

THE CONDORCET JURY THEOREM UNDER COGNITIVE HIERARCHIES: THEORY AND EXPERIMENTS

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ABSTRACT. Cognitive hierarchy models have been developed to explain systematic deviations from the equilibrium behavior in certain classes of games. This paper introduces an endogenous cognitive hierarchy model, which better explains the behavioral heterogeneity of the strategies in games for which the standard cognitive hierarchy model provides an unreasonable prediction. As in the previous models, each player in the endogenous cognitive hierarchy model is assumed to best-reply to the other players holding a belief induced by the cognitive hierarchy. Contrary to the previous models, however, players are allowed to consider the presence of opponents at their own level of cognitive hierarchy. This extension is shown to eradicate the incompatibility of cognitive hierarchy models in the classes of games. We employ the model to explain voting behavior in information aggregation problems of the Condorcet Jury Theorem. Behavioral assumption of the strategic thinking turns out to be a crucial factor in whether the asymptotic efficiency is obtained or not. We conducted laboratory experiments which show that the endogenous cognitive hierarchy model provides significant improvements upon symmetric Bayesian Nash equilibrium and the previous cognitive hierarchy models in explaining the observed behavior of voters.

1. INTRODUCTION

Since Condorcet's earlier work in 1785, mathematical support has been provided for the idea of increasing accuracy of collective decisions by including more individuals in the process. In his seminal *Essai*, Condorcet considered nonstrategic individuals voting to make a decision on a binary issue where each alternative is commonly preferred to the other one in one of the two states of the world. Each individual receives independently an imperfectly informative private signal about the true state of the world. Then under majority rule, the probability of reaching a correct decision monotonically increases with the size of the electorate and converges to certainty in the limit.

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Although allowing strategic behavior may revoke assumptions of this basic model this property survives in various circumstances of collective decision making (e.g. Austen-Smith and Banks (1996) and Feddersen and Pesendorfer (1997)). In most models, probability of making a right decision increases and converges to one as the group size increases, even when strategic players may vote against their signals. This asymptotic efficiency is often coined as the *Condorcet property*.

In the literature of the Condorcet Jury Theorem, often the signals are assumed to be binary, since the problem concerns a binary-state, binary-issue decision-making. This seemingly innocuous assumption does not reflect properly the description of information aggregation, because binary-signal assumption implies that there is only exactly two ways to update the belief over the real state of the world. Even if the state space is binary, belief can take any value in the unit interval, corresponding to the probability assigned to one of the binary states. Therefore, the right model to describe the aggregation of information on the binary state space should include continuous signal space, which correctly reflect the continuous possibility to update the belief over the binary state space.

When the strategy space and the number of voters are large, assuming complete rationality of all individuals may be too much requiring. This intuition finds support from experimental evidence and information aggregation situations are not, to say the least, exempt from it. Especially when a major aspect of the strategic models comprises the presumption that voters take into account their probability of being pivotal (Downs (1957)). As shown by Esponda and Vespa (2013) in their experimental study, hypothetical thinking in direction to extract information from others' strategies required in strategic voting models are too strong as an assumption and does not find support in laboratory experiments. Battaglini et al. (2008) report from another experimental study an increase in irrational non-equilibrium play as the size of electorate increases.³

Models of non-equilibrium strategic thinking have been proposed to explain structural deviations from equilibrium thinking in a variety of games. A sizable part of bounded rationality literature is devoted to the models of cognitive hierarchy, starting with Nagel (1995) and Stahl and Wilson (1995), which allow heterogeneity among the individuals in levels of strategic thinking. The construction of this model dictates a foundational level of cognitive hierarchy, level-0, which represents a strategically naïve initial approach to game. Then a level- k player (hereafter Lk , where $k \geq 1$) is assumed to best respond to others with a cognitive hierarchy of level $k - 1$. The construction of levels resonate with *rationalizability*, as in Bernheim (1984), reflecting the fact that the decisions made

³As Camerer (2003), chapter 7, stresses the effect of group size on behavior in strategic interactions is a persistent phenomenon, especially towards coordination.

by a level- k survive k rounds of iterated elimination of strictly dominated strategies in two-person games.

The closely related *Poisson-CH* model is introduced in Camerer et al. (2004). They allow heterogeneity in beliefs on others' levels in a particular way. A level- k type best responds to a mixture of lower levels, which is estimated by consistent truncations up to level $k - 1$ from a Poisson distribution, for each $k > 0$. The relevant Poisson distribution is either obtained from maximum likelihood estimations applied to data or calibrated from previous estimates.

The set of level-1 strategies in this *Poisson-CH* model (hereafter *CH1*) is exactly the same as that of *L1*. For higher levels, *Lk* and *CHk* differ. For example, strategies in *CHk* are not rationalizable in general.⁴ Common in these models is the assumption that level- k players do not assign any probability to the levels higher than k . This assumption captures the idea that there is a hierarchy in cognitive limits among players.

Another assumption shared by these two models is that of *overconfidence*. Both models presumes that no individual assigns a positive possibility to opposing another player with the same level of cognitive hierarchy. In this paper, we propose a new model, the model of *endogenous cognitive hierarchy (ECH)*, without imposing the overconfidence assumption. ECH builds on the Poisson-CH model by allowing individuals to best respond to others who may belong to the same level of cognitive hierarchy. While it may be reasonable to assume ignorance on higher levels due to cognitive limits, we show in this paper that exclusion of the same level opponents leads to significant consequences both theoretically and empirically.

There are three reasons why overconfidence assumption should not be taken as granted. First, we show that in a large class of games exclusion of self awareness, or imposing overconfidence, leads to a significant difference in the asymptotic behavior of the players. In these games, the prescribed strategies are extreme (distance from the symmetric Nash strategy diverges away) in both level- k and CH. We argue that such divergence is not coherent with the idea of cognitive hierarchy. Second, in our experiments, we find that ECH can explain the observed behavior much better than level- k or CH. Third, a large number of subjects, i.e., 97%, answered in the questionnaire that they do not exhibit such overconfidence, as shown in Figure 1.1. In fact, most of them think that most of others use the same reasoning, as shown in Figure A.1 in the Appendix.

We present results from a laboratory experiment designed to explore the effect of the size of electorate on accuracy of decisions under majority rule. We deviate from previous

⁴Crawford et al. (2013) provides a fine review of these models and applications.

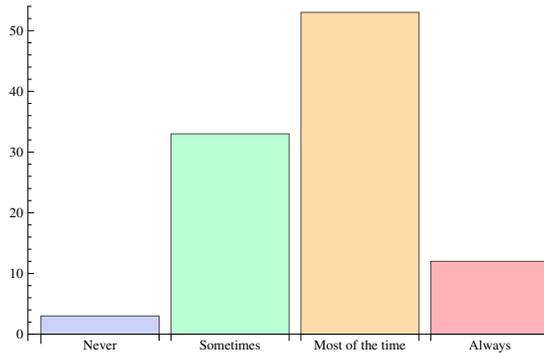


Figure 1.1. Frequencies of responses to the Question 2 in a post-experimental questionnaire: “When you made decisions, did you think that the other participants in your group used exactly the same reasoning as you did?”.

literature by setting signals to induce differing posterior probabilities on the true state of the world, contrary to binary signals.

Related Literature. Gerling et al. (2005) provides an extensive survey on the studies of collective decision making modeled similar to Condorcet jury model. Palfrey (2013) presents an ideal survey on experiments in political economy, and particularly in strategic voting. Costinot and Kartik (2007) study voting rules and show that the optimal voting rule is the same when players are sincere, playing according to Nash equilibrium, to level- k , or belonging to a mixture of these. Koriyama and Szentes (2009) provides a general analysis of optimal group size under costly information acquisition and Bhattacharya et al. (2013) tests experimentally the theoretical predictions about individual behavior and group decision under costly information acquisition. They find poor support for the comparative statics predictions of the theory.

The paper proceeds as follows. We introduce the endogenous cognitive hierarchy model formally in the following section where we discuss its properties while relating to previous models. We furthermore compare the implications of models in a stylized setting of information aggregation. In Section 3 we introduce our experimental design that carries novelties due to our modeling concerns and signal setup. Section 4 provides the results of the experiment and compares how different models fit to data. We conclude by summarizing our findings and future research possibilities after Section 5, where we further investigate the implications of both ECH and CH models. Appendices are devoted to proofs and detailed experimental data.

2. THE MODEL

Let (N, S, u) be a symmetric normal-form game where $N = \{1, \dots, n\}$ is the set of players, $S \subset \mathbb{R}$ is a convex set of pure strategies, and $u : S^n \rightarrow \mathbb{R}^n$ is the payoff function.

In the cognitive hierarchy models, each player forms a belief on the frequency with respect to the cognitive hierarchy levels of the other players. Let $g_k(h)$ denote the belief of a level- k player on the frequency that the other player belongs to level- h .

In the standard level- k model *à la* Nagel (1995), first, a naïve, nonstrategic behavior is specified. This constitutes the initial level of cognitive hierarchy, the *level-0*, or L_0 . Second, L_k for $k \in \mathbb{N}_+$ believes that all the other players belong to one level below:

$$g_k(h) = \begin{cases} 1 & \text{if } h = k - 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the cognitive hierarchy model introduced in Camerer et al. (2004), every level- k , or CH_k , best responds to a mixture of lower levels in a consistent manner. To be more specific, let $f = (f_0, f_1, \dots)$ be a distribution over \mathbb{N} that represents the composition of cognitive hierarchy levels. Then a CH_k has a distributional belief that is specified by $g_k = (g_k(0), g_k(1), \dots, g_k(k-1))$ where

$$g_k(h) = \frac{f_h}{\sum_{m=0}^{k-1} f_m} \text{ for } h = 0, \dots, k-1,$$

for all $k \in \mathbb{N}_+$. Given a global distribution of cognitive hierarchy levels, consistency requires each level to embrace a belief on distribution of other players' levels that is a truncation up to one level below. Thus, these two models share the following two assumptions.

Assumption 1. (COGNITIVE LIMIT) $g_k(h) = 0$ for all $h > k$.

Assumption 2. (OVERCONFIDENCE) $g_k(k) = 0$ for all $k > 0$.

In what follows, we introduce the *endogenous cognitive hierarchy model* of games formally, in which overconfidence is dropped while cognitive limit assumption is preserved.

2.1. Endogenous Cognitive Hierarchy Model. Fix an integer $K > 0$ that prescribes the highest level considered in the model.⁵ Then, a sequence of truncated distributions $g = (g_1, \dots, g_K)$ is uniquely defined from f . As in the cognitive hierarchy model, we focus on the symmetric equilibrium, i.e. the mixed strategy assigned to each level describes the distribution of pure strategies played in the level. A sequence of mixed strategies

⁵We assume $f_i > 0$ for all $i \leq K$. For the truncated distribution to be well-defined, it is sufficient to assume $f_0 > 0$, but we restrict ourselves to the cases where all levels are present with a positive probability.

$\sigma = (\sigma_0, \dots, \sigma_K)$ where $\sigma_k \in \Delta(S)$ for all $k \in \{1, \dots, K\}$ is an ECH equilibrium when there exists a global distribution f such that for each k , σ_k is a best response assuming that the other players' levels are drawn from the truncated distribution.

Definition 1. A sequence of symmetric strategies $\sigma = (\sigma_0, \dots, \sigma_K)$ is called **endogenous cognitive hierarchy equilibrium** when there exists a distribution f over \mathbb{N} under which

$$\text{supp}(\sigma_k) \subset \arg \max_{s_i \in S} \mathbb{E}_{s_{-i}} [u(s_i, s_{-i}) | g_k, \sigma], \quad \forall k \in \mathbb{N}_+,$$

where g_k is the truncated distribution induced by f such that

$$g_k(h) = \frac{f_h}{\sum_{m=0}^k f_m} \text{ for } h = 0, \dots, k,$$

and the expectation over s_{-i} is drawn from a distribution

$$\gamma_k(\sigma) := \sum_{m=0}^k g_k(m) \sigma_m,$$

for each player $j \neq i$.

A standard assumption is that the levels follow a Poisson distribution:

$$f^\tau(k) = \tau^k \frac{e^{-\tau}}{k!}.$$

This assumption is adopted in the literature and a detailed argument can be found in Camerer et al. (2004). We discuss the implications and limitations of the assumption later in following sections.

Next, we move to the analysis of voting situations modeled as a Condorcet jury model.

2.2. A Condorcet Jury Model. We consider a binary-state, binary decision making in the group of n players. The true state of the world takes one of the two values $\omega \in \{-1, 1\}$ with equal probabilities, and it is commonly known. The utility is a function of the realized state and the decision:

$$u(d, \omega) = \begin{cases} 0 & \text{if } d \neq \omega, \\ q & \text{if } d = \omega = 1, \\ 1 - q & \text{if } d = \omega = -1, \end{cases}$$

for each individual.⁶ Voter i receives a private signal that is normally distributed around the true state, $s_i \sim \mathcal{N}(\omega, \sigma)$. The signals are distributed independently conditional on the true state. Then, i submits a vote v_i upon receiving a signal, which is in the form of added bias, i.e., $v_i(s_i) = s_i + b_i$, so an individual i 's strategy, in fact, is to pick a bias,

⁶The assumption of symmetric prior is without loss of generality since we allow the utility loss of the two types of errors to be heterogeneous.

$b_i \in \mathbb{R}$. The decision is reached by the unbiased rule, that is by the sign function on the summation of votes: $\mu(v) = \text{sgn}(\sum_{i \in N} v_i)$. The expected utility of $v = (v_1, \dots, v_i, \dots, v_n)$ can be written as

$$\mathbb{E}_{(s_i)_{i \in N}, \omega} [u(\mu((v_i(s_i))_{i \in N}, \omega))].$$

We obtain the best response function

$$\beta(b_{-i}) = - \sum_{j \in N \setminus \{i\}} b_j + \frac{\sigma^2}{2} \ln(q/(1-q)),$$

where b_{-i} denotes the biases of all voters except i .⁷

2.2.1. *Group accuracy under cognitive hierarchies.* In this section we consider the group accuracy under cognitive hierarchy models. To do this let us fix $K = 2$, that is the highest level in the group to be 2.⁸ We first show that if the composition of levels stays the same, or in other words if the Poisson parameter that moves with the average level in the composition is fixed, increasing the jury size will increase the probability of correct decision. In other words, Condorcet property continues to hold.

Proposition 1. *Let τ be fixed. The probability of correct decision in ECH model converges to 1 as $n \rightarrow \infty$ while it goes to 1/2 in CH.*

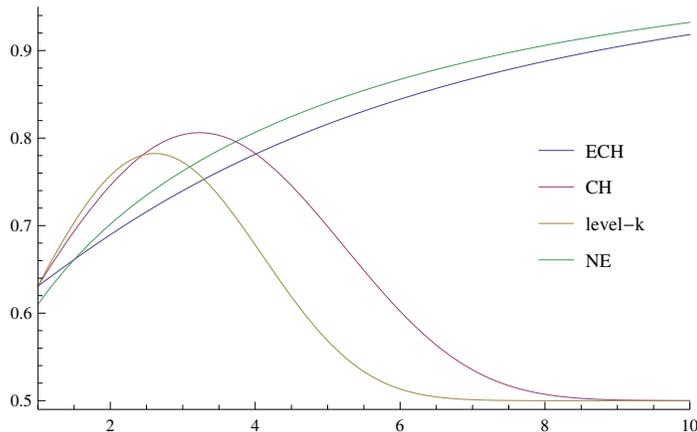


Figure 2.1. The probability of correct group decision under different models. In CH, ECH and level- k models $\sigma = 2, q = 3/4, \tau = 3$ and $b_0 = -1$ are taken, while for NE only $\sigma = 2$ and $q = 3/4$ are relevant.

⁷The algebra leading to this can be found in Appendix B.

⁸To specify an upper bound for levels is necessary for the analysis to be complete and in picking this particular value we rely on, first, the previous literature on cognitive hierarchy applications and second, ex-post, this assumption proves a good fit for our experimental results.

It is also shown that this is not the case in CH model where the accuracy is as good only as random voting, in the limit. Figure 2.1 shows the comparison of models in an exemplary modification. Proof of the proposition can be found in Appendix B.

2.3. Cognitive hierarchy as a model of complexity induced by group size. The idea of group bounded rationality may be argued to be represented best when we have decreasing sophistication as the complexity of the problem increases. In our context, we can think of the complexity as positively correlated with the size of the jury. And as shown in what follows, under certain assumptions, this leads to moving away from the Condorcet property.

Proposition 2. *Let $\tau = n^{-1}$. Then the probability of correct decision converges to 1 if $b_0 \in (-2, 2)$, collapses to 1/2 if $|b_0| > 2$ and converges to 3/4 if $b_0 \in \{-2, 2\}$.*

In fact, we conjecture that the speed of the fall in τ is negatively correlated with the bound on the size of the level-0 bias to preserve Condorcet property. In our calculations we see that if $\tau = n^{-\gamma}$, the interval $(-a, a)$ where a b_0 within secures Condorcet property is shrinking as $\gamma > 1$ is increasing. Proof of the proposition can be found in Appendix B.

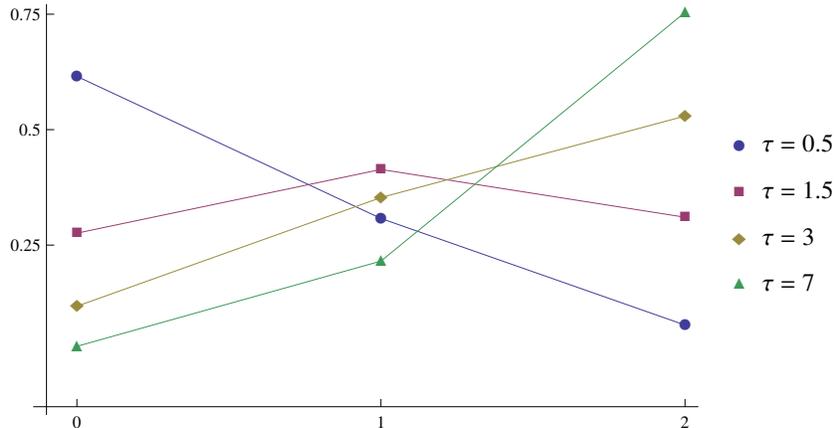


Figure 2.2. The truncated distributions for several Poisson parameters. The corresponding means are 0.46 ($\tau = 0.5$), 1.03 ($\tau = 1.5$), 1.41 ($\tau = 3$) and 1.72 ($\tau = 7$).

The conclusions of this section are based on the Poisson distribution assumption, for which truncated distributions for several parameters are shown in Figure 2.2. A foundation for this assumption given in Camerer et al. (2004) is as follows. If it is assumed, due to working memory limits, that as k increases fewer and fewer players do the next step of thinking beyond k , we may constrain $f(k)/f(k-1)$ to decline with k . If, furthermore, this decline is proportional to $1/k$, we obtain Poisson distribution.

However, since what matters is the change in the composition of levels, the relevant conclusions would not be affected by the particular choice of distribution. The reasoning is that in the case cognitive aspects of the current situation is not related at all to the complexity induced by the group size, having larger groups would comprise higher presence of naïve thinking only in absolute terms, not in proportional terms, and hence the bias can be addressed accurately in ECH due to presence of also higher levels but not in CH. In the case complexity induced by higher group size translates into more naïve thinking in proportional terms, large enough level-0 bias may worsen the outcome in ECH model as well.

3. EXPERIMENTAL DESIGN

All our -computerized- experimental sessions were run at the Experimental Economics Laboratory of Ecole Polytechnique in November and December 2013.⁹ In total we had 180 actual participants and 9 sessions. Each session with 20 subjects lasted about one hour. Earnings were expressed in points and exchanged for cash to be paid right after each session. Participants earned an average of about 21 Euros, including 5 Euros of show-up fee. Complete instructions and details can be found in an online appendix.¹⁰ The instructions pertaining to whole experiment were read aloud in the beginning of each session. Before each phase (treatment) during the experiment, the changes from the previous phase are read aloud and an information sheet including the relevant details of the game is distributed. These sheets are collected while distributing the sheets for the following phase.

We employed a within-subjects design where each subject played all four phases consecutively in a session. Following a direct-response method, in each phase there were 15 periods of play, which makes 60 periods in total that are played by each participant.¹¹ Since the question of our research relates to the strategic aspect of group decision, our experiment was presented to subjects as an abstract group decision-making task where natural language is used to avoid any reference to voting or election of any sort. In the beginning of each period, computer randomly formed groups of subjects, where the common size of the groups is determined by the current phase. Then the subjects are shown a box

⁹We utilized a z-Tree (Fischbacher (2007)) program and a website for registrations, both developed by Sri Srikandan.

¹⁰The online appendix can be found at <http://sites.google.com/site/ozkesali>.

¹¹In third phase, two groups of nine randomly chosen members are formed before each period. Having 20 participants in total, hence, two randomly chosen subjects waited during each period. The same method is applied in fourth phase as well, this time one random subject waited at each period. Subjects are told both orally and through info sheets that in the case the random lottery incentive mechanism picks a period where a subject has been waiting, the payoff from that phase for this subject will be taken as 500 points.

full of hundred cards, all colorless (gray in z-Tree). This is also when the unknown true color of the box for each group is determined randomly by the computer. Subjects were informed that the color of the box would be either blue or yellow, with equal probability. Furthermore, it was common knowledge that the blue box contained 60 blue and 40 yellow cards whereas the yellow box contained 60 yellow and 40 blue cards. After confirming to see the next page, they are shown only 10 randomly drawn cards with random locations in the box on the screen, this time with their true colors. These draws were independent among all subjects but were coming from the same box for subjects in same groups.

On the very screen subjects observed their 10 randomly drawn cards, they are required to vote for either blue or yellow. The decision for the group is reached by majority rule, which was conclusive all the time since we only had odd number of subjects in groups and abstention was not allowed. Once everyone in a group voted by clicking the appropriate button, on a following screen, subjects are shown the true color of the box, the number of the votes for blue, the number of votes for yellow and the earning for that period. Once everyone confirmed, the next period started.

	Phase 1	Phase 2	Phase 3	Phase 4
Sessions	$n = 5$	$n = 5$	$n = 9$	$n = 19$
A,B	500 : 500	800 : 300	800 : 300	800 : 300
1-7	500 : 500	900 : 200	900 : 200	900 : 200

Table 3.1. Experimental design. Each phase consists of 15 periods.

The payoffs were symmetric at the first phase of each session. The size of the groups for each period was fixed to be 5 in this phase and each subject earned 500 points in any correct group decision (i.e., blue decision when true color of the box is blue and yellow decision when true color of the box is yellow). In case of incorrect decision no points earned.

The following three phases for each treatment differed only in the size of the groups (5,9 or 19) where a biased payoff scheme is fixed. For two sessions (Sessions A and B), the correct group decision when true color of the box is blue earned each subject 800 points whereas the correct group decision when true color of the box is yellow earned 300 points. For seven sessions (Sessions 1 to 7), the correct group decision when true color of the box is blue earned each subject 900 points whereas the correct group decision when true color of the box is yellow earned 200 points. The summary of these can be found in Table 3.1. We implemented a random lottery incentive system where final payoffs at each phase is determined by the payoffs from a randomly drawn period. In the beginning of each session, during general instructions being read aloud and as part of instructions, subjects played two forced trial periods.

Each session concluded after a short questionnaire, which can be found in the online appendix.

3.1. Equilibrium Predictions. In what follows we provide the theoretical predictions given the parameter values used in our experiment. In doing so, we consider only (noisy) symmetric Nash equilibrium for now. To compare with the data, we first estimate individual strategies that are restricted to be in form of cutoff strategies. Specifically, each player is assumed to have a cutoff value, so that she votes for blue in case of observing higher number of blue cards than this cutoff and for yellow otherwise. Since we have a direct-response method, these cutoff strategies must be estimated from observed behaviors. Thus, given 12 plays in a phase after excluding the first three periods, we estimate by maximum likelihood method a cutoff strategy for each player for each phase, assuming behaviors to follow a logit probabilistic function with individual-specific error parameters that are common across phases.¹² Table 3.2 provides the cutoff strategy of the unique symmetric Nash equilibrium for each phase.¹³ We don't have experimental observations with unbiased prior and group sizes 9 and 19.

Session	Payoff	$n = 5$	$n = 9$	$n = 19$	#
1 – 7	900 : 200	4.43	4.67	4.84	140
A, B	800 : 300	4.66	4.84	4.93	40
A, B & 1 – 7	500 : 500	5	–	–	180

Table 3.2. The average symmetric equilibria cutoff strategies. Rightmost column is the number of observations.

The equilibrium strategies get closer to 5, the unbiased play, as the size of the groups increases. This feature also holds as the payoff bias decreases, for which if the group size is 5 we also have unbiased case. Table 3.3 below shows the predicted accuracy of group decisions for each phase.

Session	Payoff	$n = 5$	$n = 9$	$n = 19$
1 – 7	900 : 200	0.833	0.913	0.980
A, B	800 : 500	0.868	0.935	0.987
A, B & 1 – 7	500 : 500	0.879	–	–

Table 3.3. The average predicted accuracy of group decisions given equilibrium strategies.

¹²The scale parameter values are also endogenously determined by the maximum likelihood estimations and are used throughout the analysis of the experimental data as individual's logistic error values.

¹³These values are averages for each phase. The values calculated by using each session's average errors can be found in the Table A.1.

As discussed before, the group accuracy in each case increases as the group size increases. Also, as the prior bias decreases we see that the group accuracy increases, for which if the group size is 5 we also have unbiased case. We will investigate the experimental data in light of these comparisons at both the individual and group level. Table 3.3 gives the global averages, session specific averages can be found in detail in Table A.1.

4. EXPERIMENTAL RESULTS

In this section we present and analyze our experimental results by investigating behaviors of our subjects at both individual level and group level.

4.1. Individual behavior. In the whole experiment, only five percent of the subjects are observed to behave that seems to be different than a cutoff strategy. Figure 4.1 below shows the histograms for distributions of estimated cutoff values from those sessions we take into consideration throughout the analysis, namely Sessions 1 to 7.

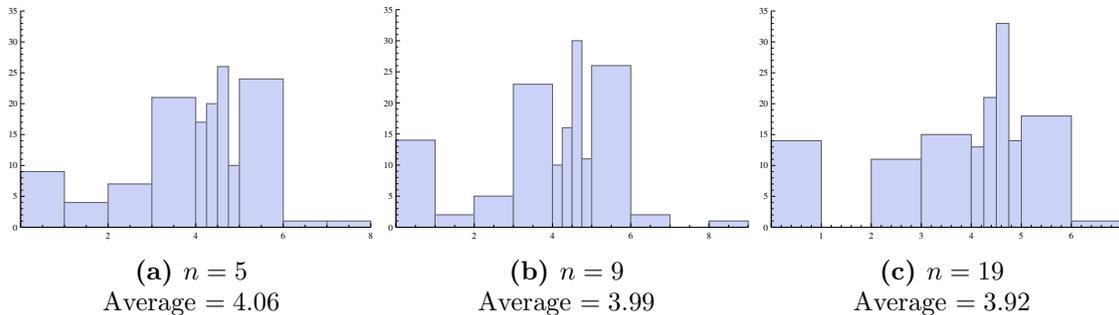


Figure 4.1. The global histograms of the cutoff strategies for each phase within Sessions 1 – 7. Hence we have, in total, 140 observations for each case where the payoffs are 900 : 200. Note that the intervals $[4, 5)$, which are the most voluminous, are divided into four equal subintervals.

The frequencies of the interval $[4, 5)$ for group sizes 5,9 and 19 are 73%, 67% and 81%, respectively. On the other hand, the frequencies of the interval $[0, 1)$ are 9%, 14% and 14%. The frequencies of cutoffs that are higher than or equal to 5 are 26%, 29% and 19%. We want to emphasize at this point that these two latter significant frequencies constitute the major subject of our analysis. Table 4.1 below provides the averages for each phase.

Session	Payoff	$n = 5$	$n = 9$	$n = 19$
1 – 7	900 : 200	4.06	3.99	3.92
A, B	800 : 300	4.26	4.46	4.34
A, B & 1 – 7	500 : 500	4.85	n/a	n/a

Table 4.1. The averages for estimated cutoff strategies.

We see that as the payoff bias increases the average biases towards the favored alternative (namely, blue) increases, which resonates with Table 3.2. This observation extends also to the comparisons including the unbiased phase.

However, the claim that the bias in strategies towards the favored alternative should decrease as groups get larger due to pivotality approach argument, which is suggested by our theoretical predictions as well, occurs to be rejected for our data. Consider the first row of Table 4.1, which shows the averages for the cutoff estimations for each biased payoff phase in Sessions 1 to 7.¹⁴ The average cutoff values in data appears to be slightly decreasing with the size of the group in the case of highly biased payoffs. Furthermore, although the comparisons in between group sizes 5 and 9, and 5 and 19 conform with theoretical predictions, the comparison between the sizes 9 and 19 differs highly. Finally, relying on the comparison within columns -especially the first one- of Table 4.1, we can say that our subjects appear to react in a way predicted by the theory to the biases in payoffs.

4.2. Group decision accuracy. Table 4.2 below shows the averages for observed accuracies of group decisions.¹⁵

Session	Payoff	$n = 5$	$n = 9$	$n = 19$
1 – 7	900 : 200	0.776	0.833	0.774
A, B	800 : 300	0.813	0.833	0.958
A, B & 1 – 7	500 : 500	0.806	n/a	n/a

Table 4.2. The averages for observed frequencies of correct group decisions.

We see some apparent discrepancies with the theoretical predictions. First thing to note is that in all cases groups do worse than theory’s predictions. Furthermore, comparing first rows of Tables 3.3 and 4.2, we see that the Condorcet property cannot be confirmed with our data. Namely, as the size of the group increases we should see an increase in the accuracy of group decisions. Although significantly different from the absolute values of theoretical predictions, we see increase in accuracy, as expected, as moving from group size 5 to 9 or 5 to 19. More important is the fact that the observed accuracy of group decision in the case of 19 players is the lowest of the three. However, if we run a difference in proportions test for the Sessions 1 – 7, the differences don’t seem to be significant.

Guarnaschelli et al. (2000) also report decreasing accuracy with larger juries under majority rule and binary signals. They conclude that group accuracy might not be as robust as to accommodate small changes in individual behavior. However, in our experimental

¹⁴For session-specific phase averages for cutoff estimations, see Table A.2.

¹⁵For session-specific phase averages for group decision accuracies, see Table A.3.

data, since we see discrepancies from theoretical predictions in individual behaviors as well we cannot relate inaccuracy observations fully to vulnerability.

The fall in the group accuracy when moving from the 800 : 300 to unbiased can be explained by the facts that the unbiased phases were the very first phases of each session and if we look within session values it disappears.¹⁶

4.3. Cognitive hierarchy models.

4.3.1. *Level-k approach.* Since a Lk player is reacting against her belief of $n - 1$ players playing a $k - 1$ level strategy, the need for playing in an opposite manner that is necessary to correct for the bias of the $Lk-1$ is amplified. So, if $L0$ strategy is to play a cutoff rule that is biased towards an alternative, the $L1$ can do best only by playing in an opposite way and in higher magnitude.¹⁷ Thus, a $L1$ voter will use strategy 10, under her belief that level-0 play is 0. The same argument applies and $L2$ plays 0. This oscillation continues, of course, to the infinity.

Level-0	Level-1	Level-2
0	10	0

Table 4.3. Strategies under level- k approach.

We will fix the strategy 0 to represent the level-0 play throughout our analysis of cognitive hierarchy models since this appears as a natural candidate: the bias towards voting for blue created by payoff scheme may tempt the easy or unsophisticated thinking of the problem. To confirm a level- k thinking approach then, we should see only 0s and 10s in our observations, a nonexistent phenomenon, so we do not attempt at explaining our experimental data with such model. Battaglini et al. (2010) are the first to observe the anomaly of level- k thinking model in Condorcet jury models.

4.3.2. *Poisson-CH model.* The Poisson-CH model postulates that a CHk player is reacting against her belief of $n - 1$ players coming from a truncated Poisson distribution of levels up to her own level. The $CH1$ play is based on beliefs that are exactly the same with the level- k approach, hence it is to have cutoff strategy 10 if $CH0$ is to play the strategy 0.

To calculate predictions for Poisson-CH model, we need to specify the Poisson parameter, τ , and the upper bound for how many levels exist. For the latter, relying on the

¹⁶Specifically, 0.885 is the average of the frequencies of correct decisions in the first phases of the two sessions (A and B).

¹⁷Not that if level-0 play is assumed to be to vote for blue all the time, a $L1$ voter will see that she is never pivotal and will be indifferent. However, since we base our analysis on cutoff strategies that take the form of logistic functions, we always have nonzero probability for being pivotal.

previous experimental findings and the fact that it is enough to see some sort of convergence, we restrict attention to the case where we have only up to 2 levels of cognitive hierarchy. For the former we find the best fitting parameter within the interval $[0, 3]$.

Session	$n = 5$		$n = 9$		$n = 19$	
	<i>CH2</i>	<i>LL</i>	<i>CH2</i>	<i>LL</i>	<i>CH2</i>	<i>LL</i>
1	3.176	-89.275	3.156	-107.466	3.101	-111.062
2	2.440	-73.222	2.330	-76.270	2.049	-83.437
3	3.040	-80.131	3.006	-89.057	2.916	-94.087
4	2.774	-78.152	2.712	-74.629	2.560	-78.758
5	3.173	-153.911	3.152	-93.234	3.097	-97.726
6	2.834	-69.145	2.779	-74.191	2.643	-79.065
7	3.040	-74.943	3.006	-84.370	2.917	-88.523

Table 4.4. The *CH2* strategies and the log-likelihood values when level-0 is 0, *CH1* is 10 and the Poisson parameter is 3 under the assumption that there are only up to two levels of cognitive hierarchy existing in the group.

Table 4.4 shows the plays of second level voters, predicted by the Poisson-CH model for sessions 1 to 7, as well as the log likelihood values if we are to explain data by the relevant specification. The log-likelihood values are observed to monotonically increase within $\tau \in [0, 3]$, so we provide in Table 4.4 the boundary case, i.e., $\tau = 3$.

4.4. Endogenous Cognitive Hierarchy model. For the analysis of ECH model, we keep the restrictions applied to the Poisson-CH. Hence, the Poisson parameter is fixed to lie in the interval $[0, 3]$ and the level distribution is restricted to include only up to level-2.

Table 4.5 shows the best fitting ECH models for the Phase 2, under aforementioned restrictions. As seen in the second column, most of the sessions have the feature of being better a fit with highest possible τ , namely 3. Sessions 2 and 4 are exceptions, where we see lower τ values that makes the model fit better, which is partly due to the relatively high frequencies of cutoff estimates that are lower than 4. In particular, we have five subjects with cutoffs lower than 3 in both Sessions 2 and 4, and two subjects with approximately 0 in the Session 2 while it is three in Session 4.

Session 1 also has two subjects with cutoff value 0. However there is only one more subject with a cutoff value lower than 3. The other sessions have at most one subject with a cutoff value of 0.

Letting τ be larger than 3, but still below 10, we have the following further observations for Phase 2. Sessions 1,3 and 7 can be explained best with the truncated Poisson ECH model with a Poisson parameter lower than 10. These are 5 for Sessions 1 and 3 with log-likelihoods -26.12 and -25.45, respectively. For Session 7, $\tau = 7$ gives the lowest *LL*

Session	τ^*	Level 0	Level 1	Level 2	LL
1	3	0	4.705	4.535	-26.291
2	2.2	0	4.636	4.425	-39.647
3	3	0	4.687	4.508	-25.739
4	2	0	4.726	4.519	-32.101
5	3	0	4.704	4.534	-23.688
6	3	0	4.652	4.464	-28.340
7	3	0	4.688	4.508	-33.976

Table 4.5. ECH model specifications when for the case $n = 5$ with log-likelihood values.

value which is -32.69. Sessions 5 and 6, hits the boundary, i.e., 10, in terms of best fitting τ , which explains the comparison between ECH and NE in Table 4.8.

When we move to the phase 3, namely the case $n = 9$, we have a similar picture. Table 4.6 provides the model specifications and log-likelihood values for phase 3. First thing to note is that only the model specification for Session 4 fits best the data when τ is lower than the bound we imposed, namely 3. This occurs at around the same value as in the previous phase, namely 2.1. This is not observed to be the case for Session 2.

Session	τ^*	Level 0	Level 1	Level 2	LL
1	3	0	5.123	4.761	-33.818
2	3	0	5.047	4.690	-33.68
3	3	0	5.111	4.756	-25.238
4	2.1	0	5.175	4.792	-35.984
5	3	0	5.123	4.761	-33.237
6	3	0	5.090	4.731	-38.772
7	3	0	5.111	4.756	-41.256

Table 4.6. ECH model specifications when for the case $n = 9$ with log-likelihood values.

Now consider the conjecture that the best fitting τ will be decreasing as the group size increases. Clearly, data from Session 2 contradict already with this and Session 4 does not represent a clear difference. As noted above, the Sessions 1 and 3 had 5 as the best fitting τ value. While it is not lower than 5 for Session 3, for Session 1 we have 4.5 now, complying with the argument.

The best fitting Poisson parameter for Session 5 is not lower in Phase 3 than its value in Phase 2. In fact, this phase continues to be better explained by NE. See Table 4.8.

On the other hand, significantly lower τ values specify the best fitting ECH models for Sessions 6 and 7. These values are 4 and 6, respectively, in Phase 3, whereas these were 10 and 7 in Phase 2.

Moving to the final phase, we do not see anymore Poisson parameter values of best fitting ECH models that are lower than our first upper bound of 3. Furthermore, none of the sessions have the complying feature, comparing to Phase 3 values. Table 4.7 is devoted to Phase 4.

Session	τ^*	Level 0	Level 1	Level 2	LL
1	3	0	5.4657	4.927	-29.06
2	3	0	5.4524	4.893	-37.529
3	3	0	5.4634	4.9	-47.63
4	3	0	5.4586	4.9	-38.449
5	3	0	5.4657	4.927	-41.454
6	3	0	5.4597	4.9	-38.114
7	3	0	5.4634	4.9	-36.027

Table 4.7. ECH model specifications when for the case $n = 19$ with log-likelihood values.

4.5. Which model fits the best? In this section we compare the performances of models in explaining our experimental data. First, note that in all sessions ECH fits better than CH. In some sessions and phases we have NE fitting better than ECH, although not plenty. But it is quick to observe that there exists high enough τ that could result in ECH dominating NE. This is due to the fact that as τ increases, the level-2 becomes dominant and at the limit only level-2 remains which is basically the symmetric Nash play.

Session	$n = 5$			$n = 9$			$n = 19$		
	CH	ECH	NE	CH	ECH	NE	CH	ECH	NE
1	-80.88	-26.07	-53.31	-101.25	-33.04	-58.31	-116.38	-21.95	-34.43
2	-70.73	-39.65	-46.52	-81.51	-33.62	-45.40	-105.27	-37.36	-51.35
3	-71.84	-25.42	-36.74	-84.11	-22.06	-31.08	-101.45	-38.12	-35.85
4	-76.82	-32.10	-53.92	-79.09	-35.98	-72.29	-88.45	-38.45	-70.05
5	-138.55	-21.46	-21.32	-80.79	-26.68	-24.06	-99.06	-35.08	-47.56
6	-60.81	-25.35	-25.11	-75.29	-37.91	-51.13	-88.34	-34.16	-40.14
7	-65.76	-32.69	-41.86	-78.42	-39.60	-49.08	-93.12	-33.34	-54.27

Table 4.8. Comparison of log-likelihoods of the models. Here, τ is restricted to $[0, 3]$ and the best fitting one is taken, for both of ECH and CH.

As discussed above also, in Phase 2, data belonging to Sessions 5 and 6 can be explained better by NE, if $\tau \leq 10$, although the difference is not large. Only Session 5 carries this comparison to Phase 3, and none to Phase 4. However, Phase 4 of Session 3 appears to be explained better by NE. Thus, we have ECH dominating the two other models in 18 cases out of 21, under current restrictions on τ .

Phase	1	2	3	4	5	6	7	ave.
n=5	0.824	0.730	0.824	0.752	0.844	0.822	0.832	0.804
n=9	0.904	0.852	0.903	0.839	0.918	0.887	0.908	0.887
n=19	0.979	0.946	0.977	0.957	0.984	0.969	0.978	0.970

Table 4.9. The predicted group accuracy for each phase in each session under best fitting ECH models.

4.5.1. *Group Accuracy under ECH.* Table 4.9 shows the group accuracy predictions of the best fitting ECH models. Comparing with the first row of Table 4.2, the last column of Table 4.9 does not appear to be a good predictor for the observed group accuracy. This holds true for the session-wise comparisons as well. But it should be noted that the differences in observations failed significance tests. Nonetheless, ECH gives predictions that are closer than NE predictions to data for any session. The average values for the latter is already presented in Table 3.3.

Furthermore, as discussed in Section 2.3, the Poisson distribution assumption plays a significant role here. As can be seen also in Figure 2.2, for all high enough values of τ , for instance, we have level-1 having higher share in the distribution compared to level-0 which clearly restricts the estimations. Removing any such assumption on distribution and estimating using maximum likelihood methods from data might give better results.¹⁸ It should be noted that current analysis avoids -in a way- overfitting that would emerge under such approach.

5. DISCUSSION ON ECH AND CH MODELS

In certain classes of games, ECH behaviors differ significantly from those of CH.

5.1. **Perfect substitution.** Suppose that the game has perfect strategic substitutability. The best reply of a player is to correct the sum of the biases caused by the other players' strategies:

$$BR_i(s_{-i}) = - \sum_{j \neq i} s_j.$$

Proposition 3. *Suppose that the game has perfect strategic substitutability. Let b_k be the bias from symmetric Nash strategy. Then, for any $b_0 \neq 0$, the ratio b_k/b_0 grows in the order of n^k for Lk and CHk while it grows in the order of n^0 for ECHk.*

Proof of the proposition can be found in Appendix B.

¹⁸We obtained some better fitting *Poisson-free* models of ECH for some sessions, these can be provided upon request.

5.2. Cournot competition. Consider a standard Cournot competition with linear demand and constant marginal cost of production which represents a game with strategic substitutability. We have that the best response function is a contraction if $n = 2$, an expansion if $n > 3$. In particular, let the best reply function be $q' = \frac{1}{2}(1 - Q_{-i})$. Since the profit function is quadratic, we have the best response function under uncertainty as a function of the expectation:

$$q' = \frac{1}{2}(1 - \mathbb{E}[Q_{-i}]).$$

The Nash equilibrium production level is $q = \frac{1}{n+1}$.

\hookrightarrow *Level-k Approach*

Let q_0 denote $L0$ play. Then $L1$'s best response is to correct for the bias created by $n - 1$ others:

$$q_1 = \frac{1}{2}(1 - (n - 1)q_0).$$

Similarly, $L2$'s best response is:

$$\begin{aligned} q_2 &= \frac{1}{2} \left(1 - (n - 1) \frac{1}{2} (1 - (n - 1)q_0) \right), \\ &= \frac{1}{2} \left(1 - \frac{1}{2}(n - 1) \right) + \frac{(n - 1)^2}{4} q_0. \end{aligned}$$

Hence $L2$ will do best by reacting in an opposite and amplified way compared to $L1$. So $L2$'s reaction will be destined to be biased at least as much as and in the direction of $L0$. Naturally, $L3$ will be tending towards hitting the other corner. Iterating along in k , a bang-bang type oscillation in strategy space is required.

\hookrightarrow *CH Model*

By definition, $CH1$ is the same as $L1$:

$$q_1 = \frac{1}{2}(1 - (n - 1)q_0).$$

For $CH2$, we have

$$\begin{aligned} q_2 &= \frac{1}{2} \left(1 - (n - 1) \left(\frac{f_0}{f_0 + f_1} q_0 + \frac{f_1}{f_0 + f_1} q_1 \right) \right), \\ &= \frac{1}{2} \left(1 - \frac{(n - 1)f_1}{2(f_0 + f_1)} \right) - (n - 1) \frac{f_0 - (n - 1)f_1}{2(f_0 + f_1)} q_0, \end{aligned}$$

where f_0 and f_1 represent the weights of $CH0$ and $CH1$ under global distribution f , respectively.

\hookrightarrow *ECH Model*

An *ECH1* player's belief assigns weighted frequencies to *ECH0* and *ECH1* that are the same with a *CH2* belief above. Hence *ECH1* best responds as

$$q_1 = \frac{1}{2} \left(1 - (n-1) \left(\frac{f_0}{f_0 + f_1} q_0 + \frac{f_1}{f_0 + f_1} q_1 \right) \right),$$

which can be rearranged as

$$\left(2 + (n-1) \frac{f_1}{f_0 + f_1} \right) q_1 = \left(1 - (n-1) \frac{f_0}{f_0 + f_1} q_0 \right).$$

Similarly found is *ECH2*'s best response:

$$q_2 = \frac{1}{2} (1 - (n-1) (g_2(0)q_0 + g_2(1)q_1 + g_2(2)q_2)).$$

That is,

$$q_2 = \frac{1 - (n-1) (g_2(0)q_0 + g_2(1)q_1)}{2 + (n-1) g_2(2)}.$$

We have symmetric Nash equilibria where $q^* = 0.0909$. Suppose now that f follows Poisson distribution with coefficient $\tau = 3$ and let $n = 10$. Tables 5.1 and 5.2 give the model specifications when $q_0 = 0.05$ and $q_0 = 0.2$ respectively.

	q_0	q_1	q_2
level- k	0.05	0.275	0
CH	0.05	0.275	0
ECH	0.05	0.101	0.092

Table 5.1. Model specifications for Cournot competition when $q_0 = 0.05$, $\tau = 3$ and $n = 10$.

	q_0	q_1	q_2
level- k	0.2	0	0.5
CH	0.2	0	0.275
ECH	0.2	0.063	0.087

Table 5.2. Model specifications for Cournot competition when $q_0 = 0.2$, $\tau = 3$ and $n = 10$.

5.3. Keynesian Beauty contest. On the other hand, in the games where the best reply function is a contraction mapping, behavior under ECH do not differ much from those under CH. Consider a standard Keynesian beauty contest game, a game with strategic complementarity, where n players simultaneously submit a number between 0 and 100, and the winner is the one who chooses a number closest to the $0 < p < 1$ times average of all submissions. The best response function is a contraction mapping defined as $BR_i(x_{-i}) =$

$\frac{p}{n-p} \sum_{j \neq i} x_j$. Suppose that the utility function is quadratic as a function of the distance from the winning number. Then the best reply under uncertainty is the expected winning number:

$$BR_i(x_{-i}) = \frac{p}{n-p} \mathbb{E} \left[\sum_{j \neq i} x_j \right].$$

hence, the unique NE is $x = 0$.

\hookrightarrow *Level-k Approach*

$L1$'s best response is:

$$b_1 = p \frac{n-1}{n-p} b_0.$$

$L2$'s best response is:

$$b_2 = p \frac{n-1}{n-p} b_1 = \left(p \frac{n-1}{n-p} \right)^2 b_0.$$

The strategies converge to Nash equilibrium levels.

\hookrightarrow *CH Model*

By definition, $CH1$ is the same as $L1$:

$$b_1 = p \frac{n-1}{n-p} b_0..$$

For $CH2$:

$$\begin{aligned} b_2 &= p \frac{n-1}{n-p} \left(\frac{f_0}{f_0 + f_1} b_0 + \frac{f_1}{f_0 + f_1} b_1 \right), \\ &= p \frac{n-1}{n-p} \left(\frac{f_0}{f_0 + f_1} + \frac{f_1}{f_0 + f_1} p \frac{n-1}{n-p} \right) b_0. \end{aligned}$$

\hookrightarrow *ECH Model*

$ECH1$'s best response is:

$$b_1 = p \frac{n-1}{n-p} \left(\frac{f_0}{f_0 + f_1} b_0 + \frac{f_1}{f_0 + f_1} b_1 \right).$$

Hence,

$$\left(1 - p \frac{n-1}{n-p} \frac{f_1}{f_0 + f_1} \right) b_1 = p \frac{n-1}{n-p} \frac{f_0}{f_0 + f_1} b_0.$$

$ECH2$'s best response is:

$$b_2 = p \frac{n-1}{n-p} (g_2(0)b_0 + g_2(1)b_1 + g_2(2)b_2).$$

Hence,

$$b_2 = \frac{p^{\frac{n-1}{n-p}} (g_2(0)b_0 + g_2(1)b_1)}{1 - p^{\frac{n-1}{n-p}}g_2(2)}.$$

The unique Nash equilibrium strategy when $p < 1$ is 0. Now suppose that f follows Poisson distribution with coefficient $\tau = 3$ and let $n = 10$. Tables 5.3 and 5.4 give the model specifications when $p = 2/3$ and $p = 1/2$ respectively.

	b_1/b_0	b_2/b_0
level- k	0.643	0.413
CH	0.643	0.470
ECH	0.310	0.221

Table 5.3. Model specifications for the Keynesian beauty contest when $p = 2/3$, $\tau = 3$ and $n = 10$.

	b_1/b_0	b_2/b_0
level- k	0.474	0.224
CH	0.474	0.287
ECH	0.184	0.115

Table 5.4. Model specifications for the Keynesian beauty contest when $p = 1/2$, $\tau = 3$ and $n = 10$.

6. CONCLUSION

We have delivered results from an experiment designed to test the effect of group size on decision accuracy. The experimental results suggest systematic heterogeneity among individual behaviors. We provide a cognitive hierarchy model that advances upon the current ones by allowing self-level consideration by individuals to account for this systematic deviation from symmetric equilibrium thinking. This model is theoretically shown to be able to accommodate our observations and performs better than previous models of cognitive hierarchy as well as symmetric Bayesian Nash equilibrium in explaining individual behavior.

The modeling consists of specification of distribution for cognitive hierarchy levels. In this paper we employed Poisson distribution for that regard. This specification hints that the bounded rationality of a group that is related to the sophistication induced by the size of the group resonates with the best fitting Poisson parameter. In other words, we

would expect to see that as group size increases, since the problem in hand would get more demanding, nonstrategic play that is seen to appear in a systematic way should be amplified and hence best fitting Poisson parameters would be lower. And if this is severe enough, we could capture the decrease in group decision accuracy. However, we don't see this in our data. Consequently, the model falls short of explaining lower group decision accuracy in larger groups. It should be noted, on the other hand, that the particular restriction of Poisson distribution assumption plays a role here and dropping that would lead to get closer to the group bounded rationality argument above, since we already see increase in tendency towards lower levels.

In current experimental design learning and ordering effects were not targets of investigation. Battaglini et al. (2010) and Bhattacharya et al. (2013) reports no observation of significant learning effects. Furthermore, we excluded first three periods of all treatments in our analysis and implemented a random matching design so that at each period subjects knew that their group is randomly formed with possibly different members.

REFERENCES

- Austen-Smith, D. and Banks, J. S. (1996). Information aggregation, rationality, and the condorcet jury theorem. *American Political Science Review*, pages 34–45.
- Battaglini, M., Morton, R. B., and Palfrey, T. R. (2008). Information aggregation and strategic abstention in large laboratory elections. *The American Economic Review*, pages 194–200.
- Battaglini, M., Morton, R. B., and Palfrey, T. R. (2010). The swing voter's curse in the laboratory. *The Review of Economic Studies*, 77(1):61–89.
- Bernheim, B. D. (1984). Rationalizable strategic behavior. *Econometrica: Journal of the Econometric Society*, pages 1007–1028.
- Bhattacharya, S., Duffy, J., and Kim, S. (2013). Information acquisition and voting mechanisms: Theory and evidence. Technical report, Working paper, University of Pittsburgh.
- Camerer, C. (2003). *Behavioral game theory: Experiments in strategic interaction*. Princeton University Press.
- Camerer, C. F., Ho, T.-H., and Chong, J.-K. (2004). A cognitive hierarchy model of games. *The Quarterly Journal of Economics*, pages 861–898.
- Costinot, A. and Kartik, N. (2007). On optimal voting rules under homogeneous preferences. Technical report, working paper.
- Crawford, V. P., Costa-Gomes, M. A., and Iriberri, N. (2013). Structural models of nonequilibrium strategic thinking: Theory, evidence, and applications. *Journal of Economic Literature*, 51(1):5–62.

- Downs, A. (1957). *An economic theory of democracy*. New York: Harper Collins Publishers.
- Esponda, I. and Vespa, E. (2013). Hypothetical thinking and information extraction: Strategic voting in the laboratory. Technical report, Mimeo, NYU Stern.
- Feddersen, T. and Pesendorfer, W. (1997). Voting behavior and information aggregation in elections with private information. *Econometrica: Journal of the Econometric Society*, pages 1029–1058.
- Fischbacher, U. (2007). z-tree: Zurich toolbox for ready-made economic experiments. *Experimental economics*, 10(2):171–178.
- Gerling, K., Grüner, H. P., Kiel, A., and Schulte, E. (2005). Information acquisition and decision making in committees: A survey. *European Journal of Political Economy*, 21(3):563–597.
- Guarnaschelli, S., McKelvey, R. D., and Palfrey, T. R. (2000). An experimental study of jury decision rules. *American Political Science Review*, pages 407–423.
- Koriyama, Y. and Szentes, B. (2009). A resurrection of the condorcet jury theorem. *Theoretical Economics*, 4(2):227–252.
- Nagel, R. (1995). Unraveling in guessing games: An experimental study. *The American Economic Review*, pages 1313–1326.
- Palfrey, T. R. (2013). Experiments in political economy. *Handbook of Experimental Economics*, 2.
- Stahl, D. O. and Wilson, P. W. (1995). On players models of other players: Theory and experimental evidence. *Games and Economic Behavior*, 10(1):218–254.

CJT UNDER COGNITIVE HIERARCHIES

APPENDIX A. PREDICTIONS AND EXPERIMENTAL RESULTS

Session	Phase 1		Phase 2		Phase 3		Phase 4		e
	σ	w	σ	w	σ	w	σ	w	
<i>A</i>	5	0.888	4.67	0.868	4.84	0.935	4.93	0.987	0.164
<i>B</i>	5	0.887	4.66	0.868	4.83	0.935	4.93	0.987	0.185
1	5	0.886	4.48	0.849	4.70	0.922	4.87	0.984	0.233
2	5	0.856	4.33	0.802	4.61	0.890	4.81	0.970	0.554
3	5	0.882	4.46	0.841	4.68	0.918	4.85	0.983	0.297
4	5	0.871	4.40	0.823	4.65	0.907	4.83	0.978	0.420
5	5	0.886	4.48	0.849	4.70	0.922	4.87	0.984	0.229
6	5	0.874	4.42	0.827	4.66	0.910	4.83	0.979	0.396
7	5	0.882	4.46	0.841	4.68	0.918	4.85	0.983	0.297
1 – 7 ave.			4.43	0.833	4.67	0.913	4.84	0.980	0.346
<i>A, B</i> ave.			4.66	0.868	4.84	0.935	4.93	0.987	0.175
All ave.		0.879							0.308

Table A.1. Nash equilibrium strategies, σ , and predicted group accuracies, w , including the unbiased phase and A and B sessions. Rightmost column gives session averages for logistic error value estimations that are used in calculating strategies.

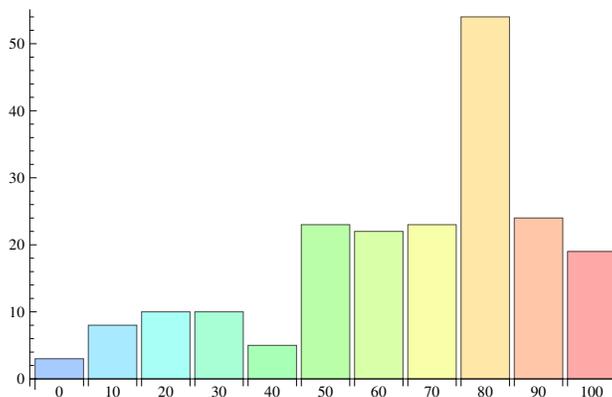


Figure A.1. Frequencies of responses to the Question 2a: “If answered ”Yes” in Question 2, what is the percentage of the other participants using the same reasoning, according to your estimation?”.

Phase	A	B	1	2	3	4	5	6	7
1	5.01	4.78	4.97	4.52	4.74	5.20	5.13	4.46	4.85
2	4.27	4.24	3.91	4.00	4.00	3.74	4.53	4.19	4.05
3	4.64	4.29	4.12	3.86	4.25	3.23	4.39	4.05	4.01
4	4.47	4.21	4.29	3.54	4.06	3.29	4.17	4.08	4.00

Table A.2. Phase averages for cutoff estimations in each session.

Phase	A	B	1	2	3	4	5	6	7
1	0.938	0.938	0.833	0.708	0.896	0.708	0.854	0.750	0.625
2	0.833	0.792	0.792	0.667	0.854	0.792	0.792	0.792	0.771
3	0.833	0.833	0.792	0.708	0.917	0.833	0.875	0.917	0.792
4	1.000	0.917	0.917	0.500	0.917	0.667	0.833	0.750	0.833

Table A.3. Phase averages for accurate group decision frequencies in each session.

APPENDIX B. PROOFS

The expected utility

$$\mathbb{E}_{(s_i)_{i \in N}, \omega} [u(\mu((v_i(s_i))_{i \in N}, \omega))]$$

can be written as

$$\frac{q}{2} \Pr \left[\sum_{1 \leq j \leq n} s_j + \sum_{1 \leq j \leq n} b_j > 0 \mid \omega = 1 \right] + \frac{1-q}{2} \Pr \left[\sum_{1 \leq j \leq n} s_j + \sum_{1 \leq j \leq n} b_j < 0 \mid \omega = -1 \right].$$

Since the sum of signals are also distributed normally, i.e. with $\mathcal{N}(\omega n, \sigma \sqrt{n})$, we have the following first order condition

$$q \phi \left(\frac{-\sum_{j \neq i} b_j - b_i - n}{\sigma \sqrt{n}} \right) - (1-q) \phi \left(\frac{-\sum_{j \neq i} b_j - b_i - (-n)}{\sigma \sqrt{n}} \right) = 0,$$

where ϕ is the normal density function which gives the best response function

$$(*) \quad \beta(b_{-i}) = - \sum_{j \in N \setminus \{i\}} b_j + \frac{\sigma^2}{2} \ln(q/(1-q)),$$

where b_{-i} denotes the biases of all voters except i .

An *ECH1* player has a belief that the frequency of *ECH0* players is $g_1(0)$ and the frequency of *ECH1* is $1 - g_1(0)$ where

$$g_1(0) = \frac{f^\tau(0)}{f^\tau(0) + f^\tau(1)} = \frac{1}{1 + \tau}.$$

Denoting the *ECH1* players' strategy by b_1^{ECH} and *ECH0* strategy by b_0 , we have

$$b_1^{ECH} = -\frac{n-1}{1+\tau n} b_0 + \frac{\sigma^2}{2} \ln(q/(1-q)),$$

using (*). So we have that, as $n \rightarrow \infty$,

$$b_1^{ECH}(b_0) \rightarrow -\frac{1}{\tau} b_0 + \frac{\sigma^2}{2} \ln(q/(1-q)).$$

Moving to second level, consider first the CH model. In what follows, we write b_1^{CH} in place of $\beta(b_0)$ in (*) above.

$$\begin{aligned} b_2^{CH} &= -(n-1)[b_0 g_1(0) + b_1^{CH} g_1(1)] + \frac{\sigma^2}{2} \ln(q/(1-q)), \\ &= -(n-1) \left(\frac{1-(n-1)\tau}{1+\tau} \right) b_0 + \left(1 - \frac{(n-1)\tau}{1+\tau} \right) \frac{\sigma^2}{2} \ln(q/(1-q)). \end{aligned}$$

For *ECH2*, note first that

$$g_2(0) = \frac{f^\tau(0)}{f^\tau(0) + f^\tau(1) + f^\tau(2)} = \frac{2}{2 + 2\tau + \tau^2},$$

similarly, that

$$g_2(1) = \frac{2\tau}{2 + 2\tau + \tau^2} \text{ and } g_2(2) = \frac{\tau^2}{2 + 2\tau + \tau^2}.$$

So an *ECH2* best responds, according to this belief as follows

$$\begin{aligned} b_i &= -(n-1)[b_0 g_2(0) + b_1^{ECH} g_2(1) + b_2 g_2(2)] + \frac{\sigma^2}{2} \ln(q/(1-q)), \\ &= \frac{-(n-1)}{2 + 2\tau + \tau^2} \left(2 - \frac{2\tau(n-1)}{1+\tau n} \right) b_0 - \frac{\tau^2(n-1)}{2 + 2\tau + \tau^2} b_2 + \left(1 - \frac{2\tau(n-1)}{2 + 2\tau + \tau^2} \right) \frac{\sigma^2}{2} \ln(q/(1-q)). \end{aligned}$$

For equilibrium, we have,

$$\begin{aligned} b_2^* \left(1 + \frac{\tau^2(n-1)}{2 + 2\tau + \tau^2} \right) &= \frac{-(n-1)}{2 + 2\tau + \tau^2} \left(2 - \frac{2\tau(n-1)}{1+\tau n} \right) b_0 + \left(1 - \frac{2\tau(n-1)}{2 + 2\tau + \tau^2} \right) \frac{\sigma^2}{2} \ln(q/(1-q)), \\ b_2^* &= -\frac{2(n-1)(\tau+1)}{(2 + 2\tau + \tau^2 n)(1+\tau n)} b_0 + \left(\frac{2 + 4\tau + \tau^2 - 2\tau n}{2 + 2\tau + \tau^2 n} \right) \frac{\sigma^2}{2} \ln(q/(1-q)). \end{aligned}$$

Where we have the first coefficient vanishing as $n \rightarrow \infty$, while the second one converges to $-\frac{2}{\tau}$:

$$b_2^{ECH}(b_0) \rightarrow -\frac{\sigma^2}{\tau} \ln\left(\frac{q}{1-q}\right).$$

B.1. Proof of Proposition 1.

Proof. CH

Given b_0 and τ , we have $(b_0, b_1^{CH}(b_0), b_2^{CH}(b_0))$ with frequencies $p_\tau = \frac{1}{2+2\tau+\tau^2}(2, 2\tau, \tau^2)$ as above. The probability of the correct decision is $(\Pr[d = 1|\omega = 1] + \Pr[d = -1|\omega = -1])/2$ where

$$\Pr[d = 1|\omega = 1] = \Pr\left[\sum_{1 \leq j \leq n} s_j + \Sigma(b_0, \tau, \sigma, n, q) > 0|\omega = 1\right],$$

if $\Sigma(b_0, \tau, \sigma, n, q)$ denotes the summation of biases under *CH*. Algebra shows that

$$\begin{aligned} (2 + 2\tau + \tau^2)\Sigma(b_0, \tau, \sigma, n, q) &= \left(2 - 2(n-1)\tau - \tau^2(n-1)\frac{1 - (n-1)\tau}{1 + \tau}\right)b_0 \\ &\quad + \left(\sigma^2\tau + \frac{\tau^2\sigma^2}{2}\left(1 - \frac{(n-1)\tau}{1 + \tau}\right)\right)\ln\left(\frac{q}{1-q}\right). \end{aligned}$$

We see that $\partial\Sigma/\partial n$ is a linear function of and has the same sign with b_0 . Furthermore, as $n \rightarrow \infty$,

$$\Sigma \rightarrow \text{sgn}(b_0)\infty$$

and

$$\Pr\left[\sum_{1 \leq j \leq n} s_j + \Sigma(b_0, \tau, \sigma, n, q) > 0|\omega = 1\right] = \frac{1}{2}\text{Erfc}\left(\frac{-n - \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}}\right)$$

where

$$\frac{-n - \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}} \rightarrow \text{sgn}(-b_0)\infty.$$

Similarly,

$$\Pr\left[\sum_{1 \leq j \leq n} s_j + \Sigma(b_0, \tau, \sigma, n, q) < 0|\omega = -1\right] = \frac{1}{2}\text{Erfc}\left(\frac{-n + \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}}\right)$$

where

$$\frac{-n + \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}} \rightarrow \text{sgn}(b_0)\infty,$$

as $n \rightarrow \infty$. Without loss of generality, assume $b_0 > 0$. Then the former probability converges to 1 whereas the latter to 0. Thus, the probability of a correct decision converges to 1/2.

ECH

Given b_0 and τ , we have $(b_0, b_1^{ECH}(b_0), b_2^{ECH}(b_0))$ with frequencies $p_\tau = \frac{1}{2+2\tau+\tau^2}(2, 2\tau, \tau^2)$ since we assume from now on that the level-2 has the correct belief. The probability of the correct decision is $(\Pr[d = 1|\omega = 1] + \Pr[d = -1|\omega = -1])/2$ where

$$\Pr[d = 1|\omega = 1] = \Pr \left[\sum_{1 \leq j \leq n} s_j > \Sigma(b_0, \tau, \sigma, n, q) | \omega = 1 \right],$$

if $\Sigma(b_0, \tau, \sigma, n, q)$ denotes minus the summation of biases under *ECH*. Algebra shows that

$$\begin{aligned} \left(-\frac{n}{2+2\tau+\tau^2} \right)^{-1} \Sigma(b_0, \tau, \sigma, n, q) &= \left(2 - \frac{(n-1)}{1+n\tau} 2\tau - \frac{2(n-1)(1+\tau)}{(2+2\tau+\tau^2n)(1+\tau n)} \tau^2 \right) b_0 \\ &\quad + \left(2\tau + \frac{2+4\tau+\tau^2-2\tau n}{2+2\tau+\tau^2n} \tau^2 \right) \frac{\sigma^2}{2} \ln \left(\frac{q}{1-q} \right), \end{aligned}$$

or, simply,

$$\Sigma(b_0, \tau, \sigma, n, q) = -\frac{2n(1+\tau)}{(1+n\tau)(2+2\tau+n\tau^2)} b_0 - \frac{n\tau(2+\tau)}{2+2\tau+n\tau^2} \frac{\sigma^2}{2} \ln \left(\frac{q}{1-q} \right).$$

As $n \rightarrow \infty$, we have

$$\Sigma(b_0, \tau, \sigma, n, q) \rightarrow -\frac{(2+\tau)\sigma^2}{2\tau} \ln \left(\frac{q}{1-q} \right).$$

Hence the probability

$$\Pr \left[\sum_{1 \leq j \leq n} s_j < \Sigma(b_0, \tau, \sigma, n, q) | \omega = -1 \right] = \frac{1}{2} \text{Erfc} \left(\frac{-n - \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}} \right),$$

as well as

$$\Pr \left[\sum_{1 \leq j \leq n} s_j > \Sigma(b_0, \tau, \sigma, n, q) | \omega = 1 \right] = \frac{1}{2} \text{Erfc} \left(\frac{-n + \Sigma(b_0, \tau, \sigma, n, q)}{\sqrt{2n\sigma}} \right),$$

are approaching 1, which means that probability of correct decision is converging to certainty under both states of the world. Note that here we have $\text{Erfc}(\frac{x}{\sqrt{2}}) = 1 - 2\Phi(x)$, where Φ is the cumulative distribution function for a standard normal variable. \square

B.2. Proof of Proposition 2.

Proof. In this case we have

$$\Sigma(b_0, \tau, \sigma, n, q)|_{\tau=1/n} = -\frac{n(1+n)}{3+2n}b_0 + \frac{1+2n}{6+4n}\sigma^2 \ln\left(\frac{q}{1-q}\right),$$

which we denote by $\Sigma(b_0, 1/n, \sigma, n, q)$.

So, first

$$\Pr\left[\sum_{1 \leq j \leq n} s_j > \Sigma(b_0, 1/n, \sigma, n, q) | \omega = 1\right] = \frac{1}{2} \operatorname{Erfc}\left(\frac{-n + \Sigma(b_0, 1/n, \sigma, n, q)}{\sqrt{2n\sigma}}\right),$$

where

$$\frac{-n + \Sigma(b_0, 1/n, \sigma, n, q)}{\sqrt{2n\sigma}} \rightarrow \operatorname{sgn}\left(-1 - \frac{b_0}{2}\right)\infty.$$

And also

$$\Pr\left[\sum_{1 \leq j \leq n} s_j < \Sigma(b_0, 1/n, \sigma, n, q) | \omega = -1\right] = \frac{1}{2} \operatorname{Erfc}\left(\frac{-n - \Sigma(b_0, 1/n, \sigma, n, q)}{\sqrt{2n\sigma}}\right),$$

where

$$\frac{-n - \Sigma(b_0, 1/n, \sigma, n, q)}{\sqrt{2n\sigma}} \rightarrow \operatorname{sgn}(-2 + b_0)\infty.$$

We know that $\operatorname{Erfc}(-\infty) = 2$, $\operatorname{Erfc}(\infty) = 0$ and $\operatorname{Erfc}(0) = 1$, hence probability of correct decision is 1 in the limit if $b \in (-2, 2)$, 0 if $|b| > 2$ and 3/4 if $b \in \{-2, 2\}$. \square

B.3. Proof of Proposition 3.

Proof. Denote the level-0 strategy as b_0 for all models.

Level-k approach

For higher levels, we have

$$\begin{aligned} b_1(n) &= -(n-1)b_0, \\ b_2(n) &= -(n-1)b_1 = -(n-1)[-(n-1)b_0] = [-(n-1)]^2 b_0, \\ &\vdots \\ b_k(n) &= [-(n-1)]^k b_0. \end{aligned}$$

Hence we have that $b_k(n)/b_0(n) \in O(n^k)$.

Cognitive Hierarchy

The proof is done by induction. Observe first that, as in level- k , $b_1(n) = -(n-1)b_0$, hence $b_1(n)/b_0(n) \in O(n)$. Now suppose for $t \in \mathbb{N}$, we have that $b_t(n)/b_0(n) \in O(n^t)$. For

$t + 1$, it holds that:

$$b_{t+1}(n) = -(n-1) \left(\frac{1}{\sum_{j=0}^t f(j)} \sum_{j=0}^t f(j)b_j(n) \right),$$

$$b_{t+1}(n)/b_0(n) = -(n-1) \left(\frac{1}{\sum_{j=0}^{t-1} f(j)} \sum_{j=0}^{t-1} f(j)b_j(n)/b_0(n) \right) - (n-1)f(t)b_t(n)/b_0(n).$$

Hence, it can be shown that the first term in the summation is dominated by the latter, which belongs to $O(n^{t+1})$.

Endogenous Cognitive Hierarchy

Observe first that we have the following:

$$b_1(n) = -(n-1) \left(\frac{f(0)}{f(0) + f(1)} b_0 + \frac{f(1)}{f(0) + f(1)} b_1(n) \right),$$

which can be simplified as;

$$b_1(n) = \frac{-(n-1)f(0)b_0}{f(0) + f(1) - (n-1)f(1)}.$$

In fact, can be shown also is that

$$b_k(n) = \frac{-(n-1)f(0)b_0(n) - (n-1) \sum_{j=1}^{k-1} f(j)b_j(n)}{\sum_{j=0}^k f(j) - (n-1)f(k)}.$$

So we have $b_k(n)/b_0 \rightarrow C$ for some constant C as $n \rightarrow \infty$. □