Too Risk Averse to Purchase Insurance?
A Theoretical Glance at the Annuity Puzzle

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Abstract

We show that, as soon there exists a positive bequest motive, sufficiently risk averse individuals should not purchase annuities. This suggests therefore a new explanation for the low level of annuitization, which is valid even if one assumes perfect markets. A model calibration accounting for temporal risk aversion generates an optimal level of annuitization of only 25% of the total savings.

Keywords: annuity puzzle, risk aversion

JEL codes: D11, D81, D91.

1 Introduction

Among the largest risk in life is the one associated with life duration. A recently retired American man of age 65 has a life expectancy of about 17.5 years. Though there is more than 22% chance that he will die within the first 10 years and more than 20% that he live more than 25 years. Savings required to sustain 10 or 25 years of retirement are very different, and one would expect that financial products specifically designed to deal with such uncertainty - annuities - should be widely developed. A number of papers have underlined the subsequent utility gains that would follow annuitization of wealth of retirement. It is generally estimated that individuals would be willing to give up to 25% of their wealth at retirement to have access to a perfect annuity market (Mitchell, Poterba, Warshawsky and Brown, 1999). Even when a bequest motives is considered, standard theoretical predictions are that individuals should annuitize the expected value of their future

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consumption. However, puzzlingly enough, empirical evidence consistently show that individuals purchase very little private annuities, in sharp contradiction with the theoretical predictions. Market imperfections or rationality biases are then suggested as explanations about this puzzle (see the Brown, 2007, as well as the following section for a review of literature). Popular explanations are for example that imperfect health insurance leads individuals to keep a substantial amount of liquidities; that annuities pricing makes them unattractive assets; or that framing effects play an important role in agents' decision to annuitize.

In this paper we emphasize one aspect of preferences which has not been considered enough in this literature on annuity: risk aversion. We explain that looking appropriately at the impact of risk aversion may provide a theoretical explanation to the annuity puzzle. Indeed, we will show that, a positive bequest motive combined with a high level of risk aversion is sufficient to predict a negative demand for annuities. The result holds when assuming a perfect annuity market.

The reason why this explanation remained unexplored is that the literature has focused on additively separable preferences, or used Epstein and Zin specification[1], both models being unadapted to study the role of risk aversion (Bommier, Chassagnon, Legrand, 2010, henceforth BCL). Relying on the standard way to look at the role of risk aversion in the expected utility framework, first introduced in Kihlstrom and Mirman (1974), we find that the demand for annuities decreases with risk aversion and eventually becomes negative when risk aversion is large enough.

The fact that annuity demand decreases -and does not increase- with risk aversion might seem couterintuitive. Annuities provide an insurance and, insurance demand is generally found to increase with risk aversion. Though, this needs not to be true when irreplaceable commodities, such as life, are at risk. As was explained by Cook and Graham (1977), rational insurance decisions aim at equalizing the marginal utility of wealth across states of nature and, when there are irreplaceable commodities, this may generate risk taking behavior. In that situation, one should expect that risk aversion decrease the demand for insurance. This was actually noticed by Drèze and Rustichini (2004), who provided an example where insurance demand may decrease with risk aversion[2].

Annuities provides a typical example where purchasing insurance may be risk increasing. Lifetime is uncertain, but living long is generally considered as a good outcome, while dying early is seen as a bad outcome. For a given amount of savings, purchasing annuities, rather than bonds, involve reducing bequest in the case of an early death (bad outcome) to increase consumption in case of survival (good outcome). Thus, for a given level of savings, annuities transfer resources from bad states of the world to good states of the world and, as such, are risk increasing. If savings

[1]Epstein and Zin preferences were for example used in Ponzetto (2003), Horneff, Maurer and Stamos (2010) and Inkmann, Lopes and Michaelides (2009).

[2]See their Proposition 9.1.

were exogenous a direct application of BCL's second proposition would imply that the demand of annuity decreases with risk aversion. In the current paper the result is obtained while saving is being made endogenous and intervivos transfers and bequests are introduced. Moreover, we prove that when risk aversion is large enough annuity demand eventually becomes negative.

Risk aversion would then provide a theoretical explanation for the annuity puzzle. Of course this does not mean that this "the" explanation of the annuity puzzle. It is obvious that markets are not perfect, and that individuals do not behave as expected utility maximizers, both aspects being likely to generate a low demand of annuities. Risk aversion is just another factor that lower the demand for annuity, adding then to the explanations which were advanced so far. The annuity puzzle is likely to have several pieces, indeed.

In order to evaluate whether risk aversion may play a significant role, we calibrated a lifecycle model where agents can invest in bonds or annuities. Calibrating risk aversion and bequest motives to plausible levels, we found that risk aversion alone would typically not generate a negative demand for annuity. Though we obtain levels of annuity demand which are lower than those provided by compulsory pension systems. Risk aversion could then be sufficient to explain why individuals may not purchase private annuities to complement the pensions that they are going to receive anyway.

The remainder of the paper is structured as follows. In Section 2 we discuss the related literature. We then expose the model and derive our theoretical predictions in Section 3. In Section 4, the model is extended to a N period setting and calibrated. Numerical simulations derive then the optimal lifecylce strategy of agent facing realistic mortality rates. Section 5 concludes.

2 Related literature

The microeconomics literature on annuity was initiated by Yaari's (1965) seminal contribution which was the first to formalize a model of intertemporal choice with lifetime uncertainty. Yaari explained that, in absence of utility of bequest, purchasing annuities would increase individual welfare. Such a results is extremely robust, as was discussed in Davidoff, Brown and Diamond (2005). Annuity contracts, even if not fairly priced, make it possible to increase lifetime consumption by lowering the amount of bequest. Thus agents who do not care for bequest but like consumption should invest all their wealth in annuities.[3] Full annuitization is no longer optimal when bequest motives are introduced. Though, Davidoff, Brown and Diamond (2005), as well as Lockwood (2010), noted that, when following Yaari's approach, bequest motives could not generate a zero demand for annuity if market were perfects, since the optimal behavior would involve annuitizing all future consumption. The low level of annuitization was then identified as a puzzle, for which

[3]See corollary 1 in Davidoff, Brown and Diamond (2005).

different explanations were suggested.

One possible explanation is that because of missing insurance products (as health or long term care insurance) people want to keep a large amount of illiquidity. Theoretically speaking, annuities do not need to be illiquid. But allowing people to sell back their annuities would magnify adverse selection problems and a market for reversible annuities may be difficult to develop. In absence of reversible annuities, and when health or long term care insurance is not possible (or not fairly priced), then the optimal strategy may then involve investing wealth in bonds or stock rather than in annuities. The papers by Sinclair and Smetters (2004), Yogo (2009), Pang and Warshawsky (2010) are contributions where this potential explanation is emphasized.

Another explanation involved the pricing of annuities, or the impossibility to invest in high return (but risky) assets through annuity products. Both factors may lead people to invest little in annuities. Mitchell, Poterba, Warshawsky and Brown (1999) and Finkelstein and Poterba (2002, 2004) showed for example that annuity pricing is far from being actuarially fair. Lockwood demonstrated that this aspect, together with bequest motives of reasonable magnitude, may be sufficient to explain the low level of annuitization. Milevsky and Young (2007) and Horneff, Maurer and Stamos (2008) looked at investment strategy when people could choose between stocks, bound and bond-indexed annuities and obtained a low level of annuitization. There, people do not purchase of annuities because of the lack of annuity products which would be backed on high-risk and high return assets. One may wonder however why the market for such products did not develop.

Last, behavioral economics may provide a whole range of explanations. For example Brown, Kling, Mullainathan and Wrobel (2008) emphasized that framing effects could be at the origin of the low demand of annuity. Brown (2007) reviewed other behavioral hypotheses, such as loss aversion, regret aversion, financial literacy or the illusion of control.

The role of risk aversion, although mentioned in several papers, has not being really discussed. The reason is that most papers used Yaari's approach, based on assumption of additive separability of preferences which makes impossible to study the role of risk aversion. A few papers did try to use Epstein and Zin's (1989) approach to disentangle risk aversion from the elasticity of substitution, but again this is not an appropriate way to study risk aversion. It was indeed shown in BCL that Epstein and Zin utility functions are not well ordered in terms of risk aversion. A simple way to study risk aversion involves remaining within the expected utility framework and increasing the concavity of the lifetime (and not instantaneous) utility function, as was initially suggested by Kihlstrom and Mirman (1974). In the case of choice with lifetime uncertainty, this approach was first used in Bommier (2006). That leads to novel predictions on number of topics, including on the relation between time discounting and risk aversion, the impact of mortality change and the

value of life. In the present paper we will use such an approach, introducing bequest motives and intervivos transfers.

3 The model

3.1 Description

We consider an agent that may live one period or two period. The probability to live two periods is $p \in (0,1)$. Such an agent cares for another person and can decide to transfer money to this person in the following ways: intervivos transfers and bequests.

To be concrete, the agent is endowed with an initial wealth W_0. Out of this wealth the agent consumes c_1. He is therefore left with wealth $W - c_1$ that he allocates either in annuities, a, or savings s. We assume perfect annuity market and a riskless rate of return $(1+R)$. After period 1 there are two possibilities. Either the agents dies and $(1+R)s$ are transmitted to the heirs, or the agent survives and he gets $(1+R)s + \frac{1+R}{p}a$ out of which he consumes c_2 and gives the remaining to the heirs through an intervivos transfer.

Ex-post, the economy is described by three variables: c_1, the first period consumption, x_2 the state of individual in period 2 (dead or alive, and if he is alive how much he consumes), and τ the money received by the heirs, either through bequests or through intervivos transfers. Looking at agents behavior involves comparing lotteries with consequences $(c_{1,}x_2, \tau) \in R^+ \times (R^+ \cup \{d\}) \times R^+$ where d denotes the death state. We constrain consumption to be non-negative, obviously, but also intergenerational transfers to be non-negative. The idea is that an agent cannot force his heir to give money, or to accept a negative bequest.

Agents are assumed to be expected utility maximizers with a utility function

$$U(c_{1,}x_2, \tau_1) = \phi\left(u_1(c_1) + u_2(x_2) + v(\tau)\right)$$

Without loss of generality we assume that $u_2(d) = 0$, $v(0) = 0$ and $\phi'(0) = 1$. We will also assume that u_1, the restriction of u_2 to R^+, and v are twice continuously differentiable, increasing and concave. The function ϕ is supposed to be twice continuously differentiable and increasing. Moreover we assume that:

$$\lim_{c \to 0} u_1'(c) = \lim_{c \to 0} u_2'(c) = +\infty$$

We assume that there exist consumption levels such that $u(c_2) > 0$. This means that for some level of period consumption, the agents prefers to live two periods than to die. We denote $c_2^* = \inf\{c_2 \geq 0 | u(c_2) \geq 0\}$, the minimum level of second period consumption such that the agent prefers to live rather than to die. With some specification we may have $c_2^* = 0$, which would mean

that life is preferable than death no matter the level of consumption. But for others specification (for example when assuming isoealistic instantaneous utility, with an elasticity smaller than one) then $c_2 > 0$. In that case, if no provided with sufficient consumption, the agent would prefer to die than to remain alive.

The function ϕ, which would not matter in determinative environment (it has no impact on ordinal preferences) is what governs risk aversion. In the expected utility framework increasing risk aversion involves taking a concave transformation of the utility function. Thus, in the above framework greater risk aversion is obtained by augmenting the concavity of the function ϕ.[4] In Yaari's approach, ϕ is assumed to be linear, which corresponds to imposing an assumption of temporal risk neutrality. An important feature of our paper is that we extend the analysis to non linear ϕ, thus making possible to explore the role of risk aversion.

The approach we follow to explore the role of risk aversion, initially first suggested by Kihlstrom and Mirman (1974) was little used in the economics literature[5], the certainty equivalent approach introduced by Selden (1978) and popularized by Epstein and Zin (1989) being found to be more convenient. We shall however refer to BCL for a formal discussion of what it means to compare risk aversion, and the relevance of these respective approaches.

3.2 Agent's program

There are two periods in the model. In the first one the agent is endowed with a wealth W_0, while in the second one his revenues only come from his savings. The agent is rational and seeks to maximize his expected intertemporal utility by choosing his consumption plan (c_1, c_2), his saving plan (s, a) in riskless bond and in annuity and his intervivos transfer τ. This transfer is transmitted to his heirs at date 2 if he survives, which occurs with probability p. If he dies, with probability $1 - p$, his saving in the riskless bond is transmitted to his heirs.

In our model, the mortality risk is the single risk faced by the agent. The return on the riskless bond is exogenous and certain, and there no income uncertainty since at the first date the agent is endowed with a constant wealth and earns nothing at the second date.

The agent's program therefore expresses as follows:

$$\max_{c_1,a,s,c_2} p\phi\left(u_1(c_1) + u_2(c_2) + v(\tau)\right) + (1-p)\phi\left(u_1(c_1) + v((1+R)s)\right) \tag{1}$$

[4] A function ϕ_1 willl be said to be more conaceve than a function ϕ_0 if $\phi_1 = v(\phi_0)$ for some convave function v.

[5] There are noteworthy expecptions, as Van der Ploeg (1993), Eden (2005), Bommier (2006) and Van den Heuvel (2008) .

with the budget constraints:

$$c_1 + a + s = W_0 \tag{2}$$

$$c_2 + \tau = (1+R)s + a\frac{1+R}{p} \tag{3}$$

$$c_1 > 0,\ c_2 > 0,\ \tau \geq 0,\ a \geq 0,\ s \geq 0 \tag{4}$$

The equation (1) is the agent's expected utility. With probability p, he lives and consumes for two periods and transmits τ to his heirs. Otherwise, he only lives for one period and transmits his savings in the riskless bonds. The equation (2) is the budget constraint of the first period. His wealth may be consumed or saved in the riskless bond s or in the annuity a. At the second period, if the agent lives, the budget constraint (3) states that the savings outcomes may be consumed or transmitted to the agent's heir. Finally, conditions in (4) simply state that consumption have to be strictly positive and transfers cannot be negative. The agent is not permitted to transmit a debt to his heirs and since he does not perceive any revenue at the second period, he is prevented from borrowing.

We deduce now the first order conditions from the previous program. However, due to the positivity constraints on transfers, these conditions have to take into account the fact that the optimum may not be interior. Several cases need therefore to be carefully handled. We define the intertemporal utility of the agent when alive U_A and when dead U_D as follows:

$$U_A = u_1(c_1) + u_2(c_2) + v(\tau)$$
$$U_D = u_1(c_1) + v((1+R)s)$$

Noting λ the Lagrange multiplier of the first period budget constraint (2) and μ the Lagrange multiplier of the second period budget constraint (3), the model generates the following first order equations:

$$\left[p\phi'(U_A) + (1-p)\phi'(U_D)\right] u_1'(c_1) = \lambda \tag{5}$$

$$p\phi'(U_A)u_2'(c_2) = \mu \tag{6}$$

$$\frac{\mu}{p} - \frac{\lambda}{1+R} \leq 0 \qquad (= 0 \text{ if } a > 0) \tag{7}$$

$$v'((1+R)s)\phi'(U_D) - \frac{1}{1-p}\left(\frac{\lambda}{1+R} - \frac{\mu}{p}\right) - \frac{\mu}{p} \leq 0 \qquad (= 0 \text{ if } s > 0) \tag{8}$$

$$pv'(\tau)\phi'(U_A) - \mu \leq 0 \qquad (= 0 \text{ if } \tau > 0) \tag{9}$$

Equations (5) and (6) equalize the marginal benefit of consuming one unit more respectively in period 1 and 2 to their respective marginal cost, which is equal to the shadow of the budget constraint. Equation (7) compares the shadow cost of the second budget constraint to the discounted

value of the shadow cost of the first period budget constraint. Equations (8) and (9) compare the marginal benefit of transmitting either when dead or when alive to his heirs to his marginal cost. Equations (7) to (9) are inequalities, as the optimal value for a, s and τ may correspond to corner solutions. These inequalities becomes equalities whenever interior solution are obtained.

3.3 Saving choices

We first consider the case where the function ϕ is linear, as is usually assumed. The result is well known and we formalize it to contrast it later on with what is obtained with agents exhibiting greater risk aversion, that is when considering concave functions ϕ.

Proposition 1 *If ϕ is linear, then the amounts invested in annuities equals the present value of the second period consumption. All what is invested in bonds is transferred to the heirs (through bequest or intervivos transfers).*

$$a = \frac{pc_2}{1+R} \text{ and } (1+R)s = \tau$$

When ϕ is linear, the agent saves in the annuity to secure his future consumption: his annuity pays off at the second period exactly his second period consumption. He saves in the riskless bond only in order to transmit a share of his wealth to his heirs. The amount transmitted through saving exactly matches the one transmitted through intervivos transfers.

Proof. When ϕ is linear $\phi'(U_A) = \phi'(U_D) = \phi'(0) = 1$. FOC and the budget constraint becomes

$$u_1'(c_1) = \lambda \tag{10}$$

$$u_2'(c_2) = \mu \tag{11}$$

$$\frac{\mu}{p} \leq \frac{\lambda}{1+R} \qquad (= \text{ if } a > 0) \tag{12}$$

$$v'((1+R)s) \leq \frac{1}{1-p}\left(\frac{\lambda}{1+R} - \frac{\mu}{p}\right) + \frac{\mu}{p} \qquad (= \text{ if } s > 0) \tag{13}$$

$$v'(\tau) \leq \frac{\mu}{p} \qquad (= \text{ if } \tau > 0) \tag{14}$$

$$c_2 + \tau = (1+R)s + a\frac{1+R}{p} \tag{15}$$

Let us show that in any case $(1+R)s = \tau$.

1. $s = 0$. The budget constraint (15) implies that $a > 0$: (12) is therefore an equality. (13) implies then that $v'(0) \leq \frac{\mu}{p}$. Suppose that $\tau > 0$: we deduce from (14) that $v'(0) \leq v'(\tau)$, which contradicts that v is concave and not linear. Thus, $(1+R)s = \tau = 0$.

2. $s > 0$. From (13), which is an equality, together with (12) and (14), we deduce that $v'((1+R)s) \geq \frac{\mu}{p} \geq v'(\tau)$ and $\tau \geq (1+R)s > 0$. The budget constraint (15) implies that $a > 0$.

(12), as (13) and (14) are therefore equalities: we deduce that $v'((1+R)s) = v'(\tau)$ and $(1+R)s = \tau$.

We always obtain $(1+R)s = \tau$, and thus also $a = \frac{pc_2}{1+R}$. ■

The above proposition shows that, when ϕ is linear, people should always purchase annuities, and intergenerational transfers which materialize either through bequest or intervivos transfers, are independent of life duration.

We shall now consider the case of agents preferences exhibiting positive temporal risk aversion, that is the case of concave functions ϕ.

Proposition 2 *If ϕ is concave and $c_2 > c_2^*$ at the optimum then*

- *either $s = \tau = 0$*
- *or*

$$(1+R)s > \tau \text{ and } a < \frac{pc_2}{1+R}$$

Proof. Let us first remark that $c_2 > c_2^*$ implies $u_2(c_2) > 0$ and $U_A - U_D > v(\tau) - v((1+R)s)$.

- $s = 0$. The budget constraint (3) implies that $a > 0$. From (8) using (7) as an equality, we deduce $v'(0)\phi'(U_D) \leq \frac{\mu}{p}$. Suppose that $\tau > 0$. We obtain from the previous inequality and (9) as an equality that $v'(0)\phi'(U_D) \leq v'(\tau)\phi'(U_A)$. Since $U_A - U_D > 0$ and ϕ is increasing and concave, $0 < \phi'(U_A) \leq \phi'(U_D)$ and thus $v'(0) \leq v'(\tau)$, contradicting the fact that v is concave and non-linear. We deduce therefore that $s = \tau = 0$.
- $s > 0$. Suppose that $(1+R)s \leq \tau$. It implies $v(\tau) - v((1+R)s) \geq 0$ and $U_A - U_D > 0$. Moreover, the budget constraint (3) implies $a\frac{1+R}{p} = c_2 + \tau - (1+R)s > 0$. Eq. (7)–(9) are equalities and yield:

$$\phi'(U_D)v'((1+R)s) = v'(\tau)\phi'(U_A),$$

which implies that $\frac{\phi'(U_D)}{\phi'(U_A)} = \frac{v'(\tau)}{v'((1+R)s)} \leq 1$ in contradiction with $U_D < U_A$ and ϕ concave. We therefore deduce that $\tau < (1+R)s$, and from the budget constraint that $a < \frac{pc_2}{1+R}$, which ends the proof.

■

As soon as the agent is temporal risk averse, and willing to leave some transfer or bequest, he should not completely annuitize his consumption. Transfers received by the heirs will depend on life duration, shorter lives being associated with greater transfers. The agent, who cannot eliminate the possibility of an early death, achieves some partial self insurance by creating a negative correlation between two aspects he thinks desirable: living long and transferring resources to his heirs.

We shall now prove that: (1) the greater the risk aversion, the more the agent will rely on this partial insurance behavior -and the less he will invest in annuities; (2) when risk aversion is large enough annuity demand becomes negative. To establish these results we need however to make slightly stronger assumptions regarding the willingness to live and to make transfers. Precisely we will make the following assumption:

Assumption A *Denote by* $c_2^{**} = \inf\{c_2 | u_2(c_2) > v(c_2)\}$ *then:*

$$u(c) > v(c) \text{ for all } c > c_2^{**}$$

$$\begin{aligned} \frac{u_1'(W_0 - \frac{c_2^{**}}{1+R})}{1+R} &< v'(c_2^{**}) \\ v'(0) &< u_2'(c_2^{**}) \end{aligned}$$

The consumption level c_2^{**} is the smallest second period consumption level that makes life worth it, once accounting for the possibility of transmitting to the heirs. Below that level of consumption, the agent would rather die and transmit all his wealth. The constant c_2^{**} is larger than c_2^*, which was defined without accounting for the possibility of making intergenerational transfers. The above assumption states that if provided with greater second period consumption than c_2^{**} he would prefer to live than to die and transmit all this consumption to his heirs. The inequality $v'(0) < u_2'(c_2^{**})$ means the bequest motive is sufficiently strong in the sense that if the agent were sure to die after period 1 he would transmit at least c_2^{**} to his heirs. We can now state the following result:

Proposition 3 *We suppose that* $\phi_k(x) = -\frac{e^{-k\,x}}{k}$ *with* $k > 0$ *and that assumption A is fulfilled. Then, the optimal annuity is a decreasing function of* k *and there exists* $k_0 > 0$ *such that for all* k *greater than* k_0*, the functions* ϕ_k *generates a null annuity.*

Proof. The proof is left in Appendix. Due to the several combinations of non-interior solutions, the proof implies distinguishing several cases. ■

This proposition states that if agents' demand is decreasing with risk aversion. Moreover we show that when agent are sufficiently risk averse, they should not purchase any annuity (annuity demand become negative). Taking properly risk aversion into account is therefore crucial since it may provide an explanation to the annuity puzzle which holds even if assuming perfect annuity market.

4 Calibration

In this section, we extend our model to a large number of retirement periods that allows us to calibrate it using realistic mortality patterns and preference parameters. Parameter values that

seem to be reasonable yield a positive annuitization level, which is however below the level of public pensions in most developed economies. Properly taking into account the risk aversion is therefore a plausible quantitative explanation to the small level of annuitization.

4.1 The model extension

As in the previous model, we assume that the agent is endowed by a wealth W_0 when he retires. We normalize the retirement date to the date 0 of the model. The mortality remains the sole risk faced by the agent and $p_{t+1|t}$ denotes the probability of being alive at date $t+1$ while being alive at date t. Thus, $1-p_{t+1|t}$ denotes the probability of dying at the end of period t. For sake of simplicity, we denote by $m_{T|0}$ the probability of living exactly until date T. The agent is supposed to be alive at date 0, so: $p_{0|-1}=1$. The relationship between $m_{T|0}$ and the probabilities $p_{t+1|t}$ expresses as follows:

$$m_{T|0} = (1-p_{T+1|T})\prod_{k=1}^{T} p_{k|k-1} \quad \text{and} \quad m_{0|0} = 1-p_{1|0}$$

While he is alive, the agent choses at each period t his consumption c_t of the single consumption good, and his savings that respect his budget constraint. The agent has two possibilities to transfer money from one period to the other one. The first one is to purchase a quantity b_t of a riskless bond at a price normalized to 1 and paying in one period $1+R$ units of good. This saving gross rate is constant and exogenous. The other solution is to purchase a quantity a_t of an annuity, at the price π_t and paying off one unit of good every period as long as the annuity holder is alive. We assume that the annuity market is perfect and the annuity price is actuarially fair. It implies that the price π_t expresses as the present value of the good unit paid every period, conditional on the agent being alive:

$$\pi_t = \sum_{i=1}^{\infty} \frac{p_{t+i|t}}{(1+R)^i} = (1+\pi_{t+1})\frac{p_{t+1|t}}{1+R} \tag{16}$$

A each date t when alive, the agent enjoys utility $u(c_t)$ from his own consumption c_t and when he dies at date T, he enjoys a consumption $v(\frac{w_{T+1}}{(1+R)^{T+1}})$ from his bequeath w_{T+1} at date $T+1$. We assume that there is no tax and the bequeath is equal to the return of savings on bonds, i.e, $(1+R)b_T$. This is a shortcut to take into account the utility of the agents' heirs, who discount their bequeath at the same rate as their savings.

When he dies at T, consumes the stream of consumption $(c_t)_{0\leq t\leq T}$ and bequeathes $(1+R)b_T$, his utility expresses as:

$$U(c,b) = -\frac{1}{\lambda}\exp\left(-\lambda\sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)$$

The agent maximizes his expected intertemporal utility by choosing his consumption stream $(c_t)_{t\geq 0}$, his bond saving $(b_t)_{t\geq 0}$ and his annuity saving $(a_t)_{t\geq 0}$, subject to a budget constraint each period. The agent's program expresses therefore as follows:

$$\max_{c,b,a} \; - \frac{1}{\lambda} \sum_{T=1}^{\infty} m_{T|0} \, \exp\left(-\lambda \left(\sum_{t=0}^{T} u(c_t) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right) \tag{17}$$

$$\text{s.t.} \quad W_0 = c_0 + b_0 + \pi_0 \, a_0 \tag{18}$$

$$(1+R)b_{t-1} + \sum_{\tau=0}^{t-1} a_\tau = c_t + b_t + \pi_t \, a_t \text{ for } t \geq 1 \tag{19}$$

It is noteworthy that there no exogenous utility discount β. The discount is endogenous and stems from λ and more precisely from the concavity of the utility aggregator $u \mapsto -\frac{1}{\lambda}\exp(-\lambda\, u)$.

The first order conditions of the previous program expresses as follows:

$$u'(c_t) \sum_{T=t}^{\infty} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=0}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right) = \lambda_t \tag{20}$$

$$\frac{m_{t|0}}{(1+R)^t} \, v'\left(\frac{b_t}{(1+R)^t}\right) \exp\left(-\lambda \left(\sum_{\tau=0}^{t} u(c_\tau) + v\left(\frac{b_t}{(1+R)^t}\right)\right)\right) = \lambda_t - (1+R)\lambda_{t+1} \tag{21}$$

$$\pi_t \lambda_t = \sum_{\tau=t+1}^{\infty} \lambda_\tau \tag{22}$$

From the annuity pricing equation, we deduce that the annuity price expresses as $\pi_t = (1 + \pi_{t+1})\frac{p_{t+1|t}}{1+R}$. This equality simply states that the price at date t of the annuity is the present value of its value at the next date. This value is equal to its payoff (1 unit of good) plus the price of the annuity at the next date, i.e. π_{t+1}. Since the FOC (22) also means that $\lambda_t \, \pi_t = \lambda_{t+1}(1 + \pi_{t+1})$, we obtain the following intertemporal relationship for the Lagrange multiplier λ_t:

$$p_{t+1|t} \, \lambda_t = \lambda_{t+1}(1+R) \tag{23}$$

The equation (23) states that the shadow cost of the budget constraint at date $t+1$ is equal to the discounted shadow cost of date t, where the discount takes the probability of dying into account.

Moreover, we take into account the finite lifetime and we denote by $T_M < \infty$ the maximal date of death for an agent alive at date 0. This date is such that the probability of being alive at the next date is null: $p_{T_M+1|T_M} = 0$. Plugging the equation (23) into (20) and (21) allows us to deduce

the following relationships:

$$u'(c_t)\sum_{T=t}^{T_M}(1-p_{T+1|T})e^{-\lambda\, v\left(\frac{b_T}{(1+R)^T}\right)}\prod_{\tau=t+1}^{T}p_{\tau|\tau-1}e^{-\lambda\, u(c_\tau)}=(1+R)u'(c_{t+1})e^{-\lambda\, u(c_{t+1})}\times \tag{24}$$

$$\sum_{T=t+1}^{T_M}(1-p_{T+1|T})e^{-\lambda\, v\left(\frac{b_T}{(1+R)^T}\right)}\prod_{k=t+2}^{T}p_{k|k-1}e^{-\lambda\, u(c_\tau)}$$

$$\frac{v'\left(\frac{b_t}{(1+R)^t}\right)}{(1+R)^{t+1}}\,e^{-\lambda\, v\left(\frac{b_t}{(1+R)^t}\right)}=u'(c_{t+1})e^{-\lambda\, u(c_{t+1})}\sum_{T=t+1}^{T_M}(1-p_{T+1|T})e^{-\lambda\, v\left(\frac{b_T}{(1+R)^T}\right)}\prod_{\tau=t+2}^{T}p_{\tau|\tau-1}e^{-\lambda\, u(c_\tau)} \tag{25}$$

The first intertemporal Euler equation (24) is valid for every date t between 0 and T_M-1. It sets equal the marginal cost of saving one unit of good today to the marginal cost of consuming one unit more tomorrow. This is an intertemporal equation for agents remaining alive. The expression reflects the fact that the consumption marginal utility at a given date depends not only on the current consumption, but also on the consumption path. The second Euler equation (25) is true for all dates t between 0 and T_M and it equalizes the marginal cost of saving one unit more today to the marginal benefit of one additional unit of bequeath tomorrow. This equation obviously concerns agents dying at the end of the period.

Together with the T_M+1 budget constraints (from date 0 to date T_M) there are $3T_M+2$ equations determining the $3T_M+2$ unknowns, i.e. the set of consumption c_t, bond holding b_t and annuity a_t for $t=0,\ldots,T_M$ with an exception for a_{T_M}, which is null, since the agent dies for sure at the end of the period.

Another expression for FOC.

$$u'(c_t)\sum_{T=t}^{T_m}m_{T|0}\exp\left(-\lambda\left(\sum_{\tau=0}^{T}u(c_\tau)+v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)=\lambda_t$$

$$\frac{m_{t|0}}{(1+R)^t}\,v'\left(\frac{b_t}{(1+R)^t}\right)\exp\left(-\lambda\left(\sum_{\tau=0}^{t}u(c_\tau)+v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)\leq\lambda_t-(1+R)\lambda_{t+1}\ (=\ \text{if } b_t>0)$$

$$p_{t+1|t}\,\lambda_t=\lambda_{t+1}(1+R)\text{ for } t\leq T_m-1 \text{ and } \lambda_{T_m+1}=0$$

We deduce:

$$p_{t+1|t}\,u'(c_t)\sum_{T=t}^{T_m}m_{T|0}\exp\left(-\lambda\left(\sum_{\tau=0}^{T}u(c_\tau)+v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)=$$

$$(1+R)u'(c_{t+1})\sum_{T=t+1}^{T_m}m_{T|0}\exp\left(-\lambda\left(\sum_{\tau=0}^{T}u(c_\tau)+v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)$$

It becomes:

$$p_{t+1|t}\, u'(c_t) \left(m_{t|0} + \sum_{T=t+1}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)\right) =$$

$$(1+R)u'(c_{t+1}) \sum_{T=t+1}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)$$

Moreover:

$$\frac{1}{(1+R)^{T_m}}\, v'\left(\frac{b_{T_m}}{(1+R)^{T_m}}\right) \leq u'(c_{T_m}) \ (= \text{ if } b_{T_m} > 0)$$

$$\frac{m_{t|0}}{(1+R)^t}\, v'\left(\frac{b_t}{(1+R)^t}\right)$$

$$\leq (1-p_{t+1|t})\, u'(c_t) \left(m_{t|0} + \sum_{T=t+1}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)\right)$$

$$\leq \frac{1-p_{t+1|t}}{p_{t+1|t}} (1+R)u'(c_{t+1}) \sum_{T=t+1}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)$$

The no-annuity economy. If the agent does not have access to the annuity market, the FOC are:

$$u'(c_t) \sum_{T=t}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=0}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right) = \lambda_t \tag{26}$$

$$\frac{m_{t|0}}{(1+R)^t}\, v'\left(\frac{b_t}{(1+R)^t}\right) \exp\left(-\lambda \left(\sum_{\tau=0}^{t} u(c_\tau) + v\left(\frac{b_t}{(1+R)^t}\right)\right)\right) = \lambda_t - (1+R)\lambda_{t+1} \tag{27}$$

We deduce:

$$\frac{m_{t|0}}{(1+R)^t}\, v'\left(\frac{b_t}{(1+R)^t}\right) \exp\left(-\lambda \left(\sum_{\tau=0}^{t} u(c_\tau) + v\left(\frac{b_t}{(1+R)^t}\right)\right)\right) =$$

$$u'(c_t) \sum_{T=t}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=0}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)$$

$$- (1+R)u'(c_{t+1}) \sum_{T=t+1}^{T_m} m_{T|0} \exp\left(-\lambda \left(\sum_{\tau=0}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)$$

$$\frac{v'\left(\frac{b_t}{(1+R)^t}\right)}{(1+R)^t}\, e^{-\lambda v\left(\frac{b_t}{(1+R)^t}\right)} = u'(c_t) \sum_{T=t}^{T_m} \frac{m_{T|0}}{m_{t|0}} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)$$

$$- (1+R)u'(c_{t+1}) \sum_{T=t+1}^{T_m} \frac{m_{T|0}}{m_{t|0}} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)$$

$$\frac{v'\left(\frac{b_t}{(1+R)^t}\right)}{(1+R)^t} = u'(c_t) \left(1 + \sum_{T=t+1}^{T_m} \frac{m_{T|0}}{m_{t|0}} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)\right)$$

$$- (1+R)u'(c_{t+1}) \sum_{T=t+1}^{T_m} \frac{m_{T|0}}{m_{t|0}} \exp\left(-\lambda \left(\sum_{\tau=t+1}^{T} u(c_\tau) + v\left(\frac{b_T}{(1+R)^T}\right) - v\left(\frac{b_t}{(1+R)^t}\right)\right)\right)$$

We deduce:

$$\frac{v'\left(\frac{b_{T_m}}{(1+R)^{T_m}}\right)}{(1+R)^{T_m}} = u'(c_{T_m}) \text{ if } b_{T_m} > 0$$
$$\frac{v'(0)}{(1+R)^{T_m}} \leq u'(c_{T_m}) \text{ if } c_{T_m} = (1+R)b_{T_m-1}$$

4.2 Implementation

To solve the model, we have to determine $3T_M + 2$ variables using $3T_M + 2$ equations. However the model is highly non-linear and even using an analytical expression for the Jacobian only provides a very slow computation and the convergence is very sensitive to initial guess. For a realistic T_M (approx. 45), this choice does not even converge.

We use therefore a backward resolution to compute the optimal values for consumption, riskless and annuity saving. However, the budget constraint at the last date T_M depends on the stock of annuity purchased from date 0 to $T_M - 1$, i.e. on $A_{T_M-1} = \sum_{\tau=0}^{T_M-1} a_\tau$. To solve this difficulty, we consider A_{T_M-1} as a parameter and we solve the model for it. The model resolution yields then a value of W_0, which is compatible with the initial choice of A_{T_M-1}. It remains then to modify the value of A_{T_M-1} to obtain the model solution for a particular initial wealth endowment W_0.

More precisely, for the first step, we use the Euler equation (24) for $t = T_M - 1$ and (25) for $t = T_M - 1$ and T_M, together with the budget constraint at date T_M, which expresses using the initial guess A_{T_M-1} as $b_{T_M-1} + A_{T_M-1} = c_{T_M} + b_{T_M}$. These four equations determine the four unknowns c_{T_M-1}, c_{T_M}, b_{T_M-1}, and b_{T_M}. For the next step, we solve simultaneously (24) and (25) at $T_M - 2$, which provide c_{T_M-2} and b_{T_M-2}. The budget constraint at date $T_M - 1$ give a_{T_M-1}, as: $(1+R)b_{T_M-2} + A_{T_M-1} - a_{T_M-1} = c_{T_M-1} + b_{T_M-1} + \pi_{T_M-1}\, a_{T_M-1}$. We deduce the stock of annuity at date $T_M - 2$, which is equal to $A_{T_M-2} = A_{T_M-1} - a_{T_M-1}$. We therefore proceed in the same way for all previous dates to deduce the optimal consumption plans and bond and annuity holdings. At the last date, once we have computed c_0 and b_0, the budget constraint at date 1 yields the annuity chosen at da te 1, i.e. a_1, as well as the stock of annuity at date 0, which is $A_0 = A_1 - a_0$. The stock of annuity at date 0 is simply the quantity of annuity chosen at date 0, i.e. $a_0 = A_0$. Using then the budget constraint at date 0, we deduce the initial wealth W_0, which is compatible with the initial choice of annuity stock A_{T_M-1}. It remains finally to find the proper A_{T_M-1} to obtain the targeted value of initial wealth.[6]

At the first step:

[6] We can show that W_0 is a continuous function of A_{T_M-1}, which implies that finding any target for W_0 is possible.

- $b_{T_M}, b_{T_M-1}, c_{T_M} > 0$:

$$\frac{v'\left(\frac{b_{T_M}}{(1+R)^{T_M}}\right)}{(1+R)^{T_M}} = u'(c_{T_M})$$
$$\frac{v'\left(\frac{b_{T_M-1}}{(1+R)^{T_M-1}}\right)}{(1+R)^{T_M}} e^{-\lambda\, v\left(\frac{b_{T_M-1}}{(1+R)^{T_M-1}}\right)} = u'(c_{T_M}) e^{-\lambda\, u(c_{T_M})}\, e^{-\lambda\, v\left(\frac{b_{T_M}}{(1+R)^{T_M}}\right)}$$
$$c_{T_M} + b_{T_M} = A + (1+R) b_{T_M-1}$$

- $b_{T_M} = 0$ and $b_{T_M-1}, c_{T_M} > 0$:

$$\frac{v'(0)}{(1+R)^{T_M}} \leq u'(c_{T_M})$$
$$\frac{v'\left(\frac{b_{T_M-1}}{(1+R)^{T_M-1}}\right)}{(1+R)^{T_M}} e^{-\lambda\, v\left(\frac{b_{T_M-1}}{(1+R)^{T_M-1}}\right)} = u'(c_{T_M}) e^{-\lambda\, u(c_{T_M})}\, e^{-\lambda\, v(0)}$$
$$c_{T_M} = A + (1+R) b_{T_M-1}$$

- $b_{T_M-1} = 0$ and $b_{T_M}, c_{T_M} > 0$:

$$\frac{v'\left(\frac{b_{T_M}}{(1+R)^{T_M}}\right)}{(1+R)^{T_M}} = u'(c_{T_M})$$
$$\frac{v'(0)}{(1+R)^{T_M}} e^{-\lambda\, v(0)} \leq u'(c_{T_M}) e^{-\lambda\, u(c_{T_M})}\, e^{-\lambda\, v\left(\frac{b_{T_M}}{(1+R)^{T_M}}\right)}$$
$$c_{T_M} + b_{T_M} = A$$

- $b_{T_M} = b_{T_M-1} = 0$ and $c_{T_M} > 0$:

$$\frac{v'(0)}{(1+R)^{T_M}} \leq u'(c_{T_M})$$
$$\frac{v'(0)}{(1+R)^{T_M}} \leq u'(c_{T_M}) e^{-\lambda\, u(c_{T_M})}$$
$$c_{T_M} = A$$

4.3 Calibration

We need to calibrate the previous model. First of all, we specify our utility functions u and v. Since u must account for the death, we choose as in Becker, Philipson and Soares (2003) the following specification for u:

$$u(c) = u_0 + \frac{c^{1-\sigma}}{1-\sigma}$$

In the previous specif, the parameter $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution and $u_0 \in \mathbb{R}$ the utility level taking death into account.

Regarding the utility of bequeath , we assume that it has the following form:

$$v(w) = \frac{(M+w)^{1-\sigma_v}}{1-\sigma_v}$$

The parameter $\sigma_v > 0$ is the inverse of the intertemporal elasticity of substitution for bequeath, while $M > 0$ accounts for the the consumption of heirs without any bequeath. The idea is that bequeath is a kind of altruism and $v(w)$ is the valuation of the agent for the utility of his heirs. In that context, the bequeath comes only on top of consumption. Remark that in the case of v, a constant is less useful since there is no comparison to death involved.

Regarding our calibration, we proceed in two ways: (i) we fix exogeneously some parameters to values that seem reasonable and (ii) we choose some parameter values to match given quantities, as the the endogenous rate of time discount and the value of a statistical life.

Exogenous calibration. First of all, we choose that the date 0 of the model corresponds to the age of 65, assuming that people retire at that age. Mortality data are US mortality data from the Berkeley Mortality Database and for example $p_{1|0}$ is the probability of dying at the age of 66. In the data, the maximal age is 110 years. People alive at teh age of 65 will live at most for 45 years. It implies that $T_M = 45$ and $p_{46|45} = 1$.

Regarding preference parameters, we chose $\sigma = 1.5$, which corresponds to a standard value of 2/3 for the intertemporal elasticity of substitution.

We choose the exogenous rate of return of savings to be equal to 3.00%, which is close to the value of the riskless short term interest rate proxied by the 3m bond.

Finally, we choose the agent's wealth when retiring to be equal approximatively to fifteen times his average consumption.

Evaluated parameters. We first choose the inverse of the elasticity of substitution for bequest σ_v to be such that the average bequest is about 20% of the agent's initial weath when he retires. This quantity is similar to what is proposed in Lockwood (2010).

We calibrate the two remaining parameters: the utility parameter u_0 and the risk aversion one λ by such as to replicate two quantities, the value of a statistical life(VSL) and the endogenous rate of time discount at the age t that we note r_t. More precisely, regarding the VSL, we replicate a VSL of approximatively 200 times the annual consumption. This value has been computed for example in Aldy and Viscusi, 2003. Moreover, we seek to obtain a rate of discount of approximatively 4.5% at the age of 65, which is the retirement age in our model.

Rate of discount. Following Epstein (1993), we define the rate of time preference ρ_0 at the retirement age (date 0) as follows:

$$\rho_0 = \left(\frac{u'(c_0)}{u'(c_1)} \frac{\frac{\partial EU}{\partial c_1}}{\frac{\partial EU}{\partial c_0}} \right)^{-1} - 1 = \frac{(1 - p_{1|0}) \exp\left(-\lambda u(c_0) - \lambda v\left(b_0\right)\right)}{\sum_{T=1}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}$$

This quantity interprets as the rate of change of marginal utility, in we offset the consumption effect. Epstein (1993) originally considers the same rate along a locally constant consumption path, in a continuous time model. We adapt the definition to a discrete time framework, without impairing the original definition.

Let us prove the second equality of the definition. The expected utility at the retirement age expresses as follows:

$$EU = \frac{1}{\lambda} - \frac{1}{\lambda} \sum_{T=0}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)$$

The derivatives with respect to c_0 and c_1 yield after some manipulation to:

$$\frac{u'(c_1)}{u'(c_0)} \frac{\frac{\partial EU}{\partial c_0}}{\frac{\partial EU}{\partial c_1}} = \frac{\sum_{T=0}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}{\sum_{T=1}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \frac{u'(c_1)}{u'(c_0)} \sum_{t=0}^{T} u(c_t) + v\left(\frac{b_T}{(1+R)^T}\right)\right)}$$

$$\text{or:} \quad 1 + \rho_0 = 1 + \frac{(1 - p_{1|0}) \exp\left(-\lambda u(c_0) - \lambda v\left(b_0\right)\right)}{\sum_{T=1}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=1}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}$$

We calibrate our parameters to obtain a value of ρ approximatively equal to 4.5%.

And we want this rate to be approx. equal to 4.5% at 65.

VSL. The value of a statistical life VSL_0 at the retirement age expresses as the opposite of the marginal rate of substitution between mortality and consumption. Noting $\mu_{1|0} = p_{1|0}^{-1} - 1$ the mortality rate at the retirement age, we prove that VSL expresses as follows:

$$VSL_0 = -\frac{\frac{\partial EU}{\partial \mu_{1|0}}}{\frac{\partial EU}{\partial c_0}} = \frac{p_{1|0}}{u'(c_0)} \frac{\frac{1}{\lambda} - \frac{1}{\lambda}\sum_{T=0}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}{\sum_{T=0}^{\infty}(1 - p_{T+1|T}) \prod_{t=0}^{T-1} p_{t+1|t} \exp\left(-\lambda \sum_{t=0}^{T} u(c_t) - \lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}$$

The quantity VSL_0 interprets as the quantity of consumption an agent is willing to give up to save one statistical life. A more rough interpretation is the value of one statistical life in terms of consumption. The first equality is the definition of VSL at the retirement age as in Johansson (2002), even he argues that many definitions have been proposed. The second equality can be proved as follows.

Proof. Substituting the expression of $\mu_{1|0}$ as a function of $p_{1|0}$, we obtain:

$$VSL_0 = -\frac{\frac{\partial EU}{\partial p_{1|0}}}{\frac{\partial EU}{\partial c_0}} \frac{\partial p_{1|0}}{\partial \mu_{1|0}} = -\frac{\frac{\partial EU}{\partial p_{1|0}}}{\frac{\partial EU}{\partial c_0}} \frac{\partial \frac{1}{1+\mu_{1|0}}}{\partial \mu_{1|0}} = p_{1|0}^2 \frac{\frac{\partial EU}{\partial p_{1|0}}}{\frac{\partial EU}{\partial c_0}}$$

Calibration	
Exogenous Parameters	
σ	1.5
R	3.00%
Estimated Parameters	
σ_v	1.39
u_0	0.275
λ	0.27
M	300

Table 1: Benchmark calibration

where EU is the expected utility of an agent who retires.

We have:

$$EU = \frac{1}{\lambda} - \frac{1}{\lambda}\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)+v\left(\frac{b_T}{(1+R)^T}\right)\right)$$

Derivatives yield:

$$\begin{aligned}
\frac{\partial EU}{\partial c_0} &= u'(c_0)\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)\\
\frac{\partial EU}{\partial p_{1|0}} &= -\frac{1}{\lambda}\frac{\partial}{\partial p_{1|0}}\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)\\
&= -\frac{1}{\lambda}\left(-\exp\left(-\lambda u(c_0)-\lambda v\left(b_0\right)\right)+\frac{1}{p_{1|0}}\sum_{T=1}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)\\
&= -\frac{1}{\lambda p_{1|0}}\left(-1+\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)\right)
\end{aligned}$$

and:

$$VSL_0 = \frac{p_{1|0}}{\lambda}\frac{1-\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}{u'(c_0)\sum_{T=0}^{\infty}(1-p_{T+1|T})\prod_{t=0}^{T-1}p_{t+1|t}\exp\left(-\lambda\sum_{t=0}^{T}u(c_t)-\lambda v\left(\frac{b_T}{(1+R)^T}\right)\right)}$$

■

We calibrate our parameters to obtain a VSL approximatively equal to 200 average consumptions.

Benchmark calibration. Our benchmark calibration is summed up in Table 1.

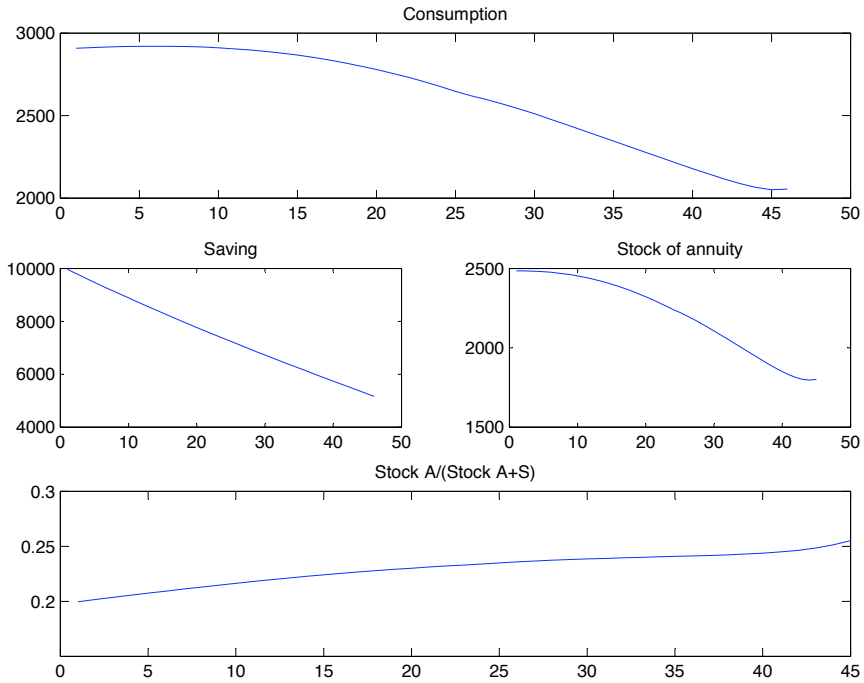

Figure 1: Results

4.4 Results

Our results are summarized in the figure 1.

The figure 1 presents four different graphs. The top one, entitled *Consumption* presents the consumption along the lifetime of the agent, after his retirement. The middle left one (*saving*) represents the evolution of savings in the riskless bond. The middle right one plots the stock of annuity owned by the agent after he retires. The last one represents the evolution of the share of annuity in the total savings (i.e., savings in th riskless bond and in annuity).

First, this last graph tells us that the level of annuitization remains small during the lifetime of the individual and increases roughly from 20% to 23% of the total savings. Taking the risk aversion properly into account together with a reasonable bequest motive (on average, 20% of the inidividual wealt is transmitted to his heirs). allow to generate a level of annuitization which cannot be rejected by the data, since this level is consistent with retirement payments.

Second, as plotted in the first graph, the model generates a decreasing shape for the consumption over time, which is consistent by the data. As underlined in many empirical studies, the shape of the consumption is decreasing for retired people.

Third, both graphs in the middle show that the saving of the retired agent, who does not receive

any income is decreasing over time.

5 Conclusion

The relation between risk aversion and annuity demand has remained unexplored in the economics literature which focused on models which are inadequate to study the role of risk aversion, either because they assume that preferences are additively separable or because they rely on Epstein and Zin utility functions. Though as soon as we account for risk aversion in a proper way, annuities demand is found to decrease with risk aversion. Moreover, annuity demand eventually becomes negative (or vanishes if we add a positivity constraint) if risk aversion is sufficiently large and individuals have some positive bequest motives. A possible reason for the low level of wealth annuitization may simply be that individual are too risk averse to purchase annuities. Intuitively they don't purchase annuities, because they don't want to take the risk of dying young without leaving bequest, which is indeed the worst scenario that one may imagine.

Calibration of our model with realistic mortality patterns and preference parameters that seem reasonable indicate that risk aversion alone is not likely to generate a negative demand for annuity, but sufficient to explain why individual do not purchase annuities on top of their pensions. Risk aversion may then provide an explanation to the annuity puzzle.

Interestingly, it's worth noting that this explanation related to risk aversion is not that far from what was called "Misleading Heuristics" and discussed as a behavioral trait in Brown (2007). In the Section 5.3 of his paper, Brown explains that agent apparently "wants to buy insurance against bad events", that is to purchase contracts which pay when the utility is low, while the theory predicts they should rather look for contracts that pay when marginal utility is high. The relation between marginal utility and utility, however, depends on risk aversion. When risk aversion is large enough, the state associated with lower utility will also be the one with higher marginal utility[7]. Buying contracts that pay when utility is low is then rational. An approach which constrains risk aversion to a given level, for example because of an assumption of additive separability, may lead to conclude that agents suffer from "misleading heuristics", while in fact they might just be more risk averse than allowed by the model. The agent which does not purchase annuities may perhaps less a mistake than the economics literature which focused on Yaari's approach.

[7]Consider for example a state dependent utility function $u(x,d)$ and two pairs (x_0,d_0) and (x_1,d_1) such that $u(x_0,d_0) < u(x_1,d_1)$. Consider now $v(x,d) = \frac{1-\exp(-k(u(x,d))}{k}$. The larger k the greater risk aversion. We have

$$\frac{\frac{\partial v}{\partial x} v(x_0,d_0)}{\frac{\partial v}{\partial x} v(x_1,d_1)} = \frac{\frac{\partial v}{\partial x} u(x_0,d_0)}{\frac{\partial v}{\partial x} u(x_1,d_1)} e^{k(u(x_1,d_1)-u(x_0,d_0))} > 1 \text{ when } k \text{ is large enough.}$$

When risk aversion is large enough the state which provide less utility, (x_0,d_0) is also associated with a higher marginal utility.

Appendix

A Proof of Proposition 3

First let's take $\phi_0(x) = \frac{1-e^{-kx}}{k}$ and show that if k is large enough then that $a = 0$.

Let us assume that for all k, $a > 0$ and see that we obtain a contradiction. When a is interior, the FOC together with the budget constraint rewrite:

$$\left[pe^{-kU_A} + (1-p)e^{-kU_D})\right] u_1'(c_1) = \lambda \tag{28}$$

$$e^{-kU_A} u_2'(c_2) = \frac{\lambda}{1+R} \tag{29}$$

$$v'((1+R)s)e^{-kU_D} \leq \frac{\lambda}{1+R} \tag{30}$$

$$v'(\tau)e^{-kU_A} \leq \frac{\lambda}{1+R} \tag{31}$$

$$W_0 = c_1 + \frac{p(c_2+\tau)}{1+R} + s(1-p) \tag{32}$$

Because of the possible existence of corner solutions for s and τ we have to consider several cases. We will see that in all possible cases we end up having a contradiction.

1. Assume that $s = \tau = 0$. We have $U_A - U_D = u_2(c_2)$ and (32) becomes $c_1 = W_0 - \frac{pc_2}{1+R}$. (28)–(30) give:

$$\left[p + (1-p)e^{ku_2(c_2)}\right] u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right) = (1+R)u_2'(c_2) \geq (1+R)v'(0)e^{ku_2(c_2)} \tag{33}$$

Suppose $c_2 < c_2^*$. The inequality (33) implies, since $u(c_2) < 0$ and $p + (1-p)e^{ku_2(c_2)} < 1$:

$$u_1'(c_1) > (1+R)u_2'(c_2) \tag{34}$$

For c_2^*, we derive from Assumption A and $c_2^* \geq c_2^{**}$ the reverse relationship $u_1'(W_0 - \frac{pc_2^*}{1+R}) < (1+R)u_2'(c_2^*)$. We deduce:

$$u_2'(c_2) > u_2'(c_2^*) \geq \frac{u_1'(W_0 - \frac{pc_2^*}{1+R})}{1+R} > \frac{u_1'(W_0 - \frac{pc_2}{1+R})}{1+R} = \frac{u_1'(c_1)}{1+R} \tag{35}$$

Inequalities (34) and (35) are incompatible, and we deduce $c_2 \geq c_2^*$ and $u_2(c_2) \geq 0$.

We deduce from (33) that $u_2'(c_2) \geq v'(0)$. We also have $v'(0) \leq \left[1 - p + pe^{-ku_2(c_2)}\right] \frac{1}{1+R} u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right) \leq \frac{1}{1+R} u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right)$.

Suppose that $c_2 < c_2^{**}$. We have using Assumption A:

$$v'(0) \geq v'(c_2^{**}) > u_1'\left(W_0 - \frac{c_2^{**}}{1+R}\right) \geq u_1'\left(W_0 - \frac{p\,c_2^{**}}{1+R}\right) \geq u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right),$$

which contradicts $v'(0) \leq \frac{1}{1+R} u_1' \left(W_0 - \frac{p\,c_2}{1+R}\right)$. We deduce that $c_2 \geq c_2^{**}$.

The optimal second period consumption exists if there exists a solution $\geq c_2^{**}$ to the following equation:

$$p + (1-p)e^{k\,u_2(c_2)} = (1+R)\,\frac{u_2'(c_2)}{u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right)}$$

The LHS is an increasing function of c_2, while the RHS is a decreasing one. A solution exists if and only if $p + (1-p)e^{k\,u_2(c_2^{**})} \leq (1+R)\,\frac{u_2'(c_2^{**})}{u_1'\left(W_0 - \frac{p\,c_2^{**}}{1+R}\right)}$. If $u_2(c_2^{**}) > 0$, we can always find a k such that the optimum does not exist. For the optimum to exist, we thus necessarily have: $c_2^* = c_2^{**} = 0$ and $u_2(0) = 0$.

Let $A > 0$ be arbitrarily large. Since $\lim_{c_2 \to 0} u_2'(c_2) = +\infty$, it is always possible to choose $\varepsilon > 0$ small enough such that:

$$(1+R)\frac{u_2'(\varepsilon)}{u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right)} > p + (1-p)e^{\frac{u_2(\varepsilon)}{u_2\left(\frac{\varepsilon}{2}\right)}\,A} > p + (1-p)e^{A}$$

$\frac{u_2(\varepsilon)}{u_2\left(\frac{\varepsilon}{2}\right)}$ is greater than 1, but also smaller than 2 in the vicinity of 0. Indeed, by Hospital rule, its limit is the limit of $2\frac{u_2'(\varepsilon)}{u_2'\left(\frac{\varepsilon}{2}\right)}$, which is smaller than 2 because u_2 is increasing and concave. We can now always choose $k \geq 0$ such that:

$$p + (1-p)e^{k\,u_2(\varepsilon)} > (1+R)\,\frac{u_2'(\varepsilon)}{u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right)} > p + (1-p)e^{k\,u_2\left(\frac{\varepsilon}{2}\right)} > p + (1-p)e^{A} \qquad (36)$$

Indeed, the two first inequalities simply mean that:

$$\frac{1}{u_2(\varepsilon/2)} \ln\left(\frac{(1+R)u_2'(\varepsilon)}{(1-p)u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right)} - \frac{p}{1-p}\right) > k > \frac{1}{u_2(\varepsilon)} \ln\left(\frac{(1+R)u_2'(\varepsilon)}{(1-p)u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right)} - \frac{p}{1-p}\right)$$

The construction of A insures the last inequality since:

$$k u_2(\frac{\varepsilon}{2}) > \frac{u_2\left(\frac{\varepsilon}{2}\right)}{u_2(\varepsilon)} \ln\left(\frac{(1+R)u_2'(\varepsilon)}{(1-p)u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right)} - \frac{p}{1-p}\right) > \frac{u_2\left(\frac{\varepsilon}{2}\right)}{u_2(\varepsilon)} \frac{u_2(\varepsilon)}{u_2\left(\frac{\varepsilon}{2}\right)} A = A$$

Eq. (36) means that c_2 exists and $\frac{\varepsilon}{2} < c_2 < \varepsilon$ together with $k\,u_2\left(\frac{\varepsilon}{2}\right) > A$. From (33), we deduce that:

$$\begin{aligned} v'(0) &\leq \left(1 - p + pe^{-k u_2(c_2)}\right) u_1'\left(W_0 - \frac{p\,c_2}{1+R}\right) \\ &\leq \left(1 - p + pe^{-A}\right) u_1'\left(W_0 - \frac{p\,\varepsilon}{1+R}\right) \end{aligned}$$

The RHS can be made arbitrarily close to $(1-p)\,u_1'\,(W_0)$, by choosing A large and ε small enough. We have therefore $v'(0)$, which can be strictly smaller than $u_1'\,(W_0)$, which contradicts Assumption A.

In consequence, we cannot have $s = \tau = 0$.

2. Assume that $\tau > 0$ and $s = 0$. Eq. (31) is an equality and we deduce from (29)–(31):

$$u_2'(c_2) = v'(\tau) \geq v'(0)e^{k(U_A - U_D)},$$

which implies $U_A - U_D = u_2(c_2) + v(\tau) < 0$, and $c_2 \leq c_2^* \leq c_2^{**}$. We have from Assumption A:

$$u_2'(c_2) \geq u_2'(c_2^{**}) > v'(0) \geq v'(\tau)$$

Is contradicts the previous relationship $u_2'(c_2) = v'(\tau)$. We cannot have $\tau > 0 = s$.

3. Assume that $s > 0$ and $\tau = 0$. Eq. (30) is an equality and we deduce from (28)–(31):

$$u_2'(c_2) = v'((1+R)s)e^{k(U_A - U_D)} = \left(p + (1-p)e^{k(U_A - U_D)}\right)\frac{1}{1+R}u_1'(c_1) \geq v'(0) \tag{37}$$

It implies that $U_A - U_D = u_2(c_2) - v((1+R)s) > 0$. We deduce from the preceding inequality and from the budget constraint stating that $c_1 = W_0 - \frac{pc_2}{1+R} - s(1-p) \leq W_0 - \frac{pc_2}{1+R}$:

$$u_2'(c_2) \geq \frac{1}{1+R}u_1'(c_1) \geq \frac{1}{1+R}u_1'\left(W_0 - \frac{pc_2}{1+R}\right)$$

Suppose that $c_2^{**} \geq c_2$. From the budget constraint $c_2 = (1+R)s + a\frac{1+R}{p} \geq (1+R)s$. From (37) and $c_2^{**} \geq (1+R)s$, we deduce that $v'(c_2^{**}) \leq v'((1+R)s) \leq \frac{1}{1+R}u_1'(c_1)$. Moreover, $c_1 = W_0 - \frac{pc_2}{1+R} - s(1-p) \geq W_0 - \frac{pc_2^{**}}{1+R} - \frac{c_2^{**}}{1+R}(1-p) = W_0 - \frac{c_2^{**}}{1+R}$. We therefore deduce that $v'(c_2^{**}) \leq \frac{1}{1+R}u_1'(W_0 - \frac{c_2^{**}}{1+R})$, which contradicts Assumption A. We thus have $c_2 \geq c_2^{**}$.

Using (37), we deduce that $(1+R)s$ is a function of c_2 that we denote ϕ and which is defined as:

$$v'(\phi(c_2))\,e^{-k\,v(\phi(c_2))} = u_2'(c_2)e^{-k\,u_2(c_2)}$$

First case. The function $c_2 \mapsto u_2(c_2) - v(\phi(c_2))$ is increasing. c_2 is defined as:

$$p + (1-p)e^{k(u_2(c_2) - v(\phi(c_2)))} = (1+R)\frac{u_2'(c_2)}{u_1'\left(W_0 - \frac{pc_2 + (1-p)\phi(c_2)}{1+R}\right)}$$

Using the same proof strategy as in the case $s = \tau = 0$, we first obtain $c_2^{**} = c_2^* = 0$ and then get a contradiction with Assumption A.

Second case. The function $c_2 \mapsto u_2(c_2) - v(\phi(c_2))$ is decreasing (in the vicinity of c_2^{**} – otherwise, the previous argument applies). First remark that it imposes $c_2^{**} > 0$. Otherwise, we have for all $c_2 \geq 0$ sufficiently small: $\phi'(c_2)v'(\phi(c_2)) \geq u'(c_2)$. After integration ($u_2(c_2^{**}) = u_2(c_2^*) = 0$), we get a contradiction with $u_2(c_2) - v(\phi(c_2)) \geq 0$.

c_2 is still defined as:

$$p + (1-p)e^{k(u_2(c_2)-v(\phi(c_2)))} = (1+R)\frac{u_2'(c_2)}{u_1'\left(W_0 - \frac{pc_2+(1-p)\phi(c_2)}{1+R}\right)}$$

$u_2(c_2) - v(\phi(c_2)) = (u_2(c_2) - v(c_2)) + (v(c_2) - v(\phi(c_2)))$ is the sum of two positive terms (since $c_2 \geq (1+R)s$) and is strictly positive id $c_2 > c_2^{**}$. If $v(c_2^{**}) - v(\phi(c_2^{**}))$ is strictly positive, it is sufficient to chose k large enough to get a contradiction (c_2 never exist – choose $k > \min_{c_2 \geq c_2^{**}} (u_2(c_2) - v(\phi(c_2)))^{-1} (1+R)\frac{u_2'(c_2^{**})}{u_1'\left(W_0 - \frac{pc_2^{**}+(1-p)\phi(c_2^{**})}{1+R}\right)}$). Therefore, $u_2(c_2^{**}) = v(c_2^{**}) = v(\phi(c_2^{**}))$.

$c_2 \mapsto u_2(c_2) - v(\phi(c_2))$ is decreasing in the vicinity of c_2^{**}, but positive. It implies that $u_2(c_2) = v(\phi(c_2))$ in the vicinity of c_2^{**}, which contradicts the definition of c_2^{**}.

4. Assume that $s > 0$ and $\tau > 0$.

   In that case:

$$v'((1+R)s)e^{k(U_A-U_D)} = v'(\tau) = u_2'(c_2) = \frac{1}{1+R}\left(p + (1-p)e^{k(U_A-U_D)}\right)u_1'(c_1) \tag{38}$$

   Suppose that $c_2 < c_2^{**}$. We have then: $u'(c_2^{**}) < u'(c_2) = v'(\tau) \leq v'(0)$, which contradicts Assumption A. We have therefore $c_2 \geq c_2^{**} \geq c_2^*$.

   The budget constraint gives $c_2 + \tau = (1+R)s + (1+R)a/p > (1+R)s$. We deduce $v(c_2) + v(\tau) \geq v((1+R)s)$ (v being concave with $v(0) = 0$ is subadditive. For all $\lambda \in [0,1]$, $v(\lambda c) = v(\lambda c + (1-\lambda)0) \geq \lambda v(c) + (1-\lambda)v(0) = \lambda v(c)$. Then $v(c_2) + v(\tau) = v((c_2+\tau)\frac{c_2}{c_2+\tau}) + v((c_2+\tau)\frac{\tau}{c_2+\tau}) \geq v(c_2+\tau) \geq v((1+R)s)$).

   We have $U_A - U_D = u(c_2) + v(\tau) - v((1+R)s) = (u_2(c_2) - v(c_2)) + (v(c_2) + v(\tau) - v((1+R)s)) \geq 0$ since both terms are positive due to our preceding remark and $c_2 \geq c_2^{**}$.

   We deduce that $v'(\tau) \geq v'((1+R)s)$ and $(1+R)s \geq \tau$.

   As previously, it is straightforward that τ and $(1+R)s$ are increasing functions of c_2 that we denote respectively $\psi(c_2)$ and $\phi(c_2)$. c_2 solves:

$$p + (1-p)e^{k(u_2(c_2)+\psi(c_2)-v(\phi(c_2)))} = (1+R)\frac{u_2'(c_2)}{u_1'\left(W_0 - \frac{p(c_2+\psi(c_2))+(1-p)\phi(c_2)}{1+R}\right)}$$

The proof is now similar to the one in 3) and we obtain one more time a contradiction.

5. We show that an increase in k implies a smaller a, supposing that we have an optimum with $s,\ \tau,$ and $a > 0$.

$$
\begin{aligned}
u_2'(c_2) &= \frac{1}{1+R}\left(p + (1-p)e^{k(U_A-U_D)}\right) u_1'(c_1) = v'((1+R)s)e^{k(U_A-U_D)} = v'(\tau) \\
W_0 &= c_1 + \frac{p(c_2+\tau)}{1+R} + (1-p)s \\
W_0 &= c_1 + s + a \\
c_2 + \tau &= (1+R)s + (1+R)\frac{a}{p}
\end{aligned}
$$

We have $c_2 \geq c_2^{**}$.

We assume an increase of k starting from an optimum where $c_1, c_2, \tau, a, s > 0$. For sake of simplicity, I note $\Delta U = U_A - U_D \geq 0$ and $S = (1+R)s$.

We have:

$$
\begin{aligned}
\frac{\partial \tau}{\partial k} &= \frac{u_2''(c_2)}{v''(\tau)}\frac{\partial c_2}{\partial k} \\
(1+R)u''(c_2)\frac{\partial c_2}{\partial k} &= (p + (1-p)e^{k\Delta U})u_1''(c_1)\frac{\partial c_1}{\partial k} + (1-p)e^{k\Delta U}u_1'(c_1)\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right) \\
u_2''(c_2)\frac{\partial c_2}{\partial k} &= e^{k\Delta U}\left(v''(S)\frac{\partial S}{\partial k} + v'(S)\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right)\right) \\
-(1+R)\frac{\partial c_1}{\partial k} &= p\left(1 + \frac{u_2''(c_2)}{v''(\tau)}\right)\frac{\partial c_2}{\partial k} + (1-p)\frac{\partial S}{\partial k} \\
\frac{\partial \Delta U}{\partial k} &= u'(c_2)\left(1 + \frac{u_2''(c_2)}{v''(\tau)}\right)\frac{\partial c_2}{\partial k} - v'(S)\frac{\partial S}{\partial k}
\end{aligned}
$$

I drop arguments and to limit ambiguity, I note $v_\tau'' = v''(\tau)$ and $\Gamma = 1 + \frac{u_2''(c_2)}{v''(\tau)} \geq 1$.

$$
\begin{aligned}
\frac{u_2''}{u_2'}\frac{\partial c_2}{\partial k} &= \frac{u_1''}{u_1'}\frac{\partial c_1}{\partial k} + \frac{(1-p)e^{k\Delta U}}{p + (1-p)e^{k\Delta U}}\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right) \\
\frac{u_2''}{u_2'}\frac{\partial c_2}{\partial k} &= \frac{v''}{v'}\frac{\partial S}{\partial k} + \left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{u_2''}{u_2'} + \frac{p}{1+R}\Gamma\frac{u_1''}{u_1'}\right)\frac{\partial c_2}{\partial k} &= -\frac{1-p}{1+R}\frac{u_1''}{u_1'}\frac{\partial S}{\partial k} + \frac{(1-p)e^{k\Delta U}}{p + (1-p)e^{k\Delta U}}\left(k\left(u_2'\Gamma\frac{\partial c_2}{\partial k} - v'\frac{\partial S}{\partial k}\right) + \Delta U\right) \\
\frac{u_2''}{u_2'}\frac{\partial c_2}{\partial k} &= \frac{v''}{v'}\frac{\partial S}{\partial k} + \left(k\left(u_2'\Gamma\frac{\partial c_2}{\partial k} - v'\frac{\partial S}{\partial k}\right) + \Delta U\right)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{u_2''}{u_2'} + \frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{(1-p)e^{k\Delta U}}{p + (1-p)e^{k\Delta U}}\,k\,u_2'\,\Gamma\right)\frac{\partial c_2}{\partial k} + \frac{1-p}{1+R}\left(\frac{u_1''}{u_1'} + k\,u_1'\right)\frac{\partial S}{\partial k} &= \frac{(1-p)e^{k\Delta U}}{p + (1-p)e^{k\Delta U}}\,\Delta U \\
\left(\frac{u_2''}{u_2'} - k\,u_2'\,\Gamma\right)\frac{\partial c_2}{\partial k} + \left(-\frac{v''}{v'} + k\,v'\right)\frac{\partial S}{\partial k} &= \Delta U
\end{aligned}
$$

We deduce:

$$a\frac{\partial c_2}{\partial k} + b\frac{\partial S}{\partial k} = \frac{(1-p)e^{k\Delta U}}{p+(1-p)e^{k\Delta U}}\,\Delta U$$
$$c\frac{\partial c_2}{\partial k} + d\frac{\partial S}{\partial k} = \Delta U$$

and:

$$\frac{ad-bc}{\Delta U}\frac{\partial c_2}{\partial k} = d\,\frac{(1-p)e^{k\Delta U}}{p+(1-p)e^{k\Delta U}} - b$$
$$\frac{ad-bc}{\Delta U}\frac{\partial S}{\partial k} = -\,c\,\frac{(1-p)e^{k\Delta U}}{p+(1-p)e^{k\Delta U}} + a$$

$$\frac{ad-bc}{\Delta U}\frac{\partial c_2}{\partial k} = \frac{(1-p)u_2'}{p+(1-p)e^{k\Delta U}}\left(\left(-\frac{v''}{v'}+k\,v'\right)\frac{1}{v'} - \frac{1}{u_1'}\left(\frac{u_1''}{u_1'}+k\,u_1'\right)\right)$$
$$\frac{ad-bc}{\Delta U}\frac{\partial S}{\partial k} = -\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\frac{(1-p)e^{k\Delta U}}{p+(1-p)e^{k\Delta U}} + \left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{(1-p)e^{k\Delta U}}{p+(1-p)e^{k\Delta U}}\,k\,u_2'\,\Gamma\right)$$

$$\frac{ad-bc}{\Delta U}\frac{\partial c_2}{\partial k} = \frac{1-p}{p+(1-p)e^{k\Delta U}}\left(-\frac{v''}{v'}\,\frac{u_2'}{v'} - \frac{u_1''}{u_1'}\,\frac{u_2'}{u_1'}\right)$$
$$\frac{ad-bc}{\Delta U}\frac{\partial S}{\partial k} = \frac{p}{p+(1-p)e^{k\Delta U}}\left(\frac{u_2''}{u_2'}+)\frac{u_2'}{u_1'}\Gamma\frac{u_1''}{u_1'}\right)$$

We compute the determinant $\Lambda = ad-bc$:

$$\begin{aligned}
\Lambda &= \left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{1-p}{1+R}\frac{u_1'}{v'}\,k\,u_2'\,\Gamma\right)\left(-\frac{v''}{v'}+k\,v'\right)\\
&- \frac{1-p}{1+R}\left(\frac{u_1''}{u_1'}+k\,u_1'\right)\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\\
&= -\frac{v''}{v'}\left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{1-p}{1+R}\frac{u_1'}{v'}\,k\,u_2'\,\Gamma\right) - \frac{1-p}{1+R}\,\frac{u_1''}{u_1'}\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\\
&+ k\,v'\left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{1-p}{1+R}\frac{u_1'}{v'}\,k\,u_2'\,\Gamma\right) - \frac{1-p}{1+R}k\,u_1'\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\\
&= -\frac{v''}{v'}\left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{1-p}{1+R}\frac{u_1'}{v'}\,k\,u_2'\,\Gamma\right) - \frac{1-p}{1+R}\,\frac{u_1''}{u_1'}\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\\
&+ k\,v'\,\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} + \frac{k}{1+R}\left(p\,e^{-k\Delta U}+1-p\right)u_1'\frac{u_2''}{u_2'} - \frac{1-p}{1+R}k\,u_1'\frac{u_2''}{u_2'}\\
&= -\frac{v''}{v'}\left(\frac{u_2''}{u_2'}+\frac{p}{1+R}\Gamma\frac{u_1''}{u_1'} - \frac{1-p}{1+R}\frac{u_1'}{v'}\,k\,u_2'\,\Gamma\right) - \frac{1-p}{1+R}\,\frac{u_1''}{u_1'}\left(\frac{u_2''}{u_2'}-k\,u_2'\,\Gamma\right)\\
&+ \frac{k}{1+R}\,p\,e^{-k\Delta U}\left(\Gamma\,u_2'\,\frac{u_1''}{u_1'} + u_1'\frac{u_2''}{u_2'}\right)
\end{aligned}$$

From the last expression deduce easily that $\Lambda < 0$ and:

$$-\frac{\Lambda}{\Delta U}\frac{\partial c_2}{\partial k} = -\frac{1-p}{p+(1-p)e^{k\Delta U}}\left(-\frac{v''}{v'}\,\frac{u_2'}{v'} - \frac{u_1''}{u_1'}\,\frac{u_2'}{u_1'}\right) < 0$$
$$-\frac{\Lambda}{\Delta U}\frac{\partial S}{\partial k} = \frac{p}{p+(1-p)e^{k\Delta U}}\left(-\frac{u_2''}{u_2'} - \frac{u_2'}{u_1'}\Gamma\frac{u_1''}{u_1'}\right) > 0$$

which means that $\frac{\partial c_2}{\partial k} < 0$ and $\frac{\partial S}{\partial k} > 0$.

For a, we have:

$$\begin{aligned} c_2 + \tau &= S + \frac{1+R}{p}\, a \\ \frac{1+R}{p}\frac{\partial a}{\partial k} &= \Gamma\, \frac{\partial c_2}{\partial k} - \frac{\partial S}{\partial k} < 0 \\ -\left(p + (1-p)e^{k\Delta U}\right) \frac{\Lambda}{\Delta U}\, \frac{1+R}{p}\frac{\partial a}{\partial k} &= p\frac{u_2''}{u_2'} + (1-p)\Gamma\frac{v''}{v'}\,\frac{u_2'}{v'} + \Gamma\frac{u_1''}{u_1'}\,\frac{u_2'}{u_1'} \end{aligned}$$

We finally have $\frac{\partial a}{\partial k} < 0$.

6. We show that an increase in k implies a smaller a, supposing that we have an optimum with s, and $a > 0$, but $\tau = 0$. Previous equations are still valid, but with $\Gamma = 1$. We also obtain that $\frac{\partial a}{\partial k} < 0$.

   Moreover the condition for non interior τ is $v'(0) \leq u'(c_2)$ remains true for a small increase in k since $\frac{\partial c_2}{\partial k} < 0$.

7. Case of $\tau = S = 0$. We have:

$$\left[p + (1-p)e^{ku_2(c_2)}\right] u_1' \left(W_0 - \frac{p\, c_2}{1+R}\right) = (1+R)u_2'(c_2) \geq (1+R)v'(0)e^{ku_2(c_2)}$$

   Deriving relative to k yields:

$$\begin{aligned} (1+R)u_2''\frac{\partial c_2}{\partial k} &= -\frac{p}{1+R}\left(p + (1-p)e^{ku_2(c_2)}\right) u_1''\frac{\partial c_2}{\partial k} + (1-p)e^{ku_2(c_2)}u_1'\left(u_2 + \frac{\partial c_2}{\partial k}\right) \\ \left((1+R)u_2'' + pu_2'\frac{u_1''}{u_1'} - (1-p)e^{ku_2(c_2)}u_1'\right)\frac{\partial c_2}{\partial k} &= (1-p)e^{ku_2(c_2)}u_1'\, u_2 \end{aligned}$$

   It is straightforward that $\frac{\partial c_2}{\partial k} < 0$ and from $c_2 = (1+R)a/p$, we deduce that $\frac{\partial a}{\partial k} < 0$

8. We show that an increase in k implies a more slack constraint on a, supposing that we have an optimum with s, and $\tau > 0$, but $a = 0$.

$$\begin{aligned} u_2'(c_2) &\leq \frac{1}{1+R}\left(p + (1-p)e^{k\Delta U}\right) u_1'(c_1) \\ u_2'(c_2) &= v'(\tau) \\ p\, u_2'(c_2) + (1-p)v'(S)e^{k\Delta U} &= \frac{1}{1+R}\left(p + (1-p)e^{k\Delta U}\right) u_1'(c_1) \\ W_0 &= c_1 + \frac{c_2 + \tau}{1+R} = c_1 + \frac{S}{1+R} \\ c_2 + \tau &= S \end{aligned}$$

The derivation relative to k yields:

$$\begin{aligned}
\frac{\partial(c_2+\tau)}{\partial k} &= \Gamma\,\frac{\partial c_2}{\partial k} \\
p\,u_2''\frac{\partial c_2}{\partial k} + (1-p)\left(v_S''\frac{\partial S}{\partial k} + v_S'\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right)\right)e^{k\Delta U} &= \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\partial c_1}{\partial k} \\
&\quad + \frac{(1-p)e^{k\Delta U}}{1+R}u_1'\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right) \\
\frac{\partial S}{\partial k} &= -\,(1+R)\frac{\partial c_1}{\partial k} = \Gamma\,\frac{\partial c_2}{\partial k} \\
\frac{\partial \Delta U}{\partial k} &= u_2'\frac{\partial c_2}{\partial k} + v_\tau'\frac{\partial \tau}{\partial k} - v_S'\frac{\partial S}{\partial k}
\end{aligned}$$

We define Λ as:

$$\begin{aligned}
\Lambda &= p\,u_2'' + (1-p)v_S''\Gamma e^{k\Delta U} + (1-p)(v_S' - \frac{1}{1+R}u_1')e^{k\Delta U}k\Gamma\,(u_2'-v_S') + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R} \\
&= p\,u_2'' + (1-p)v_S''\Gamma e^{k\Delta U} - p(1-p)\frac{(u_2'-v_S')^2e^{k\Delta U}}{p+(1-p)e^{k\Delta U}}k\Gamma + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R} < 0
\end{aligned}$$

The second expression comes from the FOC, and $v_S' - \frac{1}{1+R}u_1' = -\frac{p(u_2'-v_S')}{p+(1-p)e^{k\Delta U}}$. We deduce:

$$\begin{aligned}
\Lambda\frac{\partial c_2}{\partial k} &= -\,(1-p)\left(v_S' - \frac{1}{1+R}u_1'\right)e^{k\Delta U}\Delta U = \frac{p(1-p)e^{k\Delta U}\Delta U}{p+(1-p)e^{k\Delta U}}(u_2'-v_S') \qquad (39) \\
\frac{\partial S}{\partial k} &= -\,(1+R)\frac{\partial c_1}{\partial k} = \Gamma\,\frac{\partial c_2}{\partial k} \\
\frac{\partial \Delta U}{\partial k} &= \Gamma\,(u_2'-v_S')\,\frac{\partial c_2}{\partial k} = \Gamma\,(u_2'-v_S')^2\,\frac{p(1-p)e^{k\Delta U}\Delta U}{p+(1-p)e^{k\Delta U}}\,\frac{1}{\Lambda}
\end{aligned}$$

We have that $u_2' - v_S' \geq 0$. We deduce that c_2 and S decrease, while c_1 increases. Indeed, otherwise, $u_2' = v_\tau' < v_S'$ and $\tau \geq S$, which contradicts $c_2 = S - \tau \geq 0$.

We now want to check that the condition on a is more binding, i.e. that we have:

$$\begin{aligned}
u_2''\frac{\partial c_2}{\partial k} &\leq e^{k\Delta U}\left(v_S''\frac{\partial S}{\partial k} + v_S'\left(k\frac{\partial \Delta U}{\partial k} + \Delta U\right)\right) \\
\left(u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S'e^{k\Delta U}\Gamma\,(u_2'-v_S')\right)\frac{\partial c_2}{\partial k} &\leq e^{k\Delta U}v_S'\Delta U
\end{aligned}$$

If $u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S'e^{k\Delta U}\Gamma\,(u_2'-v_S')$ is positive, the result is true. If it is negative, we

divide the previous inequality by the definition (39) of $\frac{\partial c_2}{\partial k}$, and we get:

$$\frac{1}{\Lambda}\left(u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S' e^{k\Delta U}\Gamma\,(u_2' - v_S')\right) \leq -\frac{v_S'}{(1-p)\left(v_S' - \frac{1}{1+R}u_1'\right)}$$

$$(1-p)\left(v_S' - \frac{1}{1+R}u_1'\right)\left(u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S' e^{k\Delta U}\Gamma\,(u_2' - v_S')\right) \leq -\Lambda\, v_S'$$

$$v_S'\left(\Lambda + (1-p)u_2'' - (1-p)v_S''\Gamma e^{k\Delta U} - (1-p)kv_S' e^{k\Delta U}\Gamma\,(u_2' - v_S')\right) \leq \frac{1-p}{1+R}u_1'\left(u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S' e^{k\Delta U}\Gamma\,(u_2' - v_S')\right)$$

$$v_S'\left(u_2'' - \frac{1-p}{1+R}ku_1' e^{k\Delta U}\Gamma\,(u_2' - v_S') + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R}\right) \leq \frac{1-p}{1+R}u_1'\left(u_2'' - v_S''\Gamma e^{k\Delta U} - kv_S' e^{k\Delta U}\Gamma\,(u_2' - v_S')\right)$$

$$v_S'\left(u_2'' + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R}\right) \leq \frac{1-p}{1+R}u_1'\left(u_2'' - v_S''\Gamma e^{k\Delta U}\right)$$

It is equivalent to (using FOC for u_1'):

$$v_S'\left(v_S''\Gamma e^{k\Delta U} + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R}\right) \leq u_2''\left(\frac{1-p}{1+R}u_1' - v_S'\right)$$

$$v_S'\left(v_S''\Gamma e^{k\Delta U} + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R}\right) \leq u_2''\left(\frac{p(1-p)\,u_2' + (1-p)^2 v_S' e^{k\Delta U}}{p+(1-p)e^{k\Delta U}}u_1' - v_S'\right)$$

$$v_S'\left(v_S''\Gamma e^{k\Delta U} + \frac{p+(1-p)e^{k\Delta U}}{1+R}u_1''\frac{\Gamma}{1+R}\right) \leq pu_2''\frac{(1-p)\,u_2' - v_S' - (1-p)v_S' e^{k\Delta U}}{p+(1-p)e^{k\Delta U}}u_1'$$

This relationship is always true since the LHS is negative, while the RHS is positive. Indeed, we know that $u_2' \leq v_S' e^{k\Delta U}$, and the RHS is greater that $-\,u_1'\,v_S'\,u_2'' \geq 0$.

9. The case for $a = \tau = 0 < S$ is the same as the previous one with $\Gamma = 1$.