

The Strategic Use of Ambiguity*

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Abstract

Ambiguity can be used as a strategic device in some situations. To demonstrate this, we propose and study a framework for normal form games where players can use Knightian uncertainty strategically. In such Ellsberg games, players may use Ellsberg urns in addition to the standard objective mixed strategies. We assume that players are ambiguity-averse in the sense that they evaluate ambiguous events pessimistically. While classical Nash equilibria remain equilibria in the new game, there arise new *Ellsberg equilibria* that can be quite

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different from Nash equilibria. A negotiation game with three players illustrates this finding. Another class of examples shows the use of ambiguity in mediation. We also highlight some conceptually interesting properties of Ellsberg equilibria in two-person games with conflicting interests.

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1 Introduction

It occurs in daily life that a certain vagueness or ambiguity is in one's own favor. This might happen in very private situations where the disclosure of all information might put oneself into a bad position, but also in quite public situations when a certain ambiguity, as one intuitively feels, leads to a better outcome. Indeed, committee members at universities know only too well how useful it can be to play for time; in particular, it is sometimes extremely useful to *create* ambiguity where there was none before. Such strategic ambiguity might include a threat about future behavior ("I might do this or even the contrary if . . ."). Past presidents of Reserve Banks became famous for being intelligently ambiguous (although the debate is not decided if this was always for society's good).

Accounts of strategic ambiguity, of a more verbal nature, can be found in different fields. This includes the use of ambiguity as a strategic instrument in foreign policy, Benson and Niou (2000) and Baliga and Sjostrom (2008), and in organizational communication, Eisenberg (1984). Economists found the presence of strategic ambiguity to explain incompleteness of contracts, Mukerji (1998), policies of insurance fraud detection, Lang and Wambach (2010) and candidate behavior in electoral competition, Aragonés and Neeman (2000).

This paper takes a look at such *strategic use of ambiguity* in games. We follow the by now common distinction between Knightian uncertainty and randomness or risk, the distinction going back to Knight's dissertation and the famous Ellsberg experiments. Intuitively, we allow players to use Ellsberg urns, that is urns with (partly) unknown proportions of colored balls, instead

of merely using objective randomizing devices where all players know (ideally) the probabilities. Players can thus credibly announce: “I will base my action on the outcome of the draw from this Ellsberg urn!” Such urns are objectively ambiguous, by design; players can thus create ambiguity.

We then work under the assumption that it is common knowledge that players are ambiguity-averse: in the face of Knightian uncertainty, players are pessimistic. When evaluating the payoff of an action, they compute the minimal expected payoff according to the Knightian uncertainty involved. They then choose the option that yields the highest of all minimal expected payoffs.

In this paper, we define the new concept of an Ellsberg game. We then present three (classes of) games that show, in our opinion, that strategic Knightian uncertainty has the potential to explain interesting facts beyond the classical Nash equilibrium analysis. Our examples serve the role to highlight the potential uses of our framework. In a later paper, we plan to say more on a general theory of such *Ellsberg Games*.

Our first example highlights the use of strategic ambiguity in negotiations. We take up a beautiful example of Greenberg (2000) where three parties, two small countries in conflict, and a powerful country, the mediator, are engaged in peace negotiations. There is a nice peace outcome; the game’s payoffs, however, are such that in the unique Nash equilibrium of the game the bad war outcome is realized. Greenberg argues verbally that peace can be reached if the superpower “remains silent” instead of playing a mixed strategy. We show that peace can indeed be an equilibrium in the extended game where players are allowed to use Ellsberg urns. Here, the superpower leaves the other players uncertain about its actions. This induces the small countries to prefer peace over war.

As our second example we take a classic example by Aumann (1974) which has a nice interpretation. It illustrates how strategic ambiguity can be used by a mediator to achieve cooperation in situations similar to the prisoners’ dilemma. In this game a mediator is able to create ambiguity about the reward in case of unilateral defection. If he creates enough ambiguity, both prisoners are afraid of punishment and prefer to cooperate. The outcome thus reached is even Pareto-improving.

A remarkable consequence of these two examples is that the strategic use of ambiguity allows to reach equilibria that are not Nash equilibria in the original game, even not in the support of the original Nash equilibrium. So here is a potentially rich world to discover.

We also take a closer look at two-person 2×2 games with conflicting interests, as Matching Pennies or similar competitive situations. These games have a unique mixed Nash equilibrium. We point out two interesting features here. The first is that, surprisingly, mutual ambiguity around the Nash equilibrium distribution is not an equilibrium in competitive situations. Due to the non-linearity of the payoff functions in Ellsberg games, ambiguity around the Nash equilibrium distribution never has ambiguity as best response. Secondly, in the class of games we consider, a new and interesting type of equilibria arises. In these equilibria, both players create ambiguity. They commit to strategies like, e.g.: "I play action A with probability at least $1/2$, but I don't tell you anything more about my true distribution." It seems convincing that these strategies are easier to implement in reality than an objectively mixed Nash equilibrium strategy. In fact, an experiment by Goeree and Holt (2001) suggests that actual behavior is closer to Ellsberg equilibrium strategies than to Nash equilibrium strategies.

To our knowledge, the only contribution in game theory in this direction is by Sophie Bade (2011) and, very early, by Robert Aumann (1974). Bade defines a large class of games in which players may use what she calls *subjectively mixed strategies*, or *ambiguous acts*. Players are assumed to have ambiguity-averse preferences over all acts. The possible priors for an ambiguous act are part of the players' preferences in her setup. Bade then adds some consistency properties to exclude unreasonably divergent beliefs. In difference to her setup we let ambiguity be an objective instrument that is not derived from subjective preferences. Players can credibly commit to play an Ellsberg urn with a given and known degree of ambiguity. Bade's main result states that in two-person games, the support of "ambiguous act equilibria" is always equal to the support of some Nash equilibrium. This reminds one of Aumann's early discussion of the use of subjective randomizing devices in a Savage framework. If players evaluate Knightian uncertainty in a linear way, in other words, if they conform to the Savage axioms, no new equilibria can be generated in two-person games. This might look like a quite negative, and boring, conclusion in the sense that there is no space for interesting uses of strategic ambiguity. We do not share this point of view.

Of course, other applications of Knightian decision theory to normal form games with complete information are available. Dow and Werlang (1994), Lo (1996), Eichberger and Kelsey (2000) and Marinacci (2000) assume uncertainty about the players' actions and let the beliefs reflect this uncertainty. In

these extensions of *equilibrium in beliefs* actual strategies in equilibrium are essentially ignored. Klibanoff (1993), Lehrer (2008) and Lo (2009) determine the actual play in equilibrium and require some consistency between actions and ambiguous beliefs. Anyhow, the actions in equilibrium are still determined by objective randomizing devices. In contrast, in our model players can *create* ambiguity by the use of Ellsberg urns as randomizing devices.

The paper is organized as follows. In Section 2 we present the negotiation example. In Section 3 we develop the framework for normal form games that allows the strategic use of ambiguity. In Section 4 we apply the concept to the negotiation example, Section 5 analyzes strategic ambiguity as a mediation tool. Finally Section 6 shows how strategic ambiguity is used in competitive situations. We conclude in Section 7.

2 Ambiguity as a Threat

Our main point here is to show that strategic ambiguity can lead to new phenomena that lie outside the scope of classical game theory. As our first example, we consider the following peace negotiation game taken from Greenberg (2000). There are two small countries who can either opt for peace, or war. If both countries opt for peace, all three players obtain a payoff of 4. If one of the countries does not opt for peace, war breaks out, but the superpower cannot decide whose action started the war. The superpower can punish one country and support the other. The game tree is in Figure 1 below¹.

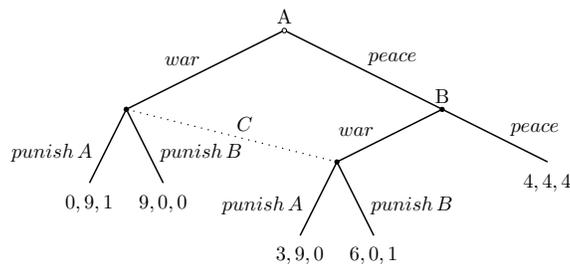


Figure 1: Peace Negotiation

¹We take the payoffs as in Greenberg's paper. In case the reader is puzzled by the slight asymmetry between country A and B in payoffs: it does not play a role for our argument. One could replace the payoffs 3 and 6 for Country A by 0 and 9.

As we deal here only with static equilibrium concepts, we also present the normal form, where country A chooses rows, country B columns, and the superpower chooses the matrix.

| | | | |
|--------------|-----------------|--------------|--|
| | <i>war</i> | <i>peace</i> | |
| <i>war</i> | 0, 9, 1 | 0, 9, 1 | |
| <i>peace</i> | 3, 9, 0 | 4, 4, 4 | |
| | <i>punish A</i> | | |

| | | |
|--------------|-----------------|--------------|
| | <i>war</i> | <i>peace</i> |
| <i>war</i> | 9, 0, 0 | 9, 0, 0 |
| <i>peace</i> | 6, 0, 1 | 4, 4, 4 |
| | <i>punish B</i> | |

Figure 2: Peace Negotiation in normal form

This game possesses a unique Nash equilibrium where country A mixes with equal probabilities, and country B opts for war; the superpower has no clue who started the war given these strategies. It is thus indifferent about whom to punish and mixes with equal probabilities as well. War occurs with probability 1. The resulting equilibrium payoff vector is $(4.5, 4.5, 0.5)$.

If the superpower can create ambiguity (and if the countries A and B are ambiguity-averse), the picture changes. Suppose for simplicity, that the superpower creates maximal ambiguity by using a device that allows for any probability between 0 and 1 for its strategy *punish A*. The pessimistic players A and B are ambiguity-averse and thus maximize against the worst case. For both of them, the worst case is to be punished by the superpower, with a payoff of 0. Hence, both prefer to opt for peace given that the superpower creates ambiguity. As this leads to a very desirable outcome for the superpower, it has no incentive to deviate from this strategy. We have thus found an equilibrium where the strategic use of ambiguity leads to an equilibrium outcome outside the support of the Nash equilibrium outcome.

We present next a framework where this intuition can be formalized.

3 Ellsberg Games

Let us formalize the intuitive idea that players can create ambiguity with the help of Ellsberg urns. An Ellsberg urn is, for us, a triple $(\Omega, \mathcal{F}, \mathcal{P})$ of a nonempty set Ω of states of the world, a σ -field \mathcal{F} on Ω (where one can take the power set in case of a finite Ω), and a set of probability measures \mathcal{P} on the measurable space (Ω, \mathcal{F}) . This *set* of probability measures represents the Knightian uncertainty of the strategy. A typical example is the classical

Ellsberg urn that contains 30 red balls, and 60 balls that are either black or yellow, and one ball is drawn at random (or rather, uncertainly). Then the state space consists of three elements $\{R, B, Y\}$, \mathcal{F} is the power set, and \mathcal{P} the set of probability vectors (P_1, P_2, P_3) such that $P_1 = 1/3$, $P_2 = k/60$, $P_3 = (60 - k)/60$ for any $k = 0, \dots, 60$.

We assume that the players of our game have access to any such Ellsberg urns; imagine that there is an independent, trustworthy laboratory that sets up such urns and reports the outcome truthfully.

We come now to the game where players can use such urns in addition to the usual mixed strategies (that correspond to roulette wheels or dice). Let $N = \{1, \dots, n\}$ be the set of players. Each player i has a finite strategy set S_i . Let $S = \prod_{i=1}^n S_i$ be the set of pure strategy profiles. Players' payoffs are given by functions

$$u_i : S \rightarrow \mathbb{R} \quad (i \in N).$$

The normal form game is denoted $G = \langle N, (S_i), (u_i) \rangle$.

Players can now use different devices. On the one hand, we assume that they have “roulette wheels” or “dices” at their disposal, i.e. randomizing devices with objectively known probabilities. The set of these probabilities over S_i is denoted ΔS_i . The players evaluate such devices according to expected utility, as in von Neumann–Morgenstern’s formulation of game theory.

Moreover, and this is the new part, players can use Ellsberg urns. As we said above, we imagine that all players have laboratories at their disposal which perform experiments that generate Knightian uncertainty. Then the players can credibly commit to base their actions on ambiguous outcomes. Technically, we model the Ellsberg urn of player i as a triple $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ as explained above.

Note that we allow the player to choose the degree of ambiguity of his urn. He tells the experimentalists of his laboratory to set up such and such an Ellsberg experiment that generates exactly the set of distributions \mathcal{P}_i . In this sense, the ambiguity in our formulation of the game is “objective”; it is not a matter of agents’ *beliefs* about the actions of other players, but rather a property of the device he is using to determine his action.

Player i acts in the game by choosing a measurable function (or Anscombe–Aumann act) $f_i : (\Omega_i, \mathcal{F}_i) \rightarrow \Delta S_i$ which specifies the classical mixed strategy played once the outcome of the Ellsberg urn is revealed. An *Ellsberg strategy* for player i is then a pair $((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$ of an Ellsberg urn and an act.

To finish the description of our Ellsberg game, we have to determine play-

ers' payoffs. We suppose that all players are ambiguity-averse: in the face of ambiguous events (as opposed to simply random events) they evaluate their utility in a cautious and pessimistic way. This behavior in response to ambiguity has been observed in the famous experiments of Ellsberg (1961) and confirmed in further experiments, for example of Pulford (2009) and Camerer and Weber (1992), see also Etner, Jeleva, and Tallon (2010) for references, and it has as well been used as a guideline by Gilboa and Schmeidler (1989) to axiomatize ambiguity-averse preferences. We follow Gilboa and Schmeidler (1989) and assume that the players evaluate the utility of an Ellsberg strategy $((\Omega_i, \mathcal{F}_i, \mathcal{P}_i), f_i)$ with a maxmin-approach. The payoff of player $i \in N$ at an Ellsberg strategy profile $((\Omega, \mathcal{F}, \mathcal{P}), f)$ is thus the minimal expected utility with respect to all different probability distributions in the set \mathcal{P} ,

$$U_i(f) := \min_{P_1 \in \mathcal{P}_1, \dots, P_n \in \mathcal{P}_n} \int_{\Omega_1} \cdots \int_{\Omega_n} u_i(f(\omega)) dP_n \dots dP_1.$$

We call the described larger game an *Ellsberg game*. An *Ellsberg equilibrium* is, in spirit of Nash equilibrium, a profile of Ellsberg strategies

$$((\Omega_1^*, \mathcal{F}_1^*, \mathcal{P}_1^*), f_1^*), \dots, ((\Omega_n^*, \mathcal{F}_n^*, \mathcal{P}_n^*), f_n^*)$$

where no player has an incentive to deviate, i.e. for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$, and all acts f_i for player i we have

$$U_i(f^*) \geq U_i(f_i, f_{-i}^*).^2$$

This definition of an Ellsberg game depends on the particular Ellsberg urn used by each player i . As there are arbitrarily many possible state spaces³, the definition of Ellsberg equilibrium might not seem very tractable. Fortunately, there is a more concise way to define Ellsberg equilibrium. The procedure is similar to the reduced form of a correlated equilibrium, see Aumann (1974) or Fudenberg and Tirole (1991). Instead of working with arbitrary Ellsberg urns, we note that the players' payoff depends, in the end, on the set of distributions that the Ellsberg urns and the associated acts induce on the set of strategies. One can then work with that set of distributions directly.

²Throughout the paper we follow the notational convention that $(f_i, f_{-i}^*) := (f_1^*, \dots, f_{i-1}^*, f_i, f_{i+1}^*, \dots, f_n^*)$. The same convention is used for profiles of pure strategies (s_i, s_{-i}) and probability distributions (P_i, P_{-i}) .

³In fact, the class of all state spaces is too large to be a well-defined set according to set theory.

Definition 1. Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A reduced form Ellsberg equilibrium of the game G is a profile of sets of probability measures $\mathcal{Q}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathcal{Q}_i \subseteq \Delta S_i$ we have

$$\begin{aligned} & \min_{P_1 \in \mathcal{Q}_1^*, \dots, P_n \in \mathcal{Q}_n^*} \int_{S_1} \cdots \int_{S_n} u_i(s) dP_n \dots dP_1 \\ & \geq \min_{P_i \in \mathcal{Q}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_{S_1} \cdots \int_{S_n} u_i(s_i, s_{-i}) dP_n \dots dP_1. \end{aligned}$$

The two definitions of Ellsberg equilibrium are equivalent in the sense that every Ellsberg equilibrium induces a payoff-equivalent reduced form Ellsberg equilibrium; and every reduced form Ellsberg equilibrium is an Ellsberg equilibrium with state space $\Omega_i = S_i$ and f_i^* the constant act. This is shown formally in the appendix. We henceforth call a set $\mathcal{Q}_i \subseteq \Delta S_i$ an Ellsberg strategy whenever it is clear that we are in the reduced form context.

Ellsberg Equilibria Generalize Nash Equilibria

Note that the classical game is contained in our formulation: players just choose a singleton $\mathcal{P}_i = \{\delta_{\pi_i}\}$ that puts all weight on a particular (classical) mixed strategy π_i .

| | | | |
|----------|---|----------|------|
| | | Player 2 | |
| | | L | R |
| Player 1 | T | 3, 3 | 0, 0 |
| | B | 0, 0 | 1, 1 |

Figure 3: Strategic Ambiguity does not unilaterally make a player better off.

Now let (π_1, \dots, π_n) be a Nash equilibrium of the game G . Can any player unilaterally gain by creating ambiguity in such a situation? The answer is no. Take the game in Figure 3 and look at the pure strategy Nash equilibrium (B, R) with equilibrium payoff 1 for both players. If player 1 introduces now unilaterally ambiguity, then he will play “T” in some states of the world (without knowing the exact probability of those states). But this does not help here because player 2 sticks to his strategy “R”, so playing “T” just leads to a payoff of zero. Unilateral introduction of ambiguity does not increase one’s own payoff. We think that this is an important property of our formulation.

Proposition 1. *Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. Then a mixed strategy profile (π_1, \dots, π_n) of G is a Nash equilibrium of G if and only if the corresponding profile of singletons $(\mathcal{P}_1, \dots, \mathcal{P}_n)$ with $\mathcal{P}_i = \{\delta_{\pi_i}\}$ is an Ellsberg equilibrium.*

In particular, Ellsberg equilibria exist. But this is not our point here. We want to show that interesting, non-Nash behavior can arise in Ellsberg games. We turn to this issue next.

4 Strategic Use of Ambiguity in Negotiations

With our new formal tools at hand, let us return to the peace negotiation example. We claim that there is the following type of Ellsberg equilibria. The superpower creates ambiguity about its decision; if this ambiguity is sufficiently large, both players fear to be punished by the superpower in case of war. As a consequence, they opt for peace.

In our game with just two actions for the superpower, we can identify an Ellsberg strategy with an interval $[P_0, P_1]$ where $P \in [P_0, P_1]$ is the probability that the superpower punishes country A. Suppose the superpower plays so with $P_0 < \frac{4}{9}$ and $P_1 > \frac{5}{9}$. Assume also that country B opts for *peace*. If A goes for war, it uses that prior in $[P_0, P_1]$ which minimizes its expected payoff, which is P_1 . This yields $U_A(\text{war}, \text{war}, [P_0, P_1]) = P_1 \cdot 0 + (1 - P_1) \cdot 9 < 4$. Hence, opting for peace is country A's best reply. The reasoning for country B is similar, but with the opposite probability P_0 . If both countries A and B go for peace, the superpower gets 4 regardless of what it does; in particular, the ambiguous strategy described above is optimal. We conclude that $(\text{peace}, \text{peace}, [P_0, P_1])$ is a (reduced form) Ellsberg equilibrium.

Proposition 2. *In Greenberg's game, the strategies $(\text{peace}, \text{peace}, [P_0, P_1])$ with $P_0 < \frac{4}{9}$ and $P_1 > \frac{5}{9}$ form an Ellsberg equilibrium.*

Note that this Ellsberg equilibrium is very different from the game's unique Nash equilibrium. In Nash equilibrium, war occurs in every play of the game; in our Ellsberg equilibrium, peace is the unique outcome⁴. By

⁴Other equilibrium concepts for extensive form games (without Knightian uncertainty) such as conjectural equilibrium Battigalli and Guaitoli (1988), self-confirming equilibrium Fudenberg and Levine (1993), and subjective equilibrium Kalai and Lehrer (1995) can also assure the peace equilibrium outcome in the example by Greenberg. Other equilibrium

using the strategy $[P_0, P_1]$ which is a set of probability distributions, the superpower *creates* ambiguity. This supports an Ellsberg equilibrium where players' strategies do not lie in the support of the unique Nash equilibrium. We also point out that the countries A and B use different *worst-case priors* in equilibrium; this is a typical phenomenon in Ellsberg equilibria that are supported by strategies which are not in the support of any Nash equilibrium of the game.

Greenberg refers to historic peace negotiations between Israel and Egypt (countries A and B in the negotiation example) mediated by the USA (superpower C) after the 1973 war. As explained by Kissinger (1982)⁵, the fact that both Egypt and Israel were too afraid to be punished if negotiations broke down partly attributed to the success of the peace negotiations. This story is supported by our Ellsberg equilibrium. We have here a first evidence that Ellsberg equilibria might capture some real world phenomena better than Nash equilibria.

5 Strategic Ambiguity as a Mediation Tool

Next we will consider a classic example from Aumann (1974). He presents a three person game where one player has some mediation power to influence his opponents' choice. The original game is given by the payoff matrix in Figure 4, where we let player 1 choose rows, player 2 choose columns and player 3 choose matrices.

| | <i>L</i> | <i>R</i> |
|----------|----------|----------|
| <i>T</i> | 0, 8, 0 | 3, 3, 3 |
| <i>B</i> | 1, 1, 1 | 0, 0, 0 |

| | <i>L</i> | <i>R</i> |
|----------|----------|----------|
| <i>T</i> | 0, 0, 0 | 3, 3, 3 |
| <i>B</i> | 1, 1, 1 | 8, 0, 0 |

l
r

Figure 4: Aumann's Example.

Player 3 is indifferent between his strategies l and r , since he gets the same

concepts for extensive form games with Knightian uncertainty are e.g. Battigalli, Cerreia-Vioglio, Maccheroni, and Marinacci (2011) and Lo (1999). Postponing the analysis of the relation of these equilibrium concepts to Ellsberg equilibrium to a later paper, we only want to stress here that in difference to the existing concepts the driving factor in Ellsberg equilibrium is that ambiguity is employed strategically and objectively.

⁵See p. 802 therein, in particular.

payoffs for both. As long as player 3 chooses l with a probability higher than $3/8$, L is an optimal strategy for player 2 regardless what player 1 does; and player 1 would subsequently play B . By the same reasoning, as long as player 3 plays r with a probability higher than $3/8$, B is optimal for player 1, and player 2 plays L then. Thus the Nash equilibria of this game are all of the form (B, L, P^*) , where P^* is any objectively mixed strategy.

This example has also been analyzed in other literature on ambiguity in games, see Eichberger and Kelsey (2006), Lo (2009) and Bade (2011). Bade chooses one ambiguous act equilibrium with maxmin expected utility preferences to show that non-Nash outcomes can be sustained in games with more than two players. We characterize all Ellsberg equilibria⁶ and provide an interpretation of the example that highlights the strategic use of ambiguity as a mediation tool.

Let us now explain how Aumann’s example can be interpreted to illustrate the strategic use of ambiguity. Suppose players 1 and 2 are prisoners, and player 3 the police officer. Let us rearrange the matrix game and put it in the form displayed in Figure 5. We swap the strategies of player 2 and rename the strategies of player 1 and 2 to $C =$ "cooperate" and $D =$ "defect" as in the classical prisoners’ dilemma. We can merge the two matrices into one, because the strategy choice of the police officer is simply the choice of a probability that influences the payoffs of prisoners 1 and 2 in case of unilateral defection from (C, C) . If he chooses $P = 1$ this corresponds to strategy l in the original game (i.e. prisoner 2 gets all the reward), $P = 0$ would be strategy r (i.e. prisoner 1 gets all the reward). The choice of the objectively mixed strategy $P = 1/2$ leads to the classical symmetric prisoners’ dilemma with a payoff of 4 in case of unilateral defection.

In this interpretation, players 1 and 2 are facing a sort of prisoners’ dilemma situation mediated by a player 3, the police officer. Given the payoffs, the police officer is most interested in cooperation between the prisoners. The police officer can influence how high the reward would be for unilateral defection by using an objective randomizing device. Nevertheless, in every Nash equilibrium of the game, the players obtain the inefficient outcome of 1.

Now suppose we let the players use Ellsberg strategies. The police officer

⁶In this special game and the preference representation chosen by Bade, the ambiguous act equilibrium coincides with one of the Ellsberg equilibria. Recall that in difference to ambiguous act equilibria, Ellsberg equilibria use ambiguity objectively.

| | | | |
|------------|-----|------------------|------------|
| | | Prisoner 2 | |
| | | C | D |
| Prisoner 1 | C | $3, 3, 3$ | $0, 8P, 0$ |
| | D | $8(1 - P), 0, 0$ | $1, 1, 1$ |

Figure 5: Mediated Prisoners' dilemma

could create ambiguity by announcing: "I'm not sure about who of you I will want to punish and who I will want to reward for reporting on your partner. I might also reward you both equally... I simply don't tell you what mechanism I will use to decide about this."

Let us exhibit Ellsberg strategies that support this behavior. If prisoner 2 expects P to be lower than $3/8$ and prisoner 1 expects P to be higher than $5/8$, they would prefer to cooperate. This behavior corresponds to the police officer playing an Ellsberg strategy $[P_0, P_1]$ with $0 \leq P_0 < \frac{3}{8}$ and $\frac{5}{8} < P_1 \leq 1$. The ambiguity averse prisoners 1 and 2 evaluate their utility with $P = P_0$ and $P = P_1$ resp. Consequently they would prefer to play (C, C) . This gives an Ellsberg equilibrium in which the prisoners cooperate.

Proposition 3. *In the mediated prisoners' dilemma, the Ellsberg strategy profiles $(C, C, [P_0, P_1])$ with $P_0 < \frac{3}{8}$ and $P_1 > \frac{5}{8}$ are Ellsberg equilibria that achieve the efficient outcome $(3, 3, 3)$.*

Again, as in Greenberg's example, it is important to see that both prisoners use different worst-case probabilities to compute their expected payoffs. Aumann (1974) has already commented on this behavior. He observes that (C, C) can be a "subjective equilibrium point" if players 1 and 2 have non-common beliefs about the objectively mixed strategy player 3 is going to use. In his analysis player 1 believes $P = 3/4$ and player 2 believes $P = 1/4$. Note that in Ellsberg equilibrium the players have the *common* belief $P \in [P_0, P_1]$.

6 Strategic Ambiguity in Competitive Situations

Let us now consider two-person games with conflicting interests; in such a situation, one usually has no strict equilibria. We take a slightly modified version of Matching Pennies as our example; the results generalize to all such

two-person 2×2 games as we explain below. The payoff matrix for this game is in Figure 6.

| | | | |
|----------|-------------|-------------|-------------|
| | | Player 2 | |
| | | <i>HEAD</i> | <i>TAIL</i> |
| Player 1 | <i>HEAD</i> | 3, -1 | -1, 1 |
| | <i>TAIL</i> | -1, 1 | 1, -1 |

Figure 6: Modified Matching Pennies I

We point out two effects that arise due to strategic ambiguity in this class of games. On the one hand, the Ellsberg equilibria are different from what one might expect first; in a game like the one above, one might intuitively guess that “full ambiguity” would be an Ellsberg equilibrium, as the natural generalization of “full randomness” (completely mixed Nash equilibrium). This is not the case.

On the other hand, we emphasize an important property of Ellsberg games (or ambiguity aversion in general): the best reply functions are no longer linear in the probabilities. As a consequence, the indifference principle of classical game theory – when two pure strategies yield the same payoff, then the player is indifferent about mixing in any arbitrary way between the two strategies – does not carry over to Ellsberg games. When a player is indifferent between two Anscombe–Aumann acts, this does not imply that she is indifferent between all mixtures over these two acts. This is due to the hedging or diversification effect provided by a (classical) mixed strategy when players are ambiguity-averse. We call this effect *immunization against strategic ambiguity*.

Immunization against Strategic Ambiguity

In our modified version of Matching Pennies, the unique Nash equilibrium is that player 1 mixes uniformly over his strategies, and player 2 mixes with $(1/3, 2/3)$. This yields the equilibrium payoffs $1/3$ and 0 . One might guess that one can get an Ellsberg equilibrium where both players use a set of probability measures around the Nash equilibrium distribution as their strategy. This is, somewhat surprisingly, at least to us, not true.

The crucial point to understand here is the following. Players can immunize themselves against ambiguity; in the modified Matching Pennies example, player 1 can use the mixed strategy $(1/3, 2/3)$ to make himself inde-

pendent of any ambiguity used by the opponent. Indeed, with this strategy, his expected payoff is $1/3$ against any mixed strategy of the opponent, and a fortiori against Ellsberg strategies as well. This strategy is also the unique best reply of player 1 to Ellsberg strategies with ambiguity around the Nash equilibrium; in particular, such strategic ambiguity is not part of an Ellsberg equilibrium.

Let us explain this somewhat more formally. An Ellsberg strategy for player 2 can be identified with an interval $[Q_0, Q_1] \subseteq [0, 1]$ where $Q \in [Q_0, Q_1]$ is the probability to play HEAD. Suppose player 2 uses many probabilities around $1/3$, so $Q_0 < 1/3 < Q_1$. The (minimal) expected payoff for player 1 when he uses the mixed strategy with probability P for “HEAD” is then

$$\begin{aligned} & \min_{Q_0 \leq Q \leq Q_1} 3PQ - P(1 - Q) - (1 - P)Q + (1 - P)(1 - Q) \\ & = \min \{Q_0(6P - 2), Q_1(6P - 2)\} + 1 - 2P \\ & = \begin{cases} Q_1(6P - 2) + 1 - 2P & \text{if } P < 1/3 \\ 1/3 & \text{if } P = 1/3 \\ Q_0(6P - 2) + 1 - 2P & \text{else.} \end{cases} \end{aligned}$$

We plot the payoff function in Figure 6.

By choosing the mixed strategy $P = 1/3$, player 1 becomes immune against any ambiguity and ensures the (Nash) equilibrium payoff of $1/3$. If there was an Ellsberg equilibrium with $P_0 < 1/2 < P_1$ and $Q_0 < 1/3 < Q_1$, then the minimal expected payoff would be below $1/3$. Hence, such Ellsberg equilibria do not exist.

Such immunization plays frequently a role in two-person games, and it need not always be the Nash equilibrium strategy that is used to render oneself immune. Consider, e.g., the slightly changed payoff matrix

In the unique Nash equilibrium, player 1 still plays both strategies with probability $1/2$ (to render player 2 indifferent); however, in order to be immune against Ellsberg strategies, he has to play HEAD with probability $3/5$. Then his payoff is $-1/5$ regardless of what player 2 does. This strategy does not play any role in either Nash or Ellsberg equilibrium. It is only important in so far as it excludes possible Ellsberg equilibria by being the unique best reply to some Ellsberg strategies.

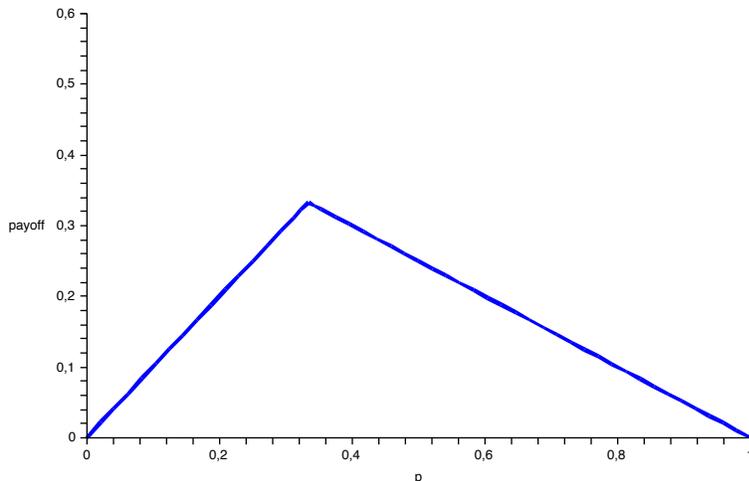


Figure 7: Player 1's (minimal expected) payoff as a function of the probability P of playing HEAD when player 2 uses the Ellsberg strategy $[1/4, 1/2]$.

| | | | |
|----------|-------------|-------------|-------------|
| | | Player 2 | |
| | | <i>HEAD</i> | <i>TAIL</i> |
| Player 1 | <i>HEAD</i> | 1, -1 | -1, 1 |
| | <i>TAIL</i> | -2, 1 | 1, -1 |

Figure 8: Modified Matching Pennies II

Ellsberg Equilibria

The question thus arises if there are any Ellsberg equilibria different from the Nash equilibrium at all. There are, and they take the following form for our first version of modified Matching Pennies (Figure 6). Player 1 plays HEAD with probability $P \in [1/2, P_1]$ for some $1/2 \leq P_1 \leq 1$ and player 2 plays HEAD with probability $Q \in [1/3, Q_1]$ for some $1/3 \leq Q_1 \leq 1/2$. This Ellsberg equilibrium yields the same payoffs $1/3$ and 0 as in Nash equilibrium. We prove a more general theorem covering this case in the appendix.

Proposition 4. *In the Modified Matching Pennies Game I, the Ellsberg equilibria are of the form $([1/2, P_1], [1/3, Q_1])$ for $1/2 \leq P_1 \leq 1$ and $1/3 \leq Q_1 \leq 1/2$.*

The typical Ellsberg equilibrium strategy takes the following form. Player 1 says :“I will play HEAD with a probability of at least 50 %, but not less.” And Player 2 replies: “I will play HEAD with at least 33 %, but not more than 50 %.”

Whereas the support of the Ellsberg and Nash equilibria is obviously the same, we do think that the Ellsberg equilibria reveal a new class of behavior not encountered in game theory before. It might be very difficult for humans to play exactly a randomizing strategy with equal probabilities (indeed, some claim that this is impossible, see Dang (2009) and references therein). Our result shows that it is not necessary to randomize exactly to support a similar equilibrium outcome (with the same expected payoff). It is just enough that your opponent knows that you are randomizing with some probability, and that it could be that this probability is one half, but not less. It is thus sufficient that the player is able to control the lower bound of his randomizing device. This might be easier to implement than the perfectly random behavior required in classical game theory.

In fact, there are experimental findings which suggest that the Ellsberg equilibrium strategy in the modified Matching Pennies game is closer to real behavior than the Nash equilibrium prediction. In an experiment run by Goeree and Holt (2001), 50 subjects play a one-shot version of the modified Matching Pennies game as in Figure 6, with the following payoffs:

| | | | |
|----------|-------------|-------------|-------------|
| | | Player 2 | |
| | | <i>HEAD</i> | <i>TAIL</i> |
| Player 1 | <i>HEAD</i> | 320, 40 | 40, 80 |
| | <i>TAIL</i> | 40, 80 | 80, 40 |

Figure 9: Modified Matching Pennies III

The Nash equilibrium prediction of the game is $((1/2, 1/2), (1/8, 7/8))$, but in the experiment row players deviate considerably from this equilibrium. 24 of the 25 row players choose to play HEAD, and only one chooses TAIL. Furthermore, four of the column players choose HEAD, the other 21 choose to play TAIL; the aggregate observation for column players is thus closer to the Nash equilibrium strategy.

Applying the Ellsberg analysis of the modified Matching Pennies game (see Proposition 6 in the appendix) to the payoffs of Goeree and Holt (2001)’s experiment yields an Ellsberg equilibrium $([1/2, 1], [1/8, 1/2])$. The aggregate observation in the experiment (that player 1 chooses HEAD with higher

probability than TAIL) is thus consistent with the Ellsberg equilibrium strategy.

7 Conclusion

This article demonstrates that the strategic use of ambiguity is a relevant concept in game theory. Employing ambiguity as a strategic instrument leads to a new class of equilibria not encountered in classic game theory. We point out that players may choose to be deliberately ambiguous to gain a strategic advantage.

In some games this results in equilibrium outcomes which can not be obtained as Nash equilibria. The peace negotiation as well as the mediated prisoners' dilemma provide examples of such Ellsberg equilibria. All examples considered in the paper suggest that the behavior in Ellsberg equilibria can be of economic relevance.

Games with more than two players offer a strategic possibility that is not possible in two-person games, because a third player is able to induce the use of different probability distributions. Although e.g. countries A and B observe the same Ellsberg strategy played by the superpower C, due to their ambiguity aversion modeled by maxmin expected utility the countries use different probability distributions to assess their utility.

However, two-person 2×2 games have Ellsberg equilibria which are conceptually different from classic mixed strategy Nash equilibria. We analyze a special class of these games, where the players have conflicting interests. These games have equilibria in which both players create ambiguity. They use an Ellsberg strategy where they only need to control the lower (or upper) bound of their set of probability distributions. We argue that this randomizing device is easier to use for a player than playing one precise probability distribution like in mixed strategy Nash equilibrium. What makes this argument attractive is that the payoffs in these Ellsberg equilibria are the same as in the unique mixed Nash equilibrium and thus the use of ambiguous strategies in competitive games is indeed an option.

We believe that the use of Ellsberg urns as randomizing devices is a tangible concept that does not involve more (or even less) sophistication than single probability distributions and, in connection with ambiguity-aversion, adds a new dimension to the analysis of strategic interaction. Ellsberg games need to be explored further to fully characterize the strategic possibilities that

ambiguity offers.

A Equivalence of the different formulations of Ellsberg equilibrium

We prove that Ellsberg equilibrium and its reduced form are equivalent. First we recap the definition of an Ellsberg equilibrium, which was stated in the text of Section 3.

Definition 2. Let $G = \langle N, (S_i), (u_i) \rangle$ be a normal form game. A profile

$$((\Omega_1^*, \mathcal{F}_1^*, \mathcal{P}_1^*), f_1^*), \dots, ((\Omega_n^*, \mathcal{F}_n^*, \mathcal{P}_n^*), f_n^*)$$

of Ellsberg strategies is an Ellsberg equilibrium of G if no player has an incentive to deviate from $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$, i.e. for all players $i \in N$, all Ellsberg urns $(\Omega_i, \mathcal{F}_i, \mathcal{P}_i)$ and all acts f_i for player i we have

$$U_i(f^*) \geq U_i(f_i, f_{-i}^*), \text{ that is}$$

$$\begin{aligned} & \min_{P_i \in \mathcal{P}_i^*, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f^*(\omega)) dP_{-i} dP_i \\ & \geq \min_{P_i \in \mathcal{P}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i} \int_{\Omega_{-i}^*} u_i(f_i(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i. \end{aligned}$$

The definition of the reduced form Ellsberg equilibrium was given in Definition 1.

Proposition 5. Definition 2 and Definition 1 are equivalent in the sense that every Ellsberg equilibrium $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ induces a payoff-equivalent reduced form Ellsberg equilibrium on $\Omega^* = S$; and every reduced form Ellsberg equilibrium \mathcal{Q}^* is an Ellsberg equilibrium $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$ with f^* the constant act.

Proof. " \Leftarrow " Let \mathcal{Q}^* be an Ellsberg equilibrium according to Definition 1. We choose the states of the world $\Omega = S$ to be the set of pure strategy profiles, thereby we see that player i uses the Ellsberg urn $(S_i, \mathcal{F}_i, \mathcal{Q}_i^*)$. We define the act $f_i^* : (S_i, \mathcal{F}_i) \rightarrow \Delta S_i$ to be the constant act that maps $f_i^*(s_i) = \{\delta_{s_i}\}$.

$\{\delta_{s_i}\} \in \Delta S_i$ is the degenerate mixed strategy which puts all weight on the pure strategy s_i . Each measure $Q_i \in \mathcal{Q}_i^*$ has an image measure under f_i^* ,

$$Q_i \circ f_i^{*-1} : \{\delta_{s_i}\} \mapsto Q_i(f_i^{*-1}(\{\delta_{s_i}\})).$$

$Q_i \circ f_i^{*-1}$ can be identified with $Q_i \in \mathcal{Q}_i^*$. Thus the reduced form Ellsberg equilibrium strategy \mathcal{Q}_i^* can be written as the Ellsberg strategy $((S, \mathcal{F}, \mathcal{Q}^*), f^*)$. This strategy is an Ellsberg equilibrium according to Definition 2.

" \Rightarrow " Let now $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ be an Ellsberg equilibrium according to Definition 2. Every $P_i \in \mathcal{P}_i^*$ induces an image measure $P_i \circ f_i^{*-1}$ on ΔS_i that assigns a probability to a distribution $f_i^*(\omega_i) \in \Delta S_i$ to occur.

To describe the probability that a pure strategy s_i is played, given a distribution P_i and an Ellsberg strategy $((\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*), f_i^*)$, we integrate $f_i^*(\omega_i)(s_i)$ over all states $\omega_i \in \Omega_i$. Thus we can define Q_i to be:

$$Q_i(s_i) := \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP_i. \quad (1)$$

Recall that \mathcal{P}_i is a closed and convex set of probability distributions. We get a measure Q_i on S_i for each $P_i \in \mathcal{P}_i \subseteq \Delta \Omega_i$. We call the resulting set of probability measures \mathcal{Q}_i^* .

$$\mathcal{Q}_i^*(s_i) := \left\{ Q_i(s_i) = \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP_i \mid P_i \in \mathcal{P}_i^* \right\}.$$

\mathcal{Q}_i^* is closed and convex, since \mathcal{P}_i^* is.

Now suppose \mathcal{Q}^* was not a reduced form Ellsberg equilibrium. Then for some player $i \in N$ there existed a set \mathcal{Q}_i of probability measures on S_i that yields a higher minimal expected utility. This means we would have

$$\begin{aligned} & \min_{Q_i \in \mathcal{Q}_i, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s) dQ_{-i} dQ_i \\ & > \min_{Q_i \in \mathcal{Q}_i^*, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s) dQ_{-i} dQ_i \end{aligned} \quad (2)$$

for some $Q_i \neq Q_i^*$. Let Q'_i be the minimizer of the first expression, then it must be that $Q'_i \notin \mathcal{Q}_i^*$. We know that Q'_i is derived from some P'_i under the equilibrium act,

$$Q'_i(s_i) = \int_{\Omega_i^*} f_i^*(\omega_i)(s_i) dP'_i. \quad (3)$$

It follows that P'_i is not element of the equilibrium Ellsberg urn $(\Omega_i^*, \mathcal{F}_i^*, \mathcal{P}_i^*)$, that is $P'_i \notin \mathcal{P}_i^*$. Now it remains to show that in the original game P'_i yields a higher minimal expected utility than using \mathcal{P}_i^* . In that case $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ is not an Ellsberg equilibrium and the proof is complete.

Let player i use P'_i in his maxmin expected utility evaluation in the original game. This yields

$$\begin{aligned} & \min_{P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f^*(\omega)) dP_{-i} dP'_i \\ &= \min_{P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) df_{-i}^*(\omega_{-i}) df_i^*(\omega_i) dP_{-i} dP'_i. \end{aligned} \quad (4)$$

Recall that we use u_i to be the utility function on S_i as well as on ΔS_i . We use equations (1) and (3) to rewrite the expression and get

$$\min_{Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ'_i. \quad (5)$$

We know by equation (2) that this is larger than the minimal expected utility over \mathcal{Q}_i^* and this gives

$$\begin{aligned} (5) &> \min_{Q_i \in \mathcal{Q}_i^*, Q_{-i} \in \mathcal{Q}_{-i}^*} \int_{S_i} \int_{S_{-i}} u_i(s_i, s_{-i}) dQ_{-i} dQ_i \\ &= \min_{P_i \in \mathcal{P}_i^*, P_{-i} \in \mathcal{P}_{-i}^*} \int_{\Omega_i^*} \int_{\Omega_{-i}^*} u_i(f_i^*(\omega_i), f_{-i}^*(\omega_{-i})) dP_{-i} dP_i. \end{aligned}$$

Going back to equation (4) we see that this contradicts the assumption that $((\Omega^*, \mathcal{F}^*, \mathcal{P}^*), f^*)$ was an Ellsberg equilibrium. Thus \mathcal{Q}^* is a reduced form Ellsberg equilibrium. \square

B Competitive 2×2 Games

We provide here the promised proposition on 2×2 games.

Proposition 6. *Consider the competitive two-person 2×2 game with payoff matrix*

| | | | |
|-----------------|----------|-----------------|-------------|
| | | <i>Player 2</i> | |
| | | <i>L</i> | <i>R</i> |
| <i>Player 1</i> | <i>U</i> | <i>a, d</i> | <i>b, e</i> |
| | <i>D</i> | <i>b, e</i> | <i>c, f</i> |

such that

$$a, c > b \text{ and } d, f < e.$$

Let $((P^*, 1 - P^*), (Q^*, 1 - Q^*))$ denote the unique Nash equilibrium. Then the Ellsberg equilibria of the game are the following:

For $P^* > Q^*$ all Ellsberg equilibria are of the form

$$([P^*, P_1], [Q^*, Q_1]) \text{ for } P^* \leq P_1 \leq 1, Q^* \leq Q_1 \leq P^*;$$

for $P^* < Q^*$ all Ellsberg equilibria are of the form

$$([P_0, P^*], [Q_0, Q^*]) \text{ for } 0 \leq P_0 \leq P^*, P^* \leq Q_0 \leq Q^*;$$

and for $P^* = Q^*$ all Ellsberg equilibria are of the form

$$(Q^*, [Q_0, Q_1]) \text{ where } Q_0 \leq Q^* \leq Q_1$$

and

$$([P_0, P_1], P^*) \text{ where } P_0 \leq P^* \leq P_1.$$

Proof. The game has a unique Nash equilibrium in which player 1 plays U with probability

$$P^* = \frac{f - e}{d - 2e + f},$$

and player 2 plays L with probability

$$Q^* = \frac{c - b}{a - 2b + c}.$$

The Nash equilibrium strategies follow from the usual analysis. The conditions on the payoffs assure that the Nash equilibrium is completely mixed, i.e. $0 < P^* < 1$ and $0 < Q^* < 1$.

Let now $[P_0, P_1]$ and $[Q_0, Q_1]$ be Ellsberg strategies of player 1 and 2, where $P \in [P_0, P_1]$ is the probability of player 1 to play U , and $Q \in [Q_0, Q_1]$ is the probability of player 2 to play L .

Let us compute the minimal expected payoff:

The minimal expected payoff of player 1 when he plays the mixed strategy P is

$$\begin{aligned} \min_{Q_0 \leq Q \leq Q_1} u_1(P, Q) &= \min_{Q_0 \leq Q \leq Q_1} aPQ + bP(1-Q) + b(1-P)Q + c(1-P)(1-Q) \\ &= \min_{Q_0 \leq Q \leq Q_1} Q(b-c + P(a-2b+c)) + bP + c - cP \\ &= \begin{cases} Q_1(b-c + P(a-2b+c)) + bP + c - cP & \text{if } P < \frac{c-b}{a-2b+c} = Q^* \\ \frac{ac-b^2}{a-2b+c} & \text{if } P = \frac{c-b}{a-2b+c} = Q^* \\ Q_0(b-c + P(a-2b+c)) + bP + c - cP & \text{else.} \end{cases} \end{aligned}$$

Note that the payoff function has a fixed value at $P = Q^*$. Depending on Q_0 and Q_1 the minimal payoff function can have six different forms. It can be strictly increasing, strictly decreasing, have flat parts or be completely constant. To determine how player 1 maximizes his minimal payoff for different Q_0 and Q_1 we look at the borders of the minimal payoff function, where $P = 0$ and $P = 1$. This gives us two functions

$$\min_{Q_0 \leq Q \leq Q_1} u_1(0, Q) = Q_1(b-c) + c \quad \text{and} \quad \min_{Q_0 \leq Q \leq Q_1} u_1(1, Q) = Q_0(a-b) + b.$$

Note that $b-c < 0$ and $a-b > 0$, that is, the minimal payoff function is decreasing with Q_1 at $P = 0$ and increasing with Q_0 at $P = 1$. When

$$Q_1 = \frac{c-b}{a-2b+c} = Q^*, \quad \text{then} \quad \min_{Q_0 \leq Q \leq Q_1} u_1(0, Q) = \frac{ac-b^2}{a-2b+c},$$

that is, the minimal payoff function is constant for $0 \leq P \leq Q^*$. The same is true for the other boundary: When

$$Q_0 = \frac{c-b}{a-2b+c} = Q^*, \quad \text{then} \quad \min_{Q_0 \leq Q \leq Q_1} u_1(1, Q) = \frac{ac-b^2}{a-2b+c},$$

that is, the minimal payoff function is constant for $Q^* \leq P \leq 1$.

With this analysis one can see immediately that when $Q_0 = Q^* = Q_1$, the minimal payoff function is constant for all $P \in [0, 1]$ thus any Ellsberg strategy $[P_0, P_1] \subseteq [0, 1]$ is a best response for player 1.

Assume that $Q_0 > Q^*$, then (since $a-b > 0$) the minimal payoff function will be strictly increasing and the best response of player 1 is $P_0 = P_1 = 1$. The opposite is true for $Q_1 < Q^*$ and thus the best response is $P_0 = P_1 = 0$.

Observe that when $Q_0 < Q^* < Q_1$, the values of both boundary functions drop below Q^* and the function takes its maximum at the kink $P = Q^*$. Therefore player 1's best response in this case is $P_0 = P_1 = Q^*$.

Two cases are still missing. The minimal expected payoff function can be flat exclusively to the left or to the right of Q^* . For all $P \in [0, \frac{c-b}{a-2b+c}] = [0, Q^*]$, Player 1's utility is constant at $\frac{ac-b^2}{a-2b+c}$ when $Q_1 = \frac{c-b}{a-2b+c} = Q^*$, and it is strictly decreasing for $P > Q^*$. Hence, all $P \leq Q^*$ are optimal for player 1. He can thus use any Ellsberg strategy $[P_0, P_1]$ with $P_1 \leq Q^*$ as a best reply. Similarly, the payoff is constant for all $P \geq Q^*$ when $Q_0 = Q^*$ (and strictly increasing for $P < Q^*$). This means that player 1's best response to a strategy $[Q_0, Q^*]$ is any strategy $[P_0, P_1] \subseteq [0, Q^*]$, and player 1's best response to a strategy $[Q^*, Q_1]$ where $Q^* \leq Q_1 \leq 1$ is any strategy $[P_0, P_1] \subseteq [Q^*, 1]$.

We repeat the same analysis for player 2. His minimal expected utility when he plays the mixed strategy Q is

$$\begin{aligned} \min_{P_0 \leq P \leq P_1} u_2(P, Q) &= \min_{P_0 \leq P \leq P_1} dPQ + eP(1-Q) + e(1-P)Q + f(1-P)(1-Q) \\ &= \min_{P_0 \leq P \leq P_1} P(e-f + Q(d-2e+f)) + eQ + f - fQ \\ &= \begin{cases} P_0(e-f + Q(d-2e+f)) + eQ + f - fQ & \text{if } Q < \frac{f-e}{d-2e+f} = P^* \\ \frac{df-e^2}{d-2e+f} & \text{if } Q = \frac{f-e}{d-2e+f} = P^* \\ P_1(e-f + Q(d-2e+f)) + eQ + f - fQ & \text{else.} \end{cases} \end{aligned}$$

Note that the payoff function has a fixed value at $Q = P^*$. Again, as for player 1, depending on P_0 and P_1 the minimal payoff function can have six different forms. We note the two functions that describe the minimal payoff function at the borders $Q = 0$ and $Q = 1$:

$$\min_{P_0 \leq P \leq P_1} u_1(P, 0) = P_0(e-f) + f \quad \text{and} \quad \min_{P_0 \leq P \leq P_1} u_1(P, 1) = P_1(d-e) + e.$$

Note that $e-f > 0$ and $d-e < 0$, that is, the minimal payoff function is increasing with P_0 in $P = 0$ and decreasing with P_1 in $P = 1$. When

$$P_0 = \frac{f-e}{d-2e+f} = P^*, \text{ then } \min_{P_0 \leq P \leq P_1} u_2(P, 0) = \frac{df-e^2}{d-2e+f},$$

that is, the minimal payoff function is constant for $0 \leq Q \leq P^*$. The same is true for the other boundary: When

$$P_1 = \frac{f-e}{d-2e+f} = P^*, \text{ then } \min_{P_0 \leq P \leq P_1} u_2(P, 1) = \frac{df-e^2}{d-2e+f},$$

that is, the minimal payoff function is constant for $P^* \leq Q \leq 1$.

Similar to the analysis of player 1 we now get the following best responses of player 2. When $P_0 = P^* = P_1$ player 2 can use any strategy $[Q_0, Q_1] \subseteq [0, 1]$, when $P_0 > P^*$ the best response is $Q_0 = Q_1 = 0$ and when $P_1 < P^*$ then $Q_0 = Q_1 = 1$. When $P_0 < P^* < P_1$ the minimal payoff function takes its maximum at the kink $Q = P^*$ and accordingly player 2's best response is $Q_0 = Q_1 = P^*$.

Finally note that Player 2's utility is constant at $\frac{df-e^2}{d-2e+f}$ for all $Q \in [0, \frac{f-e}{d-2e+f}] = [0, P^*]$ when $P_0 = \frac{f-e}{d-2e+f} = P^*$, and it is strictly decreasing for $Q > P^*$. Hence, all $Q \leq P^*$ are optimal for player 2. He can thus use any Ellsberg strategy $[Q_0, Q_1]$ with $Q_1 \leq P^*$ as a best reply. Similarly, the payoff is constant for all $Q \geq P^*$ when $P_0 = P^*$ (and strictly increasing for $Q < P^*$). This means that player 2's best response to a strategy $[P^*, P_1]$ is any strategy $[Q_0, Q_1] \subseteq [0, P^*]$, and player 2's best response to a strategy $[P^*, P_1]$ where $P^* \leq P_1 \leq 1$ is any strategy $[Q_0, Q_1] \subseteq [P^*, 1]$.

In Ellsberg equilibrium no player wants to unilaterally deviate from his equilibrium strategy. We analyze in the following which Ellsberg strategies have best responses such that no player wants to deviate.

We assume first that $Q^* < P^*$.

Three Ellsberg strategies can quickly be excluded to be part of an Ellsberg equilibrium. Suppose player 2 plays $[Q_0, Q_1]$ with $Q_0 > Q^*$, then player 1's best response is $P_0 = P_1 = 1$ and since we are looking at a strictly competitive game, player 2 would want to deviate from his original strategy to $Q_0 = Q_1 = 0$. A similar reasoning leads to the result that an Ellsberg strategy $[Q_0, Q_1]$ with $Q_1 < Q^*$ cannot be an equilibrium strategy. Thirdly, suppose player 2 plays $[Q_0, Q_1]$ with $Q_0 < Q < Q_1$, then player 1 would respond with $P_0 = P_1 = Q^*$. Since $Q^* < P^*$ player 2 would deviate from his original strategy to $Q_0 = Q_1 = 1$.

Now suppose player 2 plays $Q_0 = Q_1 = Q^*$, then player 1 can respond with any $[P_0, P_1] \subseteq [0, 1]$. Any choice with $P_0 \geq P^*$, $P_1 < P^*$ or $P_0 < P^* < P_1$ lead to contradictions similar to the cases above. The possibilities that $P_0 < P_1 = P^*$ or $P_0 = P_1 = P^*$ (which are Ellsberg equilibria) are contained in the Ellsberg equilibria that arise in the two remaining cases below.

Suppose player 2 plays $[Q^*, Q_1]$ with $Q^* \leq Q_1 \leq 1$, then if player 1 responds with $[P_0, P_1] = [P^*, P_1]$ with $P^* \leq P_1 \leq 1$ player 2 would play any strategy $[Q_0, Q_1] \subseteq [0, P^*]$ as a best response. Because $Q^* < P^*$, player 2

can choose $[Q_0, Q_1] = [Q^*, Q_1]$ with $Q^* \leq Q_1 \leq P^*$. These strategies are Ellsberg equilibria

$$([P^*, P_1], [Q^*, Q_1]) \text{ where } P^* \leq P_1 \leq 1 \text{ and } Q^* \leq Q_1 \leq P^*.$$

In the case $Q^* < P^*$ this is the only type of Ellsberg equilibrium. Note that the Nash equilibrium is contained in these equilibrium strategies.

When we assume that $P^* < Q^*$ the analysis is very similar. We skip the first four cases and only look at the cases where the minimal payoff function has flat parts. Suppose player 2 plays $[Q_0, Q^*]$ with $0 \leq Q_0 \leq Q^*$, then if we let player 1 pick $[P_0, P_1] \subseteq [P_0, P^*]$ with $0 \leq P_0 \leq P^*$, player 2's best response is any subset $[Q_0, Q_1] \subseteq [P^*, 1]$. Again, because $P^* < Q^*$, he can choose $[Q_0, Q_1] = [Q_0, Q^*]$ with $P^* \leq Q_0 \leq Q^*$ as a best response. Player 1 would not want to deviate and thus these strategies are Ellsberg equilibria

$$([P_0, P^*], [Q_0, Q^*]) \text{ where } 0 \leq P_0 \leq P^* \text{ and } P^* \leq Q_0 \leq Q^*.$$

As before, this is the only type of Ellsberg equilibrium in case $P^* < Q^*$.

Finally let $P^* = Q^*$. Repeat the considerations above having in mind the equality of the Nash equilibrium strategies. Since it was precisely the difference between P^* and Q^* that led to the Ellsberg equilibria in the above cases, we see that no Ellsberg equilibria exist where both players create ambiguity. But, in difference to the above analysis, two types of Ellsberg equilibria with unilateral ambiguity arise that could not be sustained above.

Remember that when player 2 plays $[Q_0, Q_1]$ with $Q_0 < Q^* < Q_1$ then it is optimal for player 1 to respond with $P_0 = P_1 = Q^*$. Since $P^* = Q^*$ these strategies are in equilibrium, even for $Q_0 \leq Q^* \leq Q_1$. One observes that, as long as player 2 makes sure that the mixed Nash equilibrium strategy Q^* is strictly contained in his Ellsberg strategy, player 1 will respond with Q^* and we have Ellsberg equilibria in which player 1 hedges against the ambiguity of player 2. An analogous type of Ellsberg equilibrium exists for player 2 hedging against the ambiguity of player 1 by playing $Q_0 = Q_1 = P^*$. Thus we have the following Ellsberg equilibria

$$(Q^*, [Q_0, Q_1]) \text{ where } Q_0 \leq Q^* \leq Q_1$$

and

$$([P_0, P_1], P^*) \text{ where } P_0 \leq P^* \leq P_1.$$

These equilibria do not exist in the non-symmetric case $P^* \neq Q^*$ since the hedging strategies are in general not equilibrium strategies. Due to the assumptions on the payoffs in this proposition, the hedging strategy equals the Nash equilibrium strategy of the opponent. Thereby the hedging strategy is an equilibrium strategy only when $P^* = Q^*$. \square

Remark 1. 1. In Proposition 6 we restrict to the case with (U, D) and (L, R) giving the same payoffs (b, e) for both players. Of course the Ellsberg equilibria of competitive games with more general payoffs can easily be calculated. The nice feature of our restriction is that players use the mixed Nash equilibrium strategy of their respective opponent as their hedging strategy.

2. Observe the asymmetry in the Ellsberg equilibria in the preceding proposition: no matter if $P^* < Q^*$ or $Q^* < P^*$, always it is player 2 who creates ambiguity between the Nash equilibrium strategies, player 1 never does so. This is due to the assumptions on the payoffs. If we assume that $a, c < b$ and $d, f > e$ player 1 will play between P^* and Q^* .
3. Note that Proposition 6 holds likewise for zero-sum games. Thus also zero-sum games can have non-Nash Ellsberg equilibria in which both players create ambiguity.

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