Spatial distribution of economic activities and transboundary pollution

(Very preliminary draft – please do not quote)

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Abstract – We investigate the impacts of transboundary pollution in terms of economic agglomerations by extending the short-run equilibrium results of Ottaviano's (2001) core-periphery model. Our aim is to show that the range of possible spatial configurations is enlarged when we consider parameters of environmental pollution. Besides the three standard NEG cases related to transport costs level, this will permit us to identify other cases depending on environmental parameters value.

I. Introduction

Following the pioneering Krugman (1991)'s Core-Periphery (C-P hereafter) model [1], a large literature has studied last years the endogenous spatial distribution of economics activities (for a state of the art see Fujita, Krugman, Venables [2] and Fujita, Thisse [3]). These works have highlighted the various centripetal and centrifugal forces which are acting on firms and mobile workers location when transport costs are decreasing. They confirm the tendency to geographical concentration.
Nonetheless, little attention has been devoted to the role of environmental pollution in the agglomeration or the spreading of economic activities while various empirical studies confirm that people retain environmental quality in their location choice (see [4] and [5]). The decline in the population of large industrial agglomerations has been also confirmed in a context where the secular decrease of transport costs seems to slow down.

This paper addresses these questions introducing in the Ottaviano (2001)'s model[1] [6] an environmental damage accompanying the production of differentiated manufactured goods. Previous attempts to include environmental issues in New Economic Geography (NEG) models focus on local pollution and distinguish mobile skilled workers from immobile unskilled ones. In particular, Van Marrewijk [8] shows that in the presence of pollution emissions in both industry and agriculture the range of parameters for which a C-P equilibrium is stable will be reduced compared to a model without pollution. Lange and Quaas [9] support the existence of medium-sized industrial cities in case of purely

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[1] See also Forslid and Ottaviano [7] for a quite similar resolution.

local pollution.
Our own contribution based on Ottaviano [6] – as Van Marrewijk's one [8] – generalizes the result from this papers and identifies other spatial configurations by introducing transboundary pollution. Transboundary pollution is an important feature of many contemporary pollution problems in which pollutants from production in one country damage the environment in another country: for example, this is the case with the well-known problems of acid rain originating from air pollution in Great Britain and Germany responsible to acid precipitation on forest and freshwater in Scandinavian countries.

Our aim here is twofold:

i. we authorize local and transboundary industrial pollution in the analytically solved model of Ottaviano (2001) and retain that mobile skilled workers are subject to indirect utility differential;
ii. we study short-run and long-run equilibria[2] taking into account pollution parameters besides traditional transport costs. We thus provide a formal analysis of the introduction of transboundary industrial pollution by determining the stability conditions for three types of pollution. Each type of pollution is defined in terms of the share of pollution domestically or foreignly located: low-transboundary pollution, symmetric distribution of pollution and higher-transboundary pollution. Thus, we do not consider normative aspects of transboundary pollution, like consequences of environmental policies on the agglomeration of activities[3].

The remainder of our contribution is organised as follows. Section II is devoted to the presentation of our model. Consumer behaviour and profit's maximization are described in section III. Section IV analyses the differential indirect utility of migrants. In section V different cases are envisaged according to the range of key parameters (transport costs and pollution ones). Section VI concludes.

II. Structure of the model

We develop a model sensibly similar to the Ottaviano's (2001) C-P model except for the presence of a pollution flow resulting from production and which affects the instantaneous utility function.

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[2] Following the Krugman [1] definitions: a short-run equilibrium is one in which former skilled workers, unskilled workers and firms maximize utility and profits taking location as given and a long-run equilibrium is one in which skilled workers also maximize utility with respect to location so that they have no incentive to move.
[3] See Elbers and Withagen [10] for a model explaining population dynamics and agglomeration in compliance with environmental policy, in the case of what we designed below by a "purely local pollution", and a review of the literature on this subject.

The economy consists of two countries, A and B, with fixed endowments of unskilled labor $2L$ and skilled labor H. The unskilled labor is geographically immobile and evenly distributed between locations, each hosting L unskilled workers. Skilled workers are mobile. By setting $H = 1$ by choice of units, the share of skilled workers in A, $h \in [0; 1]$, is also the number of skilled workers in A and $1 - h$ is the number of skilled workers in B. The skilled labor wage in location i; $i = \{A; B\}$ is designed by R_i.
The two countries have access to the same level of technology and each country can produce two types of goods, a differentiated, or modern, good D and a homogeneous, or traditional, good Y.

The differentiated good is produced in a monopolistic sector under increasing returns to scale using both skilled and unskilled labor[4]. Trade costs in the differentiated good are modelled as iceberg costs $\tau \geq 1$ in the Samuelson [11]'s way. A fixed number of skilled workers is used in the production of differentiated goods such that the cost function includes a fixed cost due to skilled workers and a marginal cost k corresponding to unskilled workers used in production. The total cost of production of a quantity x_i is finally given by $C(x_i) = gR_i + kx_i$, with $g = 1$.
Because of increasing returns, differentiated production and free entry hypothesis, the equilibrium is characterised by a number of firms equal to the number of varieties n_i ; $i = \{A; B\}$. Moreover, $g = 1$ induces that the number of firms equals the number of skilled workers: $n_A = h$ and $n_B = 1 - h$.

The traditional good is produced in a perfectly competitive sector under constant returns to scale using only unskilled labor as an input. Choosing traditional good as *numéraire*, price equal to one, and assuming free trade for this good (no transport costs) and a unit input coefficient equal to one, ensures that the unskilled worker wage is also equal to one.

The instantaneous utility flow of a single agent located in i depends from three variables: D_i, Y_i and E_i. These variables represent respectively the consumed quantities of both goods in i and the flow of pollution located in $i = \{A; B\}$:

$$U_i = ln\left[\left(\frac{D_i}{\alpha}\right)^{\alpha}\left(\frac{Y_i}{1-\alpha}\right)^{1-\alpha}\right] - \varphi(E_i)^2 \qquad (2.1)$$

with $\alpha \in]0; 1[$ and $\varphi > 0$.

The differentiated good D is a CES aggregate of foreign and local varieties:

[4] As noted by Ottaviano [6], this is the only difference between his model and the Krugman's model [1] and it is that assumption which allows him to provide an analytic resolution of the core-periphery model.

$$(2.2) \qquad D_i = \left\{ \int_0^{n_i} [d_{ii}(m)]^{\frac{\sigma-1}{\sigma}} dm + \int_0^{n_j} [d_{ji}(m)]^{\frac{\sigma-1}{\sigma}} dm \right\}^{\frac{\sigma}{\sigma-1}}$$

with $d_{ji}(m)$ the consumption in place i of a single variety m produced in place j, $i, j = \{A; B\}$, $\sigma \in]1; +\infty[$ represents the substitution elasticity between any two varieties and the price elasticity of demand for each variety. For convenience we state[5] that $k = \frac{\sigma-1}{\sigma}$.

As each country has got L unskilled workers necessary for the production of both goods, we have :

$$(2.3) \qquad Y_i = L - \int_0^{n_i} x_i(m) dm = L - k n_i x_i$$

where[6] $x_i(m) = d_{ii}(m) + \tau d_{ij}(m)$, $i = \{A; B\}$ is the total production by a typical firm in location i.

The pollution flows generated in $i = \{A; B\}$ from the production of the modern good are, as in Elbers and Withagen [10], directly proportional to labor input and we assume no abatement possibilities, so the pollution flows in $i = \{A; B\}$ is given by:

$$(2.4) \qquad f_i = \xi n_i;\ \xi > 0.$$

The flow of pollution located in $i = \{A; B\}$ is finally given by:

$$(2.5) \qquad E_i = \xi \left[(1-\beta) n_i + \beta n_j \right]$$

where $\beta \in [0; 1]$ is the share of pollution flow coming from outside and incrementing the domestic pollution.

Depending on the size of β, we can distinguished 3 types of pollution:

- $\beta \in \left[0; \frac{1}{2}\right[$: the pollution flows generated in one country is only partially domestically located and the share of pollution flow coming from outside and incrementing the domestic pollution is lower than the share of the pollution flows generated domestically. This case is characterized by "low-transboundary pollution". In the special case where $\beta = 0$, we are in presence of what we can call a "purely local pollution" [7].
- $\beta = \frac{1}{2}$: the pollution flow generated in one place is perfectly distributed between the two countries. In this case, designed by "symmetric distribution of pollution", the pollution doesn't matter in terms of location and in fact, as we can see later, this case corresponds to the case without pollution[8].

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[5] Lange and Quaas [9] employ a similar simplification.
[6] We use the result $x_i = x_i(m);\ \forall m \in [0; n_i]$ (Cf. III).
[7] Purely local pollution corresponds to the case studied in Van Marrewijk [8]; Lange and Quaas [9] and Elbers and Withagen [10].
[8] So, this case corresponds to the one studied in Ottaviano [6] and Forslid and Ottaviano [7]

- $\beta \in \left]\frac{1}{2};1\right]$: the pollution flows generated in one country is only partially domestically located and the share of pollution flow coming from outside and incrementing the domestic pollution is higher than the share of the pollution flows generated domestically, case designed by "higher-transboundary pollution". In the unrealistic case where $\beta = 1$, we are in presence of what we can call a "purely transboundary pollution".

Following Ottaviano [6], we impose the restriction[9] $\alpha < \frac{\sigma}{2\sigma-1}$ to ensure that the traditional sector is active in both countries.

III. Consumer Behaviour and profit's maximization: short-run equilibria

As in Ottaviano [6], utility maximisation by residents and profit maximisation by a typical firm located in $i = \{A; B\}$ give the following functions:

- Demand functions:

$$(3.1) \qquad Y_i = (1-\alpha)I_i$$

$$(3.2) \qquad d_{ji}(m) = \frac{[p_{ji}(m)]^{-\sigma}}{{q_i}^{1-\sigma}}\alpha I_i;\ i,j = \{A; B\}$$

where q_i is the local CES price index associated to (2.2) and I_i is the local income.
More specifically:

$$(3.3) \qquad q_i = \left\{\int_0^{n_i}[p_{ii}(m)]^{1-\sigma}dm + \int_0^{n_j}[p_{ji}(m)]^{1-\sigma}dm\right\}^{\frac{1}{\sigma-1}}$$

$$(3.4) \qquad I_i = n_i R_i + L$$

- First order condition for profit's maximisation give :

$$(3.5) \qquad p_{ii}(m) = k\frac{\sigma}{\sigma-1} = 1$$

$$(3.6) \qquad p_{ij}(m) = \tau$$

$$(3.7) \qquad q_i = [n_i + \rho n_j]^{\frac{1}{\sigma-1}} = [Q_{ij}]^{\frac{1}{\sigma-1}}$$

where $\rho \equiv \tau^{1-\sigma} \in]0;1]$ is the ratio of total demand by domestic residents for each foreign variety to their demand for each domestic variety.
ρ can also be seen as reflecting the "freeness of trade"; with a situation varying from autarky ($\rho = 0$) to free trade ($\rho = 1$).
The free entry condition implies:

$$(3.8) \qquad R_i = (1-k)x_i = \frac{x_i}{\sigma}$$

Using (3.2)-(3.8), it implies:

[9] That means the good Y has a large weight in utility and products variety is highly valued by consumers.

$$(3.9) \qquad x_i = \alpha \left\{ \frac{L + n_i \frac{x_i}{\sigma}}{Q_{ij}} + \frac{\rho \left[L + n_j \frac{x_j}{\sigma} \right]}{Q_{ji}} \right\}$$

Solving the system for $i = \{A; B\}$ yields:

$$(3.10) \qquad \left.\frac{x_A}{x_B}\right|_h = \frac{h + \psi(1-h)}{\psi h + (1-h)} = \left.\frac{R_A}{R_B}\right|_h \text{ where } \psi = \frac{\sigma(1+\rho^2) - (1-\rho^2)\alpha}{2\rho\sigma}$$

IV. Indirect utility differential

Given (2.1) and (2.5) the indirect utility of a skilled worker in location i is:

$$(4.1) \qquad W_i = ln\left[\frac{R_i}{(q_i)^\alpha}\right] - \varphi\xi^2\left[(1-\beta)n_i + \beta n_j\right]^2 = ln\left[\frac{R_i}{(q_i)^\alpha}\right] - \delta\left[(1-\beta)n_i + \beta n_j\right]^2$$

where $\delta \equiv \varphi\xi^2$.

φ reflects the harmfulness of the emitted pollution. ξ reflects the pollution flow rejected by the production of the industrial goods. So *ceteris paribus* when one of this two parameters rises, δ also rises, and this implies a negative impact on the indirect utility of the skilled workers.

The indirect utility differential is:

$$(4.2) \qquad f(h) \equiv W_A - W_B = ln\left.\frac{R_A}{R_B}\right|_h - \alpha ln\left.\frac{q_A}{q_B}\right|_h + \delta(1-2\beta)(1-2h)$$

This indirect utility differential can be represented as an explicit function of the share of skilled workers in location A with:

$$(4.3) \qquad f(h) = g(h) + l(h) - v(h)$$

Equation (4.3) consists of three parts:

- $g(h) = ln\frac{h+\psi(1-h)}{\psi h+(1-h)}$ measures the welfare value of the wage rate differential. This term can be positive or negative, because it reflects two opposite effects: the "market crowding effect" and the "home market effect"[10].
- $l(h) = \frac{\alpha}{\sigma-1} ln\frac{h+\rho(1-h)}{\rho h+(1-h)}$ measures the welfare values of price index differences.
- $v(h) = -\delta(1-2\beta)(1-2h)$ measures the environmental quality differential.

V. Towards a typology of long-run equilibria

In order to determine the various possible cases, we have to study the three parts of through its first derivative.

We obtain:

[10] Cf. Krugman [1].

$$(5.1) \qquad g'(h) = \frac{1-\psi^2}{\psi+h(1-h)(1-\psi)^2} \gtreqless 0 \Leftrightarrow \psi \lesseqgtr 1$$

As noted by Ottaviano [6, p.57], equation (5.1) shows how the spatial distribution of skilled workers affects their relative wages in the two locations: if $\psi < 1$ (respectively $\psi > 1$), the wages are larger (respectively smaller) in the location which is better endowed with skilled labor.

Noting $\rho_w \equiv \frac{1-\frac{\alpha}{\sigma}}{1+\frac{\alpha}{\sigma}}$, it can be seen that $\psi = 1 \Leftrightarrow [\rho = \rho_w \vee \rho = 1]$.

Moreover, $\lim_{\rho \to 0} \psi = +\infty$ and $\psi|_{\rho=\rho_w} = \frac{\sqrt{(\sigma^2-\alpha^2)}}{\sigma}$.

Since $\frac{d\psi}{d\rho} = \frac{(\sigma+\alpha)(\rho^2-\rho_w)}{2\rho^2\sigma} \gtreqless 0 \Leftrightarrow \rho \gtreqless \sqrt{\rho_w}$:

- If $\rho \in]0; \rho_w[$; $\psi > 1$;
- If $\rho \in]\rho_w; 1[$; $\psi \in \left[\frac{\sqrt{(\sigma^2-\alpha^2)}}{\sigma}; 1\right[$.

$$(5.2) \qquad l'(h) = \frac{\alpha(1-\rho^2)}{(\sigma-1)[\rho+h(1-h)(1-\rho)^2]} > 0$$

(5.2) is explained by the fact that "[…] since imported varieties incur the trade cost, the price index is always lower where there are more skilled workers and therefore more varieties are produced locally." (Ottaviano [6], p.58).

$$(5.3) \qquad v'(h) = 2\delta(1-2\beta) \gtreqless 0 \Leftrightarrow \beta \lesseqgtr \frac{1}{2}$$

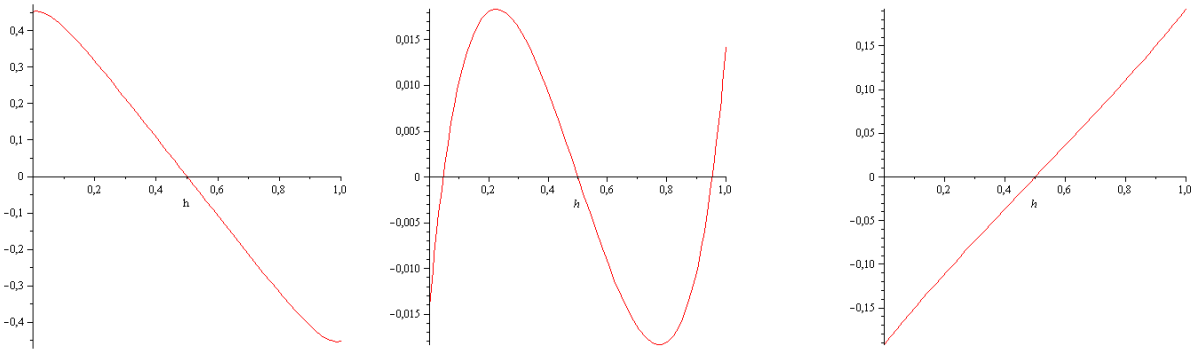
(5.3) indicates that, in compliance with intuition, the environmental quality differential increases with the number of skilled workers in an area when the pollution is a "low-transboundary pollution" and decreases with this number of skilled workers in the symmetrical case of a "higher-transboundary pollution".

5.1 Symmetric distribution of pollution: $\beta = \frac{1}{2}$

In this case $v(h) = 0, \forall h \in [0; 1]$. We are thus in presence of the traditional case studied in the new geography economy, the one's of Ottaviano [6].

In compliance with the result obtained by Ottaviano [6] [11], this case can only lead to one of the three following configurations in term of equilibrium:

[11] Cf. Ottaviano [6], Corollary 2, p.59.



(a) $\rho \in]0;\rho_s]$ (b) $\rho \in]\rho_s;\rho_b[$ (c) $\rho \in [\rho_b;1]$

- In the first case (high transport costs, $\rho \in]0;\rho_s]$; where $\rho_s{}^{\frac{\alpha}{\sigma-1}} = \left(\psi|_{\rho=\rho_s}\right)^{-1}$), there is a unique equilibrium ($h = \frac{1}{2}$) and this equilibrium is stable. Thus we have a diversified economy: both countries produce traditional and differentiated goods and they also have equally shares of skilled workers.
- In the second case (intermediate transport costs, $\rho \in]\rho_s;\rho_b[$; where $\rho_b \equiv \frac{\sigma-1-\alpha}{\sigma-1+\alpha}\rho_w$) there is three stable equilibria ($h = \left\{0;\frac{1}{2};1\right\}$), while a diversified production pattern with unequal shares of skilled workers represents an unstable equilibrium. In this case, depending on the initial share of skilled workers, both diversification and C-P patterns are possible.
- In the third case (low transport costs, $\rho \in [\rho_b;1]$), there is two stable equilibria ($h = \{0;1\}$) and an unstable equilibrium ($h = \frac{1}{2}$) and we obtain an agglomeration effect.

Demonstration : Cf. Appendix 1

5.2 Core-Periphery equilibrium with pollution.

5.2.1. Low-transboundary pollution: $\beta < \frac{1}{2}$

In this case, $v(h) = -\delta(1-2\beta)(1-2h) \gtreqless 0 \Leftrightarrow h \gtreqless \frac{1}{2}$.
Let's study the consequences on the equilibria.
We will study the case where $h = 1$. The case $h = 0$ is obtained by symmetry.
The C-P equilibrium is stable if the following condition is fulfilled:

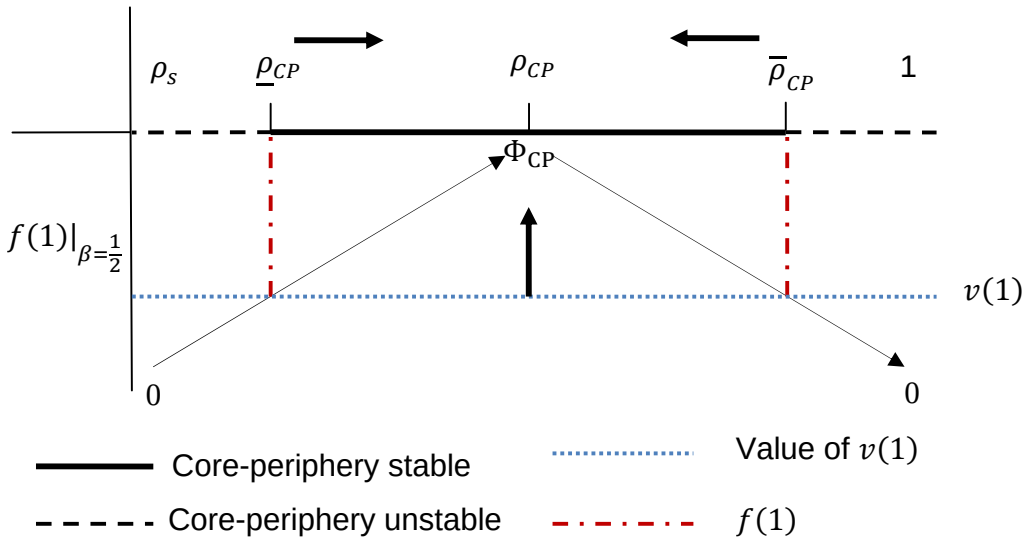
$$f(1) \geq 0 \Leftrightarrow g(1) + l(1) - v(1) \geq 0 \Leftrightarrow f(1)|_{\beta=\frac{1}{2}} \geq \delta(1-2\beta) \tag{5.4}$$

with $f(1)|_{\beta=\frac{1}{2}} = -ln\left[\psi\rho^{\frac{\alpha}{\sigma-1}}\right]$.

Regarding ρ, $f(1)|_{\beta=\frac{1}{2}}$ has the following properties[12]:

- If $\delta(1-2\beta) > \Phi_{\text{CP}}$; where $\rho_{CP}]\rho_b; \sqrt{\rho_w}[$ is the value for which $f(1)|_{\beta=\frac{1}{2}}$ reaches is only maximum $\Phi_{\text{CP}} \equiv f(1)|_{\beta=\frac{1}{2};\rho=\rho_{CP}}$; the C-P equilibrium is always unstable.
- If $\delta(1-2\beta) < \Phi_{\text{CP}}$, the centripetal forces are maximum for $\rho = \rho_{CP}$. For lower or higher values of ρ, the centripetal forces decrease. In this case, there exists an interval $\left[\underline{\rho}_{CP}; \overline{\rho}_{CP}\right]$, include in $[\rho_s; 1]$ for which (5.4) is verified and C-P equilibrium is stable. Moreover, $\frac{d\underline{\rho}_{CP}}{dv(1)} > 0$ and $\frac{d\overline{\rho}_{CP}}{dv(1)} < 0$.

Graphically:



As we can see on this graph, when $v(1)$ rises, the interval $\left[\underline{\rho}_{CP}; \overline{\rho}_{CP}\right]$ decreases and the C-P equilibrium gets less stable. If $v(1)$gets higher than Φ_{CP}, the C-P equilibrium is unstable for every value of ρ.

5.2.2. High-transboundary pollution: $\beta > \frac{1}{2}$

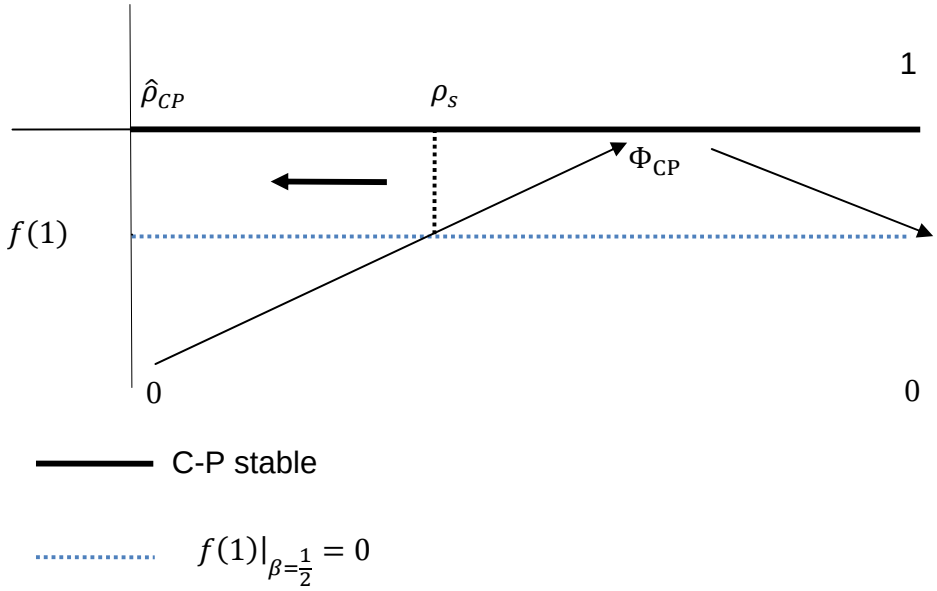
In this case, $v(h) = -\delta(1-2\beta)(1-2h) \gtreqless 0 \Leftrightarrow h \lesseqgtr \frac{1}{2}$.
Let's study the consequences on the equilibria.
The C-P equilibrium's stability condition is always given by (5.4).

Let be let be $\hat{\Phi} = \delta(1-2\beta) < 0$ and $\hat{\rho}_{CP} \in]0; \rho_s[$ the unique value for which

[12] Cf. Appendix 2.

$f(1)|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{CP}} = \hat{\Phi}$. So; the C-P equilibrium is unstable if $\rho \in]0;\hat{\rho}_{CP}[$ and stable if $\rho \in]\hat{\rho}_{CP};1]$[13].

Graphically:



5.3 Spreading equilibrium.

5.3.1. Low-transboundary pollution: $\beta < \frac{1}{2}$

The spreading equilibrium is stable if the indirect utility differential decreases for $h = \frac{1}{2}$. In other words, if the real wage earnings caused by a marginal rise of h are lower than the marginal environmental damage generated by this rise, there is no interest in agglomeration.

The spreading equilibrium is stable if the following condition is fulfilled:

$$(5.5) \qquad f'\left(\frac{1}{2}\right) < 0 \Leftrightarrow v'\left(\frac{1}{2}\right) = 2\delta(1-2\beta) > f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$$

Where $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ have the following properties[14]:

- He reaches an unique minimum $\underline{\Theta}_{Sp} < 0$ when ρ tends to 0 and an unique maximum $\overline{\Theta}_{Sp}$ for $\rho = \rho_{Sp}$; with $\rho_{Sp} \in]\rho_b;1[$.
- $\forall\hat{\Theta} \in]\underline{\Theta}_{Sp};0[$; there is an unique value $\hat{\rho}_{Sp} \in]0;\rho_b[$ for which $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{Sp}} = \hat{\Theta}$.

[13] Cf. Appendix 2.
[14] Cf. Appendix 3.

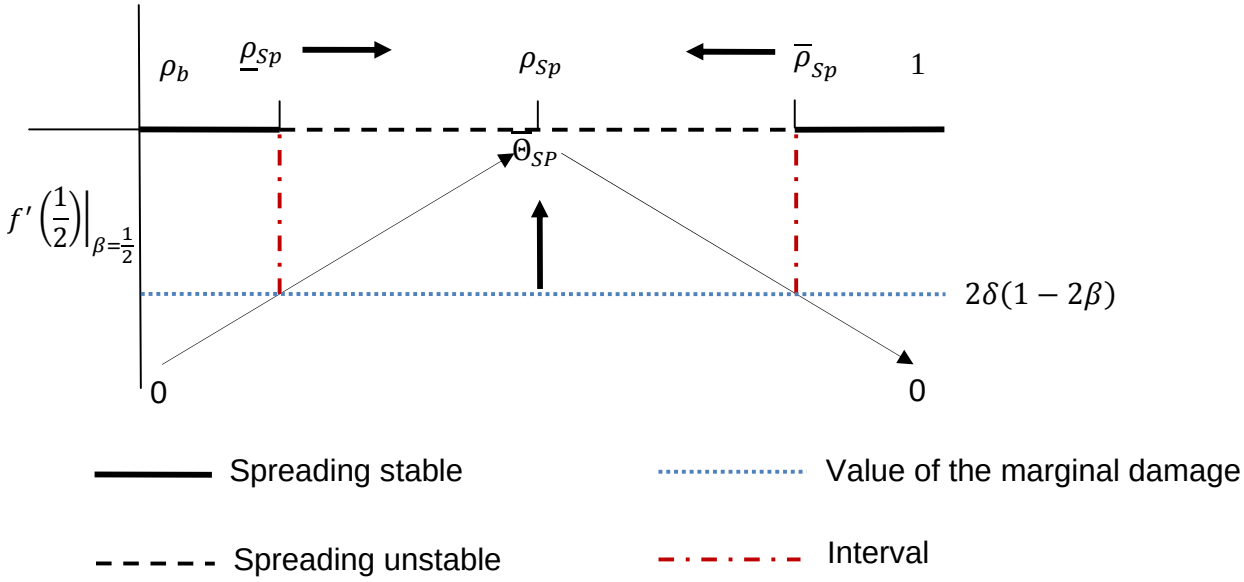
- $\forall\Theta \in \left]0;\overline{\Theta}_{Sp}\right[$; there is two values of ρ; $\underline{\rho}_{Sp}$ and $\overline{\rho}_{Sp}$; for which $\Theta = f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$.

  Moreover $\rho_b < \underline{\rho}_{Sp} < \rho_{Sp} < \overline{\rho}_{Sp} < 1$.

When $\beta \in \left[0;\frac{1}{2}\right[$; the environmental damage changes the stability properties of the spreading equilibrium in the following ways:

- If $\overline{\Theta}_{Sp} \leq 2\delta(1-2\beta)$; the spreading equilibrium is stable $\forall\rho \in \left]0;1\right]$;
- If $\overline{\Theta}_{SP} > 2\delta(1-2\beta)$; the spreading equilibrium is stable $\forall\rho \in \left]0;\underline{\rho}_{Sp}\right[\cup \left]\overline{\rho}_{Sp};1\right[$ and is unstable $\forall\rho \in \left[\underline{\rho}_{Sp};\overline{\rho}_{Sp}\right]$

Graphically, when $\overline{\Theta}_{SP} > 2\,\delta(1-2\beta)$:



As we can see on this graph, when $\delta(1-2\beta)$ rises, the interval $\left[\underline{\rho}_{Sp};\overline{\rho}_{Sp}\right]$ for which the spreading equilibrium is unstable, decreases.

5.2.2. High-transboundary pollution: $\beta > \frac{1}{2}$

The Spreading equilibrium's stability condition is always given by (5.5).
Two cases are now possible regarding the stability of the spreading equilibrium[15]:

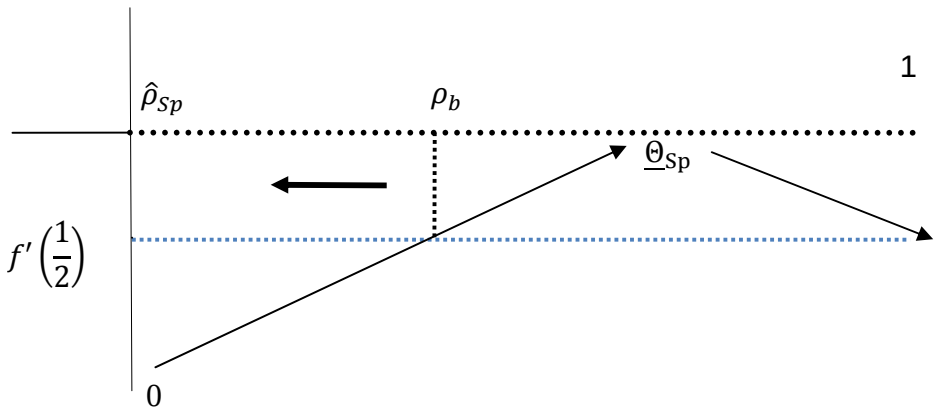
- If $\underline{\Theta}_{Sp} \geq 2\delta(1-2\beta)$; the spreading equilibrium is unstable $\forall\rho \in \left]0;1\right]$.

[15] Cf. Appendix 3.

- If $\underline{\Theta}_{Sp} < 2\,\delta(1-2\beta)$ the spreading equilibrium is stable if $\rho \in \left]0;\hat{\rho}_{Sp}\right[$ and unstable if $\rho \in \left]\hat{\rho}_{Sp};1\right]$ where $\hat{\rho}_{Sp} \in \left]0;\rho_b\right[$ the unique value for which:

$$f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{Sp}} = 2\delta(1-2\beta) < 0$$

The latter case his represented in the following graph:



····· Spreading unstable

····· $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$=0

5.4 Asymmetric equilibrium with pollution

For study the possibility of asymmetric equilibrium, we have first to know the relative behavior of $f(1)$ and $f'\left(\frac{1}{2}\right)$ regarding the value of ρ.

Firstly note that:

- $\lim_{\rho\to 0} f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} = \underline{\Theta}_{Sp} > \lim_{\rho\to 0} f(1)|_{\beta=\frac{1}{2}} = -\infty$
- $f(1)|_{\beta=\frac{1}{2};\rho=\rho_b} > f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\rho_b} = 0$
- $f(1)|_{\beta=\frac{1}{2};\rho=\rho_s} = 0 > f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\rho_s}$

$f(1)|_{\beta=\frac{1}{2}}$ and $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ being strictly increasing functions regarding ρ, we can assert that there exists an unique value $\tilde{\rho} \in \left]0;\rho_s\right[$ such that:

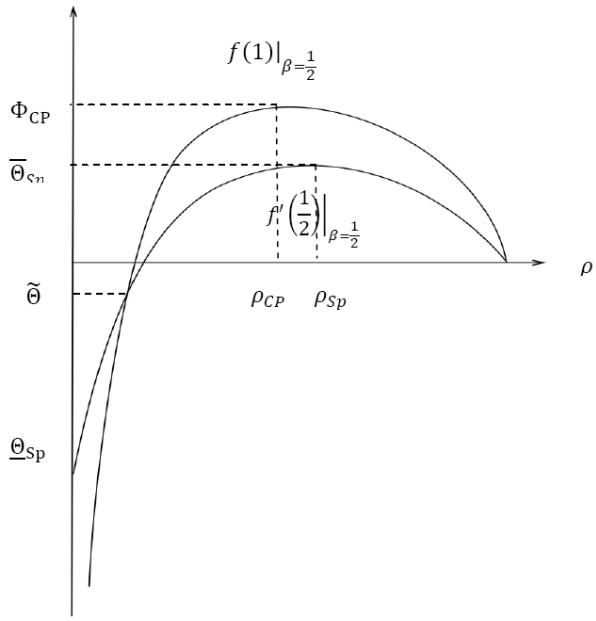
$$\widetilde{\Theta} \equiv f(1)|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}} = f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}} < 0$$

So :

- $f(1)|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}} < f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}}$ if $\rho \in]0;\widetilde{\rho}[$
- $f(1)|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}} > f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\widetilde{\rho}}$ if $\rho \in]\widetilde{\rho};\rho_s[$

Secondly, it can be demonstrates that $f(1)|_{\beta=\frac{1}{2}} > f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ $\forall\rho \in [\rho_s;1]$ (To be completed).

Finally, we can represent the two functions on the same graph:



Regarding the relative values of $f(1)|_{\beta=\frac{1}{2}}$ and $v(1)$, from one part, and $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ and $2v(1)$, from another with have to consider different cases when $\beta \in \left[0;\frac{1}{2}\right[$ and when $\beta \in \left]\frac{1}{2};1\right]$.

Case 1) $\beta \in \left[0;\frac{1}{2}\right[$

In this case :

Case1.1. If $v(1) \leq \Phi_{CP}$ and $2v(1) \leq \overline{\Theta}_{Sp}$; $\underline{\rho}_{CP} < \underline{\rho}_{Sp} < \overline{\rho}_{Sp} < \overline{\rho}_{CP}$ and we have:

$\left[0;\underline{\rho}_{CP}\right[$	$\left]\underline{\rho}_{CP};\underline{\rho}_{Sp}\right[$	$\left]\underline{\rho}_{Sp};\overline{\rho}_{Sp}\right[$	$\left]\overline{\rho}_{Sp};\overline{\rho}_{CP}\right[$	$\left]\overline{\rho}_{CP};1\right[$
$f(1) < 0$	$f(1) > 0$	$f(1) > 0$	$f(1) > 0$	$f(1) < 0$

$f'\left(\frac{1}{2}\right) < 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'\left(\frac{1}{2}\right) > 0$	$f'\left(\frac{1}{2}\right) < 0$	$f'\left(\frac{1}{2}\right) < 0$

Case1.2. If $v(1) \leq \Phi_{\mathrm{CP}}$ and $2v(1) > \overline{\Theta}_{Sp}$; we have:

$\left[0; \underline{\rho}_{CP}\right[$	$\left]\underline{\rho}_{CP}; \overline{\rho}_{CP}\right[$	$\left]\overline{\rho}_{CP}; 1\right[$
$f(1) < 0$ $f'\left(\frac{1}{2}\right) < 0$	$f(1) > 0$ $f'\left(\frac{1}{2}\right) < 0$	$f(1) < 0$ $f'\left(\frac{1}{2}\right) < 0$

Case1.3. If $v(1) > \Phi_{\mathrm{CP}}$ and $2v(1) > \overline{\Theta}_{Sp}$; $f(1) < 0$ and $f'\left(\frac{1}{2}\right) < 0$ $\forall]0; 1]$

Case 2) $\beta \in \left]\frac{1}{2}; 1\right]$

Let be $\hat{\rho}_{CP}$ the value such that $f(1)|_{\beta=\frac{1}{2}} = v(1)$ and $\hat{\rho}_{Sp}$ the value, when it exists, such that $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} = 2v(1)$.

Depending on the relative values of $f(1)|_{\beta=\frac{1}{2}}$ and $v(1)$, from one part, and $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ and $2v(1)$, from another with have the following cases:

Case2.1. If $2v(1) > \widetilde{\Theta}$ or $\widetilde{\Theta} > v(1) > 2v(1) > \underline{\Theta}_{Sp}$:

$[0; \hat{\rho}_{CP}[$	$\left]\hat{\rho}_{CP}; \hat{\rho}_{Sp}\right[$	$\left]\hat{\rho}_{Sp}; 1\right[$
$f(1) < 0$ $f'\left(\frac{1}{2}\right) < 0$	$f(1) > 0$ $f'\left(\frac{1}{2}\right) < 0$	$f(1) > 0$ $f'\left(\frac{1}{2}\right) > 0$

Case2.2. If $v(1) > \widetilde{\Theta} > 2v(1) > \underline{\Theta}_{Sp}$:

$[0; \hat{\rho}_{Sp}[$	$\left]\hat{\rho}_{Sp}; \hat{\rho}_{CP}[\right.$	$\left]\hat{\rho}_{Sp}; 1\right[$
$f(1) < 0$ $f'\left(\frac{1}{2}\right) < 0$	$f(1) < 0$ $f'\left(\frac{1}{2}\right) > 0$	$f(1) > 0$ $f'\left(\frac{1}{2}\right) > 0$

Case2.3. If $\widetilde{\Theta} > v(1) > \underline{\Theta}_{Sp} > 2v(1) >$ or $\underline{\Theta}_{Sp} > v(1)$

$[0; \hat{\rho}_{CP}[$	$\left]\hat{\rho}_{CP}; 1\right[$
$f(1) < 0$ $f'\left(\frac{1}{2}\right) > 0$	$f(1) > 0$ $f'\left(\frac{1}{2}\right) > 0$

In addition, (10.4) can be rewrite:

$$f'(h)|_{\beta=\frac{1}{2}} = \frac{ah(1-h)+b}{(h^4-2h^3)c+[c-d]h^2+dh+\psi\rho} \tag{5.6}$$

Where :

(5.7) $$a = (1-\rho)(1-\psi)[\alpha(1+\rho)(1-\psi) + (1+\psi)(\sigma-1)(1-\rho)]$$
$$= (1-\rho)(1-\psi)(\sigma-1+\alpha)[(1-\psi\rho) + \rho_z(\psi-\rho)] > 0$$

(5.8) $$b = (1-\psi^2)(\sigma-1)\rho + \alpha(1-\rho^2)\psi > 0$$

(5.9) $$c = (1-\rho)^2(1-\psi)^2 > 0$$

(5.10) $$d = (1-\rho)^2\psi + (1-\psi)^2\rho > 0$$

Note that:

(5.11) $$(h^4 - 2h^3)c + [c-d]h^2 + dh + \psi\rho > 0$$

The sign of $f'(h)|_{\beta=\frac{1}{2}}$ depends on the sign of his quadratic numerator.

Three cases are possible regarding the number of real roots from the equation $h^2 - ah + \frac{b}{a} = 0$ for $h \in]0;1[$.

Since $f\left(\frac{1}{2}+z\right)\Big|_{\beta=\frac{1}{2}}$ is an odd function on the interval $z \in \left[-\frac{1}{2}:\frac{1}{2}\right]$, the case of a one double root on $h \in [0;1]$ can immediately be excluded.

In the case of no roots on $h \in [0;1]$, the function $f(h)|_{\beta=\frac{1}{2}}$ is monotone and two cases are possible, depending on the unique sign of $f'(h)|_{\beta=\frac{1}{2}}$:

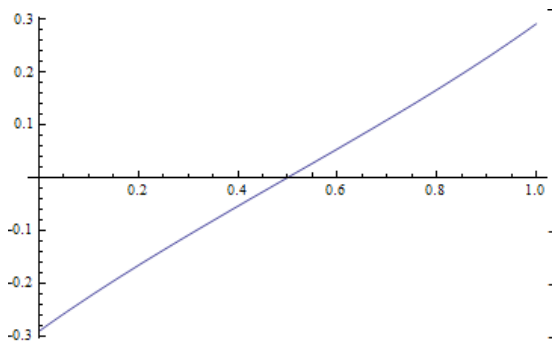


Figure A.1

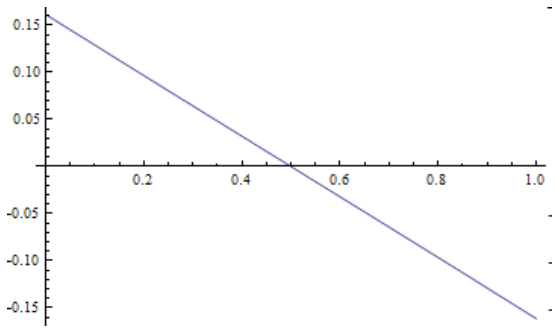


Figure A.2

In the case of two distinct roots on $h \in]0;1[$ (i.e. $\hat{h}_1 \in \left]0;\frac{1}{2}\right[$ and $\hat{h}_2 \in \left]\frac{1}{2};1\right[$ $\hat{h}_2 \in \left]\frac{1}{2};1\right[$ with $\hat{h}_2 = 1 - \hat{h}_1$), $f'(h)|_{\beta=\frac{1}{2}}$ get the same sign as $-a$ if $h \notin]\hat{h}_1;\hat{h}_2[$ and the sign of a if $h \in]\hat{h}_1;\hat{h}_2[$.

Four cases are possible regarding the respective signs of $f(1)|_{\beta=\frac{1}{2}}$ and $f'(1)|_{\beta=\frac{1}{2}}$:

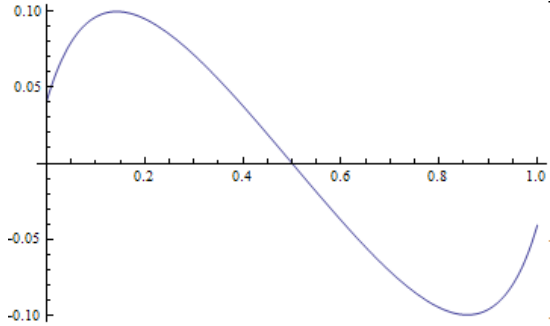


Figure A.3

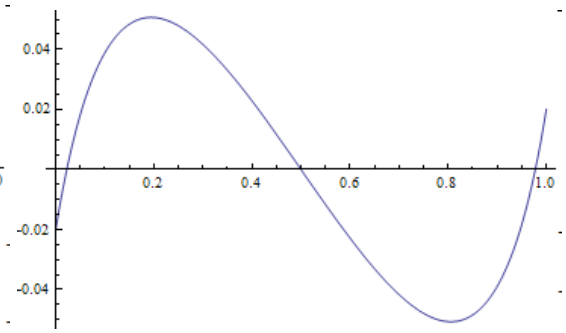


Figure A.4

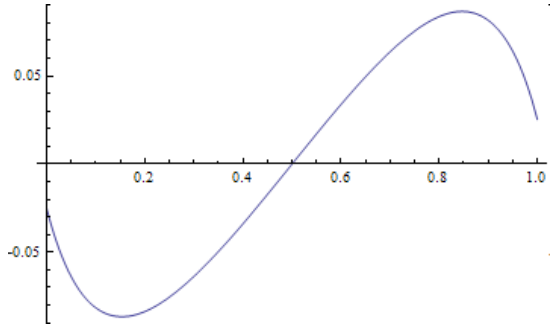


Figure A.5

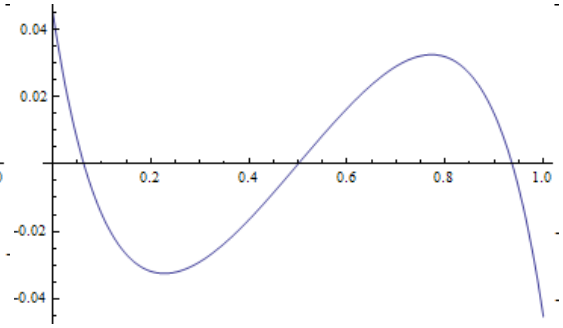


Figure A.6

In absence of pollution, case corresponding to Figure A.6 (involving two stable asymmetric equilibria) was impossible.

We can immediately see that $v\left(\frac{1}{2}+z\right)$, and consequently $f\left(\frac{1}{2}+z\right)$, are odd functions on $z \in \left[-\frac{1}{2};\frac{1}{2}\right]$. Moreover $f\left(\frac{1}{2}\right)=0$.

In the case where $\beta \neq \frac{1}{2}$, we have :

$$(10.1) \qquad f(h) = ln\frac{h+\psi(1-h)}{\psi h+(1-h)} + \frac{\alpha}{\sigma-1} ln\frac{h+\rho(1-h)}{\rho h+(1-h)} + \delta(1-2\beta)(1-2h)$$

$$(10.2) \qquad f'(h) = \frac{1-\psi^2}{\psi+h(1-h)(1-\psi)^2} + \frac{\alpha(1-\rho^2)}{(\sigma-1)[\rho+h(1-h)(1-\rho)^2]} - 2\delta(1-2\beta)$$

Moreover, $f'(h)$ can be rewrite:

$$(5.12) \qquad f'(h) = \frac{-\mu_1 h^4+2\mu_1 h^3+\mu_2 h^2+\mu_3 h+\mu_4}{(h^4-2h^3)c+[c-d]h^2+dh+\psi\rho}$$

Where:

$$\mu_1 = 2v(1)c;\ \mu_3 = a-2v(1);\ \mu_2 = -(\mu_1+\mu_3);\ \mu_4 = b-2v(1)\psi\rho$$

Note that:

$$(5.13) \qquad f'(0) = \frac{\mu_4}{(h^4-2h^3)c+[c-d]h^2+dh+\psi\rho}$$

Thus $\beta > \frac{1}{2} \Rightarrow f'(0) > 0$

The sign of $f'(h)$ depends on the sign of his numerator who is a 4th degree polynomial function. Three cases, compatible with the fact that $f(h)$ is an odd function on $z \in \left[-\frac{1}{2}:\frac{1}{2}\right]$, can be intended depending on the number of real roots of the following equation on $h \in]0;1[$

$$(5.14) \qquad -\mu_1 h^4 + 2\mu_1 h^3 + \mu_2 h^2 + \mu_3 h + \mu_4 = 0$$

They can be no roots on the interval, two distinct roots or four distinct roots.

In the case of no roots on $h \in]0;1[$ on, the possible configurations are the ones sketched by figures A.1 and A.2. However, since $\beta > \frac{1}{2} \Rightarrow f'(0) > 0$, the case of figure A.1 can be rejected in the case of a high-transboundary pollution.

In the case where the equation has two distinct roots on $h \in]0;1[$, the firsts cases are the one of figures A.3 and A.4.

Concerning the configuration corresponding to the figures A.5 and A.6.; when $\beta > \frac{1}{2}$ these cases are impossible because $f'(0) > 0$.

In the case of four distinct roots on $h \in]0;1[$, denoted $\bar{h}_i;\ i = \{1;2;3;4\}$ with $\bar{h}_1 < \bar{h}_2 < \bar{h}_3 < \bar{h}_4$ we must have :

$$(5.15) \qquad -\frac{\mu_4}{\mu_1} = \prod_{i=1}^{4} \bar{h}_i > 0$$

Noting that $\mu_1 \gtreqless 0 \Leftrightarrow \beta \lesseqgtr \frac{1}{2}$ four distinct roots can be obtained if and only if:

- When $\beta < \frac{1}{2}$ if $\mu_4 < 0 \Rightarrow f'(0) < 0$
- When $\beta < \frac{1}{2}$ if $\mu_4 > 0 \Rightarrow f'(0) > 0$

Since $f'(h)$ is of alternated signs on the right and on the left of a root, three cases can be considered accordingly to the value of β:

- When $\beta < \frac{1}{2}$ the possible case are described by the following figures :

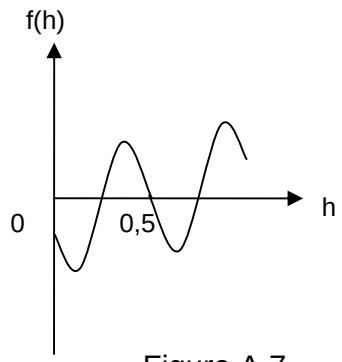


Figure A.7

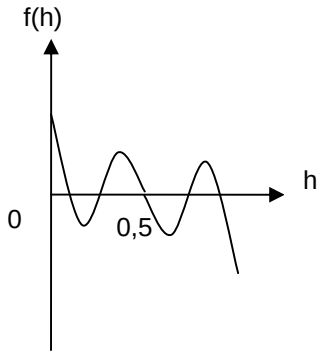


Figure A.8

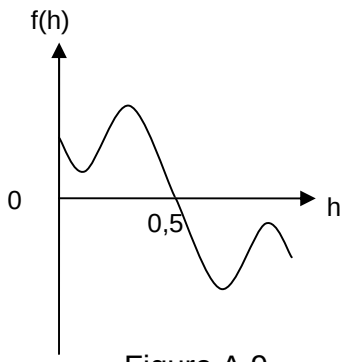


Figure A.9

- When $\beta > \frac{1}{2}$ the possible cases are described by the following figures :

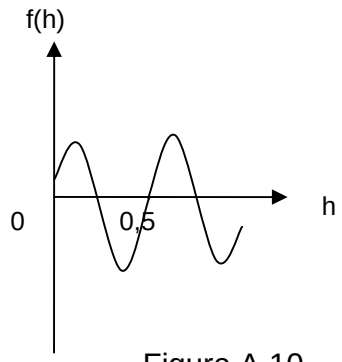


Figure A.10

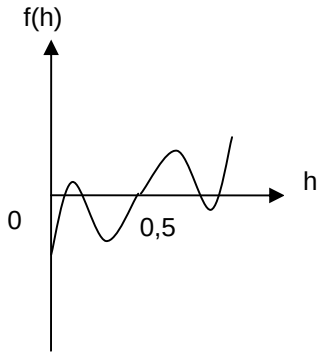


Figure A.11

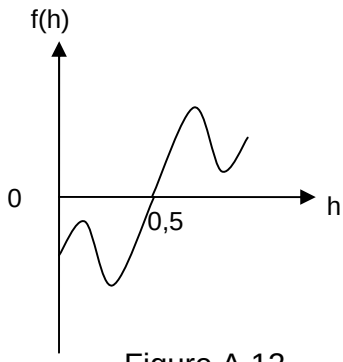


Figure A.12

In terms of equilibrium, depending on the value of $f(1)$ and $f'\left(\frac{1}{2}\right)$ the different possible cases are described in the following table:

	$f(1) < 0$	$f(1) > 0$
$f'\left(\frac{1}{2}\right) < 0$	Cases of Figures A.2; A.3 and A.9: Spreading equilibrium stable Case of Figure A.8 (only possible when $\beta < \frac{1}{2}$): Spreading and asymmetric equilibria stable	Cases of Figures A.4 and A.7: CP and Spreading Equilibria stable
$f'\left(\frac{1}{2}\right) > 0$	Cases of Figures A.6 and A.10: Asymmetric equilibria stable	Cases of Figures A.1; A.5 and A.12: C-P equilibria stable Case of Figure A.11 (only possible when $\beta > \frac{1}{2}$): C-P and asymmetric equilibria stable

So, with your specification functions, asymmetric equilibria can occur:

- ✓ Only in compliance with spreading equilibrium when $\beta < \frac{1}{2}$ in Case 1.1. and 1.2. if $\left[0; \underline{\rho}_{CP}\right[$ or $]\overline{\rho}_{CP}; 1[$ and in Case 1.3.
- ✓ Only in compliance with C-P equilibria when $\beta > \frac{1}{2}$ in Case 2.1 and 2.2. if $]\hat{\rho}_{Sp}; 1[$ and in case 2.3. if $]\hat{\rho}_{CP}; 1[$.

We have now to study the occurrence of such cases depending on the number of roots of (5.14). (To be completed).

VI. Conclusion

In this paper, we study the impact of transboundary pollution in terms of agglomeration in compliance with values of transport cost which play a central role in new economic geography models. We show that local pollution in manufacturing affects the stability of C-P equilibrium and enlarges the number of possible spatial configurations, making possible cases corresponding to asymmetric and incomplete agglomeration. Two cases must be distinguished according to the type of pollution involved. In the case of a “low-transboundary pollution”, that is the case in which the share of pollution flow coming from outside and incrementing the domestic pollution is lower than the share of the pollution flows generated in the country, environmental damages are a “centrifugal force” rending agglomeration of activities less attractive. In the case of a “high-transboundary pollution”, that is a case in which the share of pollution flow coming from outside and incrementing the domestic pollution is higher than the share of the pollution flows generated in the country, environmental damages are a “centripetal force” rending agglomeration of activities more attractive. In the two cases, there is two new configurations of equilibrium, regarding the three cases traditionally considered in the new economic geography literature, compatible with stable incomplete and asymmetric agglomerations of industrial activities.

Next step of research will be devoted to dynamic analysis and will consider stock pollution problems when the damage is a function of the stock of pollution located in one country rather than the pollution’s flow. In this framework one interesting question for further research lies on the role of environmental taxes in the dynamics of spatial agglomeration. This perspective will require the introduction of differential games.

VII. APPENDIX 1

In the case of a symmetric distribution of pollution:

$$(10.3) \qquad f(h)|_{\beta=\frac{1}{2}} = g(h) + l(h) = ln\frac{h+\psi(1-h)}{\psi h+(1-h)} + \frac{\alpha}{\sigma-1} ln\frac{h+\rho(1-h)}{\rho h+(1-h)}$$

$$(10.4) \qquad f'(h)|_{\beta=\frac{1}{2}} = g'(h) + l'(h) = \frac{1-\psi^2}{\psi+h(1-h)(1-\psi)^2} + \frac{\alpha(1-\rho^2)}{(\sigma-1)[\rho+h(1-h)(1-\rho)^2]}$$

Note that :

$$(10.5) \qquad f(1)|_{\beta=\frac{1}{2}} = -ln\left[\psi\rho^{\frac{\alpha}{\sigma-1}}\right]$$

As shown by Ottaviano [6] and Forslid and Ottaviano [7], there is a unique value of ρ, noted ρ_s and called "sustain point", with $\rho_S < 1$, solution of the equation $\psi^{-1} = \rho^{\frac{\alpha}{\sigma-1}}$ (or, equivalently, $\rho^2 - \frac{2\sigma}{(\sigma+\alpha)}\rho^{\frac{\sigma-1-\alpha}{\sigma-1}} + \rho_w = 0$), and thus for which:

$$f(1)|_{\beta=\frac{1}{2};\rho=\rho_S} = f(0)|_{\beta=\frac{1}{2};\rho=\rho_S} = 0$$

Moreover:

- $\forall\rho \in]0;\rho_s[;\ f(1)|_{\beta=\frac{1}{2}} < 0$ (condition for which the C-P is unstable);
- $\forall\rho \in]0;\rho_s[;\ f(1)|_{\beta=\frac{1}{2}} > 0$ (condition for which the C-P is stable).

$$(10.6) \qquad f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} = 4\left[\frac{1-\psi}{1+\psi} + \frac{\alpha(1-\rho)}{(\sigma-1)(1+\rho)}\right] = \frac{4(1-\rho)(\sigma-1+\alpha)(\rho-\rho_b)}{(\sigma-1)(1+\rho)(\rho+\rho_w)}$$

Where[16] $\rho_b \equiv \frac{1-\frac{1}{\sigma}-\frac{\alpha}{\sigma}}{1-\frac{1}{\sigma}+\frac{\alpha}{\sigma}}\frac{1-\frac{\alpha}{\sigma}}{1+\frac{\alpha}{\sigma}} = \rho_z\rho_w$ is called "break point" (with $\rho_z \equiv \frac{1-\frac{1}{\sigma}-\frac{\alpha}{\sigma}}{1-\frac{1}{\sigma}+\frac{\alpha}{\sigma}}$).

So:

- $\forall\rho \in]0;\rho_b[;\ f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} < 0$ (condition for which the spreading equilibrium is stable);
- $\forall\rho \in]0;\rho_b[;\ f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} > 0$ (condition for which the spreading equilibrium is unstable).

Moreover, it can be shown that $\rho_s < \rho_b$.

(10.6) implies that $\psi|_{\rho=\rho_b} = \frac{(\sigma-1)(1+\rho)+\alpha(1-\rho)}{(\sigma-1)(1+\rho)-\alpha(1-\rho)} > 1$

Moreover:

[16] Note that $\rho > \rho_b$ is always true if $\alpha > \sigma - 1$ (a « black hole » situation in the new economic geography literature), situation traditionally « ruled out ». So we suppose that $\sigma - 1 > \alpha$.

$\lim_{\alpha\to 0} f(1)|_{\beta=\frac{1}{2};\rho=\rho_b} = -ln\left[\psi\rho^{\frac{\alpha}{\sigma-1}}\right] = 0$

and $\frac{d\left[f(1)|_{\beta=\frac{1}{2}}\right]}{d\alpha} = \frac{(1-\rho^2)}{2\rho\sigma\psi} - \frac{\ln\rho}{\sigma-1} > 0$ imply $f(1)|_{\beta=\frac{1}{2};\rho=\rho_b} > 0;\ \alpha \in]0;1]$. So $\rho_b \in]0;\rho_s[$.

Finally:

$$\rho_s < \rho_b < \rho_z < \rho_w < \sqrt{\rho_w}$$

The available situations, in absence of pollution, can be illustrated in the table hereafter:

	$\rho < \rho_s$	$\rho > \rho_s$
$\rho < \rho_b$	Spreading Equilibrium Stable $\rho \in]0;\rho_s]$	Spreading and C-P Equilibria Stable $\rho \in]\rho_s;\rho_b]$
$\rho > \rho_b$	Impossible	C-P Equilibrium Stable $\rho \in [\rho_b;1]$

VIII. APPENDIX 2

We know that:

(8.1) $\quad f(1)|_{\beta=\frac{1}{2}} = \Delta_1(\rho) - \Delta_2(\rho) \gtreqless 0 \Leftrightarrow \rho \gtreqless \rho_s.$

With $\Delta_1(\rho) = -\frac{\alpha}{\sigma-1}ln(\rho)$ and $\Delta_2(\rho) = ln(\psi)$.

Where: $\Delta_1(\rho) > 0;\ \Delta'_1(\rho) = -\frac{\alpha}{(\sigma-1)\rho} < 0;\ \Delta''_1(\rho) = \frac{\alpha}{(\sigma-1)\rho^2} > 0;\ \forall\rho \in]0;1[.$

And: $\Delta_2(\rho) = ln(\psi) \gtreqless 0 \Leftrightarrow \psi \gtreqless 1.$

So:

- $\Delta_2(\rho) > 0;\ \forall\rho \in]0;\rho_w[;$
- $\Delta_2(\rho) < 0;\ \forall\rho \in]\rho_w;1[;$
- $\Delta_2(\rho) = 0;\ \forall\rho \in \{\rho_w;1\}.$

Moreover; $\Delta'_2(\rho) = \frac{1}{\psi}\frac{d\psi}{d\rho} = \frac{(\sigma+\alpha)(\rho^2-\rho_w)}{\rho(\rho^2+\rho_w)} \gtreqless 0 \Leftrightarrow \rho \gtreqless \sqrt{\rho_w}.$

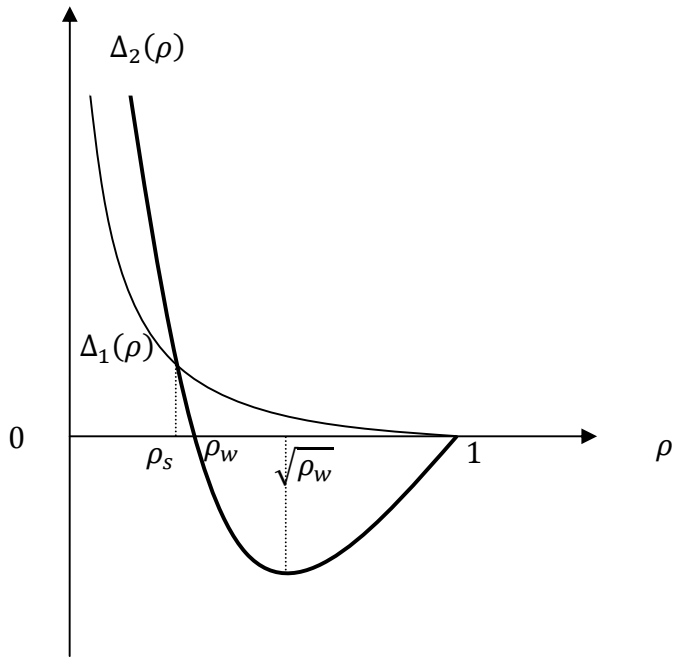
In addition: $\lim_{\rho\to 0}\Delta_1(\rho) = \lim_{\rho\to 0}\Delta_2(\rho) = +\infty.$
So, we can use L'Hospital's rule to affirm that:

(8.2) $\quad \lim_{\rho\to 0}\frac{\Delta_1(\rho)}{\Delta_2(\rho)} = \lim_{\rho\to 0}\frac{\Delta'_1(\rho)}{\Delta'_2(\rho)} = -\frac{\frac{\alpha}{(\sigma-1)\rho}}{\frac{1}{\psi}\frac{d\psi}{d\rho}} = \frac{\frac{\alpha}{(\sigma-1)\rho}}{\frac{(\sigma+\alpha)}{\rho}} = \frac{\alpha}{(\sigma-1)(\sigma+\alpha)}$

So:

$$\lim_{\rho\to 0} f(1)|_{\beta=\frac{1}{2}} = \lim_{\rho\to 0}\left[-ln(\psi) - \frac{\alpha}{\sigma-1}ln(\rho)\right]$$
$$= \lim_{\rho\to 0} ln(\psi)\left[-1 + \frac{\alpha}{(\sigma-1)(\sigma+\alpha)}\right]$$
$$= \lim_{\rho\to 0} ln(\psi)\left[\frac{\alpha-(\sigma+\alpha)(\sigma-1)}{(\sigma-1)}\right] = -\infty$$

We can now represent $\Delta_1(\rho)$ and $\Delta_2(\rho)$ on the same graph:



Finally note that:

$$\frac{\partial\left[f(1)|_{\beta=\frac{1}{2};\rho=\sqrt{\rho_w}}\right]}{\partial\rho} = -\frac{\alpha}{(\sigma-1)\sqrt{\rho_w}} < 0$$

$$\frac{\partial\left[f(1)|_{\beta=\frac{1}{2};\rho=\rho_b}\right]}{\partial\rho} = -\frac{\alpha}{(\sigma-1)\rho_b} + \frac{(\sigma+\alpha)(1-\rho_b\rho_z)}{(1+\rho_b\rho_z)\rho_b} > 0$$

$$\Leftrightarrow (\sigma-1)(\sigma+\alpha)(1-\rho_b\rho_z) - \alpha(1+\rho_b\rho_z) > 0$$

$$\Leftrightarrow \frac{(\sigma-1)(\sigma+\alpha)-\alpha}{(\sigma-1)(\sigma+\alpha)-\alpha} > \rho_b\rho_z$$

This condition his verified because $\frac{(\sigma-1)(\sigma+\alpha)-\alpha}{(\sigma-1)(\sigma+\alpha)-\alpha} > \rho_z \Leftrightarrow \alpha > \frac{\alpha}{\sigma+\alpha}$.

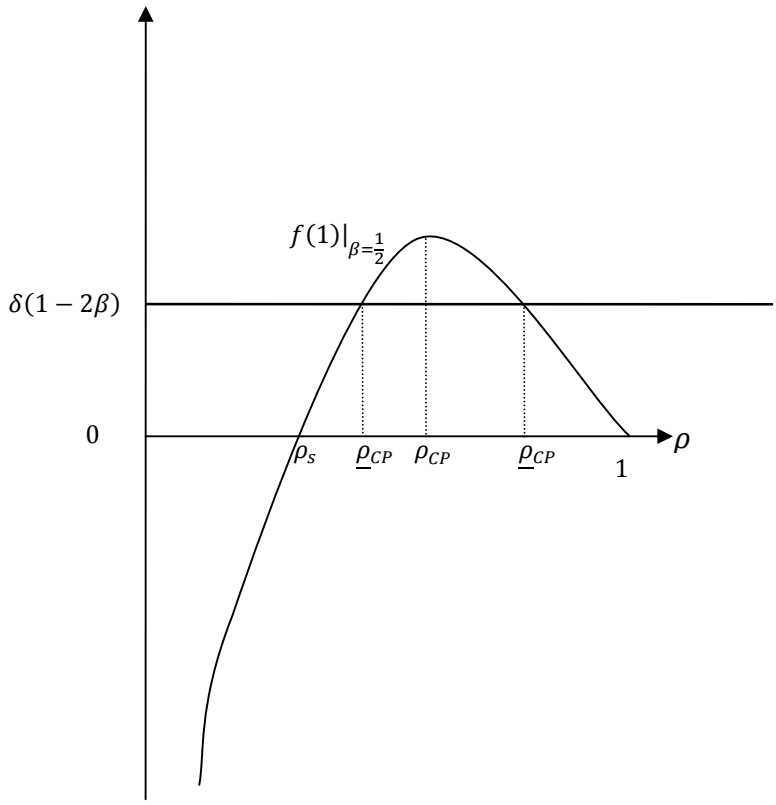
Given that the two functions are monotonically decreasing functions of ρ and given (8.1) we can assert that:

- For $\rho \in]0;\rho_s[$ and $\frac{\partial\left[f(1)|_{\beta=\frac{1}{2}}\right]}{\partial\rho} < 0$; so $f(1)|_{\beta=\frac{1}{2}}$ takes all the values between $-\infty$ and 0. Thus, $\forall\widehat{\Phi} < 0$; there is an unique value $\hat{\rho}_{CP} \in]0;\rho_s[$ for which $f(1)|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{CP}} = \widehat{\Phi}$.
- For $\rho \in]\rho_s;1]$; $f(1)|_{\beta=\frac{1}{2}}$ reach an unique maximum on the interval $]\rho_b;\sqrt{\rho_w}[$ for $\rho = \rho_{CP}$.

  Let be $\Phi_{\mathrm{CP}} \equiv f(1)|_{\beta=\frac{1}{2};\rho=\rho_{CP}} = \max_{\rho\in]0;1]}\left[f(1)|_{\beta=\frac{1}{2}}\right]$. Given the continuity of the two functions and theirs variations on $]\rho_s;1]$; $\forall\Phi \in]0;\Phi_{\mathrm{CP}}[$; there is two values of ρ for which $\Phi = f(1)|_{\beta=\frac{1}{2}}$. Let be $\underline{\rho}_{CP}$ and $\overline{\rho}_{CP}$ this values; we can assert that:

$$\rho_s < \underline{\rho}_{CP} < \rho_{CP} < \overline{\rho}_{CP} < 1$$

Graphically:



So we can assert that:

- When $\beta \in \left[0;\frac{1}{2}\right[$; three cases are possible regarding the stability of the C-P equilibrium:

- If $\Phi_{\mathrm{CP}} < \delta(1-2\beta)$; the C-P equilibrium is unstable $\forall\rho \in]0;1]$;
- If $\Phi_{\mathrm{CP}} = \delta(1-2\beta)$; the C-P equilibrium is only stable when $\rho = \rho_{CP}$;
- If $\Phi_{\mathrm{CP}} > \delta(1-2\beta)$; the C-P equilibrium is unstable $\forall\rho \in \left]0;\underline{\rho}_{CP}\right[\cup]\overline{\rho}_{CP};1[$ and is stable $\forall\rho \in \left[\underline{\rho}_{CP};\overline{\rho}_{CP}\right]$.

- When $\beta \in \left]\frac{1}{2};1\right]$; let be $\widehat{\Phi} = \delta(1-2\beta) < 0$ and $\hat{\rho}_{CP} \in]0;\rho_s[$ the unique value for which $f(1)|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{CP}} = \widehat{\Phi}$. So; the C-P equilibrium is unstable if $\rho \in]0;\hat{\rho}_{CP}[$ and stable if $\rho \in]\hat{\rho}_{CP};1]$.

IX. APPENDIX 3

Following (10.6), $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} = \frac{4(\sigma-1+\alpha)}{(\sigma-1)}\nabla_1(\rho)$, where:

(9.1) $$\nabla_1(\rho) \equiv \frac{(1-\rho)(\rho-\rho_b)}{(1+\rho)(\rho+\rho_w)}$$

First, we see that $\nabla(0) = -\frac{\rho_b}{\rho_w} = -\rho_z$ and $\nabla(\rho_b) = \nabla(1) = 0$.

So, $\underline{\Theta}_{Sp} \equiv \lim_{\rho\to 0} f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}} = -\frac{4(\sigma-1+\alpha)}{(\sigma-1)}\rho_z$ and $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\rho_b} = f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=1} = 0$

Secondly, we have:

(9.2) $$\nabla_1{}'(\rho) \equiv \frac{(1-\rho)(\rho_w+\rho_b)}{(1+\rho)(\rho+\rho_w)^2} - \frac{2(\rho-\rho_b)}{(1+\rho)^2(\rho+\rho_w)} = \frac{[2\rho_b\rho_w+\rho_w+\rho_b-2\rho(\rho_w-\rho_b)-\rho^2(\rho_w+\rho_b+2)]}{(1+\rho)^2(\rho+\rho_w)^2}$$

The sign of $\nabla_1{}'(\rho)$ depends on the sign of the following quadratic function:

(9.3) $$\nabla_2(\rho) = 2\rho_b\rho_w + \rho_w + \rho_b - 2\rho(\rho_w - \rho_b) - \rho^2(\rho_w + \rho_b + 2)$$

It can be easily seen that the quadratic equation $\nabla_2(\rho) = 0$ has two real roots with opposite signs:

$$\bar{\rho} = \frac{\sqrt{(\rho_w-\rho_b)^2} - \sqrt{(\rho_w-\rho_b)^2 + (\rho_w+\rho_b+2)(2\rho_b\rho_w+\rho_w+\rho_b)}}{(\rho_w+\rho_b+2)} < 0$$

$$\rho_{Sp} = \frac{\sqrt{(\rho_w-\rho_b)^2} + \sqrt{(\rho_w-\rho_b)^2 + (\rho_w+\rho_b+2)(2\rho_b\rho_w+\rho_w+\rho_b)}}{(\rho_w+\rho_b+2)}$$

It can be seen that:

$$\sqrt{\rho_w} < \rho_{Sp} \Leftrightarrow \rho_w(\rho_w+\rho_b+2) < \left[\sqrt{(\rho_w-\rho_b)^2} + \sqrt{(\rho_w-\rho_b)^2 + (\rho_w+\rho_b+2)(2\rho_b\rho_w+\rho_w+\rho_b)}\right]^2$$

Condition obviously verified.

So we have:

$$\rho_{CP} < \sqrt{\rho_w} < \rho_{Sp}$$

So, $\nabla_2(\rho) = 0$ for $\rho = \{\bar{\rho}; \rho_{Sp}\}$; $\nabla_2(\rho) > 0$ if $\rho \in \left]\bar{\rho}; \rho_{Sp}\right[$ and $\nabla_2(\rho) < 0$ otherwise.

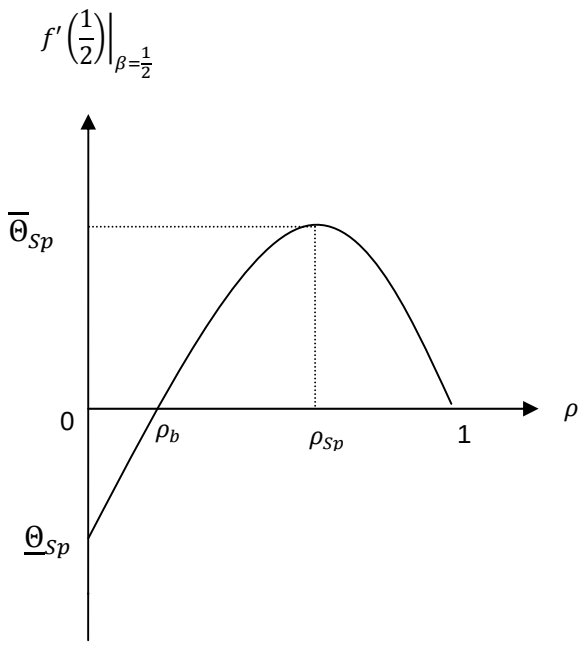
Although:

- $\nabla_2(0) = 2\rho_b\rho_w + \rho_w + \rho_b > 0$;
- $\nabla_2(\rho_b) = 2\rho_b(\rho_w - \rho_b) + (1 - {\rho_b}^2)(\rho_w + \rho_b) > 0$;
- $\nabla_2(1) = 2(\rho_b - 1)(\rho_w + 1) < 0$

Thus, we can assert that $\rho_{Sp} \in \left]\rho_b; 1\right[$.

So, $\nabla_1{}'(\rho) < 0;\ \forall\rho \in \left]0; \rho_{Sp}\right[$; $\nabla_1{}'(\rho_{Sp}) = 0$ and $\nabla_1{}'(\rho) > 0;\ \forall\rho \in \left]\rho_{Sp}; 1\right]$.

Noting $\overline{\Theta}_{Sp} \equiv f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\rho_{Sp}} = \max_{\rho\in]0;1]}\left[f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}\right]$ the maximum value of $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ we can now represent this function:



Given that the $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ is continuous, we can assert that:

- For $\rho \in \left]0; \rho_b\right[$; $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$ takes all the values between $\underline{\Theta}_{SP}$ and 0. Thus, $\forall\widehat{\Theta} \in \left]\underline{\Theta}_{Sp}; 0\right[$; there is an unique value $\hat{\rho}_{Sp} \in \left]0; \rho_b\right[$ for which $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{Sp}} = \widehat{\Theta}$.
- Given the variation on $\left]\rho_b; 1\right]$; $\forall\Theta \in \left]0; \overline{\Theta}_{Sp}\right[$; there is two values of ρ for which

$\Theta = f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2}}$. Let be $\underline{\rho}_{Sp}$ and $\overline{\rho}_{Sp}$ this values; we can assert that:

$$\rho_b < \underline{\rho}_{Sp} < \rho_{Sp} < \overline{\rho}_{Sp} < 1$$

So we can assert that:

- When $\beta \in \left[0; \frac{1}{2}\right[$; two cases are possible regarding the stability of the spreading equilibrium:
  - If $\overline{\Theta}_{Sp} \leq 2\delta(1-2\beta)$; the spreading equilibrium is stable $\forall \rho \in]0;1]$;
  - If $\overline{\Theta}_{Sp} > 2\,\delta(1-2\beta)$; the spreading equilibrium is stable $\forall \rho \in \left]0; \underline{\rho}_{Sp}\right[\cup \left]\overline{\rho}_{Sp}; 1\right[$ and is unstable $\forall \rho \in \left[\underline{\rho}_{Sp}; \overline{\rho}_{Sp}\right]$.
- When $\beta \in \left]\frac{1}{2}; 1\right]$; two cases are also possible regarding the stability of the spreading equilibrium:
  - If $\underline{\Theta}_{Sp} < 2\delta(1-2\beta)$let be $\widehat{\Theta} = \delta(1-2\beta) < 0$ and $\hat{\rho}_{Sp} \in]0; \rho_b[$ the unique value for which $f'\left(\frac{1}{2}\right)\Big|_{\beta=\frac{1}{2};\rho=\hat{\rho}_{Sp}} = \widehat{\Theta}$. So; the spreading equilibrium is stable if $\rho \in]0; \hat{\rho}_{Sp}[$ and unstable if $\rho \in \left]\hat{\rho}_{Sp}; 1\right]$.
  - If $\underline{\Theta}_{Sp} \geq 2\delta(1-2\beta)$; the spreading equilibrium is unstable $\forall \rho \in]0;1]$.

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