Contagion in financial networks : a threat index

Gabrielle Demange*

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Abstract

Financial institutions use the inter-bank market to develop relationships that protect them against liquidity risk. But these inter-bank claims also facilitate the dissemination of shocks. Under the possibility of default, the values on the inter-bank claims must be jointly determined, thereby generating spill-overs across institutions and possibly propagation of bankruptcy. Building on a simple model for the joint determination of the price of the claims as proposed by Eisenberg and Noe, the paper introduces a measure of the threat that a bank imposes on the system. Such measure may be helpful to determine how to inject cash into the banks so as to increase debt reimbursement, or to assess the contributions of the individual institutions to the risk in the system. Although the threat imposed by a bank on the system and its default level both reflect some form of weakness and are affected by the liabilities network, the measures differ. As a result, injecting cash into the banks with the largest default level may not be optimal.

1 Introduction

An intricate web of claims and obligations links the balance sheets of a wide variety of financial institutions, banks, hedge funds, and various intermediaries. Some argue that these interbank claims have played a large role in the dissemination of the financial crisis of 2007-2008. As such, interbank claims are an important concern for both bankers and regulators. There is a general call for addressing their role in the risk of the system, the famous 'systemic' risk. Following the recommendation made by G20, the new framework proposed by the Basel committee (Basel III) plans to identify some 'systemically important financial institutions' upon which higher standards will be required. Indeed regulation so far is typically defined at the unit level, determined by the balance sheet of the bank under consideration. A prominent example is the Value at Risk indicator (VaR), which is based on a statistical assessment of its payoffs independently of what is happening to other banks. Various

[2]Paris School of Economics, EHESS, Paris, France. E-Mail: demange@pse.ens.fr. I thank Jean-Edouard Colliard and Jean-Charles Rochet for helpful comments.

proposals have been made to modify this measure to account for the role of a bank in systemic risk. But the contribution of a bank to systemic risk can be understood in a variety of ways, each one leading possibly to different measures and cost evaluations. Tarashev, Borio, and Tsatsaronis (2010) distinguish between the role of a bank in spreading the losses and amplifying the default of other banks, or in participating in the systemic events. For example CoVar, a measure similar to VaR but conditional on systemic events, proposed by Adrian and Brunnermeier (2008), falls in the second category. Systemic events are defined as those in which a large fraction of simultaneous defaults arise, say due to correlated portfolios.

The purpose of this paper is to propose a measure of the cost imposed by the default of a bank in a very simplified but explicit model of inter-bank liability structure. The measure is derived from an explicit criteria, which is linked to the overall repayments within the banking system and the loss incurred by the non banking sector in case of a bankruptcy. Such kind of measure may be helpful to determine how to inject cash into the banks so as to optimally increase the reimbursement of the debts, or to assess the contributions of the individual institutions to the risk in the system.

Mutual inter-bank liabilities introduce some linkages in default when they occur. Given some assets and liabilities on the non-financial sector, the capacity for a bank to repay its inter-bank liabilities depends on the capacity of its debtors, calling for a joint determination of the repayments. Eisenberg and Noe (2000) propose a 'clearing' mechanism on the proportions of the liabilities repaid by banks -repayment ratios- to solve this loopback. In 'normal' times, liabilities are fully repaid, and the repayment ratios are equal to unity. In case a default occurs and possibly propagates, the clearing mechanism determines the repayment ratio levels in a (almost) unique way. These levels form a kind of equilibrium in which each bank in default reimburses as much as it can given a limited liability constraint. The clearing default ratio (the complement to one of its repayment ratio) reflects the weakness of a bank. Though, it is not the most appropriate indicator of the loss inflicted by its default on the payment system.

The default of a bank has an impact not only on its own repayment but also on those of its creditors which are in difficulty, and by propagation on the payments along a chain of creditors. The chain may cycle, triggering further decrease in payments and possibly the bankruptcy of some banks. The threat index of a bank measures this overall impact, as described more precisely below. Although the threat index and the default ratio of a bank are both measures of its weakness and are affected by the liabilities network, they differ in general, and are in some precise sense dual to each other. The determinants of a bank's default ratio are driven by the ability of its debtors to repay their debts. Instead the determinants of the bank's threat index are driven by the impact its default inflicts on its creditors. The discrepancy between the two indices, default ratio and threat index (partially) depends on the asset-liability structure, in particular on its asymmetry.

The threat index is useful to determine a 'target policy' that injects an amount of cash into the banks so as to improve effective payments within the payment system as much as possible. Cash should be injected into the banks with the largest threat index. As a result, due to the discrepancy between threat indices and default levels, injecting cash into the banks that appear the weakest ones, those with the largest default ratio, may be sub-optimal. Similar insights hold in alternative network models in which individual actions have an external impact on others channelled through a network. In a criminal network for example, the 'key player' to remove, the one whose arrest triggers the largest decrease in global criminal activity, may not be the one the more active (see Ballester, Calvó-Armengol, Zenou 2005).

The model is a very simplified description of a banking system. The result of the activities of a bank excluding the inter-banking relationships is summarized by a single number, called the net worth. The analysis is first conducted in the situation in which the net worth values are non-negative, as considered by Eisenberg and Noe (2000). In such a situation, default can only be due to the inter-bank liabilities. Furthermore, although default is possible, bankruptcy, in which a bank fully defaults on its obligations, never arises thanks to creditors' absolute priority. A threat index for a bank is defined as a measure of the decrease in payments within the banking system following a decrease in its net worth. The model is then extended so as to allow for a negative net worth, that is for a net liability to the non-banking sector. In some situations, bankruptcy is bound to arise: a bank fully defaults on its obligations towards the financial sector and nevertheless ends up with a debt to the non–banking sector. The clearing mechanism and the threat indices are extended so as to cope with bankruptcy. A threat index measures not only the decrease in the payments within the banking system but also the possible losses incurred by the non–banking sector due to the bankruptcy of some banks. The determinants of the threat indices are shown to differ substantially between the two situations with or without bankruptcy.

The literature considers various alternative models to investigate contagion and the impact of cross-liabilities. One alternative is a '$0-1$' model in which either a bank fulfills its obligation or totally fails, where failure is typically determined by a solvency constraint. The contagion risk of a bank is defined by the expected number (possibly weighted by their size) of failure triggered by the single initial failure of this bank. In particular, empirical studies examine the potential for contagion following a single bank's distress through financial interconnections and changes in asset prices by means of simulations. These simulations are calibrated on real payment systems (see e.g. Furfine 2003 on Fedwire) or on interbank networks (e.g. Upper and Worms 2004, Elsinger et al. 2004, Degryse and Nguyen 2004 for Germany, Austria, Belgium respectively). Cifuentes, Ferrucci, and Shin (2005) introduce a further mechanism of contagion through asset sales amplified by regulatory solvency constraints and mark-to-market rules. A difficulty is that data on bilateral exposures is limited, making difficult to assess the impact of the network structure.

Links between banks have two opposing effects on contagion. Increasing the number of links increases the opportunities for spreading liquidity shocks among counter-parties, but also facilitates the channels through which contagion spreads. This trade-off has been examined from different angles. Gai and Kapadia (2008) analyze how the network structure affects the trade-off by using a random graph model in which links are formed randomly according to some distribution. They find that financial systems exhibit a robust-yet-fragile tendency: while greater connectivity reduces the likelihood of widespread default, the impact on the financial system, should problems occur, could be on a significantly larger scale than hitherto. The result however is driven by the assumption that the total amount of assets and liabilities of a bank is kept fixed, irrespective of the network structure. In practice the amount of interbank assets and liabilities are related positively with the number of links. Intuitively, this correlation should hamper the risk-sharing benefit from forming links. Even if the relationship between asset and liability and the number of links is sufficiently strong, one may suspect that dense networks are vulnerable to contagion.

Theoretical works on the microeconomic foundations of the determinants of the structure of banking networks also examine the trade-off between insurance and risk spreading. Most of these works build on Diamond and Dybvig (1983) model in which banks are hit by liquidity shocks due to the needs of their consumers. Banks can use the inter-bank market or cross-deposits to respond to these shocks without costly liquidation. Allen and Gale (2000) provide a framework (limited to four banks) for how the degree of completeness of the inter-bank market may affect contagion risk, and concludes that a complete network is more stable to shocks than an incomplete one. In a model of spatially separated banks, Freixas, Parigi, and Rochet (2000) analyze the impact of credit lines and central bank policy on the stability of the economy subject to (local) consumers withdrawal needs possibly amplified by coordination problems.

The paper is organized as follows. Section 2 presents the model and the clearing mechanism assuming positive values for the banks' net worth, and introduces the threat indices. A target policy, which injects cash in some defaulting banks is analyzed, as well as a solidarity policy, which forces safe banks to increase their repayments beyond their nominal liabilities. Finally, some comparative statics exercises on the impact of liabilities are performed. Section 3 extends the analysis to the situation where bankruptcy is unavoidable by allowing net worth to be negative. Section 4 gathers some proofs.

2 A contagion model with default

There are n financial institutions, called banks for simplicity. Denote $N = \{1, \cdots, n\}$. The assets and liabilities within these banks are distinguished with those outside the sector. The various assets and liabilities on the non-financial sector are summarized by a single value for each bank, z_i for bank i. This value will be called the *net worth*. The inter-bank asset-liabilities are described by a $n \times n$ matrix $\ell = (\ell_{ij})$ where ℓ_{ij} represents the magnitude of i's nominal debt obligation toward

j, ℓ_{ii}=0. The total nominal liabilities of bank i are

$$\ell_i^* = \sum_j \ell_{ij}. \tag{1}$$

We are at an ex-post stage. The net worth represents the accounting values of all operations with the non-banking sector once the payoffs from previous investments are revealed. As a result, the net worth level can very well be negative due for example to portfolios in derivative assets. To simplify the exposition we start with the case considered by Eisenberg and Noe (2000) in which net worth levels are non-negative.[1] In the remainder of this section, the net worth of each bank is assumed to be positive, denoted as $z \gg 0$ where $z = (z_i)$.

Before proceeding, let us make a couple of remarks on the liability structure, sometimes referred to as a network. When dealing with a large number of banks, the pattern of their relationships is quite stable and specific, with some banks having regular and large relationship while others have none. In such a situation, the interpretation of financial interlinkages as a network, where banks are nodes and bilateral exposures are the links, is very compelling. The liability structure depends on the situation under consideration, in particular on the maturity of the debts. In payment systems, liabilities are often both ways, reflecting common clienteles for example. Not only both ℓ_{ij} and ℓ_{ji} can be simultaneously positive but they are likely to be both positive or both null. In long term arrangements, some patterns are more directed, such as the ones described in the Austrian banking system, with a partial pyramidal structure.

When a bank is indebted, $\ell_i^* > 0$, its relative liability structure describes the proportions of the loans to i, defined by

$$\pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}. \tag{2}$$

In what follows we state results as if each bank is indebted so as to facilitate the presentation. As will be clear, statements go through without assuming each bank to be indebted.

Notation. $\Pi = (\pi_{ij})$ denotes the relative liability matrix. An important matrix in the sequel is $dg(\ell^*) - \ell$ where $dg(\ell^*)$ is the diagonal $n \times n$ matrix which has ℓ_i^* on the diagonal. Denoting by $\mathbb{I}$ the $n \times n$ identity matrix, we have

$$dg(\ell^*) - \ell = dg(\ell^*)(\mathbb{I} - \Pi). \tag{3}$$

$\mathbb{1}$ denotes a vector of 1s. Given a vector say θ and S a subset of indices, θ_S denotes the vector obtained from θ by keeping the rows indexed by S. Similarly, $A_{S\times T}$ denotes the matrix obtained from a matrix A by keeping the rows indexed by i in S and the columns indexed by j in T. A^t denotes the transpose of A.

[1] They refer to cash-flow instead of net worth. We use the alternative interpretation as proposed in Shin (2008) in which the described quantities are the market values of the balance sheets in terms of the numeraire at the current date. In particular, z_i represents the market value of the bank's activities with the non-banking sector.

2.1 Clearing repayment ratio vectors

Given the positive net worth levels and the mutual liabilities, as described by z and ℓ, we consider the capacity for the banks to repay their debts. A bank in difficulty only pays a fraction of its liability. Let θ_i denote this fraction for bank i; θ_i is between 0 and 1 and called *repayment ratio*. The *default ratio* is then defined as $1 - \theta_i$. A ratio vector $\theta = (\theta_i)$ specifies the repayment ratio of each bank.

Due to the mutual liabilities, the capacity for a bank to repay its debts depends on the capacity of its debtors, calling for some consistency requirements between the repayment ratios of the various banks. The clearing mechanism introduced by Eisenberg and Noe (2000) specifies two natural requirements. The conditions bear on the net equity of the banks. Net equity refers to the accounting residual value that results from the realized operations of the bank with all other parties, outside or inside the banking sector. Formally, given repayment vector θ, i's *net equity* $e_i(\theta)$ is defined by

$$e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^*, \tag{4}$$

which is the sum of the net worth and the net payments within the banking system to i, composed of the payments received by i, $\sum_j \theta_j \ell_{ji}$, less those made by i, $\theta_i \ell_i^*$.

A vector $\theta = (\theta_i)$ in $[0,1]^n$ is said to be a *clearing repayment ratio vector* if it satisfies the following two conditions for each i

(a) limited liability: net equity is non-negative

$$e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* \geq 0. \tag{5}$$

(b) absolute priority of creditors

$$\theta_i < 1 \Rightarrow e_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* = 0. \tag{6}$$

Limited liability states that stockholders are not required to repay more than the total net asset value in the bank (i.e. more than the net worth plus the repayments from other banks). Absolute priority requires debts to be repaid in full unless net equity is null. As a result, no bank fully defaults at a clearing ratio vector thanks to the positivity of net worth: θ_i is surely positive because otherwise its equity would be positive, in contradiction with (6).

The θ_i here is called repayment ratio in reference to a clearing system but it can also be interpreted as the current price of one unit of debt of bank i.

The existence and behavior of a clearing repayment ratio vector has been proved by Eisenberg and Noe (2000). Properties 1 and 2 directly follow from their analysis so I only sketch the proofs (also some parts of the properties are extended and proved in the next section in the more general situation where the net worth can be negative).

Property 1. Clearing ratios vectors and total payments within the system *Under positive net worth levels, there is a unique clearing vector. It solves the following program*

$$\mathcal{P} \quad : \quad \max_{\theta} \sum_i \ell_i^* \theta_i$$
$$0 \leq \theta_i \leq 1 \text{ each } i \tag{7}$$
$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \leq z_i \text{ each } i. \tag{8}$$

Property 1 relies on the fact that repayment ratios are complements: the higher the repayment ratios of other banks, the more a given bank is able to repay. Complementarities imply that there is a greatest and a least clearing repayment ratio vector. It turns out that these clearing ratios coincide under the positivity of the net worth values no matter what the liability structure, ensuring the uniqueness of the clearing ratio vector.[2]

Complementarities also imply that a clearing repayment ratio vector can be found by maximizing the total payments within the banking system[3] under some constraints. The constraints are dictated by debt contracts (7), according to which a bank can only reimburse (the condition θ_i non-negative), and only up to its nominal liabilities (the condition θ_i less than 1), and limited liability (8), which requires equity to be nonnegative. Observe that absolute priority is not a constraint: it is satisfied at the optimal solution because the objective is strictly increasing in each repayment ratio. In short, the clearing repayment ratio vector maximizes the total payments within the banking system under the limited liability condition. This property will be used to define the threat indices.

Property 2. Aggregation formula, null equity set *The net equity values aggregated over a subset T of N do not depend on the liabilities within T:*

$$\sum_{i \in T} e_i(\theta) = \sum_{i \in T} z_i + \sum_{i \in T, j \notin T} \theta_j \ell_{ji} - \sum_{i \in T, j \notin T} \theta_i \ell_{ij}. \tag{9}$$

In particular, $\sum_{i \in N} e_i(\theta) = \sum_{i \in N} z_i$. Hence, at a clearing ratio vector, surely one bank fully repays its debt, $\theta_i = 1$.

A subset T is called a null equity set at ratio vector θ if the net equity of each bank in T is null : $e_i(\theta) = 0$ each i in T. Given $z \gg 0$, if T is a null equity set at some θ, then T must have liabilities outside T.

The aggregation formula (9) is trivially obtained since net equities are linear in θ and the payments within T cancel out. Hence the aggregate net equity of the banks in T is equal to their aggregate

[2]Null values for net worth introduce the possibility of multiple clearing ratios. However uniqueness is guaranteed provided the liabilities structure is sufficiently complete. These difficulties will be handled with in the next section, which deals with the more general case of possibly negative values for net worth. So we prefer to keep the presentation simple in this section, but the analysis readily extends to situations with multiple clearing ratios by assuming the clearing mechanism to select the greatest clearing ratio vector.

[3]The objective could be replaced by any function that is increasing in the θ_i.

net worth plus the net payment from banks outside T, i.e., the difference between the payments received from $N-T$ and those made by T to $N-T$. Applied to the whole set N, the aggregate net equities are equal to aggregate net worth, hence strictly positive. As a result the net equity of one bank at least is positive: N cannot be a default set. At a clearing repayment ratio vector, absolute priority requires such a bank to fully repay its debts: $\theta_i = 1$ for some i.

The existence of an outside creditor for a null equity set follows from the aggregation formula (9) on T. The aggregate net equity over T, $\sum_{i\in T} e_i(\theta)$, is null so that

$$\sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} - \sum_{i\in T, j\notin T} \theta_i \ell_{ij} = 0.$$

Net worth is positive, repayments are non-negative, so the third term on the left hand side must be positive: some member of T must have creditor a outside T. The intuition is clear. T has positive assets outside T, so, to be a null equity set, it must have some liabilities. ∎

2.2 Banks status and linear characterization of a clearing ratio vector

A clearing repayment ratio vector is characterized by a system of linear inequalities. To describe the system, it is convenient to introduce the status of the banks.

Banks status Given a clearing repayment ratio vector, the banks are naturally partitioned into safe banks, which fully repay their debts and defaulting banks, which do not. We distinguish the banks at the boundary of the two cases, safe or default. These 'fragile' banks fully repay their debt and end up with a null net equity. Specifically, given a clearing ratio vector θ, let us define S the set of safe banks, which fully repay their debts and end up with strictly positive net equity

$$S = \{i,\, \theta_i = 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* > 0\},$$

F the set of fragile banks, which just break even under full debt repayment,

$$F = \{i,\, \theta_i = 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* = 0\},$$

and D the set of defaulting banks:

$$D = \{i,\, \theta_i < 1 \text{ and } z_i + \textstyle\sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* = 0\}.$$

As stated in Property 2, there is always a safe bank. Typically there is no fragile bank, because a small perturbation of the parameters z or ℓ makes them either safe or defaulting.

Now, given the status of the banks, the clearing repayment ratio vector is of the form $(\theta_D, \mathbb{1}_{N-D})$ because the ratios of safe and fragile banks are equal to 1 by definition. For the defaulting banks, their equity is null so that a clearing repayment ratio vector solves the system of linear inequalities :

$$\theta_i \ell_i^* - \sum_{j\in D} \theta_j \ell_{ji} - \sum_{j\notin D} \ell_{ji} = z_i \text{ each } i \in D \quad (10)$$

$$\ell_i^* - \sum_{j\in D} \theta_j \ell_{ji} - \sum_{j\notin D} \ell_{ji} \leq z_i \text{ each } i \text{ not in } D. \quad (11)$$

(10) is a system of linear equations on D. It can be written in matrix form

$$(dg(\ell^*) - \ell)^t_{D\times D}\theta_D = \hat{z}_D \tag{12}$$

where $\hat{z}_i$ for i in D is defined as the sum of the net worth and the assets on the safe banks: $\hat{z}_i = [z_i + \sum_{j\in S} \ell_{ji}]$. The system has a unique solution, which is positive. This follows from the next lemma and the fact that a default set is a null equity set.[4] Recall that Π is defined as the relative liabilities matrix. The next lemma will be useful in next section.

Lemma 1 *Let $z \gg \mathbf{0}$. Let T be a null equity set at positive ratio vector θ. Then the matrix $(dg(\ell^*) - \ell)_{T\times T}$ and its transpose are invertible and their inverse have all their elements positive. The same properties hold for $(\mathbb{I} - \Pi)_{T\times T}$.*

Recall the relation (3) : $dg(\ell^*) - \ell = dg(\ell^*)(\mathbb{I} - \Pi)$, and similarly for the restrictions of the matrices[5] on $T \times T$. The properties are therefore equivalent for the two matrices $(dg(\ell^*) - \ell)_{T\times T}$, $(\mathbb{I} - \Pi)_{T\times T}$. Let us argue in term of the matrix $(\mathbb{I} - \Pi)_{T\times T}$. For a complete liability structure, the property straightforwardly follows from well known results on positive matrices because whatever strict subset T of N, the total of each row is strictly smaller than 1. In particular, the inverse of $(\mathbb{I} - \Pi)_{T\times T}$ is given by a converging infinite sum:

$$(\mathbb{I} - \Pi)^{-1}_{T\times T} = \mathbb{I}_{T\times T} + \Pi_{T\times T} + \Pi^{(2)}_{T\times T} + ... + \Pi^{(p)}_{T\times T} + ... \tag{13}$$

For an incomplete liability structure, the result extends to the subsets that are null equity sets, as stated in Lemma 1. In the Proofs Section, using the fact that each subset of a null equity set is itself a null equity set, hence has an outside creditor[6] we show that for some p the p-power $\Pi^{(p)}_{T\times T}$ has all its rows totals smaller than 1. This implies that the infinite sum on the right hand side of (13) converges and that its positive limit is the inverse to $(\mathbb{I} - \Pi)_{T\times T}$. Expression (13) is used in the next section.

From Lemma 1, the matrix $(dg(\ell^*) - \ell)^t_{D\times D}$ is invertible, and this determines in a unique way the clearing vector associated to the default set set D. Without further information on net worth values, any clearing price ratio and banks' status are possible under the restriction that the set of

[4]The fact that (12) has a unique solution is not enough to prove the uniqueness of a clearing vector: The status of the banks could possibly differ at different clearing vectors. Uniqueness is obtained by relating the ratios of the defaulting banks to those of the safe banks, as performed in next section.

[5]The Lemma extends to the situation in which some banks are not indebted because each bank in a null equity set is surely indebted. In particular the relative liabilities for a bank in T are well defined.

[6]Without an outside creditor for each subset, invertibility may fail, as in the following example

$$\Pi_{T\times T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3 has only outside creditors, thus $T = \{1, 2, 3\}$ has an outside creditor but $\{1, 2\}$ has not. Since the vector $x = (1, 1, 0)$ satisfies $x = x\Pi_{T\times T}$ the matrix $(\mathbb{I} - \Pi)_{T\times T}$ is not invertible.

defaulting banks D has an outside creditor. To see this, observe that given repayment ratios for banks in D, θ_D, the net worth can be chosen so that $(\theta_D, \mathbb{1}_{N-D})$ is a clearing repayment ratio: define the net worth for banks in D so that (10) is satisfied and the net worth for those in $N - D$ to be large enough so that these banks are indeed safe, i.e. (11) is met.

2.3 The threat indices

The repayment ratio of a bank reflects its safety. But, from the system payment perspective, the repayment ratio may not be the best indicator of the loss imposed by a bank. Specifically, let us assume that the objective of the payment system is defined by the total effective payments. Loss and benefit are thus in terms of these payments. The bank's threat index measures the loss induced by a decrease in the bank's net worth at the margin or, taking the opposite side, the index measures the benefit of rescuing the bank say in the form of additional capital. Of course, the threat index is null, when the bank is safe since a marginal decrease in its net worth has no impact on its repayment ratio, hence on the overall payments in the system.

To define these indices, we use the property that the (greatest) clearing repayment ratio vector maximizes the total payments within the banking system under the limited liability condition, as stated in Property 1. Specifically, the clearing repayment ratio vector is the solution to the following program

$$\mathcal{P} \quad : \quad \max_{\theta} \sum_i \theta_i \ell_i^*$$

$$0 \le \theta_i \le 1 \text{ each } i \quad (7)$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \le z_i \text{ each } i. \quad (8)$$

The objective of the program is the total payments within the banking system, constraints (7) are those on debt contracts, and (8) require net equity to be nonnegative, the limited liability condition. Absolute priority is not a constraint, but it is satisfied at the optimal solution because the repayment ratios are complements, as explained in the previous section.

We will be interested in the value of the program as the parameters, the net worth levels z_i or the liabilities ℓ_{ij}, vary. To make this dependence clear, $V(z, \ell)$ will denote the value of the program $\mathcal{P}$ associated to (z, ℓ), or simply $V(z)$ when only net worth levels are varying.

The formulation in terms of a linear optimization program allows us to derive in a very simple way some indices pertaining to the impact of the banks' net worth on the system. The impact of a (marginal) increase in the net worth of a bank is assessed by considering the multiplier associated to the equity constraint (8): At points of differentiability of V, the envelope theorem applies:

$$\frac{\partial V}{\partial z_i} = \mu_i$$

where μ_i denotes the multiplier associated with constraint (8). Thus, the impact of a marginal increase of one unit in i's net worth is well defined as it increases the value V of the payments within the system by μ_i units. Alternatively the impact of a marginal decrease of one unit in i's net worth is to decrease the payments by μ_i units. This is why we call the multiplier μ_i a 'threat' index.

The next proposition states that the function V is indeed differentiable at 'most' points and provides an expression for the multipliers. These properties are especially useful for deriving an 'optimal' targeting policy. The goal of such a policy is to inject a given amount of cash so as to increase the payments within the banking system as much as possible, as investigated in Section 2.5.

Proposition 1 *The value function V of program $\mathcal{P}$ is concave in (z, ℓ). It is differentiable with respect to z at each point (z, ℓ) for which there is no fragile bank. In that case the derivative vector $(\frac{\partial V}{\partial z_i})$ is the unique μ that is null on S and solves on D*

$$\ell_i^* \mu_i - \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*, \text{ for each } i \text{ in } D. \tag{14}$$

The multiplier μ_i is called i's threat index.

The proposition is proved by applying well known results on linear programming and duality. Since there are fragile banks only for a degenerate set of values for (z, ℓ), the multipliers are unique and the function V is differentiable almost everywhere. Standard complementarity relationships between primal and dual variables apply to the repayment ratios (the solutions to the primal) and the threat indices (the solutions to the dual). This explains why the threat index of a safe bank is null. Otherwise, for a bank with a repayment ratio θ_i strictly smaller than 1, the multiplier is non-null and even larger than 1. Similar results obtain for the differentiability with respect to liabilities, as investigated in Section 2.6.

A comparison of the determinants of the repayment ratios and the threat indices of the defaulting banks makes clear that they may differ and be not necessarily aligned. They satisfy respectively (10) and (14):

$$\theta_i \ell_i^* - \sum_{j \in D} \theta_j \ell_{ji} = z_i + \sum_{j \notin D} \ell_{ji}, \text{ and } \ell_i^* \mu_i - \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*, \text{ for each } i \text{ in } D.$$

Whereas the distress of a bank as measured by its repayment ratio depends on the distress of its debtors (through the ℓ_{ji}), the threat the bank imposes on the payment system depends on the threat of its creditors (through the ℓ_{ij}). Thus the impact of the liabilities structure differs except under strong symmetry of the liabilities/loans. Second, the repayment ratios are affected by the precise values taken by the net worth whereas the indices depend on these values only through the status of the banks.

Equations (14) write in matrix form $(dg(\ell^*) - \ell)_{D \times D} \mu = \ell_D^*$. This system is invertible because a default set is a null equity set (by Lemma 1). An implication is that the indices are constant over net worth levels that lead to the same set of defaulting banks D. Hence the payment function V is piece-wise linear in z, with kinks at points where there are some fragile banks.

We now provide an interpretation for the expression of the indices.

Interpreting the threat indices For the interpretation, let us work with the formulation in terms of the relative liability structure. This facilitates the comparison with some 'centrality' indices introduced in the network literature such as the Katz/Bonacich index (1953) and (1987). Dividing equation (14) by ℓ_i^*, the threat indices among the defaulting banks satisfy:

$$\mu_i = 1 + \sum_{j \in D} \pi_{ij}\mu_j, \quad \text{for each } i \text{ in } D \text{ or in matrix form } (\mathbb{I} - \Pi -_{D\times D} \mu_D = \mathbb{1}_D. \tag{15}$$

Only the relative liabilities *within* the set D determines the indices. The threat index of a distressed bank i is equal to 1 plus the weighted sum of the threat indices of its creditors weighted by the amount of the obligations of i to them. Such a relationship has a flavor of a Katz/Bonacich index within the network restricted to the defaulting banks.[7]

To interpret the indices, we consider the impact of a modification in the net worth of a bank. An increase in the net worth can be interpreted as an injection of cash. The analysis allows us to understand the boundary case where there are fragile banks.

We first assume that there is no fragile bank, so that V is differentiable at the standing point (z, ℓ) with a derivative with respect to z_i given by i's threat index $\frac{\partial V}{\partial z_i} = \mu_i$. Thus the marginal impact marginal on the total payments V of a marginal increase in the net worth δ units in z_i is given by

$$\frac{\partial V}{\partial z_i}\delta = \mu_i \delta.$$

A safe bank, which already fulfills its obligations, keeps the additional cash: there is no impact on payments ($\mu_i = 0$). A defaulting bank instead uses the additional cash (at least partially) for reimbursing its debts. Its creditors in default in turn use (part of) the additional cash to repay their debts, and so on. The initial additional cash thus triggers a sequence of additional reimbursements along the creditors which are themselves in default. For δ small enough, each amount that is received by a defaulting bank is entirely used to repay its debts. This leads to the following computation.

[7]The threat index is exactly related to the Katz/Bonacich index in some specific cases. Let G be the incidence matrix of the liabilities network, which has 1 if ℓ_{ij} is positive and 0 otherwise. Assume a liability structure in which all the positive ℓ_{ij} are equal to an identical level, and furthermore let each bank have the same number of creditors, say p, hence the same total liabilities. The matrix Π is proportional to the incidence matrix of the liabilities network : $\Pi = \frac{1}{p}G$. Then given D

$$\mu_D = \left(\mathbb{I} - \frac{1}{p}G_{D\times D}\right)^{-1}\mu_D.$$

Up to the constant 1, the index coincides with the Bonacich index for the 'attenuation' parameter $1/p$ in the liabilities network within D. The attenuation parameter defines the importance of indirect links. Contrary to the sociology framework however, it is here determined, defined by the reciprocal of the number of total creditors of a bank: the more creditors, the less the influence of indirect creditors.

The additional cash received by the defaulting bank i, which is entirely used for reimbursement, generates a first increase of δ units in the payments. This explains why μ_i is larger than 1. Each i's creditor j receives the share π_{ij} of δ, entirely used for reimbursement by those in default, thereby generating a first 'indirect' additional payment in the system equal to $(\sum_{j\in D}\pi_{ij})\delta$. The sum term is the $i-$th element of $\Pi_{D\times D}\mathbb{1}_D$. By the same argument, each of the $\pi_{ij}\delta$ units received by the defaulting i's creditor j generates $\sum_{k\in D}\pi_{jk}$ extra units of payments. So, summing over all defaulting creditors of i, the 'second' indirect additional increase equals $\sum_{j\in D}(\sum_{k\in D}\pi_{jk})\pi_{ij}\delta$, or exchanging the order of summation, $\sum_{k\in D}[\sum_{j\in D}\pi_{ij}\pi_{jk}]\delta$. Since the element in square brackets is the (i,k) element of the matrix $\Pi^{(2)}_{D\times D} = \Pi_{D\times D}\times\Pi_{D\times D}$, the 'second' indirect impact is δ times the $i-$th component of $\Pi^{(2)}_{D\times D}\mathbb{1}_D$. Iterating, the additional indirect impact along a path of p banks, each one defaulting and debtor to its successor, is δ times the $i-$th component of $\Pi^{(p)}_{D\times D}\mathbb{1}_D$. Summing all indirect impacts gives the value of $\mu_i\delta$ as the i-th component of the infinite sum $\delta\sum_{p\geq 0}\Pi^{(p)}_{D\times D} + ...)\mathbb{1}_D$. Considering all defaulting banks, we thus obtain

$$\mu_D = (\mathbb{I}_{D\times D} + \Pi_{D\times D} + \Pi^{(2)}_{D\times D} + ... + \Pi^{(p)}_{D\times D} + ...)\mathbb{1}_D$$

or $(\mathbb{I}-\Pi)_{D\times D}\mu_D = \mathbb{1}_D$, the equation (15).

The above argument explains why the indices are determined by the liability structure *within* the set D only. As long as the banks status do not change, which holds true for small enough δ, a cash injection triggers automatic increases in the payments entirely determined by the liability shares of the recipient defaulting banks, and do not depend upon other banks' liabilities or the net worth levels.

The above argument extends to the situation where there are some fragile banks, by distinguishing an increase in the net worth of a fragile bank from a decrease. Increasing the net worth in a fragile bank has no impact on the payments because it already repays its debt. If instead the net worth is decreased, its ratio is necessarily lowered and the same argument as above allows us to compute the impact on the payments. (This also works for the other fragile banks (if any) because each one can only receive less reimbursements hence behaves as a defaulting bank.) This explains why the value function V is not differentiable. Furthermore, for fragile bank i

$$\frac{\partial V}{\partial z_i^+} = 0, \frac{\partial V}{\partial z_i^-} > 1,$$

in which the expressions denote respectively the right and left derivatives of V. V is not differentiable with respect to the net worth of any defaulting bank, even non-fragile, either. The reason is that an increase in the net worth of a defaulting bank makes the fragile banks become safe due to additional repayment (for those which are in the chain of creditors) while a decrease makes these fragile banks defaulting, generating a further decrease in payments. The extremal values for the threat indices are obtained by applying expression (14) as if the default set was either $D\cup F$ (for the maximal values) or D (for the minimal ones).

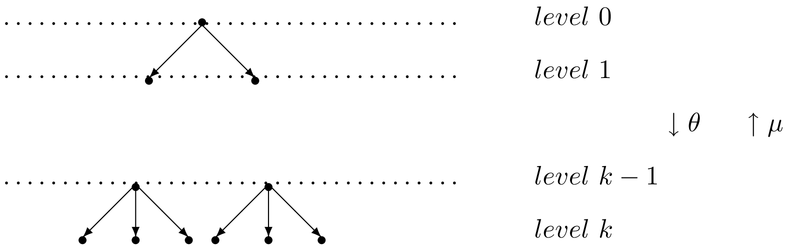

Figure 1: Pyramidal network

2.4 Examples

Pyramidal network In a pyramidal network, liabilities go in the same direction without loop, from the top or from the bottom. Consider first the situation where these liabilities are directed from the top, as represented in Figure 1, in which an arrow from i to j represents a positive liability of i towards j. This describes a situation in which chains of intermediaries collect funds for the bank at the top. Each bank collects funds from the institutions directly below it or from outside the banking system for the banks if it has no 'subordinate'. Each bank except the bank at the top lends these funds to its unique superior.

Let the level of a bank be defined as the number of links between this bank and the top.

The clearing ratio vector is unique, easily computed recursively starting from the top. The top bank, say bank 1, has no claims on other banks so its repayment ratio is determined by its net worth as the minimum of 1 and z_1/ℓ_1^*. The repayment ratio of the banks at level 1 can now be computed since they receive payments only from bank 1 and these are known. The computation proceeds: at step $s + 1$, the repayment ratios of the banks at level s are determined since the repayments of all their debtors are known. The clearing vector is unique, obtained after k steps, and the status of the banks are determined.

Knowing the status of the banks, the threat index is computed recursively in a similar way, starting from the bottom instead of the top. The index may not be unique however if there are fragile banks, which arises in the computation above when net equity is null for a ratio equal to 1.

Assume first no fragile banks. The threat index of the bank at level k is null since it has no creditor. At level $k - 1$, the index is either 0 (the bank is safe) or equal to 1 (the bank is in default). The indices are then computed recursively using expression (14): $\ell_i^* \mu_i - \sum_{j \in D} \ell_{ij} \mu_j = \ell_i^*$. The threat index of a bank only depends on those of its creditors, which have been determined at the previous step. If there are fragile banks, the computation can be performed by either considering these fragile banks as defaulting banks, so as to obtain the maximal threat indices, or by considering them as safe banks, so as to obtain the minimal indices.

Similar recursive computations can be performed in the reverse situation in which all the liabilities point toward the top. In this situation, the top has a null threat index. This situation can be qualified as less prone to contagion than the previous one because a single default cannot touch all banks.

Log-fitting model There is a lack of data on bilateral inter-bank exposures. The log-fitting method is most often used to estimate the missing data given the available information on some total exposures. The method is justified if the missing data are independent conditional on the current information. Explicit formula are obtained in the following simple situation.

Let the total amount of liabilities l_i^* and loans $l_{*i} = \sum_{j \in N} l_{ji}$ be known for each bank. Thus the sums in each row and each column of matrix ℓ are known. Without any specific information on bilateral exposures, the estimated proportions of i's liabilities are independent of i, thus equal to the overall proportions of the loans. The bilateral exposures are estimated by

$$\pi_{ij} = \frac{\ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \frac{\ell_i^* \ell_{*j}}{\sum_{j \in N} \ell_{*j}}, \ell_{ij} = \ell_i^* \pi_{ij}.$$

Since the values π_{ij} are independent of i, the expression (15) for the threat indices of defaulting banks, $\mu_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j$, implies that index μ_i is independent of i on D. Straightforward computation gives

$$\mu_i = 1 + \frac{\sum_{j \in D} \ell_{*j}}{\sum_{j \in S} \ell_{*j}} \text{ for each } i \in D. \tag{16}$$

According to this expression, the log-fitting model does not discriminate the banks within each status class. The common value for the threat index of defaulting banks is up to 1 equal to the loans distributed by the defaulting banks relative to those distributed by the safe banks.

2.5 Targeting policy

A targeting policy aims to inject a given amount of cash so as to increase the effective payments V within the banking system as much as possible. We fist consider a policy in which cash is injected into the banking system.

Cash injection The targeting policy is easily derived from Proposition 1.

Proposition 2 *A marginal injection of cash that optimally increases V is targeted towards the defaulting banks with the largest value for the threat index μ_i. These targeted banks are kept identical as long as the status of the banks remain unchanged.*

Proof The first assertion straightforwardly follows from Proposition 1. The function V is concave and $\frac{\partial V}{\partial z} = \mu$ at each z for which no bank is fragile. Since the marginal impact of cash injection towards i increases the value V of the payments within the system by μ_i units, cash should be allocated to the i with largest μ_i.

We know that μ is constant given D the set of defaulting bank, assuming no fragile banks. Thus, as long as cash injection does not modify the banks' status, that is does not transform a defaulting bank into a fragile one, the targeting banks can be kept unchanged. ■

The policy is especially simple since there is no need to modify the targets while injecting cash as long as the set of defaulting banks remains unchanged. The targets may not be the banks with the largest default ratios. More generally, the orders given by the default ratios and the threat indices may differ. Order the banks by increasing values of their default ratios $1 - \theta_i$ so that the higher the rank, the less the repayment ratio, or order them by increasing values of the threat indices μ_i, so that the higher the rank, the more profitable it is to inject cash. These orders do not necessarily coincide (but the safe banks are all at the bottom of both rankings since both their default ratios and their threat indices are null, the minimum value).

Similar insights hold in alternative models in which externalities across players are channelled though the links in a network. The key player is the player who, once removed, leads to the optimal change according to some objective which aggregates individual's behaviors. In criminal activity for example, the key player is the one who leads to the optimal change in aggregate criminal activity. Because of the externalities among players, the key player might not be the one who performs the more intense criminal activity.

The next proposition states the sub-modularity of the total payments V. We use the standard notation: z_{-i} denotes the net worth levels for banks other than i; given i's net worth z_i, (z_i, z_{-i}) denotes the net worth levels for all banks; as ℓ is fixed, the argument ℓ is omitted.

Proposition 3 *The value function V is sub-modular in the net worth levels:*

$$V(z_i', z_{-i}') - V(z_i', z_{-i}) \leq V(z_i, z_{-i}') - V(z_i, z_{-i})$$

$$z_i' \geq z_i, z_{-i}' \geq z_{-i}.$$

According to sub-modularity, the benefit of injecting cash in a given bank is larger the less net worth in each other bank, that is the more fragile the system is.

< *Comment* >

Solidarity policy Alternative policies based on different tools than cash injection can be contemplated. One tool is to force banks to pay more than their liabilities, which amounts to increase the upper-bound of 1 on the repayment ratio. Such a tool is effective only on the safe banks, which are the only ones to be able to increase their payment. Hence the targeted banks differ from those under capital injection. The associated policies can be qualified as 'solidarity policies', since they involve only transfers among banks. An optimal solidarity policy depends on the precise constraints one wants to impose to these transfers. Whatever these constraints are, one needs to assess the impact on the payments V of increasing the upper-bound on the repayment ratio. This impact is assessed by the multiplier associated to the constraint $\theta_i \leq 1$, which is the analogue of the threat multiplier. To facilitate comparison with the threat index and work with the same unit, we write the constraint $\theta_i \leq 1$ as $\theta_i \ell_i^* \leq \ell_i^*$. We call the associated multiplier λ_i the *solidarity index.* For the same reasons as for the threat indices, solidarity indices are uniquely defined when there are no fragile banks, so we assume F to be empty. Increasing the upper-bound on the ratio of a defaulting bank has no effect: λ is null on D. For the safe banks, assuming no fragile banks, the values of λ on S are given by

$$\ell_i^* \lambda_i = \sum_{j \in D} \ell_{ij} \mu_j + \ell_i^*, i \text{ in } S \text{ or } \lambda_i = 1 + \sum_{j \in D} \pi_{ij} \mu_j. \tag{17}$$

The solidarity index of a safe bank is easy to interpret. Increasing the upper-bound on the ratio of a safe bank say by one percent has a direct effect of increasing the payment by one percent of ℓ_i^*, and an indirect effect on the banks that receive these payments. This indirect effect is similar to an increase in the net worth of each creditor in the proportion given by the relative liabilities. This explains expression (17) (accounting for the fact that the threat indices μ_j are null outside D).

Remark The threat and solidarity indices, which are the multipliers of the constraints of the program $\mathcal{P}$, can be given the standard interpretation of shadow prices of resources. Here the resources of the system is given by the net worth of the banks and the liabilities. Let the system be able to buy one unit of extra net worth for bank i at price μ_i and increase i's liabilities by one unit at price λ_i. The objective of the dual is to find prices for the resources that minimize the overall value of the resources subject to the constraint that increasing the ratio of a bank is not beneficial (from the view point of the system). Here increasing i's ratio costs $(\lambda_i + \mu_i)\ell_i^*$ per 'unit' with a benefit equal to 1 (direct benefit) plus $\sum_j \ell_{ji} \mu_j$ (indirect benefit). As shown in the proof section this gives the dual of $\mathcal{P}$:

$$\mathcal{D} \quad : \min_{(\lambda,\mu) \geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^*$$

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij} \mu_j \geq 0 \text{ each } i.$$

The expressions for μ and λ follow by oberving that either λ_i or μ_i is null depending on the bank's status (except for fragile banks).

2.6 Comparative statics in liabilities

There is a concern about the impact of large cross-liabilities on the stability of the system. This section analyzes this impact on the effective payments V.

Consider first an increase in the liability between two banks, ℓ_{ij} of i to j. The increase in the effective payments has to be compared with the increase in nominal liabilities. The impact on the creditor is akin to an additional unit of net worth, the amount of which depends on how much the debtor can pay. Thus the status of both banks matter to determine the impact on the payments.

Indeed thanks to the envelope theorem, a marginal increase in ℓ_{ij} generates a marginal increase in payments given by

$$\frac{\partial V}{\partial \ell_{ij}} = \theta_i[1 - \mu_i + \mu_j]. \tag{18}$$

The interpretation is as follows. To simplify discussion let the 'unit' of increase be small enough so that its impact is equal to the marginal impact. If both banks are safe, payments increase by 1 unit as expected. To understand other cases, let us distinguish between the status of bank i.

If i is safe but j defaults, the payments increase by $1 + \mu_j$: there is an additional payment of 1 unit by i to j and this unit has the same impact on defaulting j as an additional unit of net worth, hence an additional increase by μ_j.

If i defaults, the impact here is more subtle. As i already exhausts its repayment capacity, an additional liability has no impact on its overall repayment, but there is a change in the composition of the reimbursements. Specifically the impact can be decomposed into two parts, a direct effect and a composition effect. First, i sends an additional amount θ_i to bank j, and this has a direct effect equal to $\theta_i(1 + \mu_j)$ arguing as above. Second, as bank i is constrained, it has θ_i less units to repay its original debts, as if its net worth was diminished by θ_i: the composition effect is $-\theta_i\mu_i$. The sum of the two effects is $\theta_i[1 + \mu_j - \mu_i]$, which is (18).

As for the net increase -the increase in the payments diminished of that in nominal liabilities- we see that it is positive for i safe and j in default, negative in the opposite situation, and ambiguous when both banks are defaulting.

Let us consider now an identical increase in the joint liabilities of two banks, which leaves unchanged their net liabilities: both ℓ_{ij} and ℓ_{ji} are increased by an identical amount. The marginal overall net increase in the payments is

$$(\theta_i + \theta_j) + [\mu_j - \mu_i](\theta_i - \theta_j) - 2.$$

When both banks are safe, the net increase is null as expected since the increase in cross payments cancel out. When one bank is safe, net payments never decrease: For i safe, $\theta_i = 1$ and $\mu_i = 0$, the net increase equals $(1 - \theta_j)(\mu_j - 1)$, which is non-negative since each term in the product is. The intuition is that increasing each liability calls for more transfers across banks, which results in safe banks paying more per unit of additional liability than those in default. When both banks are defaulting however, the impact on payments is ambiguous because of the recomposition effect we have identified previously. With similar repayment ratios or similar threat indices, the impact is indeed negative: the direct effects identified above, which increase repayments between the two banks, cancel out and we are left with the negative composition effect, according to which other banks get less.

Finally, let us consider an equal increase in all liabilities, which would follow for example from a softening of regulation constraints. The net increase in the payoffs is, adding up the impact of all pairs and rearranging[8]

$$-(n-1)\sum_i(1-\theta_i)+n\sum_i(1-\theta_i)(\mu_i-\overline{\mu}) \tag{19}$$

where $\overline{\mu}=\frac{1}{n}\sum_i \mu_i$ is the average value of the μ_i.

Let us illustrate in the log-fitting model (section 2.4). Recall that the multipliers of the defaulting banks are all identical. Denote by μ_d the common value and ℓ_{*j} the amount of loans distributed by j. Simple computation[9] yields that the net increase is positive if

$$\mu_d \geq \frac{n-1}{n-d} \text{ or } \frac{\sum_{j\in D}\ell_{*j}}{d} \geq \frac{d-1}{d}\frac{\sum_{j\in S}\ell_{*j}}{n-d} \tag{20}$$

where d and $n-d$ are respectively the number of defaulting and safe banks. The inequality requires the average loan per defaulting bank to be larger than the average loan per safe bank by the factor $\frac{d-1}{d}$. As seen earlier, increasing liabilities has a positive impact on a pair formed with a safe and a defaulting banks and has a negative one on a pair with defaulting banks (because their threat indices are equal). Under (20) the threat is large enough so that the positive impact dominates.

With an identical amount of loans per bank, the ℓ_{*j} are equal across j, the condition (20) is surely met: increasing liabilities is beneficial. Such a situation corresponds to a priori similar institutions, which are engaged into symmetrical inter-bank relationships. Due to shocks in their activities, they may end up in an asymmetrical situation, with some of them defaulting. However, independently of the realized net worth levels, and the subsequent status for the firms, more links are better for net reimbursements. Thus there is a benefit 'ex post', which implies an 'insurance' benefit ex ante, taking the expectation over all values of the net worth.

3 Bankruptcy

So far, we have considered a situation in which banks default but can still reimburse part of their debts. Thanks to the positive value derived from the non-banking sector (the positivity of the z_i), the net equity of each bank can be made positive for some positive repayment ratios. This section considers the possibility for a bank to have a net liability to the non-banking sector, which is represented by a negative net worth. In such a situation, even if a bank fully defaults on its banks' liabilities, it may still leave some debt to the outside sector. We call such a situation bankruptcy.

Bankruptcy is bound to arise for a low enough realization of the net worth. To see this, consider an extreme case in which a bank cannot repay its liability to the outside banking sector even if other banks fully repay their liabilities, the more favorable scenario. This arises for bank i if $z_i+\sum_j \ell_{ji}$

[8]For a given i, the sum of the net increase $\theta_i[1-\mu_i+\mu_j]-1$ over j distinct of i is $(n-1)(\theta_i-1)-\theta_i[(n-1)\mu_i-\sum_{j\neq i}\mu_j]$ which can be written as $(n-1)(\theta_i-1)+n\theta_i[\mu_i-\overline{\mu}]$. Summing over i, we obtain the net increase $-(n-1)\sum_i(1-\theta_i)-n\sum_i\theta_i(\mu_i-\overline{\mu})$ and it suffices to add the null term $n\sum_i(\mu_i-\overline{\mu})$ to obtain (19).

[9]From (16) we have $\mu_d=1+\frac{\sum_{j\in D}\ell_{*j}}{\sum_{j\in S}\ell_{*j}}$. Factoring out by $\sum_i(1-\theta_i)$, expression (19) is positive if $-(n-1)+n(\mu_d-\overline{\mu})$ is positive. The average value $\overline{\mu}$ is $\frac{d}{n}\mu_d$ so we obtain $\mu_d\geq\frac{n-1}{n-d}$, or $\frac{\sum_{j\in D}\ell_{*j}}{\sum_{j\in S}\ell_{*j}}\geq\frac{d-1}{n-d}$, and finally (20).

is negative. Whatever repayment ratio, bank i's cannot fulfill its obligations and the bank must go bankrupt.

3.1 Clearing repayment ratios

We first extend the notion of clearing ratio vector to allow for the possibility of bankruptcy. A negative net worth corresponds to a liability towards creditors outside the financial system. So we have two types of creditors, outside or inside the banking system. The bankruptcy condition that we introduce now specifies a hierarchy among the creditors, specifically the priority of the creditors outside the banking sector over those inside it.

It is convenient to introduce the *net asset value* of a bank. The net asset value is defined as the accounting sum of the net worth and the loans repayments by other banks.[10] Thus, the net equity is given by the asset value minus the repayments made by the bank. Formally, denoting bank i's net asset at θ by $a_i(\theta)$, we have

$$a_i(\theta) = z_i + \sum_j \theta_j \ell_{ji} \text{ and } e_i(\theta) = a_i(\theta) - \theta_i \ell_i^*. \tag{21}$$

The net asset value can be negative, but only if net worth z_i is negative. When the net asset value is negative, the bank is declared bankrupt. In that case, the *bankruptcy* condition introduced in the next definition requires that the bank repays nothing to its creditors within the financial sector. Thus, as explained below, under the possibility of bankruptcy, the two rules of absolute priority of creditors and bankruptcy specify a hierarchy among the creditors. First a bank defaults on its liabilities to the financial system. Second, only if full default on its interbank liabilities is not sufficient for the bank to fulfill its obligations to the outside sector, the bank is declared bankrupt and defaults on its outside creditors.

The two rules of limited liability and absolute priority of creditors are rewritten slightly because a negative equity and a null ratio are no longer excluded. As explained after the definition, they are equivalent to the conditions defined in the previous section when net worth is positive. Observe that the net asset value of a bank depends only on the ratios of other banks. The conditions that must be satisfied by a bank depend on this net asset value, especially whether it is positive or not.

Definition 1 *A vector $\theta = (\theta_i)$ in $[0, 1]^n$ is said to be a clearing repayment ratio vector if it satisfies for each i*

(a) limited liability

$$a_i(\theta) > 0 \textit{ implies } e_i(\theta) \geq 0 \tag{22}$$

[10]There is an asymmetry in our treatment of assets and liabilities within and outside the banking sector. First we do not consider the possibility of partial default to the outside sector. Second we work on net worth, allowing compensation between assets and liabilities outside the financial sector.

(b) absolute priority of creditors over stockholders

$$a_i(\theta) > 0 \textit{ implies } \theta_i > 0 \textit{ and } e_i(\theta) = 0 \textit{ if } \theta_i < 1 \tag{23}$$

(c) bankruptcy (absolute priority of outside creditors)

$$a_i(\theta) \leq 0 \textit{ implies } \theta_i = 0 \tag{24}$$

Limited liability and absolute priority apply to a bank with positive net asset value, as is surely the case if its net worth is positive, whereas the bankruptcy condition applies to a bank with a non positive net asset value.

Limited liability requires that the payments to the banks' creditors never exceed the (positive) net asset value. When net worth is positive, the asset value is surely positive, and (22) is equivalent to the non-negativity of net equity. The absolute priority of banks creditors over stockholders is interpreted as previously as requiring a bank to repay its liabilities as much as it can: bank i defaults towards the banking sector only if its asset value is less than its liabilities. The *bankruptcy* condition requires that a bank repays nothing to other banks if its net asset value is negative.

The two rules of absolute priority of creditors and bankruptcy distinguish situations with positive asset value and possible default within the banking sector from those with negative asset value and default towards both the banking and the outside sectors. They specify a hierarchy among the creditors. To make this point clear, consider a bank that has a net liability towards the non-financial sector, i.e., z_i is negative. Two cases arise according to the sign of the net asset value.

When the net asset value is positive thanks to the repayment of its loans to the financial sector, $\sum_j \theta_j \ell_{ji} \geq -z_i$, the liabilities to the outside sector are repaid. The remainder, which amounts to the net asset value, is used for reimbursing the bank's liabilities to other banks, and the left-over (possibly null) is $e_i(\theta)$, the net equity for the stockholders.

When the net asset value is non-positive, $\sum_j \theta_j \ell_{ji} \leq -z_i$, bankruptcy arises; bank's creditors within the financial system get nothing and those outside the financial system seize the repayments $\sum_j \theta_j \ell_{ji}$, which is less than their liabilities $-z_i$. Hence the outside creditors incur a loss that amounts to $-z_i - \sum_j \theta_j \ell_{ji}$, the opposite of net asset value (assuming no bankruptcy cost). Of course this loss may be passed on partially to some others institutions -insurance scheme, taxpayers- if the bank is bailed out or to stockholders in case of recapitalization. The creditors in the financial sector receive nothing.

Observe that at a clearing ratio vector, the net equity of a bank can be negative only if its asset value is negative, by the limited liability condition. In that case the bankruptcy condition requires the repayment to be null: net equity and net asset values are equal, and both negative.

Existence, complementarity

Proposition 4 *There is a greatest clearing repayment ratio. The values of net equities are the same at the clearing ratio vectors (if there are several). If aggregate net worth* $\sum_i z_i$ *(a) is positive then one bank at least fully repays its debts,* $\theta_i = 1$ *for some* i *(b) is negative then one bank at least is bankrupt,* $\theta_i = 0$ *for some* i*.*

The existence of a clearing repayment ratio vector and the fact that there is a greatest one rely on a simple but key property: $a_i(\theta)$ is independent of θ_i and $a_j(\theta)$ for $j \neq i$ are nondecreasing in θ_i. So increasing i's ratio does not affect its asset value and can only increase other banks' asset values. Existence of a clearing repayment ratio vector follows by considering a 'feasible' set. A vector θ in $[0,1]^n$ is said to be *feasible* if it satisfies $\theta_i \ell_i^* \leq \max(a_i(\theta), 0)$ for each bank i. In words, a ratio vector is feasible if each bank i reimburses its liabilities up to its asset value if positive. The limited liability and the bankruptcy constraints are thus satisfied. But the absolute priority rule does not hold in general: for a positive $a_i(\theta)$ we may perfectly have $\theta_i \ell_i^* < a_i(\theta)$ because the required inequality may hold strictly, so the ratios are 'too low' in general. However, thanks to the key property on asset values, the absolute priority condition can be satisfied for all banks at a maximal element of the feasible set,[11] which readily implies that this element is a clearing repayment ratio vector.

The key property also implies that feasible ratio vectors are complements, in the sense that taking the maximum component by component of two feasible vectors yields a feasible vector.[12] Complementarity implies that there is a greatest feasible vector, which is a clearing repayment ratio vector.

The final statements use that aggregate net equity is equal to aggregate net worth. In particular this implies the identity of the banks' net equities at all the clearing ratio vectors, as shown in the proof Section 4. This also implies *(a)* and *(b)*. Under *(a)* the net equity of one bank at least is positive, so creditors' priority requires the bank to fully repay its debts. Under *(b)*, the net equity of one bank at least is negative, which implies that the bank repays nothing. To see this note that, by the limited liability condition, the net equity of a bank can be negative only if its net asset value is negative. In that case, the bankruptcy condition requires the repayment to be null.

Uniqueness of a clearing ratio vector The property that the values of net equities are identical across clearing repayment ratio vectors has a number of implications about the status of the banks and their repayment ratios. The bankrupt banks, those with strictly net negative equity values, coincide at any clearing ratio vector. Their repayment ratios coincide as well as they must be null. Similarly the safe banks, those with strictly positive equity, coincide at any clearing ratio, with a repayment ratio equal to 1. The equality of the clearing ratios in B and S across any clearing vector

[11] By definition, a maximal element is such that increasing a component makes the ratio infeasible.

[12] Let θ and θ' be both feasible and $\theta \vee \theta' = (\max(\theta_i, \theta_i'))$ their supremum. By the key property on a_i, $a_i(\theta \vee \theta')$ is at least as large as each of the asset $a_i(\theta)$ and $a_i(\theta')$. Feasibility follows: $\max(a_i(\theta \vee \theta'), 0) \geq \max(a_i(\theta), 0) \vee \max(a_i(\theta'), 0) \geq (\theta_i \vee \theta_i')\ell_i^*$.

'propagates' to their debtors and results in a unique clearing repayment ratio vector under some conditions stated in the next proposition.

We use the following definitions. A bank is called a *universal creditor* if it has loan on any other bank and has no liability, say bank 1 is a universal creditor if $\ell_{j1} > 0$ any $j > 1$ and $\ell_{1j} = 0$. Also, recall that the (non-negative) matrix ℓ is *irreducible* if for each pair of distinct elements i, j $\ell^{(p)}(i, j)$ is positive for some p, where $\ell^{(p)}$ denotes ℓ multiplied by itself p times. The irreducibility of ℓ requires the existence of a chain of liabilities from any bank to any other one.

Proposition 5 *Assume the aggregate net worth is not null. The clearing ratio is unique in the following cases: there is a universal creditor or the matrix ℓ is irreducible.*

The fact that the clearing ratios are uniquely defined for the banks in B and S 'propagates' to all other banks if the banks are sufficiently connected -the irreducibility assumption- or if there are all linked to a universal creditor. touches all defaulting banks.

Let us give an intuition for the proof, which is detailed in Section 4. Let θ^+ be the greatest clearing ratio vector, and θ another clearing ratio vector, and T be the set on which they differ. As argued above, the bankrupt and the safe banks, respectively those with negative and positive equity values, coincide at any clearing ratio vector, and have the same ratios, respectively 0 or 1. Thus the ratios can differ only for the banks with null equity: T is a null equity set. T is not the whole set N because there is surely a safe or a bankrupt bank under the assumption of a non-null aggregate net worth, as seen in Proposition 4.

The key point is to show that T must have no outside creditor. By definition, each bank in T receives the same amount from outside banks under the two clearing ratio vectors. So thanks to the property that net equities are independent of the clearing vector, their repayment to banks outside T must be identical under the two vectors. But, because repayments can only be larger under θ^+ than under θ, this implies that T has no creditors outside T.

We show that this is impossible under either irreducibility or the presence of a universal creditor. The irreducibility of ℓ precisely requires any strict subset to have an outside creditor (note that we use here that T is not the whole set N). The universal creditor has always a ratio of 1 since it has no liability, so it is not in T, and furthermore it is an outside creditor of T.

The conditions of either irreducibility or the presence of a universal creditor are only sufficient but not necessary for uniqueness. A pyramidal structure as considered in Section 2.4 for example do not satisfy them but one easily checks that the clearing ratio is unique, computed recursively as explained before whatever the sign of the net worth levels (with the only modification of setting a ratio to zero for a bank with a negative net asset value).

The irreducibility of ℓ requires the existence of a path of liabilities between any one bank to any other bank. A related but weaker notion of connectedness is to require the existence of a path of loans-liabilities between any two banks. In graph terminology, irreducibility considers directed path in the liabilities network whereas connectedness considers non directed paths. The following example has a connected network with multiple clearing vectors.

The system has four banks. Banks 1 and 2 have a negative net worth and banks 3 and 4 a positive one: $z = (-1, -1, 1.5, 1.5)$. The liabilities matrix is:

$$\ell = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

All nominal liabilities are of the same amount 1. Bank 1 has a liability to bank 2 and vice versa. Bank 3 has a liability to banks 1 and 4, and bank 4 to 2 and 3. We show that any ratio vector in which 3 and 4 fully repay their debts and 1 and 2 have equal ratios is a clearing vector. Let θ with $\theta_1 = \theta_2, \theta_3 = \theta_4 = 1$. Bank 1's equity, $e_1(\theta) = -1 + \theta_2 + \theta_3 - \theta_1$, is null so bank 1 is allowed to default; bank 3's equity, $e_3(\theta) = 1.5 + \theta_4 - 2\theta_3 = 0.5$, is positive so bank 3 is safe (and similarly for banks 2 and 4 by symmetry).

Observe that the liability matrix is reducible: $\{1, 2\}$ have no liabilities towards 3 and 4. But $\{1, 2\}$ has 3 and 4 as debtors and the liabilities-loans graph is connected. At a clearing vector, banks 1 and 2 may appear as almost bankrupt or almost safe as their ratio may take any value between 0 and 1 (but whatever value their equity is null).

3.2 Defining threat indices

As seen earlier, bankruptcy is unavoidable if aggregate net worth is negative. More generally, bankruptcy is unavoidable when, whatever the ratio vector, one bank at least has negative net equity. In that case no ratio vector satisfies the constraints of the program $\mathcal{P}$. As a result, the clearing ratio vector is surely not a solution to $\mathcal{P}$. This section shows how to modify the program in such a way that the clearing vector appears to maximize a meaningful objective. The threat indices are then defined as measuring the incremental benefit in the objective due to an additional unit of cash in the banks.

By definition, a negative net equity represents a loss. This loss is borne by the creditors if the bank is declared bankrupt, or by the stockholders and whatever entity called to help such as creditors or taxpayers if the bank is bailed out. Under some conditions, a clearing repayment ratio vector solves a program in which these losses are accounted for. Specifically, given a cost c per unit of loss, the objective of the program is given by the payments flow net of the cost of the losses. The program, parameterized by the cost c, writes

$$\mathcal{Q}_c \quad : \quad \max_{(\theta,\delta)} \sum_i \theta_i \ell_i^* - c(\sum_i \delta_i)$$

$$0 \leq \theta_i \leq 1,\ 0 \leq \delta_i \text{ each } i \tag{25}$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \leq z_i + \delta_i \text{ each } i \tag{26}$$

The inequality (26) can also be written using net equity as $\delta_i + e_i(\theta) = 0$. At a solution, the inequality is surely binding for a bank i with a positive δ_i. Thus, a positive δ_i is equal to the opposite of the negative net equity, which, as explained above, represents a loss to bank i's creditors or alternatively an additional amount of cash in the bank.

Observe that the constraints of the two programs $\mathcal{P}$ and $\mathcal{Q}_c$ differ. In particular, $\mathcal{P}$ is not the program $\mathcal{Q}_0$ obtained for c null. (In $\mathcal{Q}_0$, the equity constraint can be relaxed at no cost; ratios equal to 1 are feasible and maximize the flows). On the contrary, when net worth levels are positive, the solutions to $\mathcal{P}$ coincide with those to $\mathcal{Q}_c$ for large enough c, as stated in the next proposition. The intuition is that in that case bankruptcy is avoidable, and that it is optimal to avoid it when the cost associated to bankruptcy losses is large enough. More generally, under some conditions, even when bankruptcy must occur, the clearing repayment ratio vector solves $\mathcal{Q}_c$ for c large enough.

Proposition 6 *Let the aggregate net worth z be positive, $\sum_{i\in N} z_i > 0$, and each liability ℓ_{ij} be positive, i and j distinct. The (unique) clearing repayment ratio vector solves $\mathcal{Q}_c$ for c large enough. The threat indices, defined as the multipliers associated to the equity constraint (26), satisfy*

$$\mu_k = c \text{ for } k \text{ with } \delta_k > 0 \tag{27}$$

$$\mu_i - \sum_{j\in D} \pi_{ij}\mu_j = 1 + c\sum_{k\in B} \pi_{ik} \text{ for } i \text{ in } D. \tag{28}$$

The key point in the proof is to show that it is not optimal to inject simultaneously cash into a bank and have it repay, that is δ_i and θ_i cannot be both positive at an optimal solution. If it was the case, i's net equity would be negative (because δ_i is positive) but the bank would repay part of its liabilities, which is impossible at a clearing ratio vector. This key point is proved under the additional assumption of a positive aggregate net worth.

Without an additional assumption, the solution to $\mathcal{Q}_c$ may not be a clearing repayment ratio vector even for large c. Let for example each bank have a negative net equity under full repayment: $e_i(\mathbb{1}) = z_i + \sum_j \ell_{ji} - \ell_i^* < 0$. The optimal solution is to have banks to repay fully their debts and to inject in each of them exactly the amount necessary to do so: $\delta_i = -e_i(\mathbb{1})$. The payment flows within the financial system is clearly maximal. Furthermore the aggregate injection is minimal. To see this, note that summing (26) over all the banks requires $\sum_i(\delta_i + z_i) \geq 0$ at a feasible solution. Since aggregation yields $\sum_i \delta_i = -\sum_i e_i(\mathbb{1}) = -\sum_i z_i$, the aggregate injection is minimal.

What a large enough cost means can be made more precise. The proof shows that it suffices that $c\pi_{ij}$ be larger than 1 for any pair i, j of distinct banks. Arguing as in the previous section, a bank's threat index measures the incremental benefit in the objective due to an additional unit of cash in the bank. As c becomes large, the objective becomes dominated by the cost associated with capital injection. This explains why the behavior of the threat indices largely depends whether there are bankrupt banks or not. Without bankrupt banks, an empty set B, expression (28) coincides with (14) found in the previous section. Threat indices reflect the impact of the banks' net worth on the payment system and do not depend on c because there is no capital injection.

With bankrupt banks instead, although the solution to $\mathcal{Q}_c$ stays constant for large enough c, the threat indices adjust to the value of c. Specifically, a bank's threat index is increasingly driven by the impact of its net worth on the losses of the bankrupt banks. Working with the threat indices per cost c, $\hat{\mu}_i = \mu_i/c$, we obtain at the limit for c increasingly large

$$\hat{\mu}_i - \sum_{j\in D} \pi_{ij}\hat{\mu}_j = \sum_{k\in B} \pi_{ik} \text{ for each } i \text{ in } D. \tag{29}$$

This expression can be interpreted as in the previous section by considering the additional flow of repayments that an increase in the net worth of a defaulting bank induces. We show that the threat index measures how much of the additional flow reaches the bankrupt banks, thereby diminishing the loss. Indeed, for c extremely large, and if there are bankrupt firms, capital injection is what matters ultimately and not the payments flowing along the defaulting banks.

To show this, observe that the right hand side of (29) is the proportion of i's liabilities towards bankrupt banks. The vector of relative liabilities of banks in D to banks in B, $\sum_{k\in B} \pi_{ik}$ for i in D, can be written as $\Pi_{D\times B}\mathbb{1}_B$. Thus equation (29) in matrix form is $(\mathbb{I}-\Pi)_{D\times D}\hat{\mu}_D = \Pi_{D\times B}\mathbb{1}_B$ which gives the following expression for $\hat{\mu}_D$:

$$\hat{\mu}_D = \Pi_{D\times B}\mathbb{1}_B + \Pi_{D\times D}\Pi_{D\times B}\mathbb{1}_B + \Pi^{(2)}_{D\times D}\Pi_{D\times B}\mathbb{1}_B ... + \Pi^{(p)}_{D\times D}\Pi_{D\times B}\mathbb{1}_B ... \tag{30}$$

The terms in the sum correspond to amounts received by bankrupt banks, directly or indirectly through a chain of defaulting banks, following an increase in the net worth values of defaulting banks. Let defaulting bank i receive an additional unit of cash. The unit is entirely used for reimbursement. Each bankrupt bank k receives π_{ik}, thereby generating a direct total flow into bankrupt banks equal to $\sum_{k\in B} \pi_{ik}$. This term is the i-th component of the vector $\Pi_{D\times B}\mathbb{1}_B$, the first element in the sum on the right hand side of (30). Non bankrupt banks also receive additional payment, π_{ij} for j, and for those which are defaulting, they will pass this to their creditors: defaulting j pays an amount of $\pi_{ij}\sum_{k\in B} \pi_{jk}$ to the bankrupt banks. Hence there is a total of $\sum_{j\in D} \pi_{ij} \sum_{k\in B} \pi_{jk}$ reaching the bankrupt banks through an intermediary defaulting bank. This term is equal to the i-th component of $\Pi_{D\times D}\Pi_{D\times B}\mathbb{1}_B$, the second element in the sum on the right hand side of (30). Iterating, the amount received by the bankrupt banks after flowing through a chain of p defaulting banks is the i-th component of $\Pi^{(p)}_{D\times D}\Pi_{D\times B}\mathbb{1}_B$. Finally, the total amount received by bankrupt banks is obtained by summing over all p, which gives the right hand side of (30).

An alternative interpretation of expression (30) is in stochastic term. Interpret Π as a transition matrix in which element π_{ij} is the probability of reaching j from i (by definition the sum $\sum_j \pi_{ij}$ is equal to 1). In this interpretation, the element i, j of the matrix $\Pi^{(p)}_{D\times D}$ is the probability of reaching j from i in p steps while staying all along in D, and the i-th component of the vector $\pi_{D\times B}\mathbb{1}_B = (\sum_{k\in B}\pi_{ik})_{i\in D}$ is the probability of reaching in one step an element of B from i. Thus the i-th element of $\Pi^{(p)}_{D\times D}\Pi_{D\times B}\mathbb{1}_B$, which is $\sum_j \pi^{(p)}_{D\times D}(i,j)(\sum_{k\in B}\pi_{ik})$, is the probability of reaching a bankrupt bank for the first time in $p+1$ steps starting from i and staying all along in D, i.e., never reaching a safe nor a bankrupt bank. Such an interpretation of μ could be useful because it allows to rely on standard probability techniques.

3.3 Concluding remarks

We have defined and investigated a threat index in an explicit model of interbank liabilities. Our analysis is at the ex post stage. Given the liabilitities and the realized values for the banks' net worth, it computes how some criteria based on debt repayments would be changed following some cash injection. The threat indices may substantially differ from the default levels. As a result, injecting cash into the banks that appear the weakest ones, those with the largest default ratio, may be sub-optimal.

The threat index reflects an externality imposed by a defaulting bank on the system. While the default level of a bank depends on its assets and the safety of its debtors, its threat index depends on its liabilities and the safety of its creditors. A bank thus may not assess properly the externality it imposes on the system when it decides on its interbank relationships, since it is concerned with the safety of its debtors and not with that of its creditors. This raises the issue of the adequate regulatory tools. To address it, an ex ante perspective in which investment or liabilities are choosen is needed.

4 Proofs

Proof of Lemma 1 It suffices to prove the result for one of the matrices. Consider $\mathbb{I} - \Pi$. By construction $\sum_{j\in N}\pi_{ij} = 1$ so we have $\sum_{j\in T}\pi_{ij} \leq 1$ for each i in T. If this inequality holds strictly for each i in T, the result follows from well known results on productive matrix (alternatively matrix $\mathbb{I} - \Pi$ is dominant diagonal). If the inequality does not hold strictly for each row, we show that an iterate of the matrix has all its sum totals smaller than 1: $\Pi^{(p)}_{T\times T}\mathbb{1}_T \ll \mathbb{1}_T$. The result then follows: Multiplying this inequality by $\Pi_{T\times T}$ yields $\Pi^{(p+1)}_{T\times T}\mathbb{1}_T \ll \Pi_{T\times T}\mathbb{1}_T \leq \mathbb{1}_T$, hence by induction the inequality $\Pi^{(p')}_{T\times T}\mathbb{1}_T \ll \mathbb{1}_T$. holds for all p' larger than p'. This implies that the sum $\mathbb{I}_{T\times T} + \Pi_{T\times T} + \Pi^{(2)}_{T\times T} + ... + \Pi^{(p)}_{T\times T} + ...$ converges. The limit is the inverse of matrix $I - \Pi_T$ and is positive.

To show that the inequality $\Pi^{(p)}_{T\times T}\mathbb{1}_T \ll \mathbb{1}_T$ holds for some p, it suffices to show that for each i in T there is some q for which the strict inequality $\sum_{j\in T}\pi^{(q)}_{ij} < 1$: if the inequality holds for q it holds for all q' larger than q (as we have seen above) and p can be taken to be the maximum of the q that work for each i in T.

We argue by contradiction: there is i in T for which the equality $\sum_{j\in T}\Pi^{(q)}_{T\times T}(i,j)=1$ holds for each q where $\Pi^{(q)}_{T\times T}(i,j)$ denotes the element i,j of the matrix $\Pi^{(q)}_{T\times T}$. Interpret Π as a transition matrix in which element π_{ij} is the probability of reaching j from i. The element $\Pi^{(q)}(i,j)$ is the probability of reaching j from i in q steps and $\Pi^{(q)}_{T\times T}(i,j)$ is the probability of reaching j from i in q steps while staying all along in T. Thus the equality $\sum_{j\in T}\Pi^{(q)}_{T\times T}(i,j)=1$ for each q implies that all the paths from i to j are included in T. Let C be composed with all the elements that can be reached from i. By construction, C has no outside creditor. Furthermore, from the argument above, C is included in T. Hence C is a null equity set (as a subset of the null equity set T) and Property 2 gives the desired contradiction: C must have an outside creditor. ■

Proof of Proposition 1 The program $\mathcal{P}$ has a finite solution: the feasible set is non-empty (because it contains $\theta=\mathbf{0}$ since z is positive) and is compact. From well known results on linear programming, the multipliers of the constraints are the solutions to the dual program of $\mathcal{P}$, and furthermore, the values of the primal and dual coincide.

We first explicit the dual. Recall program $\mathcal{P}$:

$$\mathcal{P} \quad : \quad \max_\theta \sum_i \ell^*_i\theta_i$$

$$0\le\theta_i\le 1 \text{ each } i \quad (7)$$

$$\theta_i\ell^*_i-\sum_j\theta_j\ell_{ji}\le z_i \text{ each } i \quad (8)$$

Program $\mathcal{P}$ is of the form: maximize $\ell^*.\theta$ under $A\theta\le b$, $\theta\ge 0$ where A is the $2n\times n$ matrix and b is the $2n$-vector

$$A=\begin{pmatrix} dg(\ell^*)-\ell^t \\ dg(\ell^*)\end{pmatrix},\ b=\begin{pmatrix} z\\ \ell*\end{pmatrix}$$

Recall that the dual of $\max \ell^*.\theta$ under $A\theta\le b$ and $\theta\ge 0$ is $\min b.\gamma$ under $A^t\gamma\ge\ell^*$, and $\gamma\ge 0$.

We show that the dual program of $\mathcal{P}$ is

$$\mathcal{D} \quad : \min_{(\lambda,\mu)\ge 0}\sum_i\mu_i z_i+\sum_i\lambda_i\ell^*_i$$

$$(\lambda_i+\mu_i-1)\ell^*_i-\sum_j\ell_{ij}\mu_j\ge 0 \text{ each } i. \quad (31)$$

and furthermore that the constraints of the dual (31)

Apply the duality theorem to $\mathcal{P}$. Write the $2n$-vector γ as $\begin{pmatrix}\mu\\ \lambda\end{pmatrix}$. The objective of the dual $b.\gamma$ is $\sum_i z_i\mu_i+\sum_i\lambda_i\ell^*_i$ and the constraints

$$\begin{pmatrix} dg(\ell^*)-\ell & dg(\ell^*)\end{pmatrix}\begin{pmatrix}\mu\\ \lambda\end{pmatrix}\ge\ell^*$$

Spelling out the i-th constraint of the dual yields

$$\ell_i^* \mu_i - \sum_j \ell_{ij} \mu_j + \ell_i^* \lambda_i \geq \ell_i^*$$

which is (31). These constraints are binding:

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij} \mu_j = 0 \text{ each } i. \tag{32}$$

for each i. By contradiction suppose $(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ for some i. If $\lambda_i > 0$, λ_i can be decreased without affecting the other constraints and the objective is decreased. If $\lambda_i = 0$, we have $\mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ so μ_i must be strictly positive. Decreasing μ_i is feasible because constraint (31) is not binding for i by assumption and a decrease in μ_i relaxes the constraints for the banks distinct from i. But a decrease in μ_i results in a decrease in the objective, a contradiction again.

Now, let S be the set of safe banks, for which (8) is strict. By the slackness conditions, $\mu_i = 0$ for i in S. Equation (32) immediately gives that the solidarity indices satisfy (17). Let us assume that there are no fragile banks. All banks that are not in S are in D with a repayment ratio strictly smaller than 1. By the slackness conditions, their solidarity indices λ_i are null. Using $\mu_i = 0$ for i in S and $\lambda_i = 0$ for i in D, equations (32) write as $\mu_i \ell_i^* - \sum_{j\in D} \ell_{ij}\mu_j = \ell_i^*$: this proves (14). The fact that the system (14) has a unique solution, which is furthermore positive, follows from Lemma 1. ■

Proof of Proposition 3 The sub-modularity of V with respect to z can be proved iteratively by increasing the values of each component of z_i: it suffices to show that given a pair i, j and z_{-ij} a payoff vector for banks other than i and j, V satisfies

$$V(z_i', z_j', z_{-ij}) - V(z_i, z_j', z_{-ij}) \leq V(z_i', z_j, z_{-ij}) - V(z_i, z_j, z_{-ij}) \text{ for } z_i' \geq z_i, z_j' \geq z_j.$$

For a differentiable function V, the sub-modularity is satisfied if the partial derivative $\frac{\partial V}{\partial z_i}$ is non-increasing in z_j: This follows from the following expression:

$$V(z_i', z_j, z_{-ij}) - V(z_i, z_j, z_{-ij}) = \int_{z_i}^{z_i'} \frac{\partial V}{\partial z_i}(t, z_j, z_{-ij})dt. \tag{33}$$

Here V is not differentiable at all points but the partial derivative of V exists almost everywhere, $\frac{\partial V}{\partial z_i}(t, z_j, z_{-ij})$ is given by the unique multiplier μ_i at (t, z_j, z_{-ij}) when there is no fragile bank, which is the case but for a finite number of points as t runs in the interval (z_i, z_i'). Since V is continuous, the integral expression (33) is still valid. V is thus sub-modular if for each i the multiplier μ_i at (t, z_j, z_{-ij}) is a non-decreasing function of z_j, j distinct from i for fixed t and fixed z_{-ij} (actually μ_i is also decreasing in z_i from the concavity of V)

Let us start at z_j and increase it progressively up to z_j'. We use a $'$ to denote values corresponding to z_j'.

Consider D the set of defaulting banks at z. Assume not fragile bank. The clearing ratio vector θ' is given by (12), $(dg(\ell^*) - \ell)_{D\times D}^t \theta_D' = \hat{z}_D'$, as long as each θ_i' for i in D is strictly less than 1. θ'

is a non-decreasing function of the z_j because the inverse of $(dg(\ell^*)-\ell)^t_{D\times D}$ is a positive matrix by Lemma 1. Hence $\theta' \geq \theta$ (for the banks not in D their ratios is constant equal to 1). Since D remains the set of defaulting banks, the index μ' satisfies the same (invertible) system (14): μ' is equal to μ. As z_j is increased further, some clearing ratios for i in D may hit the upper bound 1, making them fragile. Increasing by any small amount z_j, these banks become safe and the set D is reduced to some D': $D' \subset D$.

The multiplier μ is null outside D and μ' is null outside D' so it suffices to show that $\mu'_{D'} \leq \mu_{D'}$ to prove $\mu' \leq \mu$. The vectors μ_D and $\mu'_{D'}$ are given respectively by $(dg(\ell^*)-\ell)_{D\times D}\mu_D = \ell^*_D$ and $(dg(\ell^*)-\ell)_{D'\times D'}\mu'_{D'} = \ell^*_{D'}$. The matrix $(dg(\ell^*)-\ell)_{D\times D}$ restricted to the rows indexed by elements in D' is $(dg(\ell^*)-\ell)_{D'\times D} = (dg(\ell^*)-\ell)_{D'\times D'}, -\ell_{D'\times(D-D')})$. Thus the equation $(dg(\ell^*)-\ell)_{D\times D}\mu_D = \ell^*_D$ restricted to the rows indexed by elements in D' writes

$$(dg(\ell^*)-\ell)_{D'\times D'}\mu_{D'} - \ell_{D'\times(D-D')}\mu_D = \ell^*_{D'}.$$

The above equality implies $(dg(\ell^*)-\ell)_{D'\times D'}\mu_{D'} \geq \ell^*_{D'}$. Hence $\mu_{D'} \geq (dg(\ell^*)-\ell)^{-1}_{D'\times D'}\ell^*_{D'}$ because the inverse of $M_{D'\times D'}$ exists and is non-negative (thanks to Lemma 1). By definition $(dg(\ell^*)-\ell)^{-1}_{D'\times D'}\ell^*_{D'}$ is equal to $\mu'_{D'}$ so we obtain $\mu'_{D'} \leq \mu_{D'}$, the desired result. ∎

Proof of Proposition 4 The existence of a clearing ratio vector relies on the feasible set introduced in the text: A vector θ in $[0,1]^n$ is said to be *feasible* if it satisfies $\theta_i\ell^*_i \leq \max(a_i(\theta),0)$ for each bank i. The limited liability and the bankruptcy constraints are satisfied. To prove that a maximal element θ satisfies the absolute priority rule condition, we only need to consider a bank with a positive asset. Because increasing i's ratio does not affect its asset value and can only increase other banks' asset values, if i's ratio is less than 1, it must be that $\theta_i\ell^*_i$ is equal to $\max(a_i(\theta),0)$. Since the value $a_i(\theta)$ is positive, the equality writes $\theta_i\ell^*_i = a_i(\theta)$. So either θ_i is equal to 1 or equity is null: the creditor's absolute priority rule is satisfied.

We first prove that the net equity of each bank is the same at each clearing ratio vector, in case of multiplicity. For that, it suffices to compare the values of net equity with those for the greatest clearing repayment ratio vector. Let θ^+ be the greatest clearing repayment ratio vector, and θ be another clearing repayment ratio vector. Recall that $e_i(\theta^+) = a_i(\theta^+) - \theta^+_i\ell^*_i$ and $a_i(\theta^+) = z_i + \sum_j \theta^+_j\ell_{ji}$ denote respectively i's equity and asset values at vector θ^+, and similarly $e_i(\theta)$ and $a_i(\theta)$ at θ. Note that $a_i(\theta^+) \geq a_i(\theta)$ each i since $\theta^+ \geq \theta$. To prove the equalities $e_i(\theta^+) = e_i(\theta)$ for each i, it is enough to show

$$e_i(\theta^+) \geq e_i(\theta) \text{ for each } i \tag{34}$$

thanks to the aggregation formula: Since the sum of the net equity values over all banks is equal to the sum of their net worth values whatever repayment ratio vector, summing all inequalities (34) over i implies that the right and left hand sides are equal, hence each inequality is binding: net equity values are the same under both repayment ratio vectors.

To prove (34), we consider various cases depending on the values of θ_i and θ_i^+. (The first two cases are the same as in Eisenberg and Noe 2000).

For $\theta_i = 1$, θ_i^+ is also equal to 1, thus inequality $e_i(\theta^+) \geq e_i(\theta)$ follows from the inequality $a_i(\theta^+) \geq a_i(\theta)$ and $\theta_i^+ = \theta_i$.

For $0 < \theta_i < 1$, i's net equity under θ is null; since surely $0 < \theta_i^+$, i's net equity under θ^+ can only be non-negative, so $e_i(\theta^+) \geq e_i(\theta) = 0$.

For $\theta_i = 0$, $a_i(\theta) = e_i(\theta)$, and $a_i(\theta)$ cannot be positive by absolute priority. Hence $0 \geq e_i(\theta)$. If $\theta_i^+ > 0$, $e_i(\theta^+) \geq 0 \geq e_i(\theta)$; if $\theta_i^+ = 0$, net equities are given by the net asset values, hence again $e_i(\theta^+) \geq e_i(\theta)$.

Let us now consider the possibility of multiple clearing vectors and prove the uniqueness under the assumptions stated in the proposition. Recall that there is a greatest clearing ratio θ^+. Let θ be another clearing repayment ratio vector. To prove uniqueness, it is sufficient to show that the non-negative vector $\theta^+ - \theta$ is null. Let T be the set for which $\theta_i^+ - \theta_i > 0$. Because net equities are identical across clearing repayment ratio vectors, bankrupt banks (with a negative equity value) and safe banks (with a positive equity value) coincide at any clearing ratio vector. Hence $\theta_i^+ - \theta_i > 0$ can only hold for banks with null net equity: T is a null equity set. Furthermore, under the assumption that aggregate net worth is not null, there are surely some bankrupt banks (if the flow is negative) or safe banks (if the flow is negative): T is not the whole set N. We show that no bank in T has a creditor outside T.

The vector of banks' net equities is given by $e(\theta) = z - (dg(\ell^*) - \ell)^t\theta$ at a repayment ratio vector θ. Since net equities are identical across clearing repayment ratio vectors, we have $(dg(\ell^*) - \ell)^t(\theta^+ - \theta) = 0$. The aggregation formula over the null equity set T implies $e_i(\theta^+) = e_i(\theta^+) = 0$, which yields

$$\sum_{j \notin T, i \in T} (\theta_j^+ - \theta_j)\ell_{ji} = \sum_{j \notin T, i \in T} (\theta_i^+ - \theta_i)\ell_{ij}$$

By definition, the repayment ratios differ on T and only on T: $\theta_i^+ - \theta_i$ is null for i in $N - T$ and positive for i in T. So the left hand side is null, and the nullity of right hand side requires $\ell_{ij} = 0$ for i in T and j not in T, i.e., no bank in T has a creditor outside T.

Uniqueness follows in the following cases.

(1) The liability matrix ℓ is irreducible. In that case any subset T has an outside creditor.[13]

(2) There is a universal creditor. The universal creditor has surely a positive equity hence is an outside creditor of a null equity set. ■

[13] An alternative proof, less elaborate, is to apply Perron-Frobinius theorem. The identity $(dg(\ell^*) - \ell)^t(\theta^+ - \theta) = 0$ that the vector $\theta^+ - \theta$ is a non-negative eigenvector of $dg(\ell^*) - \ell)^t$ associated to the largest eigenvalue 1. Applying Perron-Frobinius theorem to the irreducible matrix, $\theta^+ - \theta$ is a strictly positive vector, in contradiction with T a strict subset of N.

Proof of Proposition 6 Write the program $\mathcal{Q}_c$ as

$$\begin{aligned} \mathcal{Q}_c \quad &: \quad \max_{\theta,\delta\geq 0}[\sum_i \ell_i^*\theta_i] - c[\sum_i \delta_i] \\ \theta_i\ell_i^* - \sum_j \theta_j\ell_{ji} &\leq z_i + \delta_i \text{ each } i \quad (26) \\ \theta_i\ell_i^* &\leq \ell_i^* \text{ each } i. \end{aligned}$$

In matrix form the objective is $l^*.\theta - c\mathbb{1}.\delta$ and the constraints write as $A\begin{pmatrix}\theta \\ \delta\end{pmatrix} \leq b$ where A is the $2n \times 2n$ matrix and b is the $2n$-vector

$$A = \begin{pmatrix} dg(\ell^*) - \ell^t & -I \\ dg(\ell^*) & 0 \end{pmatrix}, \ b = \begin{pmatrix} z \\ \ell* \end{pmatrix}.$$

Applying the duality theorem as in the proof of Proposition 1, the dual is

$$\mathcal{D}_c \quad : \min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i\ell_i^*$$

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j \geq 0 \text{ each } i \tag{35}$$

$$\mu_i \leq c \text{ each } i. \tag{36}$$

From well known results, if (θ, δ) and (λ, μ) are feasible respectively for the primal and the dual, and satisfy the complementary slackness conditions, each one is a solution respectively for the primal and the dual. The slackness conditions are (using notation $\delta_i + e_i(\theta) \geq 0$ for nequality (26))

$$\begin{cases} \theta_i = 0 & \text{or} \quad (\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j = 0 \\ \theta_i = 1 & \text{or} \quad \lambda_i = 0 \\ \delta_i = 0 & \text{or} \quad \mu_i = c \\ \delta_i + e_i(\theta) = 0 & \text{or} \quad \mu_i = 0 \end{cases}$$

Consider a solution (θ, δ) to $\mathcal{Q}_c$. We prove that, for c large enough, θ is a clearing repayment ratio vector under the assumptions stated in the proposition, i.e., the positivity of all interbank liabilities of ℓ and the positivity of aggregate net worth. Define

$$S = \{i, \delta_i = 0 \text{ and } e_i(\theta) > 0\}, \ B = \{i, \delta_i > 0\} \text{ and } D = N - B \cup S.$$

For a bank in B, i's constraint (26) must be binding, $\delta_i + e_i(\theta) = 0$, because otherwise the objective of $\mathcal{Q}_c$ could be increased. Thus i's net equity is strictly negative.

Note that D is the set of banks i for which δ_i is null (because i is not in B) and net equity $e_i(\theta)$ is null (because otherwise i would be in S): $D = \{i, \delta_i = 0, \text{ and } e_i(\theta) = 0\}$.

From this, we have that $z_i + \delta_i$ is null for each bank in B or in D. Under the assumption of a positive aggregate net worth, this implies that S is non-empty: Otherwise the sum $\sum_{i \in N}(z_i + \delta_i)$ would be null, in contradiction with $\sum_i z_i > 0$ and $\delta \geq 0$.

We check that the θ is a clearing repayment ratio vector with S as the set of safe banks and B as the set of bankrupt banks.

S is the set of banks for which injection is null and i's equity is strictly positive. This is clearly true: otherwise the ratio can be increased while still satisfying all constraints, thereby increasing the objective.

A bank in D has $e_i(\theta) = 0$, so the clearing conditions are satisfied for it whatever its ratio.

B is the set of banks for which injection is positive. We have to prove that i's ratio is null. In other words we need to exclude the situation where the bank repays some debts and receives some cash, i.e. both $\theta_i > 0$ and $\delta_i > 0$. By contradiction, if $\theta_i > 0$, then $\lambda_i + \mu_i - 1 - \sum_j \pi_{ij}\mu_j = 0$ (by the first slackness conditions and working on relative liabilities). $\delta_i > 0$ implies $\mu_i = c$ by the third slackness condition Hence $c - 1 \leq \sum_j \pi_{ij}\mu_j$ because λ_i is non-negative. Let $\underline{\pi}$ be the minimum value of the off-diagonal elements π_{ij}. We show that $\sum_j \pi_{ij}\mu_j \leq c(1 - \underline{\pi})$. First each μ_j is not greater than c; second there is surely a bank k that is in S hence for which μ_k is null; using $\pi_{ik} \geq \underline{\pi}$ we obtain $\sum_j \pi_{ij}\mu_j \leq c(1 - \underline{\pi})$. Hence

$$c - 1 \leq c(1 - \underline{\pi})$$

which is impossible for c large enough. This ends the proof. ∎

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