \\ \text{and} \\ G(ij) := \{a \in A : (a, (i, j)) \in G\}. \end{array} \right\} \quad (1)$$

Thus, in network $G$,

---

[3] The discrete metric $d_N$ is given by

$$d_N(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

[4] By allowing loops we can represent a network having no connections between distinct nodes as a network consisting entirely of loops at each node.

$G(a)$ is the *set of node pairs* connected by arc $a$,
and
$G(ij)$ is the *set of arcs* from node $i$ to node $j$.

If for some arc $a \in A$, $G(a)$ is empty, then arc $a$ is not used in network $G$. Also, if for some node $i \in N$, $G(ij)$ and $G(ji)$ are empty for all $j$, then node $i$ is isolated.

#### 2.1.2 Regular Directed Networks

We will assume throughout that the set of all possible arc types $A$ is given by $R^L$. Thus each arc type $a \in R^L$ is a real vector of attributes with $L$ components and each component represents a particular attribute. Given a connection $(a, (i, j))$, where

$$a = (a_1, \ldots, a_L) \in R^L,$$

we will agree that if a particular component, say the $k^{th}$ component, of the arc type vector $a$ is zero (i.e., $a_k = 0$), then the attribute represented by that component is inactive in the connection $(a, (i, j))$. Moreover, we will agree that if all components of the arc type $a$ in the connection $(a, (i, j))$ are zero (i.e., if $a$ is the zero vector in $R^L$), then the connection $(a, (i, j)) = (0, (i, j))$, while present, is inactive.

Our focus here will be on directed networks where all connections are uniquely present, and thus on networks where all node pairs are connected via one and only one connection. We will call such networks *regular networks*. Using the notation introduced in expression (1), network $G$ is regular if and only if $|G(ij)| = 1$ for all node pairs $(i, j)$.[5] Stated formally, the set of all regular networks, $\mathbb{G}_R$, is given by

$$\mathbb{G}_R := \left\{ G \in P_f(R^L \times (N \times N)) : \forall (i, j) \in N \times N, \ |G(ij)| = 1 \right\}. \tag{2}$$

The set of regular networks $\mathbb{G}_R$ is a vector space with zero element given by the zero network, $G_0$. In the zero network all connections are inactive (i.e., for all $(i, j)$, $(a, (i, j)) \in G_0$ if and only if $a = 0$). Addition and scalar multiplication in $\mathbb{G}_R$ are defined as follows:

(1) (Addition) Given regular networks $G$ and $G'$, network $\overline{G} = G + G'$ is the unique set of connections

$$(\overline{a}_{ij}, (i, j)) := (a_{ij} + a'_{ij}, (i, j)) := (a_{ij}, (i, j)) + (a'_{ij}, (i, j)), \tag{3}$$

where for each node pair $(i, j) \in N \times N$ , $(a_{ij}, (i, j))$ and $(a'_{ij}, (i, j))$ are the unique connections between nodes $i$ and $j$ in networks $G$ and $G'$ respectively.

[5] $|G(ij)|$ denotes the cardinality of the set of arcs from node $i$ to node $j$. If $G(ij) = \emptyset$, then $|G(ij)| = 0$. Viewing $G(\cdot)$ as a set-valued mapping from node pairs, $N \times N$ into subsets of arcs, in a regular network $G$, $G(\cdot)$ is nonempty and single-valued, and hence, $G(\cdot)$ is an arc-valued function with domain $N \times N$.

(2) (scalar multiplication by reals) Given regular networks $G$ and real number $t$, network $\widetilde{G} = tG$ is the unique set of connections

$$(\widetilde{a}_{ij}, (i,j)) := (ta_{ij}, (i,j)) := t(a_{ij}, (i,j)), \tag{4}$$

where for each node pair $(i,j) \in N \times N$ , $(a_{ij}, (i,j))$ is the unique connection between nodes $i$ and $j$ in networks $G$.

Given regular networks $G$ and $G'$, the notion of a convex combination of $G$ and $G'$,

$$\widehat{G} := tG + (1-t)G', \tag{5}$$

falls into place as the unique set of connections

$$(\widehat{a}_{ij}, (i,j)) := (ta_{ij} + (1-t)a'_{ij}, (i,j)) := t(a_{ij}, (i,j)) + (1-t)(a'_{ij}, (i,j))$$

where for each node pair $(i,j) \in N \times N$ , $(a_{ij}, (i,j))$ and $(a'_{ij}, (i,j))$ are the unique connections between nodes $i$ and $j$ in networks $G$ and $G'$ respectively.

Each regular network $G$ also has a unique matrix representation $[G]$ given by

$$[G] := \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} := \begin{pmatrix} \cdots a^1 \cdots \\ \vdots \\ \cdots a^i \cdots \\ \vdots \\ \cdots a^n \cdots \end{pmatrix}$$

where for each node $i \in N$, $a^i := (a_{i1}, \ldots, a_{in})$ is the $i^{th}$ row of $[G]$, also denoted by $[G]^i$, and where for each $(i,j) \in N \times N$, $a_{ij} \in R^L$ is the $ij^{th}$ entry in matrix $[G]$,also denoted by $[G]_{ij}$. Given regular network $G$, $a_{ij} = [G]_{ij}$ if and only if $(a_{ij}, (i,j))$ is the unique connection between nodes $i$ and $j$ in network $G$. Note that if regular network $\widehat{G}$ is a convex combination of regular networks $G$ and $G'$, given by $tG + (1-t)G'$, then the matrix $t[G] + (1-t)[G']$ is the matrix representation, $[\widehat{G}]$, of $\widehat{G}$.

We will equip the vector space of regular networks $\mathbb{G}_R$ with the *variation norm* given by

$$\|G - G'\|_V := \sum_i \sum_j \|a_{ij} - a'_{ij}\|_E \, , \tag{6}$$

where $\|\cdot\|_E$ denotes the Euclidean distance norm in $R^L$.[6] Using this norm we can define a notion of closedness in the set of regular networks. In particular, a subset of networks $\mathbb{G}$ contained in $\mathbb{G}_R$ is closed if and only if the limit of any sequence of networks $\{G^n\}_n$ in $\mathbb{G}$ is contained in $\mathbb{G}$. Thus, $\mathbb{G}$ is closed if and only if $\{G^n\}_n \subseteq \mathbb{G}$ and $\|G^n - G\|_V \to 0$ implies that $G \in \mathbb{G}$. The subset $\mathbb{G}$ is bounded if and only if there is a positive real number $M$ such that $\|G\|_V \leq M$ for all $G \in \mathbb{G}$. Finally, $\mathbb{G}$ is compact if and only if it is closed and bounded.

[6]Thus, the vector space of regular networks with arc set $R^L$ is a complete, separable, normed linear space, or a Banach space.

### 2.1.3 Feasible Sets of Regular Directed Networks

In formulating our game of network and coalition formation, it will often be useful to restrict attention to a particular *feasible* subset of the regular networks.

**Definition 2** (*Feasible Networks*)

*A subset of networks $\mathbb{G}$ in $\mathbb{G}_R$ is a candidate for feasible set provided it is nonempty, compact, and convex.*

### 2.1.4 Feasible Sets: Examples and Comments

In the examples to follow we will exhibit several types of restrictions on the set of networks $\mathbb{G}_R$ leading to feasible sets which are useful in applications. Usually these restrictions take the form of feasibility constraints on the set of arc types which can be used in making connections between node pairs. In particular, for each node pair $(i, j) \in N \times N$, let $A(ij)$ be a nonempty, compact, convex subset of $R^L$ containing the feasible set of arc types which can be used in making a connection from node $i$ to node $j$. Consider set of networks is given by

$$\mathbb{G} := \{G \in \mathbb{G}_R : \forall (i, j) \in N \times N,\ G(ij) \subseteq A(ij)\ \}. \tag{7}$$

Then $\mathbb{G}$ is a candidate for feasible set. We have the following examples.

(1) (Trading Networks) Suppose that nodes represent traders and that for each pair of potential trading partners $(i, j) \in N \times N$, the feasible set of arc types $A(ij)$ is given by the closed, bounded, convex set

$$A(ij) := \underbrace{[0, \overline{p}_{bij}] \times [0, \overline{q}_{bij}]}_{\text{buy price-quantity}} \times \underbrace{[0, \overline{p}_{sij}] \times [0, \overline{q}_{sij}]}_{\text{sell price-quantity}} \subset R^4.$$

The connection $(a_{ij}, (i, j))$ in trading network $G \in \mathbb{G}$, with $a_{ij} = ((p_{bij}, q_{bij}), (0, 0)) \in A(ij)$, means that in network $G$ trader $i$ offers to buy up to $q_{bij}$ units from trader $j$ for per unit price $p_{bij}$, while $(a_{ji}, (j, i)) \in G$ with $a_{ji} = ((0, 0), (p_{sji}, q_{sji})) \in A(ji)$ means that trader $j$ offers to sell up to $q_{sji}$ units to trader $i$ for per unit price $p_{sji}$.

If, for example, connection $(a_{ij}, (i, j))$ is contained in trading network $G$, with $a_{ij} = ((p_{bij}, q_{bij}), (p_{sij}, q_{sij})) \in A(ij)$, then trader $i$ is simultaneously offering to buy up to $q_{bij}$ units from trader $j$ for per unit price $p_{bij}$ and to sell up to $q_{sij}$ units to trader $j$ for per unit price $p_{sij}$. If trader $i's$ offer to trader $j$ is such that $p_{bij} > p_{sij}$, then trader $j$ has a riskless arbitrage opportunity.[7] In particular, trader $j$ can sell a unit to trader $i$ for $p_{bij}$ and then immediately buy it back for less - for $p_{sij}$. Thus after the transaction, trader $j$ would still have a unit but also would have an arbitrage profit of $p_{bij} - p_{sij} > 0$. Thus, for the connection $a_{ij} = ((p_{bij}, q_{bij}), (p_{sij}, q_{sij}))$ to be *no-arbitrage*, it must be true in general that

$$p_{bij} \leq p_{sij}.$$

[7] Trader $i$ is providing trader $j$ with a free lunch.

The difference, $p_{sij} - p_{bij}$ is trader $i's$ bid ask spread. Note also that if connections

$$(a_{ij}, (i,j)) \text{ with } a_{ij} = ((p_{bij}, q_{bij}), (0,0)) \in A(ij)$$
and
$$(a_{ji}, (j,i)) \text{ with } a_{ji} = ((0,0), (p_{sji}, q_{sji})) \in A(ji),$$

are present in trading network $G$, then a trade will be executed if $p_{bij} \geq p_{sji}$.

(2) (Arc Allocation Networks) Suppose that $A = R$ (the real numbers) and that for each node $i \in N$,

$$A(ij) = [b_i, s_i] \text{ for all } j \in N,$$

for real numbers $b_i$ and $s_i$, $b_i \leq s_i$. Given real number $L_i$ and $H_i$, $L_i \leq H_i$, consider the set of networks given by

$$\mathbb{G} = \{G \in \mathbb{G}_R : \forall (i,j) \in N \times N,\ G(ij) \subseteq A(ij) \text{ and } L_i \leq \sum_{j \in N} a_{ij} \leq H_i\}. \tag{8}$$

If the numbers $b_i$, $s_i$, $L_i$ and $H_i$ are such that $\mathbb{G}$ is nonempty, then $\mathbb{G}$, a compact, convex set of regular networks, is a candidate for feasible set.

Suppose further that for each node $i \in N$, $A(ij) = [0,1]$ and that $L_i = H_i = 1$ for all $i \in N$. Then the resulting set of networks, given by

$$\mathbb{G} = \{G \in \mathbb{G}_R : \forall (i,j) \in N \times N,\ G(ij) \subseteq [0,1] \text{ and } \sum_{j \in N} a_{ij} = 1\}, \tag{9}$$

is nonempty, compact, and convex set of regular networks consisting of Markov networks. If $G$ is a Markov network, then connections $(a_{ij}, (i,j))$ in $G$ have a probabilistic interpretation. In particular, because $G(ij) = \{a_{ij}\}$ and

$$a_{ij} \in [0,1] \text{ and } \sum_{j \in N} a_{ij} = 1,$$

the arc type $a_{ij}$ in connection $(a_{ij}, (i,j))$ can be thought of as the probability that individual $i$ initiates an interaction or a connection with $j$.

(3) (Club Networks and Time Allocations) An interesting class of regular networks are the regular club networks. Suppose the set of nodes is given by $N = I \cup C$ where $I$ is a finite set of individuals and $C$ is a finite set of clubs. The vector space of regular club networks $\mathbb{K}_R$ is then given by

$$\mathbb{K}_R := \left\{G \in P_f(R^L \times (I \times C)) : \forall (i,c) \in I \times C,\ |G(ic)| = 1\right\},$$

where $P_f(R^L \times (I \times C))$ denotes the collection of all nonempty, closed subsets of $R^L \times (I \times C)$. A connection $(a_{ic}, (i,c)) \in G$ means that in club network $G$ individual $i \in I$ is a member of club $c \in C$, and in club $c$, takes action $a_{ic} \in R^L$. Now consider the nonempty, compact, convex subset of regular club networks given by

$$\mathbb{K} = \{G \in \mathbb{K}_R : \forall (i,c) \in I \times C,\ G(ic) \subseteq [0,1] \text{ and } \sum_{c \in C} a_{ic} = 1\}.$$

Because $G(ic) = \{a_{ic}\}$ and

$$a_{ic} \in [0,1] \text{ and } \sum_{c \in C} a_{ic} = 1,$$

the connection $(a_{ic}, (i,c))$ in club network $G \in \mathbb{K}$ can be given the interpretation that in club network $G$, individual $i$ allocates $a_{ic}$ percent of his time to club $c$. We will return to this example later in the paper.

## 2.2 Players and Coalitions

We will make a distinction between the set of players (or decision makers) and the set of nodes. In particular, we will not assume that the set of players and the set of nodes are necessarily one and the same. For example, some nodes may be club locations while other nodes may be individuals who choose clubs as well as the actions they take as members of clubs (e.g., the set of individuals in example (3) above).

Because changing one network to another network very often involves groups of players acting in concert, coalitions will play a central role in our model. Let $D$ denote the set of players (a set not necessarily equal to $N$ the set of nodes) with typical element denoted by $d$ and let $P(D)$ denote the collection of all coalitions (i.e., nonempty subsets of $D$) with typical element denoted by $S$. We will denote by $m$ the cardinality of the set of players $D$ (i.e., $|D| = m$). Depending on the rules of network formation, it will often be useful to restrict attention to a particular feasible subset of coalitions.

**Definition 3** (*Feasible Coalitions*)

*Given finite player set $D$, a feasible set of coalitions is a nonempty subset $\mathcal{F}$ of the collection of all coalitions $P(D)$.*

### 2.2.1 Feasible Sets of Coalitions: Examples and Comments

(1) Suppose that the feasible set of coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| = 2\}.$$

Thus, all feasible coalitions consist of two players. The set $\mathcal{F}_2$ is an appropriate feasible set if, for example, the set of nodes and the set of players are one in the same (i.e., $D = N$) and the rules of network formation are bilateral. Under bilateral rules changing a connection (i.e., changing the arc type) requires the efforts of both players involved in the connection. For example, if the feasible set of coalitions is given by $\mathcal{F}_2$ and it is coalition $S's$ turn to move, where $S = \{i,j\} \in \mathcal{F}_2$, then any connections involving nodes $i$ and $j$ would be subject to change, and change would be the consequence of the efforts by players (nodes) $i$ and $j$ *and* the stochastic influences of nature.[8]

[8] The requirement that only players involved in a connection be involved in changing the connection is similar to (but not identical to) the Jackson-Wolinsky rules (Jackson-Wolinsky (1996)). However,

(2) Suppose that the feasible set of coalitions is given by

$$\mathcal{F}_1 = \{S \in P(D) : |S| = 1\}.$$

Thus, all feasible coalitions consist of one player. The set $\mathcal{F}_1$ is an appropriate feasible set if, for example, the set of nodes and the set of players are one in the same (i.e., $D = N$) and the rules of network formation are noncooperative (e.g., see Bala-Goyal (2000)). Under noncooperative rules, changing a connection (i.e., changing an arc) requires the efforts of the initiating player in the connection.[9] For example, if the feasible set of coalitions is given by $\mathcal{F}_1$ and it is coalition $S's$ turn to move, where $S = \{i\} \in \mathcal{F}_1$, then any connections initiated by player (node) $i$ would be subject to change, and change would be the consequence of efforts by player (node) $i$ and the stochastic influences of nature.

(3) Consider the following club network example which illustrates another aspect of the usefulness of making a distinction between nodes and players. First, suppose that the set of nodes is given by $N = I \cup C$, where $I$ is a finite set of individuals and $C$ is a finite set of clubs, and assume that the set of club networks is given by

$$\mathbb{K} = \{G \in \mathbb{K}_R : \forall (i, c) \in I \times C,\ G(ic) \subseteq A(ic)\},$$

where for each individual-club pair $(i, c) \in I \times C$, $A(ic) \subset R^L$ is the compact, convex subset of feasible arc types, including the zero vector, which can be used in making a connection from individual $i$ to club $c$.

Next, suppose that the set of players is given by $D = P(I)$. Thus, a player $d \in D$ is a group or coalition of individuals. Finally, assume that the feasible set of *player* coalitions is given by

$$\mathcal{F}_1 = \{S \in P(D) : |S| = 1\}.$$

Thus, each player coalition consists of 1 player, but each player is a group of individuals.

Finally, suppose as in Page and Wooders (2009b) that there is noncooperative free mobility. Thus, if it is coalition $S's$ turn to move and $S \in \mathcal{F}_1$ is given by $S = \{d\} = \{I'\}$ where player $d$ is the group of individuals $I' \in P(I)$, then any club connections involving any individual $i$ in $I'$ (i.e., any club activities carried out by any individual in group $I'$ in any club) would be subject to change, and any change would be the consequence of the efforts of individuals (nodes) in group $I'$ and the stochastic influences of nature.

under the Jackson-Wolinsky rules, a connection can be *removed* from a network if and only if *one or both* players involved in the connection agree to remove the connection (arc subtraction is unilateral), and a connection can be added to a network (or an exiting connection changed) if and only if both players involved in the connection agree to add or change the connection (arc addition or modification is bilateral). In our dynamic model here, while a connection can be made inactive through the efforts of the players involved in the connection *and nature*, it cannot be added or removed completely from the network (see Page (2009a) for a complete treatment of dynamic, stochastic games of network formation where such changes are allowed).

[9] Let $(a, (i, j))$ be a connection in network $G$ where the set of nodes is equal to the set of players. In the connection $(a, (i, j))$, player $i$ is the initiating player.

## 2.3 States, Feasible Actions, and Payoffs

### 2.3.1 States

We shall take as the state space the set $\Omega := (\mathbb{G}\times\mathcal{F})$ of all feasible network-coalition pairs. Each state in $(\mathbb{G}\times\mathcal{F})$ has the following interpretation: if $(G, S)$ is the current state, then $G$ is the current status quo network of social interactions and it is coalition $S$'s turn to propose a new network. Thus, if $(G, S)$ is the current state, then nature will accept substantive proposals by members of coalition $S$ concerning what the new status quo network should be. We will refer to the coalition whose turn it is to move as the status quo coalition.

Equipping $\mathcal{F}$ with the discrete metric $d_{\mathcal{F}}$ (i.e., $d_{\mathcal{F}}(S', S) = 0$ if $S' = S$, and $d_{\mathcal{F}}(S', S) = 1$ if $S' \neq S$), the state space $(\mathbb{G}\times\mathcal{F})$ is a compact metric space under the metric $d_\Omega$ given by

$$d_\Omega((G', S'), (G, S)) := \left\|G' - G\right\|_v + d_{\mathcal{F}}(S', S).$$

Letting $B(\Omega) := B(\mathbb{G} \times \mathcal{F})$ be the Borel $\sigma$-field generated by the metric $d_\Omega$, we will equip our state space $(\mathbb{G}\times\mathcal{F}, B(\mathbb{G} \times \mathcal{F}))$ with a probability measure

$$\mu = \nu \times \eta \tag{10}$$

where the probability measure $\eta$ on player coalitions is such that $\eta(S) > 0$ for all $S \in \mathcal{F}$ and where the probability measure $\nu$ on networks is such that the set of countably many disjoint atoms[10] is given by

$$\{\mathbb{A}_{\alpha 1}, \mathbb{A}_{\alpha 2}, \ldots\} = \{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty} \subset \mathbb{G}. \tag{11}$$

Thus, we have as our state space, the probability space

$$(\Omega, B(\Omega), \mu) = (\mathbb{G}\times\mathcal{F}, B(\mathbb{G} \times \mathcal{F}), \nu \times \eta), \tag{12}$$

a compact metric space with metric $d_\Omega = h + d_{\mathcal{F}}$. Because $\mathbb{G}$ is a compact metric space, $B(\mathbb{G} \times \mathcal{F}) = B(\mathbb{G}) \times \mathbb{B}(\mathcal{F})$ where $\mathbb{B}(\mathcal{F})$ is the set of all subsets of $\mathcal{F}$ (including the empty set).

*In order to save writing and spare the reader, when no confusion is possible, we will use the notation*

$$(\Omega, B(\Omega)) = (\mathbb{G}\times\mathcal{F}, B(\mathbb{G} \times \mathcal{F})) \tag{13}$$

*for our state space and the notation*

$$\omega = (G, S) \tag{14}$$

*for elements of the state space.*

[10] A set of networks $\mathbb{A}_{\alpha k} \in B(\mathbb{G})$ is an atom of the probability space $(\mathbb{G}, B(\mathbb{G}), \nu)$ if $\nu(\mathbb{A}_{\alpha k}) > 0$ and for all subsets $\mathbb{B} \subseteq \mathbb{A}_{\alpha k}$, $\mathbb{B} \in B(\mathbb{G})$, $\nu(\mathbb{B}) = \nu(\mathbb{A}_{\alpha k})$ or $\nu(\mathbb{B}) = 0$. The set of networks $\mathbb{G}$ contains at most countably many disjoint atoms, $\{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty}$ and $\mathbb{G}$ can be written as

$$\mathbb{G} = \mathbb{NA} \cup \left[\cup_{k=1}^{\infty} \mathbb{A}_{\alpha k}\right],$$

where the set $\mathbb{NA}$ contains no atoms. We say that the probability space $(\mathbb{G}, B(\mathbb{G}), \nu)$ is atomless or nonatomic if it contains no atoms.

### 2.3.2 Feasible Actions

In our game each player's action takes the form of a network recommendation or network proposal. In particular, given current state $\omega \in \Omega$, each player $d \in D$ has available a nonempty subset of network proposals $\Phi_d(\omega) \subseteq \mathbb{G}$ that can be put forth by player $d$ for consideration by nature. However, only players who are members of the status quo coalition (i.e., the coalition whose turn it is to move) are allowed to propose *substantive* changes and each such proposal must be consistent with the rules of network formation. In particular, if $G' \in \Phi_d(G, S)$ is proposed by player $d \in S$ (and therefore, by a member of the status quo player coalition $S$), then the proposed network $G'$ must be such that under the rules of network formation it is possible for coalition $S$ or some subcoalition $S' \subseteq S$, to which player $d$ belongs to change the status quo network $G$ to network $G'$. Moreover, because players who are not members of the status quo coalition are not allowed to propose substantive changes, these players (i.e., players $d \notin S$) can only propose that the status quo network be maintained. Formally, we will assume that

**A-1** (*convexity and continuity of the constraint mappings*)

all constraint correspondences, $\Phi_d(\cdot)$,are such that,

(i) for all states $\omega = (G, S)$,

$$\left.\begin{array}{c}\text{(a) } \Phi_d(G,S) \text{ is convex with } G \in \Phi_d(G,S), \\ \text{and} \\ \text{(b) } \{G\} = \Phi_d(G,S) \text{ for all } d \notin S,\end{array}\right\} \tag{15}$$

(ii) $\Phi_d(\cdot)$ has a closed graph,

$$Gr\Phi_d(\cdot) := \{(\omega, G) : G \in \Phi_d(\omega)\}. \tag{16}$$

Thus, under A-1(i)(a) each player $d$ in each state has available a convex set of network proposals, including the status quo network and under A-1(i)(b) if the player is not part of the status quo coalition, then the status quo is the *only* network proposal available to that player. Moreover, if network $G' \in \Phi_d(G, S)$ is proposed by player $d \in S$, then under the rules of network formation, it must be feasible for player $d$, working alone or together with some subcoalition $S' \subseteq S$ (including possibly all members of $S$), to change the status quo network $G$ to the proposed network $G'$.

We will denote by $\Phi(\cdot)$ the aggregate constraint correspondence,

$$\omega \to \Phi(\omega) := \Pi_{d \in D} \Phi_d(\omega). \tag{17}$$

Assumption A-1(ii) implies that the aggregate constraint correspondence has a closed graph.

### 2.3.3 The Rules of Network Formation and Feasible Sets of Actions: Examples and Comments

In light of our discussions of feasible networks and feasible player coalitions, our objective in this section is to give some examples of player constraint mappings, $\Phi_d(\cdot)$, corresponding to various state spaces and rules of network formation.

The rules of network formation must specify for any move of the game which connections can be changed and which players can change them. Thus in order to formalize the rules, we must first specify for each player $d$ in each status quo coalition $S$ (i.e., in each coalition whose turn it is to move) the subset of node pairs $E_{dS} \subseteq N \times N$ whose connecting arc types player $d$ can substantively and feasibly change by working with coalition $S$ under the rules of network formation. For example, suppose that players and nodes are one and the same and that the feasible set of player coalitions is $\mathcal{F}_1$. If the rules of network formation are unilateral or noncooperative and it is coalition $\{j\}$'s turn to move, then for player (or node) $i$ the arc types connecting node pairs in $E_{i\{j\}}$ given by

$$E_{i\{j\}} = \begin{cases} \{i\} \times N & \text{if } i \in \{j\} \\ \emptyset & \text{if } i \notin \{j\} \end{cases}$$

are in play. Alternatively, suppose that players and nodes are one and the same and that the feasible set of player coalitions is $\mathcal{F}_2$. If the rules are bilateral (i.e., Jackson-Wolinsky like) and it is coalition $\{l, k\}$'s turn to move, then for player (or node) $i$ the arc types connecting node pairs in $E_{i\{l,k\}}$ given by

$$E_{i\{l,k\}} = \begin{cases} [\{i\} \times \{l, k\}] \cup [\{l, k\} \times \{i\}] & \text{if } i \in \{l, k\} \\ \emptyset & \text{if } i \notin \{l, k\} \end{cases}$$

are in play. Finally, suppose that players and nodes are one and the same and that the feasible set of player coalitions is an arbitrary collection of coalitions $\mathcal{F}$, possibly even $P(D)$. If the rules are bilateral but Jackson-van den Nouweland like[11] and it is coalition $S$'s turn to move, then for player (or node) $i$ the arc types connecting node pairs in $E_{iS}$ given by

$$E_{iS} = \begin{cases} [\{i\} \times S] \cup [S \times \{i\}] & \text{if } i \in S \\ \emptyset & \text{if } i \notin S \end{cases}$$

are in play.

In general, if under the rules of rules of network formation the set of node pairs whose arc types are in play for player $d$ in state

$$\omega = (G, S) \in \mathbb{G} \times \mathcal{F},$$

[11] Under Jackson-Wolinsky like rules, in any one move of the game, each player in the two-player status quo coalition can propose only pairwise changes involving the other player in the coalition. Alternatively, under Jackson-van den Nouweland like rules (2005), in any one move of the game, each player in the multi-player status quo coalition, can propose pairwise changes involving any of the other players in the status quo coalition. Thus, under Jackson-Wolinsky like rules, only pairwise changes involving a single pair of nodes can be made during any one move of the game, while under Jackson-van den Nouweland like rules, many such pairwise changes can be made in any one move of the game.

is given by the set $E_{dS}$, then each player's proposal constraint mapping is given by

$$\Phi_d(G,S) = \begin{cases} \{G' \in \mathbb{G} : \forall (i,j) \notin E_{dS},\, G'(ij) = G(ij)\} & \text{if } d \in S \\ \{G\} & \text{if } d \notin S. \end{cases}$$

Again assuming that players and nodes are the same, an interesting variation on the examples so far and one that is similar to the network model of Ballester, Calvo-Armengol, and Zenou (2006), is to require that the structure of connections between distinct players (i.e., node pairs $(i,j)$ with $i \neq j$) remain fixed, while allowing players to vary only their loop connections. We can then think of player $i's$ choice of a loop arc type $a_{ii}$ connecting $i$ to $i$ as player $i's$ choice of a node type (for example, $a_{ii}$ might be an effort level, a level of spending on public goods, a contract choice). Suppose then that the feasible set of networks consists of regular networks where all nodes are uniquely and feasibly connected, but with fixed connections between distinct nodes. Denoting this feasible set by $\mathbb{L}$, we have

$$G \in \mathbb{L} \text{ if and only if } \forall (i,j) \in N \times N,$$

$$\begin{gathered} |G(ij)| = 1 \text{ and } G(ij) \subseteq A(ij) \subset R^L, \\ \text{and} \\ \forall\ G \text{ and } G' \text{ in } \mathbb{L} \text{ and } \forall (i,j) \in N \times N \text{ with } i \neq j, \\ G(ij) = G'(ij) = \{a_{ij}\} \in A(ij). \end{gathered}$$

Here, for each node pair $(i,j)$, $A(ij)$ is compact, convex, and contains the zero vector. Thus, each network $G$ in $\mathbb{L}$ is uniquely identified by its loop profile $a := \{a_{ii}\}_{i \in N}$. To emphasize this fact, we denote the networks in $\mathbb{L}$ by $G_a$, where $a := \{a_{ii}\}_{i \in N}$, and we write the matrix representation of $G_a$ as

$$[G_a] = [\overline{G}] + [a]$$

where $\overline{G}$ is a fixed network with inactive loop connections (i.e., in $\overline{G}$, $(a_{ii},(i,i)) = (0,(i,i))$ for all $i$) and fixed, unique connections between distinct nodes and $[a]$ is a diagonal matrix where $a$ is loop profile corresponding to the network $G_a \in \mathbb{L}$.

Under noncooperative rules, if state space is $\Omega = \mathbb{L} \times \mathcal{F}_1$, then each player's proposal constraint mapping is given by

$$\Phi_i(G_a, j) = \begin{cases} \{G' \in \mathbb{L} : \forall (i',j') \neq (i,i),\, G'(i'j') = G(i'j')\} & \text{if } i = j \\ \{G_a\} & \text{if } i \neq j. \end{cases} \tag{18}$$

### 2.3.4 Payoffs

If the current state is $\omega = (G,S)$ (i.e., if the status quo network is $G$ and it is coalition $S$'s turn to move) and if players propose $m$-tuple of networks $G_D \in \Phi(\omega)$, then player $\overline{d}'s$ payoff is given by

$$r_{\overline{d}}(\omega, G_D) := r_{\overline{d}}(\omega, (G_{\overline{d}}, G_{-\overline{d}})).$$

The following notational conventions will be useful in stating our assumptions concerning player payoff functions. For each connection $(a_{lk},(l,k))$ in the network

$G_d \in \Phi_d(\omega)$ proposed by player $d$, the components of the arc type vector $a_{lk} \in R^L$ connecting nodes $l$ and $k$ describe (or quantify) the attributes of this connection. Many of our assumptions concerning payoff function are about the sensitivity of player payoffs to these attributes and the degree of complementarity between arc attributes across players. Abusing our notation a bit, we will denote the arc type vector by $G_d(lk)$. This will allow us to notationally keep track of the network in which the connection resides and the nodes which are being connected. Thus, for connection $(a_{lk}, (l,k)) \in G_d$,

$$a_{lk} = G_d(lk) \in R^L.$$

We will denote the $r^{th}$ component of the vector $a_{lk}$ in connection $(a_{lk}, (l,k)) \in G_d$ by,

$$a_{lkr} = G_d(lk)_r \in R,$$

$r \in L := \{1, 2, \ldots, L\}$.

We will use various combinations of assumptions concerning player payoff functions listed below in proving our results concerning Nash equilibrium in pure stationary strategies. We begin with the most basic. We will assume throughout that,

**A-2(1)** (*measurability and continuity of payoffs*)

each player $d \in D$ has a payoff function

$$r_d(\cdot,\cdot) : \Omega \times \mathbb{G}^m \to [-M, M] \tag{19}$$

such that

(i) for each state $\omega \in \Omega$, $r_d(\omega, \cdot)$ is continuous on $\mathbb{G}^m$, and

(ii) for each $m$-tuple of network proposals $G_D = (G_d)_{d \in D} \in \mathbb{G}^m$, $r_d(\cdot, G_D)$ is $B(\Omega)$-measurable.

Strengthening A-2(1), sometimes for each player we will assume that

**A-2(2)** (*strict concavity, $C^2$ smoothness, and strict diagonal property*)

(iii) (*strict concavity*) for each state $\omega \in \Omega$, and each $(m-1)$-tuple of network proposals by players $d' \neq d$, the payoff function

$$r_d(\omega, (\cdot, G_{-d})) : \Phi_d(\omega) \to [-M, M],$$

is *strictly* concave in $G_d$ on $\Phi_d(\omega)$;

(iv) (*$C^2$ smoothness)* for each $\omega \in \Omega$ there is an open set $\Phi^o(\omega)$ containing $\Phi(\omega)$ such that for each player $d \in D$, the payoff function

$$r_d(\omega, \cdot) : \Phi^o(\omega) \to [-M, M],$$

has continuous second order partial derivatives with respect to the components $G_d(lk)_r$, $r \in L$, of the arc type vector $G_d(lk) = a_{lk} \in R^L$;

(v) (*strict diagonal property*) for each state $\omega \in \Omega$, the payoff function

$$r_d(\omega, \cdot) : \Phi(\omega) \to [-M, M],$$

is such that

$$-\left|\frac{\partial^2 r_d(\omega, (G_d, G_{-d}))}{\partial (G_d(hq)_r)^2}\right| + \sum_{d' \in N \backslash \{d\}} \sum_{lks \in N \times N \times L \backslash \{hqr\}} \left|\frac{\partial^2 r_d(\omega, (G_d, G_{-d}))}{\partial G_d(hq)_r \partial G_{d'}(lk)_s}\right| < 0.$$

By the strict diagonal property, the absolute value of the second order partial derivative of the $d^{th}$ player's payoff function $r_d(\omega, \cdot)$ with respect to any component of the arc type connecting any node pair $(h, q)$ in any network proposed by the $d^{th}$ player is strictly larger than the absolute value of the second order cross partial derivative with respect to this component, $a_{hpr} = G_d(hq)_r$, and any component, $a_{lks} = G_{d'}(lk)_s$, of any arc type connecting any other node pair $(l, k)$ in any network proposed by any other player $d'$ $(\neq d)$.

Finally, sometimes for each player we will assume that

**A-2(3)** (*affinity*)

(vi) for each state $\omega \in \Omega$, $r_d(\omega, (\cdot, \cdot))$ is affine on $\Phi(\omega)$.[12]

## 2.4 The Law of Motion

Given the profile of player proposals $G_D \in \Phi(\omega)$ and given the current state, $\omega \in \Omega$, *nature* then *chooses* the next state (i.e., the next network-coalition pair) according to probabilistic transition law, $q(\cdot|\omega, G_D)$ defined on the state space $(\Omega, B(\Omega))$. We will assume that

**A-3** (*measurability, stochastic continuity, absolute continuity, and affinity of the law of motion*)

(i) (*measurability*) for all $E \in B(\Omega)$, the function

$$(\omega, G_D) \to q(E|\omega, G_D)$$

is measurable over the graph of $\Phi(\cdot)$;

(ii) (*stochastic continuity*) for all $d_\Omega$-closed $F \in B(\Omega)$,

(a) the function

$$G_D \to q(F|\omega, G_D)$$

[12] Thus, in each state, $\omega \in \Omega$,

$$r_d(\omega, \alpha G_D + (1-\alpha)G'_D) = \alpha r_d(\omega, G_D) + (1-\alpha) r_d(\omega, G'_D),$$

on $\Phi(\omega)$.

is continuous on $\Phi(\omega)$ for all $\omega \in \Omega$; or

(b) the function

$$(\omega, G_D) \to q(F|\omega, G_D)$$

is continuous on the graph of $\Phi(\cdot)$

(iii) (*absolute continuity and uniform integrable boundedness*) for all state proposal pairs, $(\omega, G_D)$, contained in the graph of $\Phi(\cdot)$

(a) the probability measures $q(\cdot|\omega, G_D)$ are absolutely continuous with respect the probability measure $\mu = \nu \times \eta$ defined on $(\Omega, B(\Omega))$ (i.e., $q(\cdot|\omega, G_D) \ll \mu$ for all $(\omega, G_D) \in Gr\Phi(\cdot)$).

(b) the densities $f(\cdot|\omega, G_D)$ of $q(\cdot|\omega, G_D)$ with respect to the dominating probability measure $\mu$ are integrably bounded, that is, there exists a $\mu$-integrable function $h(\cdot) : \Omega \to R$ such that for all $(\omega, G_D) \in Gr\Phi(\cdot)$,

$$0 \leq f(\omega'|\omega, G_D) \leq h(\omega') \text{ for all } \omega' \in \Omega.$$

(iv) (*affinity*) for all events $E \in B(\Omega)$ and states $\omega \in \Omega$, the function

$$G_D \to q(E|\omega, G_D)$$

is affine on $\Phi(\omega)$.[13]

It should be noted that (A-3)(ii)(a) is stronger than the usual weak continuity assumption. Under weak continuity, we would have for any sequence $\{(G_D^n)\}_n$ in $\Phi(\omega)$ with

$$(G_D^n) \underset{h}{\to} (\overline{G}_D) \in \Phi(\omega),$$

and any $d_\Omega$-closed $F \in B(\Omega)$,

$$\limsup_n q(F|\omega, G_D^n) \leq q(F|\omega, \overline{G}_D)$$
or equivalently,
$$\int_\Omega f(\omega')q(d\omega'|\omega, G_D^n) \to \int_\Omega f(\omega')q(d\omega'|\omega, \overline{G}_D),$$

for any bounded, continuous function $f(\cdot)$. Under (A-3)(ii)(a), however, we have strengthened weak continuity so that for any sequence $\{G_D^n\}_n$ in $\Phi(\omega)$ with

$$G_D^n \to \overline{G}_D \in \Phi(\omega),$$

[13] Thus, for each measurable set of network-coalition pairs, $E \in B(\Omega)$, and for each state, $\omega \in \Omega$,

$$q(E|\omega, \alpha G_D + (1-\alpha)G'_D) = \alpha q(E|\omega, G_D) + (1-\alpha)q(E|\omega, G'_D),$$

on $\Phi(\omega)$.

and any $d_\Omega$-closed $F \in B(\Omega)$,

$$\lim_n q(F|\omega, G_D^n) = q(F|\omega, \overline{G}_D)$$

or equivalently (by Delbaen's Lemma (1974)),

$$\textstyle\int_\Omega v(\omega')q(d\omega'|\omega, G_D^n) \to \int_\Omega v(\omega')q(d\omega'|\omega, \overline{G}_D),$$

for any bounded, *measurable* function $v(\cdot)$.

## 2.5 Plans and Pure Stationary Strategies

### 2.5.1 Plans and Strategies

A *plan* $\pi_d = (\pi_d^1, \pi_d^2, \ldots)$ for player $d \in D$ is a sequence of history dependent conditional probability measures on $(\mathbb{G}, B(\mathbb{G}))$. Under plan $\pi_d$ in period $n$ given the history of states and action $m$-tuples (i.e., the $(n-1)$-sequence of network-coalition pairs and $m$-tuples of network proposals) $H^{n-1} := \left(\omega^1, G_D^1, \omega^2, G_D^2, \ldots, \omega^{n-1}, G_D^{n-1}\right)$, and given the current (period $n$) state $\omega^n = (G^n, S^n)$, player $d$ chooses $a$ network proposal according to the conditional probability measure

$$\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{20}$$

Here, $\mathcal{P}\left(\Phi_d(\omega^n)\right)$ is the set of all probability measures with support contained in $\Phi_d(\omega^n)$.[14] Let $\mathcal{H}^{n-1}$ denote set of all $(n-1)$-histories and let

$$\Pi_d^n := \Pi_{\Phi_d}(\mathcal{H}^{n-1} \times \Omega, \mathcal{P}(\mathbb{G}))$$

denote the set of all measurable functions, $(H^{n-1}, \omega^n) \to \pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}(\mathbb{G})$ such that $\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right)$ for all $\omega^n \in \Omega$. Formally, the set of plans for player $d$ is given by

$$\Pi_d^\infty := \prod_{n=1}^{\infty} \Pi_d^n.$$

A *Markov plan* $\psi_d = (\psi_d^1, \psi_d^2, \ldots)$ for player $d \in D$ is a sequence of state-dependent conditional probability measures on $(\mathbb{G}, B(\mathbb{G}))$. Under Markov plan $\psi_d$ in period $n$ given the current (period $n$) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player $d$ chooses a network proposal according to the conditional probability measure

$$\psi_d^n(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{21}$$

Let

$$\Sigma_d^n := \Sigma_{\Phi_d}(\Omega, \mathcal{P}(\mathbb{G})) := \Sigma_{\Phi_d}$$

[14]For any set $\mathbb{E} \subseteq \mathbb{G}$ we shall denote by $\mathcal{P}\left(\mathbb{E}\right)$ the set of all probability measures with support contained in $\mathbb{E}$.

denote the set of all measurable functions, $\omega \to \psi_d^n(\cdot|\omega) \in \mathcal{P}(\mathbb{G})$ such that $\psi_d^n(\cdot|\omega^n) \in \mathcal{P}(\Phi_d(\omega^n))$ for all $\omega^n \in \Omega$. The set of Markov plans for player $d$ is given by

$$\Sigma_d^\infty := \prod_{n=1}^{\infty} \Sigma_d^n.$$

A *stationary Markov plan* $(\sigma_d, \sigma_d, \ldots)$ for player $d \in D$ - or as we shall call it here - a *stationary strategy* for player $d \in D$ - is a constant sequence of state-dependent conditional probability measures on $(\mathbb{G}, B(\mathbb{G}))$. Under stationary strategy $(\sigma_d, \sigma_d, \ldots)$ given the current (period $n$) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player $d$, in each and every period $n$, chooses a network proposal according to the conditional probability measure

$$\sigma_d(\cdot|\omega^n) \in \mathcal{P}(\Phi_d(\omega^n)). \tag{22}$$

Rather than write $\sigma_d(\cdot|\omega)$ we will sometimes write $\sigma_d(\omega)$.

A *pure stationary strategy* for player $d \in D$ is a stationary Markov strategy $(\sigma_d, \sigma_d, \ldots)$ such that for some function

$$f_d(\cdot) : \Omega \to \mathbb{G} \text{ with } f_d(\omega) \in \Phi_d(\omega) \text{ for all } \omega \in \Omega,$$

$$\sigma_d(f_d(\omega)|\omega) = 1 \text{ for all } \omega \in \Omega. \tag{23}$$

Thus under *pure stationary strategy* $(\sigma_d, \sigma_d, \ldots)$ in any state $\omega \in \Omega$, the conditional probability measure for player $d$ assigns probability 1 to the network proposal $f_d(\omega) \in \Phi_d(\omega)$. Such a conditional probability measure is usually called a Dirac Young measure and is denoted by, $\delta_{f_d(\omega)}$. Rather than represent a pure stationary strategy for player $d \in D$ using a conditional probability measure $\sigma_d(\cdot|\omega)$ concentrating all its probability mass on a particular state dependent network $f_d(\omega)$ (i.e., the pure strategy function), we will often instead represent a pure stationary strategy for player $d \in D$ using the underlying function $f_d(\cdot)$. Thus, a pure stationary strategy for player $d \in D$ will often times be described as a constant sequence of functions $(f_d, f_d, \ldots)$ such that for all $\omega \in \Omega$, $f_d(\omega) \in \Phi_d(\omega)$ and $\sigma_d(f_d(\omega)|\omega) = 1$.

Finally, a *pure stationary bang-bang strategy* for player $d \in D$ is a stationary Markov strategy $(\sigma_d, \sigma_d, \ldots)$ such that for some function

$$f_d(\cdot) : \Omega \to \mathbb{G} \text{ with } f_d(\omega) \in ext\Phi_d(\omega) \text{ for all } \omega \in \Omega,$$

$$\sigma_d(f_d(\omega)|\omega) = 1 \text{ for all } \omega \in \Omega.$$

Thus under pure stationary bang-bang strategy $(\sigma_d, \sigma_d, \ldots)$ in any state $\omega \in \Omega$, the conditional probability measure for player $d$ assigns probability 1 to the network proposal $f_d(\omega) \in ext\Phi_d(\omega)$, where $ext\Phi_d(\omega)$ is the set of extreme regular networks in the compact, convex set of regular networks $\Phi_d(\omega)$. A network $\widehat{G} \in \Phi_d(\omega)$ is an extreme network if there does not exist networks $G'$ and $\overline{G}$ in $\Phi_d(\omega)$ such that for some $t \in (0,1)$, $\widehat{G} = tG' + (1-t)\overline{G}$. As with stationary strategies, rather than represent a pure stationary bang-bang strategy for player $d \in D$ using a conditional probability measure $\sigma_d(\cdot|\omega)$ concentrating all its probability mass on a particular extreme network $f_d(\omega)$, we will often instead represent a pure stationary bang-bang strategy for player $d \in D$ using the underlying function $f_d(\cdot)$. Thus, a pure stationary bang-bang strategy for player $d \in D$ will often times be described as a constant sequence of functions $(f_d, f_d, \ldots)$ such that for all $\omega \in \Omega$, $f_d(\omega) \in ext\Phi_d(\omega)$ and $\sigma_d(f_d(\omega)|\omega) = 1$.

## 2.6 Players' Expected Payoffs in Pure Strategies

Given pure stationary strategy $f_D(\cdot) := (f_d(\cdot))_{d \in D}$, if the current state is $\omega \in \Omega$ then player $d$'s immediate expected payoff is

$$r_d(\omega, f_D(\omega)) = \int_{\Phi(\omega)} r_d(\omega, G_D) d\sigma_D(G_D|\omega) \tag{24}$$

where for all $\omega \in \Omega$, $f_d(\omega) \in \Phi_d(\omega)$ and $\sigma_d(f_d(\omega)|\omega) = 1$ for all players $d \in D$ and $\sigma_D(\omega) := \sigma_D(\cdot|\omega)$ is the product measure $\times_d \sigma_d(\cdot|\omega)$ with support equal to $f_D(\omega) \in \Phi(\omega) := \Pi_{d \in D}\Phi_d(\omega)$.

If network proposal $m$-tuple $G_D$ is chosen according to pure stationary strategies $f_D(\omega)$ in state $\omega$, then nature chooses the next network-coalition pair (i.e., the next state) according to the law of motion (i.e., the probability measure) $q(\cdot|\omega, G_D) = q(\cdot|\omega, f_D(\omega))$.

Let

$$r_d^n(f_D)(\omega) := \begin{cases} r_d(\omega, f_D(\omega)) & \text{for } n = 1 \\ \\ \int_\Omega r_d(\omega', f_D(\omega'))q^{n-1}(\omega'|\omega, f_D(\omega)) & \text{for } n \geq 2, \end{cases} \tag{25}$$

denote the $n^{th}$ period expected payoff to player $d$ under pure stationary strategy $f_D(\cdot)$ starting at network-coalition pair $\omega = (G, S)$ given law of motion $q(\cdot|\cdot, \cdot)$. Here, for $n \geq 2$, $q^n(\cdot|\omega, f_D(\omega))$ is defined recursively by

$$\left.\begin{aligned} & q^n(E|\omega, f_D(\omega)) \\ & = \int_\Omega q^{n-1}(E|\omega', f_D(\omega'))q(\omega'|\omega, f_D(\omega)) \\ & = \int_\Omega q(E|\omega', f_D(\omega'))q^{n-1}(\omega'|\omega, f_D(\omega)). \end{aligned}\right\} \tag{26}$$

The discounted expected payoff to player $d$ over an infinite time horizon under pure stationary strategy $f_D(\cdot) \in \prod_{d \in D} \Sigma_{\Phi_d}$ starting at state $\omega$ is then given by

$$E_d(f_D)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(f_D)(\omega). \tag{27}$$

In general, the discounted expected payoff to player $d$ over an infinite time horizon under plan $\pi_D = (\pi_d)_{d \in D} \in \Pi^\infty := \prod_{d \in D} \Pi_d^\infty$ starting in state $\omega$ is then given by

$$E_d(\pi_D)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(\pi_D)(\omega). \tag{28}$$

# 3 Dynamic Network and Coalition Formation Games and Nash Equilibrium

## 3.1 Nash Equilibrium in Pure Stationary Strategies

A dynamic network and coalition formation game is given by

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^{\infty})_{d \in D}.$$

A dynamic network and coalition formation game starting at state $\omega \in \Omega$ is given by

$$\Gamma_\omega := (\Omega, E_d(\cdot)(\omega), \Pi_d^{\infty})_{d \in D}.$$

**Definition 4** (*Nash Equilibrium*)

*A pure stationary strategy $(f_d^*(\cdot))_{d \in D}$ is a Nash equilibrium of the dynamic network and coalition formation game $\Gamma$ if for all starting network-coalition pairs $\omega = (G, S) \in \mathbb{G} \times \mathcal{F}$ and all players $d \in D$,*

$$E_d(f_d^*, f_{-d}^*)(\omega) \geq E_d(\pi_d, f_{-d}^*)(\omega) \text{ for all } \pi_d \in \Pi_d^{\infty}.$$

Thus, a pure stationary strategy $(f_d^*(\cdot))_{d \in D}$ is a Nash equilibrium of dynamic network and coalition formation game $\Gamma$ if it is a Nash equilibrium for the game $\Gamma_\omega$ for all starting states.

**Theorem 1** (*The Existence of Nash Equilibrium in Pure Stationary Strategies: Smooth and Strictly Concave Models*)

*Under assumptions [A-1], [A-2 (1) and (2)]-[A-3(i), (ii)(a), and (iii)(a)] the dynamic network and coalition formation game*

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^{\infty})_{d \in D}.$$

*has a Nash equilibrium in pure stationary strategies.*

Our approach to proving existence essentially follows the approach introduced in the seminal papers Nowak and Raghavan (1992) and Nowak (2007).[15] But because we assume that players' discount factors $\beta_d$ are heterogeneous, and more importantly, because our stochastic continuity assumptions concerning the law of motion are weaker than those of Nowak-Raghavan and Nowak (their assumptions imply our assumptions), we include a proof in the last section of the paper. As in NRN (Nowak-Raghavan (1992) and Nowak (2007)) the basic objectives of our proofs are to show that there exists an $m$-tuple of pure stationary strategies $f_D(\cdot) := (f_d^*(\cdot))_{d \in D}$ and an $m$-tuple of uniformly bounded, $B(\Omega)$-measurable value functions, $w_D^*(\cdot) :=$

[15] There is also the deep and difficult problem of computing stationary equilibria in stochastic games. Recently, Herings and Peeters (2004) have introduced an algorithm (in a non-network setting) based on a stochastic tracing procedure which provides a solution to the computational problem for the stationary case in most stochastic games (but with finite state and action spaces).

$(w_d^*(\cdot))_{d\in D}$, $w_d^*(\cdot) : \Omega \rightarrow [-M, M]$, such that for each player $d \in D$ and for all states $\omega \in \Omega$,

$$w_d^*(\omega) = r_d(\omega, f_D^*(\omega)) + \beta_d \int_\Omega w_d^*(\omega') q(\omega'|\omega, f_D^*(\omega)).$$

### 3.2 Nash Equilibrium in Pure Stationary Bang-Bang Strategies

Moving from the strictly concave model underlying Theorem 1 to an affine model we obtain the following result for pure stationary bang-bang, network formation strategies.

**Theorem 2** (*The Existence of a Nash Equilibrium in Pure Stationary Bang-Bang Strategies: Affine Models*)

*Under assumptions [A-1], [A-2 (1) and (3)]-[A-3 (i), (ii)(a), (iii)(a), and (iv)] the dynamic network and coalition formation game*

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D}.$$

*has a Nash equilibrium in pure stationary bang-bang strategies.*

### 3.3 An Example: Dynamic Club Networks and Bang-Bang Time Allocation Strategies

Consider a club network example similar to example (3) in section 2.1.4. In particular, let $D = \{d_1, \ldots, d_m\}$ be a set of players and $C = \{c_1, \ldots, c_n\}$ be a set of clubs, and let the set of regular club networks be given by

$$\mathbb{K}_R := \left\{G \in P_f(R^L \times (D \times C)) : \forall (d, c) \in D \times C,\ |G(dc)| = 1\right\},$$

where $P_f(R^L \times (D \times C))$ denotes the collection of all nonempty, closed subsets of $R^L \times (D \times C)$. Next consider the feasible set of regular club networks given by

$$\mathbb{K} = \{G \in \mathbb{K}_R : \forall (d, c) \in D \times C,\ G(dc) \subseteq [0, 1] \text{ and } \sum_{c\in C} G(dc) = 1\}.$$

We will interpret the connection $(a_{dc}, (d, c)) := (G(dc), (d, c))$ in club network $G \in \mathbb{K}$ to mean that in club network $G$, player $d$ allocates $a_{dc}$ percent of his time to club $c$. We will call these types of regular club networks, regular club time allocation networks.

Each club time allocation network $G \in \mathbb{K}$ has a unique alternative representation as the union of regular *player club time allocation networks.* For each player $d$, a regular club time allocation network $g_d$ is a nonempty closed subset of $[0, 1] \times (\{d\} \times C)$ such that for all clubs $c \in C$,

$$|g(dc)| = 1,\ g(dc) \subseteq [0, 1], \text{ and } \sum_{c\in C} g(dc) := \sum_{c\in C} a_{dc} = 1.$$

Let $\mathbb{K}_d$ denote the collection of all regular time allocation networks for player $d$. Each time allocation network $G \in \mathbb{K}$, can be written as

$$G = \cup_{d' \in D} g_{d'}, \text{ where } g_{d'} \in \mathbb{K}_{d'}.$$

Thus, any club time allocation network $G$ has unique representation as the union, $\cup_{d' \in D} g_{d'}$, of a finite collection of player club time allocation networks $(g_{d'})_{d'}$, and conversely, the union of any finite collection of regular player club time allocation networks $(g_{d'})_{d'}$ is a regular club time allocation network. Finally, note that each player's set of regular club time allocation networks, $\mathbb{K}_d$, is nonempty, convex, and compact.

First, assume that the rules of network formation are noncooperative and that the feasible set of player coalitions is given by $\mathcal{F}_1$. Thus the state space is given by $\Omega = \mathbb{K} \times \mathcal{F}_1$. Given state $\omega = (G, d_j) \in \mathbb{K} \times \mathcal{F}_1$, player $d_i's$ proposal constraint mapping is given by

$$\Phi_{d_i}(G, d_j) = \begin{cases} \{G' \in \mathbb{K} : \forall\ c_k \in C \text{ and } d_{j'} \neq d_j,\ G'(d_{j'}c_k) = G(d_{j'}c_k)\} & \text{if } d_i = d_j \\ \{G\} & \text{if } d_i \neq d_j. \end{cases}$$

Using player club time allocation networks, player $d_i's$ proposal constraint mapping can be written as,

$$\Phi_{d_i}(G, d_j) = \begin{cases} \{G' := (g'_{d_j}, g_{-d_j}) \in \mathbb{K} : g'_{d_j} \in \mathbb{K}_{d_j}\} & \text{if } d_i = d_j \\ \{G\} & \text{if } d_i \neq d_j, \end{cases}$$

where $G = (g_d, g_{-d})$ is the regular club time allocation network $G$ with player club time allocation representation $\cup_d g_d$. Thus, each player's constraint mapping, $\Phi_{d_i}(\cdot)$, satisfies assumption A-1.

Next, assume that each player's payoff function is measurable in states and continuous and affine in players' network proposals. Thus, player payoff functions satisfy assumptions A-2 (1) and (3).

Finally, assume that the law of motion is given by the conditional probabilities, $q(d\omega'|\omega, G_D)$, satisfying assumptions A-3 (i), (ii)(a), (iii)(a), and (iv).

By Theorem 2, our dynamic game of club network formation has a Nash equilibrium in pure stationary bang-bang strategies. Thus, there exists value functions $w^*_D(\cdot) = (w^*_{d_i}(\cdot))_{d_i \in D}$ and bang-bang stationary strategies $f^*_D(\cdot) = (f^*_{d_i}(\cdot))_{d_i \in D}$ such that for all players $d_i$ in all states $\omega$,

$$u_{d_i}(\omega, (f^*_{d_i}(\omega), f^*_{-d_i}(\omega)))(w^*_{d_i}) = \max_{G' \in \Phi_{d_i}(\omega)} u_{d_i}(\omega, (G', f^*_{-d_i}(\omega)))(w^*_{d_i})$$

$$= \max_{G' \in ext\Phi_{d_i}(\omega)} u_{d_i}(\omega, (G', f^*_{-d_i}(\omega)))(w^*_{d_i}),$$

where

$$u_{d_i}(\omega, (f^*_{d_i}(\omega), f^*_{-d_i}(\omega)))(w^*_{d_i})$$

$$= r_{d_i}(\omega, (f^*_{d_i}(\omega), f^*_{-d_i}(\omega))) + \beta_{d_i} \int_{\mathbb{K}} w^*_{d_i}(\omega') q(d\omega'|\omega, (f^*_{d_i}(\omega), f^*_{-d_i}(\omega))).$$

Because the equilibrium stationary strategies, $f_D^*(\cdot) = (f_{d_i}^*(\cdot))_{d_i \in D}$, are bang-bang, we can say a bit more about the way in which players try to influence the process of club network formation in equilibrium via their proposal strategies. First, note that in state $\omega = (G, d_j)$ all players other than player $d_j$ are constrained to propose the status quo network. In particular, for all $d_i \neq d_j$ and for all possible status quo networks $G \in \mathbb{K}$,

$$\{G\} = \Phi_{d_i}(G, d_j) = ext\Phi_{d_i}(G, d_j).$$

Thus, for all players $d_i \neq d_j$ and for all possible status quo networks $G \in \mathbb{K}$,

$$f_{d_i}^*(G, d_j) = G \in ext\Phi_{d_j}(G, d_j)$$

Second, because in state $\omega = (G, d_j)$ it is player $d_j$ turn to move, we have under noncooperative rules that $\Phi_{d_j}(G, d_j)$ consists of club networks with all possible club time allocations for player $d_j$, but with the club time allocations of all other players, $d_i \neq d_j$, fixed at their status quo network values. The extreme points of this set then are the club networks where player $d_j$ allocates all his time to one club. In particular, letting $G_{d_j}^{c_k}$ denote the club network where player $d_j$ allocates 100 percent of his time to some club $c_k \in C$, and letting the club time allocations of all other players, $d_i \neq d_j$, remain fixed at their status quo network values, then for all possible status quo networks, $G \in \mathbb{K}$, we have for player $d_j$ in state $\omega = (G, d_j)$,

$$ext\Phi_{d_j}(G, d_j) = \{G_{d_j}^{c_1}, \ldots, G_{d_j}^{c_n}\}.$$

Thus, in equilibrium when it is player $d_j's$ turn it is to move, player $d_j$ no matter what the status quo networks $G \in \mathbb{K}$, proposes a club network

$$f_{d_i}^*(G, d_j) = G_{d_j} \in \{G_{d_j}^{c_1}, \ldots, G_{d_j}^{c_n}\},$$

in which he spends 100 percent of his time is a single club, $c_k \in C$.

# 4 Emergent Markov Processes of Network and Coalition Formation

## 4.1 Equilibrium Transitions

Under equilibrium stationary strategies, $f_D^*(\cdot) = (f_{d_i}^*(\cdot))_{d_i \in D}$, the emergent Markov process of network and coalition formation,

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty},$$

is governed by the equilibrium Markov transition,[16]

$$p^*(E|\omega) = q(E|\omega, f_D^*(\omega)) = \int_E dq(\omega'|\omega, f_D^*(\omega))$$

[16] Law of motion $\omega \to p^*(\cdot|\omega)$ is a Markov transition if for each $\omega$, $p^*(\cdot|\omega)$ is a probability measure and for each $E \in B(\Omega)$,

$$p^*(E|\cdot) : \Omega \to [0, 1]$$

is measurable. Here, $\omega = (G, S)$ is a realization of the process $W_n^* = (G_n^*, S_n^*)$ for some $n$.

Thus,

$$\Pr\left\{W^*_{n+1} \in E | W^*_n = \omega\right\} = p^*(E|\omega)$$
and
$$\Pr\left\{W^*_n \in E | W^*_0 = \omega\right\} = p^{*n}(E|\omega) = q^n(E|\omega, f^*_D(\omega)),$$

where the $n$-step transition $p^{*n}(\cdot|\cdot)$ is defined recursively as follows: for all $\omega \in \Omega$ and $E \in B(\Omega)$,

$$p^{*n}(E|\omega) = \int_\Omega p^*(E|\omega')p^{*n-1}(d\omega'|\omega) = \int_\Omega p^{*n-1}(E|\omega')p^*(d\omega'|\omega) \tag{29}$$

for $n = 1, 2, \ldots$, and $p^{*0}(\cdot|\omega) = \delta_\omega(\cdot)$ is the Dirac measure at $\omega$.

## 4.2 Absorbing Sets and Invariant and Ergodic Probability Measures

A set $E \in B(\Omega)$ is called a $p^*$-absorbing set if $p^*(E|\omega) = 1$ for all network-coalition pairs $\omega \in E$. Let

$$\mathcal{L}^* := \mathcal{L}(p^*(\cdot|\cdot)) \subseteq B(\Omega) \tag{30}$$

denote the collection of all $p^*$-absorbing sets. Note that the set of all absorbing sets is closed under countable unions and intersections. A $p^*$-absorbing set $E \in \mathcal{L}^*$ is said to be *indecomposable* if it does not contain the union of two disjoint absorbing sets.

A probability measure $\lambda(\cdot)$ on the state space of feasible network-coalition pairs $(\Omega, B(\Omega))$ is invariant for Markov transition $p^*(\cdot|\cdot)$ (i.e., is $p^*$-invariant) if

$$\lambda(E) = \int_\Omega p^*(E|\omega)d\lambda(\omega) \text{ for all } E \in B(\Omega). \tag{31}$$

Thus, if probability measure $\lambda(\cdot)$ is $p^*$-invariant, then for any set of network-coalition pairs $E \in B(\Omega)$, if the current status quo network-coalition pair $\omega_n = (G_n, S_n)$ is chosen according to probability measure $\lambda(\cdot)$ - so that the probability that $\omega_n$ lies in $E$ is just $\lambda(E)$ - then the probability that next period's network-coalition pair $\omega_{n+1} = (G_{n+1}, S_{n+1})$ lies in $E$ is also $\lambda(E) = \int_\Omega p^*(E|\omega)d\lambda(\omega)$. Denote by $\mathcal{I}^*$ the collection of all $p^*$-invariant measure.

A $p^*$-invariant measure $\lambda(\cdot)$ is said to be $p^*$-ergodic if $\lambda(E) = 0$ or $\lambda(E) = 1$ for all $E \in \mathcal{L}^*$. Denote by $\mathcal{E}^*$ the collection of all $p^*$-ergodic measures. Because the $p^*$-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{I}^*$ of $p^*$-invariant measures (see Theorem 19.25 in Aliprantis and Border (1999)), each measure $\lambda(\cdot)$ in $\mathcal{I}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.

## 4.3 Recurrence, Irreducibility, and Maximal Harris Sets

Given set $E \in B(\Omega)$, the number of *visitations* to $E$ by the process $\{W^*_n\}_n = \{(G^*_n, S^*_n)\}_{n=1}^\infty$ is given by

$$\eta^*_E := \sum_{n=1}^\infty I_E(W^*_n), \tag{32}$$

while the expected number of visitations starting from network-coalition pair $\omega = (G, S)$ is given by

$$G^{*}(\omega, E) := E_{\omega}^{*}[\eta_{E}^{*}] = \sum_{n=1}^{\infty} p^{*n}(E|\omega). \tag{33}$$

The *hitting time* of network-coalition formation process $\{W_n^*\}_n$ for set $E \in B(\Omega)$ is given by

$$\tau_E^* := \inf\{n \geq 1 : W_n^* \in E\}. \tag{34}$$

Following in Tweedie (2001),

$$L^{*}(\omega, E) := \Pr\{\tau_E^* < \infty | W_0^* = \omega\} = \Pr\{\cup_{n=1}^{\infty}(W_n^* \in E | W_0^* = \omega)\} \tag{35}$$

is the probability of hitting (or reaching) in finite time the set of network-coalition pairs $E$ starting from network-coalition pair $\omega \in \Omega$ given transition $p^*(\cdot|\cdot)$.

Finally, the probability with which the network-coalition formation process $\{W_n^*\}_n$ visits $E \in B(\Omega)$ infinitely often (denoted by i.o.) is given by

$$\left.\begin{array}{r} Q^{*}(\omega, E) := \Pr\{W_n^* \in E \text{ i.o.} | W_0^* = \omega\} \\ = \Pr\{\cap_{m=1}^{\infty} \cup_{n=m}^{\infty}(W_n^* \in E | W_0^* = \omega)\} \text{ for all } \omega \in \Omega. \end{array}\right\} \tag{36}$$

By the Orey (1971), if for any $E \in B(\Omega)$,

$$L^{*}(\omega, E) = 1 \text{ for all } \omega \in \Omega, \text{ then } Q^{*}(\omega, E) = 1 \text{ for all } \omega \in \Omega. \tag{37}$$

The network-coalition formation process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is said to be *$\psi$-irreducible* if for some nontrivial, $\sigma$-finite measure $\psi(\cdot)$ on $B(\Omega)$,

$$\psi(E) > 0 \text{ implies } L^{*}(\omega, E) > 0 \text{ for all } \omega \in \Omega.$$

Thus if the process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is $\psi$-irreducible, then it hits all the "important" sets of network-coalition pairs (i.e., the sets $E$ such that $\psi(E) > 0$) with positive probability starting from any network-coalition pair in the state space $\Omega = \mathbb{G} \times \mathcal{F}$.

The network-coalition formation process $\{W_n^*\}_n$ governed by $p^*(\cdot|\cdot)$ is said to be *$\psi$-recurrent* if for some nontrivial, $\sigma$-finite measure $\psi(\cdot)$ on $B(\Omega)$,

$$\psi(E) > 0 \text{ implies } Q^{*}(\omega, E) = 1 \text{ for all } \omega \in \Omega.$$

Thus, if a set of network-coalition pairs $E \in B(\Omega)$ is recurrent (i.e., $\psi$-recurrent) then the network-coalition formation process $\{W_n^*\}_n$ visits $E$ infinitely often.

A set of network-coalition pairs (i.e., a set of states) $H \in B(\Omega)$ is called a *Maximal Harris set* if for some nontrivial, $\sigma$-finite measure $\psi(\cdot)$ on $B(\Omega)$ such that $\psi(H) > 0$,

$$\psi(A) > 0 \text{ implies } L^{*}(\omega, A) = 1 \text{ for all } \omega \in H,$$
$$\text{and}$$
$$L^{*}(\omega, H) = 1 \text{ implies that } \omega \in H.$$

Note that Maximal Harris sets are absorbing and indecomposable. Moreover, if $H$ and $H'$ are distinct Maximal Harris sets, then they are disjoint.

A set of network-coalition pairs $T \in B(\Omega)$ is *transient* if $T$ is the disjoint union of countably many *uniformly transient sets* $U_j$, that is, sets $U_j \in B(\Omega)$ such that $T = \cup_j U_j$ and for each set there is a finite constant $M_j$, such that for all network-coalition pairs $\omega \in U_j$,

$$E^*_\omega[\eta^*_{U_j}] = \sum_{n=1}^{\infty} p^{*n}(U_j|\omega) < M_j. \tag{38}$$

A set of network-coalition pairs $E \in B(\Omega)$ is said to be $p^*$-*inessential* if

$$Q^*(\omega, E) = 0 \text{ for all } \omega \in \Omega. \tag{39}$$

Thus, a set of states $E$ is inessential if the probability that the network-coalition formation process visits the set $E$ infinitely often is zero stating from any state. If a set of states is inessential, then if the process visits the state at all, it leaves the state for good after finitely many moves. The union of countable many inessential states is called an *improperly* $p^*$-*essential set*. Any other set is called *properly* $p^*$-*essential*.

## 4.4 The Fundamental Conditions for Stability: Drift and Global Uniform Countable Additivity

Given the Markov transition $\omega \to p^*(\cdot|\omega)$ what can be said concerning stability? Quite a bit if the Markov transition $p^*(\cdot|\cdot)$ satisfies the following two conditions:

*The Tweedie Conditions (2001)*:

there exists a measurable set of network-coalition pairs $C \subseteq \Omega$, a nonnegative measurable function

$$V(\cdot) : \Omega \to [0, \infty],$$

and a finite $b$ such that (i) (the drift condition) for all $\omega \in \Omega$

$$\int_\Omega V(\omega')dp^*(\omega'|\omega) \leq V(\omega) - 1 + bI_C(\omega),$$

and (ii) (uniform countable additivity) for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n\to\infty} \sup_{\omega \in C} p^*(B_n|\omega) = 0.$$

We say that the Markov transition $p^*(\cdot|\cdot)$ satisfies *global uniform countable additivity* if for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),

$$\lim_{n\to\infty} \sup_{\omega \in \Omega} p^*(B_n|\omega) = 0, \tag{40}$$

and we say that the Tweedie conditions are satisfied globally if the Tweedie conditions hold with $C = \Omega$.

In Section 5 below, using some beautiful results by Meyn and Tweedie (1993), Tweedie (2001), and Costa and Dufour (2005), we will show that if the emergent Markov transition $p^*(\cdot|\cdot)$ governing the equilibrium process of network and coalition formation is *globally uniformly countable additive*, then the equilibrium process possesses some striking stability properties - analogous to those demonstrated in Page and Wooders (2007) for static abstract games of network formation.

To begin let us strengthen our stochastic continuity assumption from A-3(ii) (a) to A-3(ii) (b). Under assumptions A-3(ii) (b), we have for all $d_\Omega$-closed sets $F \in B(\Omega)$ of network-coalition pairs, the function

$$(\omega, G_D) \to q(F|\omega, G_D)$$

is continuous in $(\omega, G_D)$ over the graph of $\Phi(\cdot)$.

**Theorem 3** *(Setwise Convergence on Closed Sets and Global Uniform Countable Additivity)*

*Given that the state space $(\Omega, B(\Omega))$ of networks and coalitions is a compact metric space, if the law of motion is such that $q(F|\cdot,\cdot)$ is continuous on the graph of $\Phi(\cdot)$ for all $d_\Omega$-closed sets $F$ of network-coalition pairs (i.e., if A-3(ii) (b) holds), then*

$$p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$$

*is globally uniformly countable additive.*

**Proof.** Let $M(\Omega)$ denote the Banach space of bounded measurable functions on $(\Omega, B(\Omega))$, equipped with the sup norm and let $rca(\Omega)$ denote the Banach space of finite signed Borel measures on $(\Omega, B(\Omega))$. First, observe that the set of probability measures

$$\Pi_\Phi := \{q(\cdot|\omega, G_D) : (\omega, G_D) \in Gr\Phi(\cdot)\}$$

is sequentially compact in the $\sigma(rca(\Omega), M(\Omega))$ topology. This follows because $Gr\Phi(\cdot)$ is a compact metric space and because by Delbaen's Lemma (1974),

$$(\omega^n, G_D^n) \to (\overline{\omega}, \overline{G}_D))$$

implies that

$$\int_\Omega v(\omega')q(d\omega'|\omega^n, G_D^n) \to \int_\Omega v(\omega')q(d\omega'|\overline{\omega}, \overline{G}_D)) \text{ for all } v(\cdot) \ \in M(\Omega).$$

By Corollary 2.2 in Lasserre (1998), therefore,

$$\lim_{k\to\infty} \sup_{(\omega,G_D)\in Gr\Phi(\cdot)} \int_\Omega v_k(\omega')q(d\omega'|\omega, G_D) = 0 \tag{41}$$

whenever $v_k(\cdot) \downarrow 0$, $v_k(\cdot) \in M(\Omega)$.

To see that (41) implies global uniform countable additivity (40), consider a sequence $\{B_k\}_k \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_k \downarrow \emptyset$) and let $v_k(\cdot) := I_{B_k}(\cdot)$, where

$$I_{B_k}(\omega) = \begin{cases} 1 & \text{if } \omega \in B_k \\ 0 & \text{if } \omega \notin B_k. \end{cases}$$

We have $I_{B_k}(\cdot) \downarrow 0$, $I_{B_k}(\cdot) \in M(\Omega)$ and

$$\int_\Omega v_k(\omega')q(d\omega'|\omega, G_D) = q(B_k|\omega, G_D).$$

Finally, for each $k$ let $(\omega^k, G_D^k) \in Gr\Phi(\cdot)$ be such that

$$q(B_k|\omega^k, G_D^k) = \sup_{(\omega, G_D) \in Gr\Phi(\cdot)} q(B_k|\omega, G_D).$$

We have for all $\omega \in \Omega$,

$$p^*(B_k|\omega) = q(B_k|\omega, f_D^*(\omega)) \leq q(B_k|\omega^k, G_D^k) \to 0.$$

■

**Remarks 1:** *Alternatively, global uniform countable additivity will be guaranteed if instead of assuming A-3(ii) (b) (stochastic continuity), we assume that the densities $f(\cdot|\omega, G_D)$ of $q(\cdot|\omega, G_D)$ with respect to the dominating probability measure $\mu$ are uniformly integrably bounded (i.e., that A-3 (iii) (b) holds). Under this assumption, we have for any sequence $\{B_n\}_n \subset B(\Omega)$ decreasing to $\emptyset$ (i.e., $B_n \downarrow \emptyset$),*

$$p^*(B_n|\omega) = \textstyle\int_{B_n} dq(\omega'|\omega, f_D^*(\omega))$$

$$= \textstyle\int_{B_n} f(\omega'|\omega, f_D^*(\omega))d\mu(\omega')$$

$$\leq \textstyle\int_{B_n} h(\omega')d\mu(\omega') \to 0 \text{ as } B_n \downarrow \emptyset.$$

Let A-3′ denote the altered set of assumptions A-3 (i.e., altered either by strengthening A-3 (ii) (a) to A-3 (ii) (b) or by maintaining to A-3 (ii) (a) and adding A-3 (iii) (b)).[17] By Theorems 1 and 2, under assumptions [A-3'], there is an equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of network and coalition formation and by Theorem 3 it is globally uniformly countably additive. Moreover, letting $C = \Omega$, $V(\omega) = 1$ for all $\omega \in \Omega$, and $b = 2$, the drift condition is also satisfied. Thus, by strengthening our assumptions concerning the law of motion $q(\cdot|\cdot, \cdot)$ slightly beyond what is required to guarantee the existence of an equilibrium Markov transition, $p^*(\cdot|\cdot)$, we are able to conclude in Theorem 3 that the Tweedie conditions are satisfied globally (i.e., with $C = \Omega$).

[17] Under assumption A-3 (iii) (b), neither A-3 (ii) (a) or (b) is required for global uniform countable additivity. But A-3 (ii) (a) is required for existence.

# 5 Basins of Attraction, Invariance, and Ergodicity

We now have our first result concerning stochastic basins of attraction and the stability of the emergent network-coalition formation process

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

governed by $p^*(\cdot|\cdot)$.

**Theorem 4** (*Basins of Attraction: The Finite Decomposition of the State Space*)

*Under assumptions [A-1], [A-2] and [A-3'], the emergent network-coalition formation process*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$ generates a finite decomposition of the state space of network-coalition pairs $\Omega = \mathbb{G} \times \mathcal{F}$ into a finite number of disjoint basins of attraction and a disjoint transient set. In particular, this decomposition is given by*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T, \tag{42}$$

*where each $H_i$ is a maximal Harris set and $T$ is transient. Moreover,*

$$L^*(\omega, \cup_i H_i) = 1 \tag{43}$$

*for every network-coalition pair $\omega \in \Omega$.*

By Theorem 4, the emergent network-coalition formation process $\{W_n^*\}_n$ is such that starting at any network-coalition pair not contained in a basin of attraction, the process will reach some basin $H_i$ in a finite number of moves with probability 1, and once there will stay there with probability 1. An analogous conclusion is reached in Page and Wooders (2007) for static, abstract games of network formation over finitely many networks. There it is shown that no matter what rules of network formation prevail, given any profile of player preferences the feasible set of networks contains a finite, disjoint collection of sets each set representing a *strategic* basin of attraction in the sense that if the game is repeated - each time starting at the status quo network reached in the previous play of the game - the process of network formation generated by repeating this static game will reach a network contained in some strategic basin and once there will stay there.

**Proof.** Because the Tweedie conditions hold globally under our assumptions [A-1], [A-2] and [A-3'], by Theorem 2 in Tweedie (2001), the state space $\Omega$ admits a *finite* decomposition

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

where each $H_i$ is indecomposable and Maximal Harris and $T$ is transient. Moreover, by Theorem 2 in Tweedie (2001), this Harris decomposition is such that

$$L^*(\omega, \cup_{i=1}^{N} H_i) = 1$$

for all $\omega \in \Omega$. ■

**Theorem 5** (*Invariance and Ergodicity of the Process of Network and Coalition Formation)*

*Suppose assumptions [A-1], [A-2] and [A-3'] hold. Let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*be the emergent network-coalition formation process governed by the equilibrium Markov transition* $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$*, and let*

$$\Omega = \left(\cup_{i=1}^{N} H_i\right) \cup T,$$

*be the corresponding finite Harris decomposition.*

*The following statements are true:*

*(1) Corresponding to each basin of attraction* $H_i$*, there is a unique* $p^*$*-invariant probability measure* $\lambda_i(\cdot)$ *with* $\lambda_i(H_i) = 1$*. Moreover, for each network-coalition pair* $\omega = (G, S)$*,*

$$p^{*(n)}(E|\omega) := \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L^*(\omega, H_i)\lambda_i(E \cap H_i), \text{ for all } E \in B(\Omega). \tag{44}$$

*where* $p^{*k}(E|\omega)$ *is defined recursively, see (29).*

*(2) The set of all ergodic probability measures is given by*

$$\mathcal{E}^* = \{\lambda_i(\cdot)\}_{i=1}^{N}.$$

*Moreover, a probability measure* $\lambda(\cdot)$ *on* $(\Omega, B(\Omega))$ *is* $p^*$*-invariant, i.e.* $\lambda(\cdot) \in \mathcal{I}^*$*, if and only if* $\lambda(\cdot)$ *is given by*

$$\lambda(E) = \sum_{i}^{N} \lambda(H_i)\lambda_i(E \cap H_i), \text{ for all } E \in B(\Omega). \tag{45}$$

*(3)* $\mathcal{E}^*$ *is a singleton (i.e.,* $\mathcal{E}^* = \{\lambda(\cdot)\}$*) if and only if the network-coalition formation process* $\{W_n^*\}_n$ *is* $\psi$*-irreducible, in which case for each network-coalition pair* $\omega = (G, S)$ *and for every set of network-coalition pairs* $E \in B(\Omega)$

$$\frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \lambda(E).$$

**Proof.** (1) Under our assumptions [A-1], [A-2] and [A-3'] (see the proof of Theorem 3 above), $p^*(\cdot|\cdot)$ satisfies the Tweedie conditions globally. As a result, the first statement in part (1) is an immediate consequence of Lemma 5 in Tweedie (2001). The second statement also follows from the Tweedie conditions holding globally and Theorem 1 in Tweedie (2001) (also, see Chapter 13 in Meyn and Tweedie (1993)).

(2) Again because the Tweedie Conditions are satisfied globally, the first statement in part (2) follows from Lemma 2 in Tweedie (2001), Theorem 2.18 part (1) in Costa and Dufour (2005), Theorem 3.8 in Costa and Dufour, and the proof of Proposition 5.3 in Costa and Dufour. The second statement in part (2), that $\lambda(\cdot) \in \mathcal{I}^*$ implies (45), follows from the proof of Proposition 5.3 in Costa and Dufour (2005). The fact that (45) implies $\lambda(\cdot) \in \mathcal{I}^*$ follows from observation (but also, see Theorem 19.25 in Aliprantis and Border (1999) and Theorem 2 in Villareal (2004)).

(3) Finally, because the Tweedie Conditions are satisfied globally, necessary and sufficient conditions for $\mathcal{E}^*$ to be a singleton, given in terms of $\psi$-irreducibility follow from Theorem 3 in Tweedie (2001). The convergence result in part (3) follows from the convergence result in part (1) of the Theorem and the fact that if there is only one basin of attraction $H$ (i.e., one maximal Harris set), then by Theorem 4, $L^*(\omega, H) = 1$ for all $\omega \in \Omega$. ■

Note that the probability measures in $\mathcal{E}^*$ are *orthogonal*, that is, for all $i$ and $i'$ in $\{1, 2, \ldots, N\}$ with $i \neq i'$,

$$\lambda_i(\Omega \backslash H_i) = \lambda_{i'}(H_i) = 0.$$

## 5.1 Ergodic Properties of the Strategic Values

For each starting network-coalition pair $\omega = (G, S) \in \Omega$, $w_d^*(\omega)$ is the strategic value to player $d$ of following his part of the stationary equilibrium strategies $f_D^*(\cdot)$, given that all other players follow their parts of the strategy. Because $f_D^*(\cdot)$ is Nash, we know this is the best that player $d$ can do relative to all other strategies, even those that are history dependent. Strategies $f_D^*(\cdot)$ together with the trembles of nature determine the equilibrium Markov process of network and coalition formation via the transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$. The questions we wish to address in this section concern the properties of players' strategic values across time and states given the equilibrium process of network and coalition formation.

We begin by considering time averages. Let

$$p^{*(n)} w_d^*(\omega) = \frac{1}{n} \sum_{k=1}^{n} \int_{\Omega} w_d^*(\omega') p^{*k}(d\omega'|\omega) = \int_{\Omega} w_d^*(\omega') p^{*(n)}(d\omega'|\omega),$$

where recall,

$$w_d^*(\omega) = E_d(f_D^*)(\omega) := \textstyle\sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(f_D^*)(\omega)$$

$$= r_d(\omega, f_D^*(\omega)) + \beta_d \textstyle\int_{\Omega} w_d^*(\omega') dq(\omega'|\omega, f_D^*(\omega))$$

and

$$p^{*(n)}(E|\omega) = \tfrac{1}{n} \textstyle\sum_{k=1}^{n} p^{*k}(E|\omega) = \tfrac{1}{n} \sum_{k=1}^{n} \int_{\Omega} p^*(E|\omega') p^{*k-1}(d\omega'|\omega).$$

Here, $p^{*k}(E|\omega)$ is the probability that process reaches the set of network-coalition pairs $E$ starting at network-coalition pair $\omega = (G, S)$ in $k$ periods or moves.

The function $p^{*(n)} w_d^*(\cdot)$ specifies for each starting network-coalition pair, player $d$'s $n$-period time average expected strategic value (i.e., the average value of following his part of the stationary equilibrium strategies $f_D^*(\cdot)$ for $n$ moves). We can think of $\lim_n p^{*(n)} w_d^*(\cdot)$ therefore as specifying for each starting network-coalition pair, player $d$'s time average expected value.

By part (1) of Theorem 5 above, we have *for all* $\omega \in \Omega$ and $E \in B(\Omega)$

$$p^{*(n)}(E|\omega) = \frac{1}{n}\sum_{k=1}^{n} p^{*k}(E|\omega) \xrightarrow{n} \sum_{i=1}^{N} L^*(\omega, H_i)\lambda_i(E \cap H_i) = \lambda_\omega(E), \tag{46}$$

where $\lambda_\omega(\cdot) \in \mathcal{I}^*$ for all $\omega \in \Omega$ and $\{\lambda_i(\cdot) : i = 1, 2, \ldots, N\} = \mathcal{E}^*$. Because $p^{*(n)}(\cdot|\omega)$ converges setwise for all $\omega$, by Delbaen's Lemma (1974) we have *for all* $\omega \in \Omega$

$$p^{*(n)} w_d^*(\omega) \to \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega') d\lambda_i(\omega'). \tag{47}$$

Thus, we obtain one of the fundamental principles of equilibrium dynamics: the equality of time averages and state averages.

**Theorem 6** *(The Equality of Time Average Values and State Average Values)*

*Under assumptions [A-1], [A-2] and [A-3'] the emergent network-coalition formation process*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*governed by the equilibrium Markov transition* $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$ *is such that:*

*(1) for each player* $d$ *starting at any network-coalition pair* $\omega = (G, S)$ *contained in a basin of attraction* $H_i$ *the time average value of the equilibrium strategies* $f_D^*$ *is equal to state average value of the equilibrium strategies, that is, for all basins of attraction* $H_i$ *and for all initial states* $\omega = (G, S) \in H_i$,

$$\underbrace{\lim_n p^{*(n)} w_d^*(\omega)}_{time\ average} = \underbrace{\int_{H_i} w_d^*(\omega') d\lambda_i(\omega').}_{state\ average} \tag{48}$$

*Moreover, for all initial states* $\omega = (G, S) \in \Omega$,

$$\lim_n p^{*(n)} w_d^*(\omega) = \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega') d\lambda_i(\omega') \tag{49}$$

*(2) For all invariant measures* $\lambda(\cdot) \in \mathcal{I}^*$

$$\int_\Omega g_d^*(\omega') d\lambda(\omega') = \int_\Omega w_d^*(\omega') d\lambda(\omega'), \tag{50}$$

*where*

$$g_d^*(\omega) := \sum_{i=1}^{N} L^*(\omega, H_i) \int_{H_i} w_d^*(\omega') d\lambda_i(\omega') \text{ for all } \omega \in \Omega. \tag{51}$$

**Proof.** (1) Part (1) is an immediate consequence of part (1) of Theorem 5, Delbaen's Lemma (1974), and the fact that for all basins $H_i$ and all states $\omega \in H_i$, $L^*(\omega, H_i) = 1$.

(2) Let invariant probability measure $\lambda(\cdot) = \sum_{i=1}^N \lambda(H_i)\lambda_i(\cdot) \in \mathcal{I}^*$ be given. We have

$$\int_\Omega w_d^*(\omega')d\lambda(\omega') = \sum_{i=1}^N \lambda(H_i) \int_\Omega w_d^*(\omega')d\lambda_i(\omega') = \sum_{i=1}^N \lambda(H_i) \int_{H_i} w_d^*(\omega')d\lambda_i(\omega')$$

and

$$\int_\Omega g_d^*(\omega')d\lambda(\omega') = \sum_{i=1}^N \lambda(H_i) \int_\Omega g_d^*(\omega')d\lambda_i(\omega') = \sum_{i=1}^N \lambda(H_i) \int_{H_i} g_d^*(\omega')d\lambda_i(\omega')$$

Letting $\int_{H_i} w_d^*(\omega')d\lambda_i(\omega') := w_d^*(H_i)$, we have

$$\int_{H_i} g_d^*(\omega')d\lambda_i(\omega') = \int_{H_i} \left[\sum_{i=1}^N L^*(\omega', H_i)w_d^*(H_i)\right] d\lambda_i(\omega').$$

Moreover, because for all $\omega' \in H_i$, $L^*(\omega', H_i) = 1$ and $L^*(\omega', H_{i'}) = 0$, for all $i' \neq i$,

$$\int_{H_i} \left[\sum_{i=1}^N L^*(\omega', H_i)w_d^*(H_i)\right] d\lambda_i(\omega') = w_d^*(H_i) = \int_{H_i} w_d^*(\omega')d\lambda_i(\omega').$$

Thus we have for each $i$

$$\int_{H_i} g_d^*(\omega')d\lambda_i(\omega') = \int_{H_i} w_d^*(\omega')d\lambda_i(\omega'),$$

and thus,

$$\int_\Omega g_d^*(\omega')d\lambda(\omega') = \sum_{i=1}^N \lambda(H_i) \int_{H_i} g_d^*(\omega')d\lambda_i(\omega')$$

$$= \sum_{i=1}^N \lambda(H_i) \int_{H_i} w_d^*(\omega')d\lambda_i(\omega')$$

$$= \int_\Omega w_d^*(\omega')d\lambda(\omega').$$

■

Also see Birkhoff's Ergodic Theorems (pointwise and mean), for example, Theorems 2.3.4 and 2.3.5 in Hernandez-Lerma and Lasserre (2003)).

By part (1) of Theorem 5, each player's time average value $\lim_n p^{*(n)}w_d^*(\omega) = g_d^*(\omega)$ is constant with respect to the starting network-coalition pair on each basin of attraction. In particular,

$$\lim_n p^{*(n)}w_d^*(\omega) = \int_\Omega w_d^*(\omega')d\lambda(\omega') = \int_{H_i} w_d^*(\omega')d\lambda_i(\omega') \text{ for all } \omega \in H_i.$$

By part (2) of Theorem 5, for any given invariant probability measure each player's average of time averages over the entire state space is equal to his state average over the entire state space with respect to the given measure.

# 6 Strategic Stability and Dynamic Consistency

Again let $f_D^*(\cdot) = (f_d^*(\cdot))_{d\in D}$ be a stationary equilibrium of the dynamic network-coalition formation game with corresponding equilibrium Markov transition $p^*(\cdot|\cdot) =$

$q(\cdot|\cdot, f_D^*(\cdot))$, and let

$$\Omega = \left(\cup_{i=1}^N H_i\right) \cup T,$$

be the finite state space decomposition generated by $p^*(\cdot|\cdot)$ with basins of attraction $\{H_1, \ldots, H_N\}$ and transient set $T$. Finally, let $\mathcal{E}^* = \{\lambda_i(\cdot)\}_{i=1}^N$ be the corresponding set of ergodic probability measures with $\lambda_i(H_i) = 1$ for all $i$.

Each player's strategy

$$\omega = (G, S) \to f_d^*(G, S)$$

governs the way in which player $d$ tries to influence the process of network and coalition formation across time and for each given status quo coalition $S$, $f_d^*(\cdot, S)$ is a *deterministic* equilibrium transition on networks governing player $d$'s network proposal process. For each status quo coalition $S$, we will refer to the equilibrium transitions, $(f_d^*(\cdot, S))_{d \in D}$, as the *S-proposal transitions* and we will refer to the induced equilibrium Markov network-coalition transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$, as the *state transition*.

To begin, let

$$\mathcal{L}(f_d^*(\cdot, S)) := \mathcal{L}(\delta_{f_d^*(\cdot, S)})$$

denote the set of absorbing sets corresponding to player $d$'s $S$-proposal transition $f_d^*(\cdot, S)$. Here, as before, $\delta_{f_d^*(\cdot, S)}$ denotes the Dirac Young measure with probability mass concentrated on $f_d^*(\cdot, S)$. If the set of networks $\mathbb{E}$ is an absorbing set for player $d$ under $S$-proposal transition $f_d^*(\cdot, S)$, then for any status quo network $G \in \mathbb{E}$, it is optimal for player $d \in S$ to propose either the status quo network or a particular new network $G'$ in $\mathbb{E}$ with probability 1. Thus,

$$\mathbb{E} \in \mathcal{L}(f_d^*(\cdot, S)) \text{ if and only if } f_d^*(G, S) \in \mathbb{E} \text{ for all } G \in \mathbb{E}.$$

Moreover, by assumption A-1(i)(b) if $d \notin S$, then player $d$ is constrained to propose only the status quo network. Thus, if for some player $d$ in coalition $S$, $\mathbb{E} \in \mathcal{L}(f_d^*(\cdot, S))$, then for *any* player $d$ not in coalition $S$ (i.e., $d \in D \backslash S$), $f_d^*(G, S) \in \mathbb{E}$ for all status quo networks $G \in \mathbb{E}$.[18] If in addition, $\mathbb{E}$ is absorbing for all players in $S$, that is, if $\mathbb{E} \in \cap_{d \in S} \mathcal{L}(f_d^*(\cdot, S))$, then for all status quo networks $G \in \mathbb{E}$, it is optimal for *all* players (even those not in coalition $S$) to propose a network contained in $\mathbb{E}$. Note, however, that unless $\mathbb{E}$ is a singleton (i.e., $\mathbb{E} = \{G\}$ for some network $G \in \mathbb{G}$), players may not agree on their individual network proposals. However, if $\mathbb{E}$ is absorbing for all members of $S$ then at least all members will agree that their proposals should be contained in $\mathbb{E}$. Thus, we can think of the sets in $\cap_{d \in S} \mathcal{L}(f_d^*(\cdot, S))$ as being *strategically stable* for coalition $S$ - as long as coalition $S$ is the status quo coalition. We will denote by $\mathcal{L}(f_S^*)$ the intersection $\cap_{d \in S} \mathcal{L}(f_d^*(\cdot, S))$ and we will refer to $\mathcal{L}(f_S^*)$ as an $S$-strategically stable set.

[18] In fact, for all states $\omega = (G, S)$ and for all players $d \notin S$, the singleton sets $\{G\}$ are absorbing for the $S$ - proposal transitions

$$(f_d^*(\cdot, S))_{d \in D \backslash S}.$$

Let $\mathcal{C}$ be a subcollection of the feasible coalitions $\mathcal{F}$. We will say that a set of networks $\mathbb{E}$ is $\mathcal{C}$-strategically stable if it is $S$-strategically stable for all coalitions $S \in \mathcal{C}$, that is, if

$$\mathbb{E} \in \cap_{S \in \mathcal{C}} \mathcal{L}(f_S^*) := \mathcal{L}_{\mathcal{C}}(f_S^*),$$

and we will say that $\mathbb{E}$ is strategically stable if $\mathcal{C} = \mathcal{F}$.[19] Thus, if $\mathbb{E}$ is $\mathcal{C}$-strategically stable, then in any status quo state $\omega = (G, S)$ with $G \in \mathbb{E}$ and $S \in \mathcal{C}$, all players in $S$ will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in $S$ will be constrained (under the rules of network formation) to propose the status quo network $G$ - also a network in $\mathbb{E}$. Moreover, the same will be true in any other state $\omega' = (G', S')$ with $G' \in \mathbb{E}$ and $S' \in \mathcal{C}$, that is, all players in $S'$ will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in $S$ will be constrained to propose the status quo network $G$.

Finally, suppose the $\mathcal{C}$-strategically stable set of networks $\mathbb{E}$ is such that nature chooses with probability 1 network-coalition pairs from $\mathbb{E} \times \mathcal{C}$ starting from any status quo network-coalition pair contained in $\mathbb{E} \times \mathcal{C}$; that is, suppose that in addition to $\mathbb{E}$ being $\mathcal{C}$-strategically stable, that $\mathbb{E} \times \mathcal{C}$ is absorbing for the state transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$. We will refer to a $\mathcal{C}$-strategically stable set of networks $\mathbb{E}$ as being *$\mathcal{C}$-dynamically consistent* if $\mathbb{E} \times \mathcal{C}$ is absorbing for $p^*(\cdot|\cdot)$. Thus, a set of networks $\mathbb{E} \in \mathcal{L}_{\mathcal{C}}(f_S^*)$ is $\mathcal{C}$-dynamically consistent if $\mathbb{E} \times \mathcal{C} \in \mathcal{L}^*$, where as before $\mathcal{L}^*$ is the collection of absorbing sets corresponding to the state transition $p^*(\cdot|\cdot)$ (see expression (30)).

We have the following formal definitions.

**Definitions 5** (*$\mathcal{C}$-Strategic Stability and $\mathcal{C}$-Dynamic Consistency*)

*(1) ($\mathcal{C}$-Strategic Stability)*

*A set of networks $\mathbb{E} \in B(\mathbb{G})$ is $\mathcal{C}$-strategically stable (i.e., $\mathbb{E} \in \mathcal{L}_{\mathcal{C}}(f_S^*)$) if in all states $(G, S) \in \mathbb{E} \times \mathcal{C}$ each player $d \in S$ proposes a network in $\mathbb{E}$, that is, if*

$$\text{for all } (G, S) \in \mathbb{E} \times \mathcal{C},\ f_d^*(G, S) \in \mathbb{E} \text{ for all } d \in S.$$

*(2) ($\mathcal{C}$-Dynamic Consistency)*

*A $\mathcal{C}$-strategically stable set of networks $\mathbb{E} \in B(\mathbb{G})$ is $\mathcal{C}$-dynamically consistent if in all states $(G, S) \in \mathbb{E} \times \mathcal{C}$ nature chooses states in $\mathbb{E} \times \mathcal{C}$ with probability 1, that is, if*

$$\text{for all } (G, S) \in \mathbb{E} \times \mathcal{C},\ p^*(\mathbb{E} \times \mathcal{C} | G, S) = 1.$$

*(3) (Strategic Stability and Dynamic Consistency)*

*An $\mathcal{F}$-strategically stable set of networks $\mathbb{E} \in B(\mathbb{G})$ is dynamically consistent if it is $\mathcal{F}$-dynamically consistent.*

[19] Thus, $\mathcal{L}_{\mathcal{C}}(f_S^*) := \cap_{S \in \mathcal{C}} \left[\cap_{d \in S} \mathcal{L}(f_d^*(\cdot, S))\right]$

The following result gives necessary conditions for dynamic consistency. The proof is straightforward.

**Theorem 7** *(Dynamic Consistency and Invariance)*

*Suppose assumptions [A-1], [A-2] and [A-3'] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$.*

*If $\mathbb{E} \in B(\mathbb{G})$ is dynamically consistent, then starting at any network-coalition pair contained in $E := \mathbb{E} \times \mathcal{F}$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $E \cap H_i$, where $H_i$ is a basin of attraction and once there will remain there. Moreover, there exists a $p^*$-invariant probability measure which assigns positive measure to $E \cap H_i$.*

Note that $E \cap H_i$ is absorbing for the state transition $p^*(\cdot|\cdot)$; that is, $E \cap H_i \in \mathcal{L}^*$. Moreover, note that it is possible for $E$ to intersect more than one basin of attraction, but because each basin of attraction is indecomposable, each basin of attraction can intersect only one such set $E := \mathbb{E} \times \mathcal{F}$ where $\mathbb{E}$ is dynamically consistent. It is also possible for $E$ to intersect the transient set - but it is not possible for $E$ to be a subset of the transient set. If $E$ intersects basins $H_i$ and $H_{i'}$, and $\lambda(\cdot)$ is a $p^*$-invariant measure such that $\lambda(E) = 1$, then by part (2) of Theorem 6 above we have,

$$\lambda(E) = \sum_{i''}^{N} \lambda(H_{i''}) \lambda_{i''}(E \cap H_{i''}) = \lambda(H_i)\lambda_i(E \cap H_i) + \lambda(H_{i'})\lambda_{i'}(E \cap H_{i'}).$$

Thus, under any $p^*$-invariant measure $\lambda(\cdot)$ the measure of any absorbing set $E$ is a weighted sum of the probability masses the invariant measures $\lambda(\cdot)$ assigns to each basin $H_i$.

## 6.1 Dynamic Path dominance Core and Dynamic Pairwise Stability

One way to extend the definition of the path dominance core introduced in Page and Wooders (2007) to the dynamic setting considered here is as follows:

**Definition 6** (*The Dynamic Path Dominance Core*)

*A network $G^* \in \mathbb{G}$ is in the dynamic path dominance core if the set $\{G^*\}$ is dynamically consistent, that is,* $\{G^*\} \in \mathcal{L}_{\mathcal{F}}(f_S^*)$ and $\{G^*\} \times \mathcal{F} \in \mathcal{L}^*$.

Thus a network $G^*$ is in the dynamic path dominance core, then in state $(G^*, S)$, $S \in \mathcal{F}$, all members of the status quo coalition $S$ propose network $G^*$ and nature, while perhaps choosing a different status quo coalition, $S' \in \mathcal{F}$, follows the proposal of coalition $S$ and chooses $G^*$ as the next status quo network. Thus, if the network-coalition formation process reaches a state $(G^*, S)$ where the status quo network is in the path dominance core, then all future movements of the process are movements from one feasible status quo coalition to another.

We have the following result giving necessary conditions for a network to be in the path dominance core (i.e., to be dynamically consistent).

**Theorem 8** *(The Dynamic Path Dominance Core and Invariance)*

*Suppose assumptions [A-1], [A-2] and [A-3'] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*be the emergent network-coalition formation process governed by the equilibrium Markov transition* $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$.

*If network* $G^* \in \mathbb{G}$ *is in the dynamic path dominance core, that is, if* $\{G^*\}$ *is dynamically consistent, then starting at any network-coalition pair contained in* $\{G^*\} \times \mathcal{F}$*, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs* $(\{G^*\} \times \mathcal{F}) \cap H_i$*, where* $H_i$ *is a basin of attraction and once there will remain there. Moreover, there exists a* $p^*$*-invariant probability measure which assigns positive measure to* $(\{G^*\} \times \mathcal{F}) \cap H_i$*.*

Note that if for some network $G^* \in \mathbb{G}$ and some coalition $S^* \in \mathcal{F}$, $\{G^*\} \in \mathcal{L}(f_{S^*}^*)$ and $\{(G^*, S^*)\} \in \mathcal{L}^*$, so that $\{G^*\}$ is $\{S^*\}$-dynamically consistent, this does not necessarily imply that $G^*$ is in the dynamic path dominance core, even if $\{(G^*, S^*)\}$ is a basin of attraction, because $\{G^*\}$ may not be dynamically consistent. Why? Because while nature will choose with probability 1 the network-coalition pair $(G^*, S^*)$ if the status quo is $(G^*, S^*)$, if the status quo coalition is not $S^*$, that is, if the status quo state is $(G^*, S')$ for some coalition $S' \in \mathcal{F}$ not equal to $S^*$, some players in $S'$ may propose a network other than $G^*$ (i.e., it may be the case that $G^* \notin \mathcal{L}(f_d^*(\cdot, S'))$ for some player $d \in S'$) and in turn nature may choose a state other than $(G^*, S^*)$. Moreover, if $G^*$ is not strategically stable, but nonetheless $\{G^*\} \times \mathcal{C} \in \mathcal{L}^*$ for some subset of coalitions $\mathcal{C} \subseteq \mathcal{F}$, then if the equilibrium network-coalition formation process reaches any state $(G^*, S) \in \{G^*\} \times \mathcal{C}$, the process will remain in the set $\{G^*\} \times \mathcal{C}$ - despite network proposals to the contrary by players in coalitions in $\mathcal{C}$. In such a case, the state transition overrides the wishes of the players. This leads to the following alternative notion of dynamic path dominance core.

**Definition 6'** (*The State Transition Core*)

*(1) (State Transition Core) A network* $G^* \in \mathbb{G}$ *is in the state transition core if the set of states* $\{G^*\} \times \mathcal{F} \in B(\Omega)$ *is an absorbing set for the state transition* $p^*(\cdot|\cdot)$.

*(2) (Weak State Transition Core) A network* $G^* \in \mathbb{G}$ *is in the weak state transition core if the set of states* $\{G^*\} \times \mathcal{C} \in B(\Omega)$ *is an absorbing set for the state transition* $p^*(\cdot|\cdot)$ *for some subset of coalitions* $\mathcal{C} \subseteq \mathcal{F}$.

Under the definition of weak state transition core, for any basin of attraction $H_{i*}$ of the form $H_{i*} = \{(G^*, S^*)\}$, $G^*$ is in the weak state transition core.[20] Moreover, if for some state transition absorbing set $E$, $E \cap H_{i*}$ is nonempty but $E$ is disjoint from the other basins, then starting at any network-coalition pair in $E$, the process will reach in finite time with probability 1 the network-coalition pair $(G^*, S^*)$ and will remain there.

Finally, note that if $p^*(\{G^*\} \times \mathcal{C}|G^*, S) = 1$ for all $S \in \mathcal{C} \subseteq \mathcal{F}$, then because the law of motion

$$q(\cdot|(G, S), G_D)$$

is absolutely continuous with respect the probability measure $\mu = \nu \times \gamma$ for all $((G, S), G_D) \in Gr\Phi(\cdot)$, $G^*$ must be an atom of the probability measure $\nu$, that is,

$$\{G^*\} \in \{\mathbb{A}_{\alpha 1}, \mathbb{A}_{\alpha 2}, \ldots\} = \{\mathbb{A}_{\alpha k}\}_{k=1}^{\infty} \subset \mathbb{G}.$$

To extend the definition of the pairwise stability introduced in Jackson and Wolinsky (1996) to the dynamic setting considered here, we begin by specializing the feasible set of coalitions to coalitions of size no greater than 2.

**Definition 7** (*Dynamic Pairwise Stability*)

*Suppose the feasible set of coalitions is given by*

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

*(i.e., all feasible coalitions consist of at most two players). Then a network $G^* \in \mathbb{G}$ is dynamically pathwise stable if the set $\{G^*\}$ is dynamically consistent, that is, if $\{G^*\} \in \mathcal{L}_{\mathcal{F}_2}(f_S^*)$* and *$\{G^*\} \times \mathcal{F}_2 \in \mathcal{L}^*$.*

We have the following characterization

**Theorem 9** *(Dynamic Pairwise Stability and Invariance)*

*Suppose assumptions [A-1], [A-2] and [A-3'] hold and let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

*be the emergent network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, f_D^*(\cdot))$.*

*If network $G^* \in \mathbb{G}$ is dynamically pairwise stable, that is, if $\{G^*\}$ is dynamically consistent, then starting at any network-coalition pair contained in $\{G^*\} \times \mathcal{F}_2$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $(\{G^*\} \times \mathcal{F}_2) \cap H_i$, where $H_i$ is a basin of attraction and once there will remain there. Moreover, there exists a $p^*$-invariant probability measure which assigns positive measure to $(\{G^*\} \times \mathcal{F}_2) \cap H_i$.*

[20] An alternative name for this notion of dynamic path dominance core might be the unhappy core - especially if the state transition consistently goes against the network proposals of the players.

Our conclusion that for some basin of attraction $H_i$, $(\{G^*\} \times \mathcal{F}_2) \cap H_i$ is contained in the support of some $p^*$-invariant measure is similar to the conclusion reached by Jackson and Watts (2002) for a stochastic process of network formation over a finite set of linking networks governed by Markov chain generated by myopic players. They reach their conclusion by considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain. This method is similar to a method introduced into games by Young (1993) which in turn is based on some very general perturbation methods found in Freidlin and Wentzell (1984). Here we have reached a similar conclusions without using perturbation methods.

### 6.2 An Example (Continued): Dynamic Club Networks and Bang-Bang Time Allocation Strategies

Consider again the affine model of the dynamic club time allocation game introduced in subsection 3.3. Recall that in this example, $D = \{d_1, \ldots, d_m\}$ is the set of players, $C = \{c_1, \ldots, c_n\}$ is the set of clubs, and

$$\mathbb{K}_R := \left\{ G \in P_f(R^L \times (D \times C)) : \forall (d, c) \in D \times C,\ |G(dc)| = 1 \right\},$$

is the set of regular club networks. Also, recall that the set of regular club time allocation networks given by

$$\mathbb{K} = \{G \in \mathbb{K}_R : \forall (d, c) \in D \times C,\ G(dc) \subseteq [0, 1] \text{ and } \sum_{c \in C} G(dc) = 1\},$$

where a connection $(G(dc), (d, c)) := (a_{dc}, (d, c))$ in club network $G \in \mathbb{K}$ means that in club network $G$, player $d$ allocates $a_{dc}$ percent of his time to club $c$.

Each club time allocation network $G \in \mathbb{K}$ has a unique alternative representation as the union of regular player club time allocation networks, $G = \cup_{d' \in D} g_{d'}$, where for each player $d$, a regular club time allocation network $g_d$ is a nonempty closed subset of $[0, 1] \times (\{d\} \times C)$ such that for all clubs $c \in C$,

$$|g(dc)| = 1,\ g(dc) \subseteq [0, 1], \text{ and } \sum_{c \in C} g(dc) := \sum_{c \in C} a_{dc} = 1.$$

Let $\mathbb{K}_d$ denote the collection of all regular player club time allocation networks for player $d$.

Under noncooperative club network formation rules, we take as the feasible set of player coalitions, $\mathcal{F}_1$, and thus the state space is given by $\Omega = \mathbb{K} \times \mathcal{F}_1$. Given state $\omega = (G, d_j) \in \mathbb{K} \times \mathcal{F}_1$, player $d_i's$ proposal constraint mapping is given by

$$\Phi_{d_i}(G, d_j) = \begin{cases} \{G' := (g'_{d_j}, g_{-d_j}) \in \mathbb{K} : g'_{d_j} \in \mathbb{K}_{d_j}\} & \text{if } d_i = d_j \\ \{G\} & \text{if } d_i \neq d_j, \end{cases}$$

where $G = (g_d, g_{-d})$ is the regular club time allocation network $G$ with player club time allocation representation $\cup_d g_d$. Thus, each player's constraint mapping, $\Phi_{d_i}(\cdot)$, satisfies assumption A-1.

Next, assume that each player's payoff function is measurable in states and continuous and affine in players' network proposals. Thus, player payoff functions satisfy assumptions A-2 (1) and (3).

Finally, assume that the law of motion, $q(d\omega'|\omega, G_D)$, is given by the product of conditional probabilities,

$$q(d\omega'|\omega, G_D) := \varphi(dG'|\omega)p(d'_j|\omega, G_D),$$

such that for all feasible coalitions $\{d_j\} \in \mathcal{F}_1$, $p(d_j|\cdot,\cdot)$ is continuous on the graph of the proposal constraint mapping $\Phi(\cdot)$ and affine in proposals on $\Phi(\omega)$ for all $\omega \in \Omega$, and such that the probability measures $\varphi(dG'|\omega)$, $\omega \in \Omega$, are absolutely continuous with respect to $\nu$ and have integrably bounded densities. Recall that $\nu$ is the dominating probability measure on $(\mathbb{G}, B(\mathbb{G}))$, the measurable space of feasible networks. Thus, the law of motion satisfies A-3'

By Theorem 2, because our dynamic game of club network formation is affine and continuous, it has a Nash equilibrium in pure stationary bang-bang strategies. Thus, there exists stationary strategies $f^*_D(\cdot) = (f^*_{d_i}(\cdot))_{d_i \in D}$ such that for all players $d_i$ in all states $\omega = (G, d_j)$,

$$f^*_{d_i}(G, d_j) \in ext\Phi_{d_i}(G, d_j) = \begin{cases} \{G^{c_{j1}}_{d_j}, \ldots, G^{c_{jn}}_{d_j}\} & \text{if } d_i = d_j \\ \{G\} & \text{if } d_i \neq d_j, \end{cases}$$

where $G^{c_{jk}}_{d_j}$ denotes the club network where player $d_j$ allocates 100 percent of his time to some club $c_{jk} \in C$ and the club time allocations of all other players, $d_i \neq d_j$, remain fixed at their status quo network values. In general, then, for all players, $d_i$, and all states, $\omega = (G, d_j)$,

$$f^*_{d_i}(G, d_j) \in \{G^{c_{j1}}_{d_j}, \ldots, G^{c_{jn}}_{d_j}; G\}.$$

The alternative representation of the set $\{G^{c_{j1}}_{d_j}, \ldots, G^{c_{jn}}_{d_j}; G\}$ using player club time allocation networks is given by

$$\{G^{c_{j1}}_{d_j}, \ldots, G^{c_{jn}}_{d_j}; G\} = \{(g^{c_{j1}}_{d_j}, g_{-d_j}), \ldots, (g^{c_{jn}}_{d_j}, g_{-d_j}); (g_{d_j}, g_{-d_j})\},$$

where the player club time allocation network $g^{c_{jk}}_{d_j} \in \mathbb{K}_{d_j}$ is such that player $d_j$ allocates *all* of his time to club $c_{jk} \in C$ and where the other club time allocation networks, corresponding to the other players ($\neq d_j$), are as they are in the status quo club network, $G = (g_{d_j}, g_{d_{-j}})$. We will refer to a player club time allocation network $g^{c_j}_{d_j} \in ext\mathbb{K}_{d_j}$, $c_j \in C$, (where player $d_j$ allocates all his time to club $c_j$) as an extreme player club time allocation network. Moreover, we will refer to any club time allocation network $G \in \mathbb{K}$ as an extreme club time allocation network (or simply as an extreme club network - or single-membership club network) if $G$ can be written as $G = (g^{c_1}_{d_1}, \ldots, g^{c_m}_{d_m})$ where for each player $d_j$, $g^{c_j}_{d_j} \in ext\mathbb{K}_{d_j}$, $c_j \in C$.

Our last result is about the types of club networks which arise in the long run in continuous and affine dynamic club formation games.The proof of this result follows from our definition of dynamic consistency and Theorem 6.

**Theorem 10** *(Pure Stationary Bang-Bang Strategies and the Long Run Stability of Single-Membership Club Networks)*

*Suppose assumptions [A-1], [A-2 (1) and (3)]-[A-3 (i), (ii)(a), (iii)(a)(b), and (iv)] hold. Let $f_D^*(\cdot)$ be a Nash equilibrium in pure stationary bang-bang strategies for the noncooperative game of club network formation played over club time allocation networks, and let*

$$q(d\omega'|\omega, f_D^*(\omega)) = \varphi(dG'|\omega)p(d_j'|\omega, f_D^*(\omega)) \; \forall \omega = (G, d_j)$$

*be the induced equilibrium Markov transition governing the process of club network formation. Finally, let*

$$\Omega = \left(\cup_{i=1}^N H_i\right) \cup T,$$

*be the corresponding finite Harris decomposition.*

*If in each state, $\omega = (G, d_j) = ((g_{d_j}, g_{-d_j}), d_j)$, the support of the probability measure $\varphi(dG'|(G, d_j))$ is contained in*

$$\{G_{d_j}^{c_{j1}}, \ldots, G_{d_j}^{c_{jn}}; G\} = \{(g_{d_j}^{c_{j1}}, g_{-d_j}), \ldots, (g_{d_j}^{c_{jn}}, g_{-d_j}); (g_{d_j}, g_{-d_j})\},$$

*then the set of extreme club networks $\mathbb{X} \in B(\mathbb{G})$ is dynamically consistent and therefore starting at any network-coalition pair contained in $X := \mathbb{X} \times \mathcal{F}$, the network-coalition formation process will reach in finite time with probability 1 a nonempty subset of network-coalition pairs $X \cap H_i$, where $H_i$ is a basin of attraction and once there will remain there. Moreover, there exists a p*-invariant probability measure which assigns positive measure to $X \cap H_i$.*

It follows from Theorem 10 that starting at any club network-coalition pair $\omega \in \Omega$,

$$L^*(\omega, \cup_{i=1}^N (X \cap H_i)) = 1.$$

Thus, in the long run stable club structures are those where players devote all their time to a single club. The intuition behind this conclusion is that because player payoffs are affine in club network proposals, rather than concave, for example, players have no incentive to diversify their club network proposals.

# 7 Proofs of Theorems 1 and 2

## 7.1 Preliminaries

To begin let $\mathcal{V}$ be the set of all $\mu$-equivalence classes of $B(\Omega)$-measurable functions, $v(\cdot) : \Omega \to [-M, M]$ called value functions. Because the state space of network-coalition pairs, $\Omega$, is a compact metric space, the space of $\mu$-equivalence classes of $\mu$-integrable functions, $\mathcal{L}_1(\Omega, B(\Omega), \mu)$, is separable. As a consequence the set of value functions $\mathcal{V}$ is a compact, convex, and metrizable subset of $\mathcal{L}_\infty(\Omega, B(\Omega), \mu)$ for the weak star topology $\sigma(\mathcal{L}_\infty, \mathcal{L}_1)$. Letting

$$\mathcal{V}^m = \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{m:=|D| \text{ times}},$$

$\mathcal{V}^m$ equipped with the product topology $\sigma_m(\mathcal{L}_\infty, \mathcal{L}_1)$ is also compact, convex, and metrizable.

Given status quo state $\omega \in \Omega$, $m$-tuple of probability measures $\sigma = (\sigma_d) \in \Pi_{d \in D}\mathcal{P}(\Phi_d(\omega))$, and $m$-tuple of value functions $v = (v_d) \in \mathcal{V}^m$ define

$$u_d(\omega, \sigma)(v_d) := (1 - \beta_d) r_d(\omega, \sigma) + \beta_d \int_\Omega v_d(\omega') dq(\omega'|\omega, \sigma).$$

The proof will proceed in 5 steps:

*Step 1*: Let

$$V(\omega, \sigma)(v) := \textstyle\sum_d \left( u_d(\omega, (\sigma_d, \sigma_{-d}))(v_d) - \max_{\eta \in \mathcal{P}(\Phi_d(\omega))} u_d(\omega, (\eta, \sigma_{-d}))(v_d) \right), \tag{52}$$

and consider the correspondence $\omega \to N_v(\omega)$ where

$$N_v(\omega) := \{\sigma : V(\omega, \sigma)(v) = 0\}. \tag{53}$$

Note that $\sigma = (\sigma_d) \in N_v(\omega)$ if and only if for each player $d \in D$,

$$u_d(\omega, (\sigma_d, \sigma_{-d}))(v_d) \geq u_d(\omega, (\eta, \sigma_{-d}))(v_d) \text{ for all } \eta \in \mathcal{P}(\Phi_d(\omega)).$$

Thus, $\omega \to N_v(\omega)$ is the Nash correspondence. Given stochastic continuity assumption [A-3](ii)(a) it follows from Delbaen's Lemma (1974) that

$$(G_d) \to \int_\Omega v_d(\omega') dq(\omega'|\omega, (G_d))$$

is continuous for any $v_d(\cdot) \in \mathcal{V}$. Thus, for $\omega \in \Omega$ and $v_d(\cdot) \in \mathcal{V}$

$$\sigma \to u_d(\omega, \sigma)(v_d) \text{ and } \sigma \to V(\omega, \sigma)(v)$$

are continuous on $\Pi_{d \in D}\mathcal{P}(\Phi_d(\omega))$ with respect to the compact and metrizable topology of weak convergence of probability measures. Thus, for all $\omega \in \Omega$ and $v(\cdot) \in \mathcal{V}^m$, $N_v(\omega)$ is a nonempty, compact subset of $\Pi_{d \in D}\mathcal{P}(\Phi_d(\omega))$ and by Theorem 6.4 in Himmelberg (1975) $N_v(\cdot)$ is measurable.

*Step 2:* Consider the induced payoff correspondence given by

$$P_v(\omega) := \{(U_d) \in R^m : (U_d) = (u_d(\omega, \sigma)(v_d)) \text{ for some } \sigma \in N_v(\omega)\}. \tag{54}$$

By Theorem 6.5 in Himmelberg (1975) the payoff correspondence $\omega \to P_v(\omega)$ is measurable with nonempty, compact values, and by Theorem 9.1 in Himmelberg (1975) the correspondence

$$\omega \to coP_v(\omega) \tag{55}$$

is measurable with nonempty, compact convex values.

*Step 3*: The Nowak-Raghavan Lemma.

Let $\Sigma(coP_v(\cdot))$ be the set of all $\mu$-equivalence classes of measurable selectors of $\omega \rightarrow coP_v(\omega)$, $v \in \mathcal{V}^m$. The Nowak-Raghavan Lemma states that the payoff selection correspondence $v \rightarrow \Sigma(coP_v(\cdot))$ is upper semicontinuous with nonempty convex, weakly compact values. Convexity, weak compactness, and nonemptiness are straightforward. We need only prove upper semicontinuity. To this end, let $Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$ denote the graph of the payoff selection correspondence and let $\{(U^n(\cdot), v^n(\cdot))\}_n$ be a sequence in $Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$ converging weakly to $(U^*(\cdot), v^*(\cdot))$. In order to establish that the payoff selection correspondence is upper semicontinuous we must show that $(U^*(\cdot), v^*(\cdot)) \in Gr\left\{\Sigma(coP_{(\cdot)}(\cdot))\right\}$, that is, we must show that $U^*(\omega) \in coP_{v^*}(\omega)$ a.e. $[\mu]$.

The proof of this lemma proceeds in three steps:

*First*, given state $\omega \in \Omega$ and sequence $v^n(\cdot) \rightarrow v^*(\cdot)$, let $\{\sigma^n(\omega)\}_n$ be a sequence in $\Pi_{d \in D}\mathcal{P}\left(\Phi_d(\omega)\right)$ such that $\sigma^n(\omega) \in N_{v^n}(\omega)$ for all $n$. Without loss of generality, suppose that $\sigma^n(\omega) \rightarrow \sigma^*(\omega) \in \Pi_{d \in D}\mathcal{P}\left(\Phi_d(\omega)\right)$ with respect to the topology of weak convergence of probability measures. Then for all players $d \in D$,

$$u_d(\omega, \sigma^n(\omega))(v_d^n) \rightarrow u_d(\omega, \sigma^*(\omega))(v_d^*).$$

To see this, observe the following:

$$\begin{aligned} & |u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^*)| \\ \leq & \underbrace{|u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^n)|}_{A^n} + \underbrace{|u_d(\omega, \sigma^*(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^*)|}_{B^n}. \end{aligned}$$

$$\begin{aligned} & \underbrace{|u_d(\omega, \sigma^n(\omega))(v_d^n) - u_d(\omega, \sigma^*(\omega))(v_d^n)|}_{A^n} \\ & \leq M\beta_d \left| \int_\Omega \int_{\Phi(\omega)} dq(\omega'|\omega, (G_d'))d\sigma^n((G_d')|\omega) \right. \\ & \left. - \int_\Omega \int_{\Phi(\omega)} dq(\omega'|\omega, (G_d'))d\sigma^*((G_d')|\omega) \right| \\ & = M\beta_d \left| \int_{\Phi(\omega)} q(\Omega|\omega, (G_d'))d\sigma^n((G_d')|\omega) \right. \\ & \left. - \int_{\Phi(\omega)} q(\Omega|\omega, (G_d'))d\sigma^*((G_d')|\omega)) \right|. \end{aligned}$$

$$\underbrace{|u_d(\omega,\sigma^*(\omega))(v_d^n) - u_d(\omega,\sigma^*(\omega))(v_d^*)|}_{B^n}$$

$$= \beta_d \Big| \int_\Omega \int_{\Phi(\omega)} v_d^n(\omega')dq(\omega'|\omega,(G'_d))d\sigma^*((G'_d)|\omega)$$

$$- \int_\Omega \int_{\Phi(\omega)} v_d^*(\omega')dq(\omega'|\omega,(G'_d))d\sigma^*((G'_d)|\omega)\Big|$$

$$= \beta_d \Big| \int_{\Phi(\omega)} \int_\Omega v_d^n(\omega')dq(\omega'|\omega,(G'_d))d\sigma^*((G'_d)|\omega)$$

$$- \int_{\Phi(\omega)} \int_\Omega v_d^*(\omega')dq(\omega'|\omega,(G'_d))d\sigma^*((G'_d)|\omega)\Big| .$$

By Delbaen's Lemma, $q(\Omega|\omega,(G_d))$ is continuous in $(G_d)$. Thus, since $\sigma^n(\cdot|\omega) \to \sigma^*(\cdot|\omega)$ with respect to weak convergence of probability measures,

$$M\beta_d \int_{\Phi(\omega)} q(\Omega|\omega,(G'_d))d\sigma^n((G'_d)|\omega) \to M\beta_d \int_{\Phi(\omega)} q(\Omega|\omega,(G'_d))d\sigma^*((G'_d)|\omega)),$$

so that $A^n \to 0$.

Next, given that the probability measures $q(\cdot|\omega,(G_d))$ are absolutely continuous with respect to probability measure $\mu$, $v^n(\cdot) \to v^*(\cdot)$ weakly implies that for each $(G_d)$

$$\int_\Omega v_d^n(\omega')dq(\omega'|\omega,(G_d)) \to \int_\Omega v_d^*(\omega')dq(\omega'|\omega,(G_d)).$$

In particular, we have for each $(\omega,(G_d))$

$$F^n(\omega,(G_d)) := \int_\Omega v_d^n(\omega')dq(\omega'|\omega,(G_d)) = \int_\Omega v_d^n(\omega')f(\omega'|\omega,(G_d))d\mu(\omega')$$

$$\to \int_\Omega v_d^*(\omega')f(\omega'|\omega,(G_d))d\mu(\omega') = \int_G v_d^*(\omega')dq(\omega'|\omega,(G_d)) := F^*(\omega,(G_d)),$$

where $f(\omega'|\omega,(G_d))$ is the density of $q(\omega'|\omega,(G_d))$ with respect to $\mu$. Thus, by the Dominated Convergence Theorem

$$\beta_d \int_{\Phi(\omega)} F^n(\omega,(G_d))d\sigma^*((G_d)|\omega) \to \beta_d \int_{\Phi(\omega)} F^*(\omega,(G_d))d\sigma^*((G_d)|\omega)),$$

so that $B^n \to 0$.

Therefore, we conclude that if $v^n(\cdot) \to v^*(\cdot)$ and $\sigma^n(\omega) \to \sigma^*(\omega)$, then for all players $d \in D$,

$$u_d(\omega,\sigma^n(\omega))(v_d^n) \to u_d(\omega,\sigma^*(\omega))(v_d^*).$$

*Second*, we have $U^n(\cdot) \to U^*(\cdot)$ weakly where for all $n$, $U^n(\cdot) \in \Sigma(coP_{v^n}(\cdot))$, and $v^n(\cdot) \to v^*(\cdot)$ weakly where for all $n$, $v^n(\cdot) \in \mathcal{V}^m$. By Proposition 1 in Page (1991), we can assume without loss of generality that for some $\mu$ null set $N$ (i.e., $\mu(N) = 0$)

$$\frac{1}{n}\sum_{k=1}^{n} U^k(\omega) \to U(\omega) \text{ and } U(\omega) \in coLs\{U^n(\omega)\} \text{ for all } \omega \in \Omega \backslash N.$$

Here "*co*" denotes convex hull and $Ls\{U^n(\omega)\}$ is the set of limit point of the sequence $\{U^n(\omega)\}_n$. Now let $U^*(\cdot)$ be a measurable selector of $coLs\{U^n(\cdot)\}$ such that $U^*(\omega) = U(\omega)$ for all $\omega \in \Omega \backslash N$. Thus, $U^*(\omega) \in coLs\{U^n(\omega)\}$ for all $\omega \in \Omega$. By Theorem 8.2 in Wagner (1977) $U^*(\cdot)$ has a Caratheodory representation

$$U^*(\omega) = \sum_{i=0}^{m} \alpha^{*i}(\omega) U^{*i}(\omega)$$

where the $R^m$-valued functions $U^{*0}(\cdot), U^{*1}(\cdot), \ldots, U^{*m}(\cdot)$ are measurable selectors of $Ls\{U^n(\cdot)\}$ and the nonnegative functions $\alpha^{*0}(\cdot), \alpha^{*1}(\cdot), \ldots, \alpha^{*m}(\cdot)$ are measurable with $\sum_{i=0}^{m} \alpha^{*i}(\omega) = 1$ for all $\omega$. Thus, for each $i$ and each $\omega$, $U^{n_k}(\omega) \to U^{*i}(\omega)$ for some subsequence $\{U^{n_k}(\omega)\}_k$.

*Third*, the proof that the payoff selection correspondence $v \to \Sigma(coP_v(\cdot))$ is upper semicontinuous, will be complete if we show that $U^{*i}(\omega) \in coP_{v^*}(\omega)$. To accomplish this, we need the following Lemma (*): If $U^n(\omega) \to U^{*i}(\omega)$ where $U^n(\omega) \in coP_{v^n}(\omega)$ for all $n$ and $v^n(\cdot) \to v^*(\cdot)$ weakly, then $U^{*i}(\omega) \in coP_{v^*}(\omega)$.

Proof of Lemma (*): Again by Theorem 8.2 in Wagner (1977) each $U^n(\cdot)$ has a Caratheodory representation

$$U^n(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega) U^{ni}(\omega)$$

where each $U^{ni}(\omega) \in P_{v^n}(\omega)$. Thus, for each $n$, there exists $\sigma_D^{ni}(\omega) \in N_{v^n}(\omega)$ such that $U^{ni}(\omega) = (u_d(\omega, \sigma_D^{ni}(\omega))(v_d^n))$. Without loss of generality assume that

$$(\alpha^{n0}(\omega), \alpha^{n1}(\omega), \ldots, \alpha^{nm}(\omega)) \underset{n}{\rightarrow} (\alpha^{*0}(\omega), \alpha^{*1}(\omega), \ldots, \alpha^{*m}(\omega))$$

and

$$(\sigma_D^{n0}(\omega), \sigma_D^{n1}(\omega), \ldots, \sigma_D^{nm}(\omega)) \underset{n}{\rightarrow} (\sigma_D^{*0}(\omega), \sigma_D^{*1}(\omega), \ldots, \sigma_D^{*m}(\omega)).$$

Now we have

$$U^{ni}(\omega) = (u_d(\omega, \sigma_D^{ni}(\omega))(v_d^n)) \underset{n}{\rightarrow} (u_d(\omega, \sigma_D^{*i}(\omega))(v_d^*)) \in P_{v*}(\omega).$$

Thus,

$$U^n(\omega) = \textstyle\sum_{i=0}^{m} \alpha^{ni}(\omega) U^{ni}(\omega) = \sum_{i=0}^{m} \alpha^{ni}(\omega)(u_d(\omega, \sigma_D^{ni}(\omega))(v_d^n))$$

$$\underset{n}{\rightarrow} \textstyle\sum_{i=0}^{m} \alpha^{*i}(\omega)(u_d(\omega, \sigma_D^{*i}(\omega))(v_d^*)) = U^{*i}(\omega) \in coP_{v^*}(\omega),$$

and we can conclude that

for all $\omega$,

$$U^*(\omega) = \textstyle\sum_{i=0}^{m} \alpha^{*i}(\omega) U^{*i}(\omega) \in coP_{v^*}(\omega),$$

completing the proof of the Nowak-Raghavan Lemma.

*Step 4:* Applying the Kakutani-Glicksberg Fixed Point Theorem (1952) to $v \to \Sigma(coP_v(\cdot))$ we obtain an $m$-tuple of value functions

$$v(\cdot) = (v_d(\cdot)) \in \mathcal{V}^m$$

such that

$$v(\omega) \in coP_v(\omega) \text{ for all } \omega \in \Omega \backslash N \text{ where } \mu(N) = 0.$$

Let $v^*(\cdot) = (v_d^*(\cdot)) \in \mathcal{V}^m$ be a measurable selection of $coP_v(\cdot)$ such that $v^*(\omega) = v(\omega)$ for all $\omega \in \Omega \backslash N$. Thus, $v^*(\omega) \in coP_v(\omega)$ for all $\omega \in \Omega$ and because $coP_v(\omega) = coP_{v^*}(\omega)$ for all $\omega \in \Omega$, we have $v^*(\omega) \in coP_{v^*}(\omega)$ for all $\omega \in \Omega$.

With Step 4 showing that there exists $v^*(\cdot) = (v_d^*(\cdot)) \in \mathcal{V}^m$, such that

$$v^*(\omega) \in coP_{v^*}(\omega) \text{ for all } \omega \in \Omega,$$

we are ready to begin proofing our main Theorems. The proof of both Theorems 1 and 2 reduce to showing under the additional assumptions of Theorems 1 and 2 that either $P_{v^*}(\omega)$ is a singleton for all $\omega \in \Omega$ (Theorem 1) or that $coP_{v^*}(\omega) = P_{v^*}(\omega)$ for all $\omega \in \Omega$ (Theorem 2).

## 7.2 The Existence of Nash Equilibrium in Pure Stationary Strategies (Strictly Concave Case)

Under assumptions *[A-1], [A-2 (1) and (2)]-[A-3(i), (ii)(a), and (iii)(a)]*, our it follows from Lemma 2 in Nowak (2007) that the Nash correspondence (53),

$$\omega \to N_{v^*}(\omega) := \{\sigma : V(\omega, \sigma)(v^*) = 0\},$$

is single-valued and contains a *unique pure strategy*. This implies that the Nash payoff correspondence (54),

$$\omega \to P_{v^*}(\omega)$$
$$P_{v^*}(\omega) := \{(U_d) \in R^m : (U_d) = (u_d(\omega, \sigma)(v_d^*)) \text{ for some } \sigma \in N_{v^*}(\omega)\}$$

is also single-valued.

By the Measurable Implicit Function Theorem (Theorem 7.1 in Himmelberg (1975)), there exists a measurable selection of $N_{v^*}(\cdot)$, that is, a measurable function

$$\omega \to f_D^*(\omega) \in N_{v^*}(\omega) \text{ for all } \omega \in \Omega$$

such that for each player $d \in D$ and $\omega \in \Omega$

$$v_d^*(\omega) = u_d(\omega, f_D^*(\omega))(v_d^*) := (1 - \beta_d) r_d(\omega, f_D^*(\omega)) + \beta_d \int_\Omega v_d^*(\omega') dq(\omega' | \omega, f_D^*(\omega)).$$

For $d \in D$, let $w_d^*(\cdot) := \frac{v_d^*(\cdot)}{1-\beta_d}$. Substituting, we have for all $\omega \in \Omega$

$$w_d^*(\omega) = r_d(\omega, f_D^*(\omega)) + \beta_d \int_\Omega w_d^*(\omega') dq(\omega' | \omega, f_D^*(\omega)). \text{ (**)}$$

By classical results on discounted dynamic programming (e.g., Blackwell (1965)), we conclude from (**) that (i) for all players $d \in D$ and all starting states $\omega \in \Omega$

$$w_d^*(\omega) = E_d(f_D^*)(\omega) := \sum_{n=1}^{\infty} \beta_d^{n-1} r_d^n(f_D^*)(\omega),$$

and therefore that (ii) for all players $d \in D$ and all starting states $\omega \in \Omega$

$$E_d(f_d^*, f_{-d}^*)(\omega) \geq E_d(\pi_d, f_{-d}^*)(\omega) \text{ for all } \pi_d \in \Pi_d^\infty.$$

■

## 7.3 The Existence of Nash Equilibrium in Pure Stationary Bang-Bang Strategies (The Affine Case)

Under assumptions *[A-1], [A-2 (1) and (3)]-[A-3 (i), (ii)(a), (iii)(a), and (iv)]*, our it follows from Corollary 1.4 in Balder (1997) that the Nash correspondence (53),

$$\omega \rightarrow N_{v^*}(\omega) := \{\sigma : V(\omega, \sigma)(v^*) = 0\},$$

is nonempty, closed, convex-valued and contains *pure strategies*. This implies, together with affinity, that the Nash payoff correspondence (54),

$$\omega \rightarrow P_{v^*}(\omega)$$
$$P_{v^*}(\omega) := \{(U_d) \in R^m : (U_d) = (u_d(\omega, \sigma)(v_d^*)) \text{ for some } \sigma \in N_{v^*}(\omega)\}$$

is also nonempty, closed, convex-valued. Thus,

$$P_{v^*}(\omega) = coP_{v^*}(\omega) \text{ for all } \omega \in \Omega.$$

By Corollary 1.4 in Balder (1997) and the Measurable Implicit Function Theorem (Theorem 7.1 in Himmelberg (1975)), there exists a measurable selection $f_D^*(\cdot)$ of $N_{v^*}(\cdot)$ such that

$$\omega \rightarrow f_D^*(\omega) \in ext\Phi(\omega) \text{ for all } \omega \in \Omega,$$

such that for each player $d \in D$ and $\omega \in \Omega$

$$v_d^*(\omega) = u_d(\omega, f_D^*(\omega))(v_d^*) := (1 - \beta_d) r_d(\omega, f_D^*(\omega)) + \beta_d \int_\Omega v_d^*(\omega') dq(\omega' | \omega, f_D^*(\omega)).$$

The rest of the proof is exactly as the proof of Theorem 1. ■

# References


[1] Aliprantis, C. D. and Border, K. C. (1999) *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer-Verlag, Berlin-Heidelberg.

[2] Amir, R. (1991) "On Stochastic Games with Uncountable State and Action Spaces," In T. E. S. Raghavan, et al. (eds.), *Stochastic Games and Related Topics in Honor of Professor L. S. Shapley*, Kluwer Academic: Dordrecht.

[3] Amir, R. (1996) "Continuous Stochastic Games of Capital Accumulation with Convex Transitions," *Games and Economic Behavior* 15, 11-131.

[4] Amir, R. and Lambson, V. (2003) "Entry, Exit, and Imperfect Competition in the Long Run," *Journal of Economic Theory* 110, 191-203.

[5] Aubin, J. P. (2007) *Mathematical Methods of Game and Economic Theory*, revised edition, Dover, Mineola.

[6] Bala, V. and Goyal, S. (2000) "A Noncooperative Model of Network Formation," *Econometrica* 68, 1181-1229.

[7] Balder, E. J. (1997) "Remarks on Nash Equilibria for Games with Additively Coupled Payoffs," *Economic Theory* 9, 161-167.

[8] Ballester, C., Calvo-Armengol, A. and Zenou, Y. (2006) "Who's Who in Networks. Wanted: The Key Player," *Econometrica* 74, 1403-1417.

[9] Blackwell, D. (1965) "Discounted Dynamic Programming," *Annals of Mathematical Statistics* 36, 226-235.

[10] Chakrabarti, S. (1999) "Markov Equilibria in Discounted Stochastic Games," *Journal of Economic Theory* 85, 294-327.

[11] Chakrabarti, S. (2008) "Pure Strategy Equilibrium in Stochastic Games with Concave Transition Probabilities," Department of Economics IUPUI, typescript.

[12] Chwe M. (1994) "Farsighted Coalitional Stability," *Journal of Economic Theory* 63, 299-325.

[13] Costa, O. L. V. and Dufour, F. (2005) "On the Ergodic Decomposition for a Class of Markov Chains," *Stochastic Processes and Their Applications* 115, 401-415.

[14] Delbaen, F. (1974) "Continuity of Expected Utility," In J. Dreze, J. Mirrlees, and E. Malinvaud (eds.), *Allocation under Uncertainty: Equilibrium and Optimality*, Macmillan: London.

[15] Doeblin, W. (1937) "Sur les Proprietes Asymtotiques de Mouvement Regis par Certains Types de Chains Simples," Bull. math. Soc. Roumaine Sci. 39, No.1, 57-115, No. 2, 3-61.

[16] Doeblin, W. (1940) "Elements d'une Theorie generale des Chaines Simple Constantes de Markoff," Ann. sci. Ecole norm. sup., III. Ser., 57, 61-111.

[17] Duffie, D, Geanakoplos, Mas-Colell, A., and McLennan, A. (1994) "Stationary Markov Equilibria," *Econometrica* 62, 745-781.

[18] Dutta, B. and S. Mutuswami (1997) "Stable Networks," *Journal of Economic Theory* 76, 322-344.

[19] Dutta, B., Ghosal, S., and Ray, D. (2005) "Farsighted Network Formation," *Journal of Economic Theory* 122, 143-164.

[20] Freidlin, M., and Wentzell, A. (1984) *Random Perturbations of Dynamical Systems*, pp 161-211, Springer-Verlag, New York.

[21] Glicksberg, I. L. (1952) "A Further Generalization of the Kakutani Fixed Point Theorem with Application to Nash Equilibrium Points," *Proceedings of the American Mathematical Society* 3, 170-174.

[22] Herings, P. J.-J., and Peeters, Ronald, J. A. P. (2004) "Stationary Equilibria in Stochastic Games: Structure, Selection, and Computation," *Journal of Economic Theory* 118, 32-60.

[23] Hernandez-Lerma, O, and Lasserre, J. B. (2003) *Markov Chains and Invariant Probabilities*, Birkhauser Verlag, Basel.

[24] Himmelberg, C. J. (1975) "Measurable Relations," *Fundamenta Mathematicae* LXXXVII, 53-72.

[25] Himmelberg, C. J., Parthasarathy, T., and VanVleck, F. S. (1976) "Optimal Plans for Dynamic Programming Problems," *Mathematical of Operations Research* 1, 390-394.

[26] Jackson, M. O. and van den Nouweland, A. (2005) "Strongly Stable Networks," *Games and Economic Behavior* 51, 420-444.

[27] Jackson, M. O. and A. Wolinsky (1996) "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74.

[28] Jackson, M. and Watts, A. (2002) "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 106, 265-295.

[29] Konishi, H. and Ray, D. (2003) "Coalition Formation as a Dynamic Process," *Journal of Economic Theory* 110, 1-41.

[30] Lasserre, J. B. (1998) "Weak Convergence of Probability Measures: A Uniform Principle," *Proceedings of the American Mathematical Society* 126, 3089-3096.

[31] Mertens, J.-F. and Parthasarathy, T. (1987) "Equilibria for Discounted Stochastic Games," CORE Discussion Paper 8750, Universite Catholique de Louvain.

[32] Mertens, J.-F. and Parthasarathy, T. (1991) "Nonzero-sum Stochastic Games," In T. E. S. Raghavan, et al. (eds.), *Stochastic Games and Related Topics in Honor of Professor L. S. Shapley*, Kluwer Academic: Dordrecht.

[33] Meyn, S. P. and Tweedie, R. L. (1993) *Markov Chains and Stochastic Stability*, Springer-Verlag, London.

[34] Nowak, A. S. (2003) "On a New Class of Nonzero-Sum Discounted Stochastic Games Having Stationary Nash Equilibrium Points," *International Journal of Game Theory* 32, 121-132.

[35] Nowak, A. S. (2007) "On Stochastic Games in Economics," *Mathematical Methods of Operations Research* 66, 513-530.

[36] Nowak, A. S. and Raghavan, T. E. S. (1992) "Existence of Stationary Correlated Equilibria with Symmetric Information for Discounted Stochastic Games," *Mathematics of Operations Research* 17, 519-526.

[37] Nummelin, E. (1984) *General Irreducible Markov Chains and Non-Negative Operators*, Cambridge University Press, Cambridge.

[38] Orey, S. (1971) *Lecture Notes on Limit Theorems for Markov Chains on a General State Space*, Van Nostrand Reinhold, London.

[39] Page, F. H., Jr. (1991) "Komlos Limits and Fatou's Lemma in Several Dimensions," *Canadian Mathematical Bulletin* 34, 109-112.

[40] Page, F. H., Jr. (2009a) "Network Formation Games, Endogenous Dynamics, and Strategic Social Interactions," CAEPR Working Paper No. 2009-000.

[41] Page, F. H., Jr. (2009b) "Strategic Social Interactions and Endogenous Markov Supernetworks," CAEPR Working Paper No. 2009-000.

[42] Page, F. H., Jr. and Wooders, M. (2005) "Strategic Basins of Attraction, the Farsighted Core, and Network Formation Games," FEEM Working Paper 36.05.

[43] Page, F. H., Jr. and Wooders, M. (2007) "Networks and Clubs," *Journal of Economic Behavior and Organization* 64, 406-425.

[44] Page, F. H., Jr. and Wooders, M. (2009a) "Strategic Basins of Attraction, the Path Dominance Core, and Network Formation Games," *Games and Economic Behavior,* 66, 462-487.

[45] Page, F. H., Jr. and Wooders, M. (2009b) "Club Networks with Multiple Memberships and Noncooperative Stability," *Games and Economic Behavior,* in press, doi:10.1016/j.geb.2009.07.006.

[46] Page, F. H., Jr., Wooders, M., Kamat, S. (2005), "Networks and Farsighted Stability," *Journal of Economic Theory* 120, 257-269.

[47] Parthasarathy, K. R. (1967) *Probability Measures on Metric Spaces*, Academic Press, New York.

[48] Skyrms, B. and Pemantle, R. (2000) "A Dynamic Model of Social Network Formation," Proceedings of the National Academy of Sciences 97, 9340-9346.

[49] Tweedie, R. L. (2001) "Drift Conditions and Invariant Measures for Markov Chains," *Stochastic Processes and Their Applications* 92, 345-354.

[50] Villareal, Cesar, E. (2004) "Ergodicity and Harris Decomposition," *Technical Report PISIS*-2004-03, Graduate Program in Systems Engineering, UANL, San Nicolás de los Garza, México, August.

[51] Wagner, D. H. (1977) "Survey of Measurable Selection Theorems," *SIAM Journal of Control and Optimization* 15, 859-903.

[52] Watts, A. (2001) "A Dynamic Model of Network Formation," *Games and Economic Behavior* 34, 331-341.

[53] Young, H. P. (1993) "The Evolution of Conventions," *Econometrica* 61, 57-84.