Existence and Uniqueness of a Fixed Point for the Bellman Operator in Deterministic Dynamic Programming*

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February 19, 2012

Abstract

We study existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming. Without any topological assumption, we show that the Bellman operator has a unique fixed point in a restricted domain, that this fixed point is the value function, and that the value function can be computed by value iteration.

Keywords: Dynamic programming, Bellman operator, value function, fixed point.

JEL Classification: C61

*This is an extensively revised version of the paper presented as "The Bellman Operator as a Monotone Map" at the 11th SAET Conferance in Faro, 2011. I would like to thank all participants at this conference and the Workshop in Honor of Cuong Le Van in Exeter, 2011, for comments and discussions. In particular, I am grateful to Larry Blume for his encouraging comments and suggestions, and to Juan Pablo Rincón-Zapatero, V. Filipe Martins-da-Rocha, Yiannis Vailakis, and Cuong Le Van for their helpful comments and discussions. Financial support from the Japan Society for the Promotion of Science is gratefully acknowledged.

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1 Introduction

Dynamic programming is one of the most fundamental tools in economic analysis. This has been particularly true since the publication of the influential book by Stokey and Lucas (1989). In this book and earlier studies, however, models with unbounded returns were not fully covered, though such models are extremely common in economics, especially in macroeconomics. This problem has been treated in several important contributions, including Alvarez and Stokey (1998), Le Van and Morhaim (2002), and Rinćon-Zapatero and Rodríguez-Palmero (2003, 2007, 2009).

Building on the work by the last pair of authors, Martins-da-Rocha and Vailakis (2010) recently established one of the most general results on existence and uniqueness of a solution to the Bellman equation—or a fixed point of the Bellman operator—applicable to models with unbounded returns. Among the assumptions of their result are the following:

(i) The state space is $\mathbb{R}^n_+$.

(ii) The feasibility correspondence is continuous and compact-valued.

(iii) The return function is continuous.

(iv) Except at the origin, a return of $-\infty$ can be avoided by following some continuous (suboptimal) policy.

Using these and other assumptions, Martins-da-Rocha and Vailakis (2010) showed that the Bellman operator is a local contraction to apply their general fixed point theorem. This is a powerful approach that guarantees not only the existence and uniqueness of a fixed point within a restricted domain, but also the continuity of the value function and the convergence of value iteration from any initial function in that domain.

In this paper we show that the assumptions listed above are in fact unnecessary for establishing the existence and uniqueness of a fixed point for the Bellman operator. Indeed, no topological assumption is required given the remaining assumptions used by Martins-da-Rocha and Vailakis (2010). More precisely, under weaker versions of those remaining assumptions, we obtain the following conclusions: (a) the Bellman operator has a unique fixed point in a restricted domain; (b) this fixed point is the value function; and (c) the value function can be computed by value iteration starting from the lower boundary of the restricted domain.

Although the uniqueness part of this result can be shown by extending some of Stokey and Lucas's (1989) arguments,[1] the existence and convergence parts require an additional tool. In the case of Martins-da-Rocha and Vailakis (2010), it is their fixed point theorem on local contractions for both existence and convergence (as well as uniqueness). In our case, we exploit the monotonicity of the Bellman operator and apply the Knaster-Tarski fixed point theorem (e.g., Aliprantis and Border, 2006) for existence, and to develop additional monotonicity-based arguments for convergence.[2]

Unlike the previous contributions mentioned above, we establish no regularity property of the value and policy functions. However, many properties of these functions can be shown separately under additional assumptions. For example, if the return function is upper semi-continuous, then the value function is also upper semi-continuous; see Le Van and Morhaim (2002). If the return function is concave, then the value function is also concave and thus continuous except on the boundary of the state space.[3] Such arguments can easily be added to our analysis. Moreover, on a practical level, it is useful to know that the value function can be computed by value iteration regardless of its regularity properties.

The rest of the paper is organized as follows. In the next section, we describe our framework and state our main result, which is proved in Section 4. In Section 3, we present two examples. The first one is trivial but has a continuum of fixed points, illustrating the importance of restricting the domain of the Bellman operator. The second example shows that value iteration may fail to converge to the value function unless the initial function is chosen appropriately.

[1] We do not entirely follow their approach since we prove uniqueness along with existence and convergence.

[2] The monotonicity arguments of Bertsekas and Shreve (1978, Chapter 5) are not applicable to our setting since they require the return function to be everywhere positive or everywhere negative. Le Van and Vailakis (2011) also use a monotonicity argument, but they require the Bellman operator to be concave to ensure uniqueness of a fixed point. Both Bertsekas and Shreve (1978) and Le Van and Vailakis (2011) deal with stochastic models, which are beyond the scope of this paper.

[3] For yet another example, if the return function is strictly concave, then the optimal policy correspondence is single-valued and thus continuous (provided that it is upper hemicontinuous).

2 The Main Result

Let X be a set. Let Γ be a nonempty-valued correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{2.1}$$

Let $u : D \to [-\infty, \infty)$. In the optimization problem introduced below, X is the state space, Γ is the feasibility correspondence, u is the return function, and D is the domain of u.

Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}. \tag{2.2}$$
$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{2.3}$$

Let $\beta \geq 0$. Given $x_0 \in X$, consider the following optimization problem:

$$\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \operatorname*{L}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \tag{2.4}$$

where $\mathrm{L} \in \{\underline{\lim}, \overline{\lim}\}$ with $\underline{\lim} = \liminf$ and $\overline{\lim} = \limsup$. Since $u(x, y) < \infty$ for all $(x, y) \in D$, the objective function is well-defined for any feasible path.

For $\{x_t\}_{t=0}^{\infty} \in \Pi$, we define

$$S(\{x_t\}_{t=0}^{\infty}) = \operatorname*{L}_{T \uparrow \infty} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}). \tag{2.5}$$

The value function $v^* : X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} S(\{x_t\}_{t=0}^{\infty}), \qquad x_0 \in X. \tag{2.6}$$

Note that $v^*(x_0)$ is unaffected if $\Pi(x_0)$ is replaced by $\Pi^0(x_0)$,[4] where

$$\Pi^0 = \{\{x_t\}_{t=0}^{\infty} \in \Pi : S(\{x_t\}_{t=0}^{\infty}) > -\infty\}, \tag{2.7}$$
$$\Pi^0(x_0) = \{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0) : \{x_t\}_{t=0}^{\infty} \in \Pi^0\}, \qquad x_0 \in X. \tag{2.8}$$

[4] We follow the convention that $\sup \emptyset = -\infty$.

Let V be the set of functions from X to $[-\infty, \infty)$. The Bellman operator B on V is defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x,y) + \beta v(y)\}, \qquad x \in X, v \in V. \tag{2.9}$$

Given $v \in V$, it need not be the case that $Bv \in V$. A fixed point of B is a function $v \in V$ such that $Bv = v$.

Let $v, w \in V$. We define the partial order $\leq$ on V in the usual way:

$$v \leq w \quad \Longleftrightarrow \quad \forall x \in X,\ v(x) \leq w(x). \tag{2.10}$$

It is immediate from (2.9) that B is a monotone operator:

$$v \leq w \quad \Rightarrow \quad Bv \leq Bw. \tag{2.11}$$

If $v \leq w$, we define the order interval $[v, w]$ by

$$[v, w] = \{f \in V : v \leq f \leq w\}. \tag{2.12}$$

We are ready to state the main result of this paper:

Theorem 2.1. *Suppose that there exist $\underline{v}, \overline{v} \in V$ such that*

$$\underline{v} \leq \overline{v}, \tag{2.13}$$
$$B\underline{v} \geq \underline{v}, \tag{2.14}$$
$$B\overline{v} \leq \overline{v}, \tag{2.15}$$
$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi^0, \quad \varliminf_{t \uparrow \infty} \beta^t \underline{v}(x_t) \geq 0, \tag{2.16}$$
$$\forall \{x_t\}_{t=0}^{\infty} \in \Pi, \quad \varlimsup_{t \uparrow \infty} \beta^t \overline{v}(x_t) \leq 0. \tag{2.17}$$

Then the following conclusions hold:
(a) *The Bellman operator B has a unique fixed point in $[\underline{v}, \overline{v}]$.*
(b) *This fixed point is the value function v^*.*
(c) *The sequence $\{B^n \underline{v}\}_{n=1}^{\infty}$ converges to v^* pointwise.*

Proof. See Section 4. □

Theorem 2.1 does not require any of the assumptions (i)–(iv) listed in the introduction. In fact, no topological assumption is required given conditions (2.13)–(2.17), as far as conclusions (a)–(c) are concerned. Rinćon-Zapatero and Rodríguez-Palmero (2003) offer several nontrivial, economically relevant examples satisfying stronger versions of these conditions. A detailed comparison between Theorem 2.1 and Martins-da-Rocha and Vailakis's (2010) result is available in an earlier version of this paper (Kamihigashi, 2011).

If there exist $\underline{v}, \overline{v} \in V$ satisfying (2.13)–(2.15), then the Bellman operator B has a fixed point in $[\underline{v}, \overline{v}]$ by the Knaster-Tarski fixed point theorem (see Section 4 for a precise argument). But if (2.16) and (2.17) are violated, then B can have multiple fixed points in $[\underline{v}, \overline{v}]$; see Section 3.1 for an example.

If (2.16) is strengthened by replacing Π^0 with Π, then it essentially follows from Stokey and Lucas (1989, Theorem 4.3) that any fixed point of B in $[\underline{v}, \overline{v}]$ coincides with v^*. However, this strengthened version of (2.16) is almost never satisfied if u is unbounded below. Conditions similar to (2.16) have been used to solve this problem since Le Van and Morhaim (2002).

In conclusion (c), we have convergence to v^* only from $\underline{v}$. Our argument for (c) is based on the observation that the limit of the increasing sequence $\{B^n \underline{v}\}_{n=1}^{\infty}$ is the supremum of the sequence. This allows us to interchange this supremum and another supremum (see (4.16)–(4.18)) to show that the limit is the value function v^*. The case of the decreasing sequence $\{B^n \overline{v}\}_{n=1}^{\infty}$, which also converges pointwise, is not symmetric since the sup and inf operators are in general not interchangeable. See Section 3.2 for an example satisfying (2.13)–(2.17) in which $\lim_{n\uparrow\infty} B^n \overline{v} \neq v^*$.[5]

3 Counterexamples

3.1 Multiple Fixed Points

The Bellman operator B can have multiple fixed points in $[\underline{v}, \overline{v}]$ if (2.16) and (2.17) are violated. To see this, suppose that $\beta > 0$, $X = \mathbb{Z}_+$, and

$$\forall i \in X, \qquad \Gamma(i) = \{i+1\}, \quad u(i, i+1) = 0. \tag{3.1}$$

At each state $i \in X$, there is only one feasible choice $(i+1)$ with a return of zero. Thus $v^*(i) = 0$ for all $i \in X$. Let $\alpha > 0$. Define $\underline{v}, \overline{v} \in V$ by $\underline{v}(i) = -\alpha\beta^{-i}$ and $\overline{v}(i) = \alpha\beta^{-i}$ for all $i \in X$. Then (2.13) holds. Since $\underline{v}(i) = \beta\underline{v}(i+1)$ and $\overline{v}(i) = \beta\overline{v}(i+1)$ for all $i \in X$, (2.14) and (2.15) hold with equality. This observation alone shows that B has multiple fixed points

[5] See Strauch (1966, p. 880) for a related example of an undiscounted stochastic model.

in $[\underline{v}, \overline{v}]$. In fact, for any $a \in [-\alpha, \alpha]$, the function v defined by $v(i) = a\beta^{-i}$ for all $i \in X$ is a fixed point of B. Therefore B has a continuum of fixed points. Note that there is only one feasible path from state 0, which is given by $\{x_t\}_{t=1}^{\infty} = \{t\}_{t=1}^{\infty}$. Then $\beta^t \underline{v}(x_t) = -\alpha$ and $\beta^t \overline{v}(x_t) = \alpha$ for all $t \in \mathbb{Z}_+$; i.e., (2.16) and (2.17) are violated in this example.

3.2 Nonconvergence to v^*

Even under (2.13)–(2.17), the sequence $\{B^n \overline{v}\}_{n=1}^{\infty}$ may not converge to v^*. To see this, let $\alpha > 0$ and suppose that $\alpha < \beta < 1$. Consider the example depicted in Figure 1; more precisely, assume the following:

$$X = \{(i,j) : i, j \in \mathbb{Z}_+, j \leq i\}, \tag{3.2}$$

$$\Gamma((i,j)) = \begin{cases} \{(i',0) : i' \in \mathbb{Z}_+\} & \text{if } (i,j) = (0,0), \\ \{(i,j)\} & \text{if } i = j \neq 0, \\ \{(i, j+1)\} & \text{if } j < i, \end{cases} \tag{3.3}$$

$$u((i,j),(i',j')) = \begin{cases} -\alpha & \text{if } (i,j) = (i',j') = (0,0), \\ -\beta^{-i} & \text{if } (i,j) = (i',j') \neq (0,0), \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Then the value function v^* can be computed directly:[6]

$$v^*((i,j)) = \begin{cases} -\alpha/(1-\beta) & \text{if } (i,j) = (0,0), \\ -\beta^{-j}/(1-\beta) & \text{otherwise.} \end{cases} \tag{3.5}$$

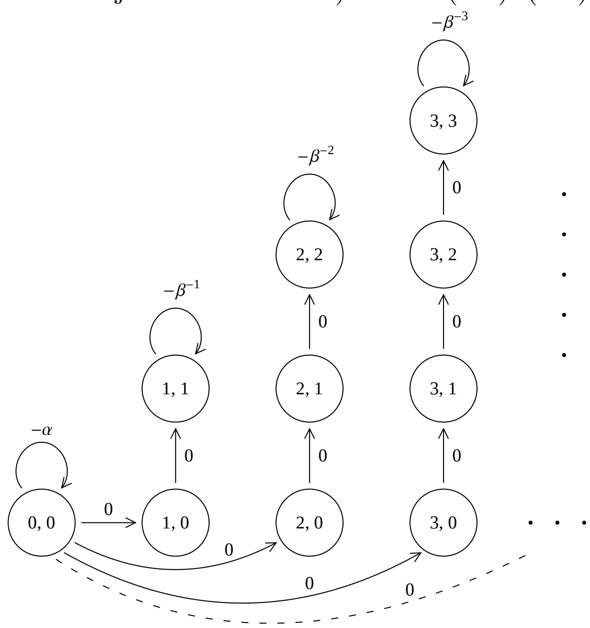
Let $\underline{v} = v^*$ and $\overline{v} = 0$. Then $\underline{v} \leq \overline{v}$ and $B\underline{v} = \underline{v}$. Since $u \leq 0$, we have $B\overline{v} \leq \overline{v}$. Thus (2.13)–(2.15) hold. As any feasible path eventually becomes constant, (2.16) and (2.17) hold with equality. Hence Theorem 2.1 applies.

Consider the sequence $\{\overline{v}_n\}_{n=1}^{\infty} \equiv \{B^n \overline{v}\}_{n=1}^{\infty}$. If $(i,j) \neq (0,0)$, there is only one feasible transition from (i,j), so that $\overline{v}_n((i,j))$ can be computed directly:

$$\overline{v}_n((i,j)) = \begin{cases} -\beta^{-j} \sum_{k=0}^{n-(i-j)-1} \beta^k & \text{if } i > 0 \text{ and } n \geq i - j + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

[6] Let $i \in \mathbb{N}$. Then at state (i,i), we have $v^*((i,i)) = -\beta^{-i}/(1-\beta)$. Note that $v^*((i, i-k)) = \beta^k v^*((i,i))$ for $k = 1, \ldots, i$; thus $v^*((i,j)) = -\beta^{i-j} v^*((i,i)) = -\beta^{-j}/(1-\beta)$. It remains to compute $v((0,0))$. If $x_t = (0,0)$ for all $t \in \mathbb{Z}_+$, then $S(\{x_t\}_{t=0}^{\infty}) = -\alpha/(1-\beta)$. If $x_1 = (i,0)$ with $i > 0$, then $S(\{x_t\}_{t=0}^{\infty}) = \beta v^*((i,0)) = -\beta/(1-\beta) < -\alpha/(1-\beta)$. Hence it is never optimal to leave state $(0,0)$, so that $v^*((0,0)) = -\alpha/(1-\beta)$.

Figure 1: States $(i,j) \in X$ (circles), feasible transitions (arrows), and associated returns (values adjacent to arrows) under (3.2)–(3.4)



This formula works for $(i,j) = (0,0)$ as well; i.e., $\overline{v}_n((0,0)) = 0$ for all $n \in \mathbb{N}$.[7]

Now letting $\overline{v}^* = \lim_{n\uparrow\infty} \overline{v}_n$,[8] we see that $\overline{v}^*((i,j)) = v^*((i,j))$ for all $(i,j) \in X \setminus \{(0,0)\}$, but $\overline{v}^*((0,0)) = 0 > v^*((0,0))$; i.e., the sequence $\{\overline{v}_n\}_{n=1}^{\infty}$ fails to converge to v^* at $(0,0)$.

Interestingly, the sequence $\{B^n\overline{v}^*\}_{t=1}^{\infty}$ restarted from $\overline{v}^*$ converges to v^*. Indeed, $(B^n\overline{v}^*)((0,0)) = -\alpha(1 + \beta + \cdots + \beta^{n-1}) \to v^*((0,0))$ as $n \uparrow \infty$.

4 Proof of Theorem 2.1

The proof consists of three lemmas and a concluding argument. The proof of the first lemma slightly generalizes an argument of Stokey and Lucas (1989, Theorem 4.3). The second lemma essentially shows that $B^T v$ with $T \in \mathbb{N}$ and $v \in V$ is the value function of the T period problem with the value

[7]To see this, define $\overline{v}_0 = \overline{v} = 0$. Then $\overline{v}_0((0,0)) = 0$. Let $n \in \mathbb{Z}_+$. With $\overline{v}_n$ given by (3.6), we have $\overline{v}_{n+1}((0,0)) = \beta \sup_{i\in X} \overline{v}_n((i,0)) = 0$ since $\overline{v}_n((i,0)) = 0$ for all $i \geq n$. By induction, $\overline{v}_n((0,0)) = 0$ for all $n \in \mathbb{N}$.

[8]In this paper the limit is taken pointwise: $\overline{v}^*(x) = \lim_{n\uparrow\infty} \overline{v}_n(x)$ for all $x \in X$.

of the terminal stock x_T given by $v(x_T)$. This result extends the classical idea of Bertsekas and Shreve (1978, Section 3.2) to our setting. The last lemma is less trivial than the first two. The concluding argument applies the Knaster-Tarski fixed point theorem and combines the first and last lemmas.

Lemma 4.1. *Let $\overline{v} \in V$ satisfy (2.17). Let $v \in V$ be a fixed point of B with $v \leq \overline{v}$. Then $v \leq v^*$.*

Proof. Let $x_0 \in X$. If $v(x_0) = -\infty$, then $v(x_0) \leq v^*(x_0)$. Consider the case $v(x_0) > -\infty$. Let $\epsilon > 0$. Let $\{\epsilon_t\}_{t=0}^{\infty} \subset (0, \infty)$ be such that $\sum_{t=0}^{\infty} \beta^t \epsilon_t \leq \epsilon$. Since $v = Bv$, for any $t \in \mathbb{Z}_+$ and $x_t \in X$, there exists $x_{t+1} \in \Gamma(x_t)$ such that

$$v(x_t) \leq u(x_t, x_{t+1}) + \beta v(x_{t+1}) + \epsilon_t. \tag{4.1}$$

We pick $x_1 \in \Gamma(x_0), x_2 \in \Gamma(x_1), \ldots$ so that (4.1) holds for all $t \in \mathbb{Z}_+$. Then $\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)$. By repeated application of (4.1) we have

$$\begin{aligned} v(x_0) &\leq u(x_0, x_1) + \beta v(x_1) + \epsilon_0 && (4.2)\\ &\leq u(x_0, x_1) + \beta[u(x_1, x_2) + \beta v(x_2) + \epsilon_1] + \epsilon_0 && (4.3)\\ &\;\;\vdots && (4.4)\\ &\leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) + \epsilon, \quad \forall T \in \mathbb{N}. && (4.5) \end{aligned}$$

Since $v(x_0) > -\infty$, we have $\beta^T v(x_T) > -\infty$ for all $T \in \mathbb{N}$. It follows that

$$v(x_0) - \epsilon - \beta^T v(x_T) \leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}). \tag{4.6}$$

Applying $\underline{\lim}_{T\uparrow\infty}$ to both sides, recalling (2.5) and (2.6), we have

$$v(x_0) - \epsilon - \overline{\lim_{T\uparrow\infty}} \beta^T v(x_T) \leq \underline{\lim_{T\uparrow\infty}} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) \leq S(\{x_t\}_{t=0}^{\infty}) \leq v^*(x_0). \tag{4.7}$$

By (2.17) we have $v(x_0) - \epsilon \leq v^*(x_0)$. Since this is true for any $\epsilon > 0$, we have $v(x_0) \leq v^*(x_0)$. Since x_0 was arbitrary, we obtain $v \leq v^*$. □

For any $v \in V$, define $v_1 = Bv$; for each $n \in \mathbb{N}$, provided that $v_n \in V$, define $v_{n+1} = Bv_n$. The following remark follows from (2.11).

Remark 4.1. Let $v, w \in V$ satisfy $v \leq w$ and $Bw \leq w$. Then for all $n \in \mathbb{N}$, we have $v_n \leq w$ and thus $v_n \in V$.

Lemma 4.2. *Let $\overline{v} \in V$ satisfy (2.15). Let $v \in V$ satisfy $v \leq \overline{v}$. Then for any $T \in \mathbb{N}$, we have $v_T \in V$ and*

$$\forall x_0 \in X, \quad v_T(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) \right\}. \tag{4.8}$$

Proof. Note from (2.15) and Remark 4.1 with $w = \overline{v}$ that $v_n \in V$ for all $n \in \mathbb{N}$. For any $x_0 \in X$, we have

$$v_1(x_0) = \sup_{x_1 \in \Gamma(x_0)} \{u(x_0, x_1) + \beta v(x_1)\} \tag{4.9}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \sup_{\{x_t\}_{t=2}^{\infty} \in \Pi(x_1)} \{u(x_0, x_1) + \beta v(x_1)\} \tag{4.10}$$

$$= \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \{u(x_0, x_1) + \beta v(x_1)\}, \tag{4.11}$$

where (4.10) holds since $\{u(x_0, x_1) + \beta v(x_1)\}$ is independent of $\{x_t\}_{t=2}^{\infty}$,[9] and (4.11) follows by combining the two suprema (see Kamihigashi, 2008, Lemma 1). It follows that (4.8) holds for $T = 1$.

Now assume (4.8) for $T = n \in \mathbb{N}$. For any $x_0 \in X$, we have

$$v_{n+1}(x_0) = \sup_{x_1 \in \Gamma(x_0)} \{u(x_0, x_1) + \beta v_n(x_1)\} \tag{4.12}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \Bigg\{ u(x_0, x_1) \tag{4.13}$$

$$+ \beta \sup_{\{x_{i+1}\}_{i=1}^{\infty} \in \Pi(x_1)} \left\{ \sum_{i=0}^{n-1} \beta^i u(x_{i+1}, x_{i+2}) + \beta^n v(x_{n+1}) \right\} \Bigg\}$$

$$= \sup_{x_1 \in \Gamma(x_0)} \sup_{\{x_{i+1}\}_{i=1}^{\infty} \in \Pi(x_1)} \left\{ \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) \right\} \tag{4.14}$$

$$= \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \left\{ \sum_{t=0}^{n} \beta^t u(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) \right\}, \tag{4.15}$$

[9] This step uses the assumption that Γ is nonempty-valued.

where (4.13) uses (4.8) for $T = n$, (4.14) holds since $u(x_0, x_1)$ is independent of $\{x_{i+1}\}_{i=1}^{\infty}$, and (4.15) follows by combining the two suprema (see Kamihigashi, 2008, Lemma 1). It follows that (4.8) holds for $T = n + 1$. By induction, (4.8) holds for all $T \in \mathbb{N}$. □

Lemma 4.3. *Let $\underline{v}, \overline{v} \in V$ satisfy (2.13)–(2.16). Then $\underline{v}^* \equiv \lim_{T\uparrow\infty} \underline{v}_T \geq v^*$.*[10]

Proof. Note from (2.13)–(2.15), (2.11), and Remark 4.1 that $\{\underline{v}_T\}_{T=1}^{\infty}$ is an increasing sequence in V. Thus for any $x_0 \in X$, we have

$$\begin{aligned}
\underline{v}^*(x_0) &= \sup_{T\in\mathbb{N}} \underline{v}_T(x_0) && (4.16)\\
&= \sup_{T\in\mathbb{N}} \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi(x_0)} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} && (4.17)\\
&= \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi(x_0)} \sup_{T\in\mathbb{N}} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} && (4.18)\\
&\geq \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi^0(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \left\{ \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T) \right\} && (4.19)\\
&\geq \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi^0(x_0)} \left\{ \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \varliminf_{T\uparrow\infty} \beta^T \underline{v}(x_T) \right\} && (4.20)\\
&\geq \sup_{\{x_t\}_{t=1}^{\infty}\in\Pi^0(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) = v^*(x_0), && (4.21)
\end{aligned}$$

where (4.17) uses Lemma 4.2, (4.18) follows by interchanging the two suprema (see Kamihigashi, 2008, Lemma 1), (4.19) holds since $\Pi^0(x_0) \subset \Pi(x_0)$ (recall (2.8)) and $\mathrm{L}_{T\uparrow\infty} a_T \leq \sup_{T\in\mathbb{N}} a_T$ for any sequence $\{a_T\}$ in $[-\infty, \infty)$, (4.20) follows from the properties of $\varliminf$ and $\varlimsup$,[11] and the inequality in (4.21) uses (2.16). It follows that $\underline{v}^* \geq v^*$. □

To complete the proof of Theorem 2.1, suppose that there exist $\underline{v}, \overline{v} \in V$ satisfying (2.13)–(2.17). The order interval $[\underline{v}, \overline{v}]$ is partially ordered by $\leq$

[10] Here $\underline{v}_T = B^T \underline{v}$ for all $T \in \mathbb{N}$ and $\underline{v}^*(x) = \lim_{T\uparrow\infty} \underline{v}_T(x)$ for all $x \in X$.

[11] We have $\varliminf(a_t + b_t) \geq \varliminf a_t + \varliminf b_t$ and $\varlimsup(a_t + b_t) \geq \varlimsup a_t + \varliminf b_t$ for any sequences $\{a_t\}$ and $\{b_t\}$ in $[-\infty, \infty)$ whenever both sides are well-defined (e.g., Michel, 1990, p. 706).

(recall (2.10)). Given any $F \subset [\underline{v}, \overline{v}]$, we have $\sup F \in [\underline{v}, \overline{v}]$ because

$$\forall x \in X, \qquad (\sup F)(x) = \sup\{f(x) : f \in F\} \in [\underline{v}(x), \overline{v}(x)]. \tag{4.22}$$

Since B is a monotone operator, and since $B([\underline{v}, \overline{v}]) \subset [\underline{v}, \overline{v}]$ by (2.13)–(2.15) and (2.11), it follows that B has a fixed point v in $[\underline{v}, \overline{v}]$ by the Knaster-Tarski fixed point theorem (e.g., Aliprantis and Boder, 2006, p. 16). Since $\underline{v} \leq v = Bv$, we have $\underline{v}_n \leq v$ for all $n \in \mathbb{N}$ by Remark 4.1; thus $\underline{v}^* \leq v$.[12] Since $v \leq v^*$ by Lemma 4.1, and since $v^* \leq \underline{v}^*$ by Lemma 4.3, it follows that $v \leq v^* \leq \underline{v}^* \leq v$. Hence $v = \underline{v}^* = v^*$. Therefore v^* is a unique fixed point of B in $[\underline{v}, \overline{v}]$; this establishes (a) and (b). Finally (c) holds since $\underline{v}^* = v^*$.

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[12] See footnote 10 for the definitions of $\underline{v}_n$ and $\underline{v}^*$.

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