# Ambiguity Aversion with Three or More Outcomes

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*Ambiguous choice problems which involve three or more outcome values can reveal aspects of ambiguity and ambiguity aversion which cannot be displayed in classic two-outcome Ellsberg urn problems, and hence are not always captured by models designed to accommodate them. These aspects include Allais-type difficulties in preferences over purely subjective acts, attitudes toward different sources involving different amounts of ambiguity, and attitudes toward ambiguity at different outcome levels (or in gains versus losses). This paper presents a few such examples, and examines the standard models' predictions and performance in such cases.*

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## Three Decision Problems

Consider the following decision problems. The *Slightly-Bent Coin Problem* involves two sources of ambiguity. One source is a coin which has been slightly bent in an unknown direction. It still has some well-defined probability, in the sense that if it were to be flipped millions of times, there is some fixed value to which the proportion of heads would converge – you just don't know what that value is, and you only get to flip once. In this sense it exhibits exactly the same type of ambiguity as displayed by the event black (or yellow) in the classic Three-Color Ellsberg Urn – repeated sampling with replacement would also yield some fixed limiting proportion of black draws, but again, you don't know what that proportion is. The only difference is that since the coin is only slightly bent, you know that *its* unknown proportion is very close to one half. The other source of ambiguity in the problem is an urn containing a ball, which could be either black or white. The mechanics of the coin flip does not depend upon the contents of the urn, and the coin is flipped and the ball drawn simultaneously. The bets are based on the outcome of the flip and the color of the ball.

SLIGHTLY-BENT COIN PROBLEM

| | BET I | | | | BET II | |
|---|---|---|---|---|---|---|
| | black | white | | | black | white |
| heads | +$8,000 | $0 | vs. | heads | $0 | $0 |
| tails | –$8,000 | $0 | | tails | –$8,000 | +$8,000 |

The next problem, the *Thermometer Problem*, involves bets on the temperature in Timbuktu at noon next May Day. The thermometer is more than just a very accurate digital thermometer – it's a perfectly accurate *analogue* thermometer, which can exactly report any value in the continuum. Divide the continuum of feasible temperatures into an extremely large number of equal-length intervals. Bet 1 yields its prize of $6,000 if the temperature $t$ lands in the left $^{45}/_{100}$ portion of any interval and $0 otherwise, and Bets 2, 3 and 4 have a similar structure.

THERMOMETER PROBLEM

| | | |
|---|---|---|
| BET 1 ($6K if $t$ in left $^{45}/_{100}$ of any interval) | vs. | BET 2 ($3K if $t$ in left $^{90}/_{100}$ of any interval) |
| BET 3 ($6K if $t$ in left $^{1}/_{1,000}$ of any interval) | vs. | BET 4 ($3K if $t$ in left $^{2}/_{1,000}$ of any interval) |

The final problem, which we call the *Ambiguity at Low vs. High Outcomes Problem*, consists of a choice between bets on two Ellsberg-type urns. Each bet can be contrasted with the purely objective bet $b_0$ = ($0,⅓;$$c$,⅓;$100,⅓), where $$c$ is your certainty equivalent of an objective 50:50 lottery over \$0:\$100. Urn I differs from $b_0$ in having ambiguity across its middle and lower outcome values, whereas Urn II differs from $b_0$ in having ambiguity across its middle and upper outcome values.

AMBIGUITY AT LOW VS. HIGH OUTCOMES PROBLEM – THREE-COLOR VERSION
(\$C = CERTAINTY EQUIVALENT OF OBJECTIVE 50:50 \$100 BET)

$$b_0 \quad \overbrace{\$0}^{\text{1 ball}} \quad \overbrace{\$c}^{\text{1 ball}} \quad \overbrace{\$100}^{\text{1 ball}}$$

$$\begin{array}{ccc} & \text{URN I} & \\ \overbrace{\begin{array}{cc}\text{black} & \text{white}\\ \$0 & \$c\end{array}}^{\text{2 balls}} & \overbrace{\begin{array}{c}\text{red}\\ \$100\end{array}}^{\text{1 ball}} \end{array} \quad \text{vs.} \quad \begin{array}{cc} & \text{URN II} \\ \overbrace{\begin{array}{c}\text{red}\\ \$0\end{array}}^{\text{1 ball}} & \overbrace{\begin{array}{cc}\text{black} & \text{white}\\ \$c & \$100\end{array}}^{\text{2 balls}} \end{array}$$

## I. Introduction

The concept of *objective uncertainty* dates back at least to 17th Century French gamblers such as Pascal and Fermat, and mathematicians have since developed the theory of probability far beyond what is needed (or could even be applied) by economists or other decision theorists. Although humans faced situations of *subjective uncertainty* (plagues, earthquakes…) long before the invention of dice or roulette wheels, the formal recognition and specification of subjective uncertainty as a distinct concept is much more recent.[1] More recent still is the formal development of *subjective probability,* in which the theory of probability can be applied to an individual's beliefs – and hence their decisions – in subjective settings, and which has typically been posited jointly with expected utility risk preferences and termed *subjective expected utility*.[2]

While the combination of subjective probability theory with classical expected utility theory would seem to constitute the ideal framework for the analysis of choice under uncertainty, a still more recent phenomenon has caused researchers to question the empirical validity of the subjective probability hypothesis. These are the well-known thought experiments proposed by Daniel Ellsberg (1961). So-called *Ellsberg urns* present situations of objective, subjective and *mixed objective/subjective* uncertainty, and most individuals'

[1] E.g., Keynes (1921), Knight (1921).
[2] E.g., Savage (1954), Anscombe and Aumann (1963).

preferences for bets on such urns seem to systematically violate the existence of well-defined subjective probabilities. This feature of preferences has been termed *ambiguity aversion*. Economists and others have responded to this phenomenon by developing models – typically generalizations of subjective expected utility – designed to accommodate ambiguity aversion, and such models have been usefully applied to the analysis of economic behavior.

Although the major models of ambiguity aversion are of course defined over multiple-outcome prospects, Ellsberg's classic 1961 article only involved bets on a single pair of outcome values (Ellsberg used \$0 and \$100). But even in the world of *objective* uncertainty, preferences over bets on just a single pair of values – that is, preferences over a family of objective lotteries of the form $\{(\$100,p;\$0,1\text{–}p)\,|\,p\in[0,1]\}$ – can reveal very little about an individual's attitudes toward risk, not even whether they are risk neutral, risk averse or risk loving.

Ellsberg's two-outcome, *mixed-uncertainty* examples did a little better, in allowing an individual's aversion or preference for ambiguity to reveal itself, and the major models of ambiguity aversion have provided different ways to represent this. However, just as with the objective case, there are aspects of ambiguity and ambiguity aversion which can only reveal themselves in three or more outcome choice problems, such as the ones presented at the beginning of this paper.[3]

The purpose of this paper is to consider three such aspects. Since the major models were motivated by and designed to accommodate Ellsberg's two-outcome examples, it is perhaps no surprise that in most cases, they predict neutrality toward these new aspects. This too has an analogue in the objective case: preferences over the two-outcome family $\{(\$100,p;\$0,1\text{–}p)\,|\,p\in[0,1]\}$ can be successfully modeled by the hypothesis of expected value maximization, but this hypothesis would predict neutrality toward aspects of risk and risk aversion which can arise once three outcomes are allowed.

The following section reviews the classic Ellsberg urn examples and some of the models which have developed in response to them. Sections III, IV and V present aspects of ambiguity and ambiguity aversion which can arise in a world with three or more outcomes, and examine how the major models perform in such cases. Section VI concludes.

[3] Ellsberg (2001) himself offered some three-outcome examples (pp.148,189,208,215,225)), but these seem to have escaped the attention of modern researchers. Three-outcome examples involving subjective uncertainty also appear in Luce and Raiffa (1957,Ch.13), Diecidue and Wakker (2001), Zhang (2002), Nau (2006), Lo (2008), Ergin and Gul (2009) and Wakker (2010), and have been experimentally examined by MacCrimmon and Larsson (1979) Tversky and Kahneman (1992), Fennema and Wakker (1996), Fox and Tversky (1998), Wu and Gonzalez (1999), Abdellaoui, Vossmann and Weber (2005), Diecidue, Wakker and Zeelenberg (2007), L'Haridon and Placido (2008,2010), Hey, Lotito and Maffioletti (2010) and Baillon, L'Haridon and Placido (2011). However, to my knowledge none of the issues addressed in this paper have appeared in the literature.

## II. Classic Urns and Major Models

### *A. Classic Ellsberg Urn Problems*

In his 1961 article and 1962 PhD thesis,[4] Ellsberg presented a class of decision problems involving both subjective and objective uncertainty, which seem to contradict the classic subjective expected utility hypothesis as axiomatized and formalized in Savage (1954). The example now known as the *Three-Color Ellsberg Paradox* involves an opaque urn containing 90 balls. Exactly 30 of these balls are known to be red, and each of the other 60 is either black or yellow, but the exact numbers are unknown, and could be anywhere from 0:60 to 60:0. A ball is to be drawn from the urn, and the decision maker is presented with two pairs of bets based on the color of the ball, as illustrated below:[5]

THREE-COLOR ELLSBERG PARADOX

| | 30 balls | 60 balls | | |
|---|---|---|---|---|
| | red | black | yellow | |
| $a_1$ | \$100 | \$0 | \$0 | $\succ$ |
| $a_2$ | \$0 | \$100 | \$0 | |
| $a_3$ | \$100 | \$0 | \$100 | $\prec$ |
| $a_4$ | \$0 | \$100 | \$100 | |

Ellsberg conjectured, and subsequent experimenters have found,[6] that most individuals would prefer bet $a_1$ over bet $a_2$, and bet $a_4$ over bet $a_3$, which we will refer to as *Ellsberg preferences* in this choice problem. The example is termed a "paradox" since such preferences directly (and systematically) contradict the subjective probability hypothesis – if the individual did assign subjective probabilities to the events $\{\text{red}, \text{black}, \text{yellow}\}$, then the strict preference ranking $a_1 \succ a_2$ would reveal the strict subjective probability ranking prob(red) > prob(black), but the strict ranking $a_3 \prec a_4$ would reveal the strict ranking prob(red) < prob(black).

The widely accepted reason for these rankings is that while the bet $a_1$ guarantees a known probability 1/3 of winning the \$100 prize, the probability of winning offered by $a_2$ is

[4] Ellsberg's thesis has since been published as Ellsberg (2001).

[5] Ellsberg (1961, pp.653-656; 2001, pp.42-45,155-158). Ellsberg (2001, pp.137) gives an equivalent version with the payoffs \$100:\$0 replaced by \$0:–\$100. Ellsberg (1961, p.651-652,659; 2001, pp.43-44,136-137,201-202) refers to similar work by Chipman (1960). See also Ellsberg's reminiscences in (2001, p.244, n.1).

[6] See the surveys of Camerer and Weber (1992), Kelsey and Quiggin (1992), Camerer (1995), Siniscalchi (2008), Al-Najjar and Weinstein (2009), Hey, Lotito and Maffioletti (2010), Etner, Jeleva and Tallon (2012) and Gilboa and Marinacci (2012).

unknown, and could be anywhere from 0 to 2/3. Although the range [0,2/3] has 1/3 as its midpoint, and there is no reason to expect any asymmetry, individuals seem to prefer the known to the unknown probability. Similarly, bet $a_4$ offers a guaranteed 2/3 chance of winning, whereas the probability offered by $a_3$ could be anywhere from 1/3 to 1. Again, individuals prefer the known-probability bet. Ellsberg referred to bets $a_2$ and $a_3$ as involving *ambiguity*, and a preference for known-probability over ambiguous bets has come to be known as *ambiguity aversion*.

Ellsberg's article contained two additional widely-cited examples. In the left-hand example below, known as the *Two-Urn Ellsberg Paradox*,[7] Urn I contains 100 red and green balls in unknown proportions, and Urn II contains exactly 50 black and 50 yellow balls. Again, typical preferences are for the known-probability bets $b_2$ and $b_4$ over their unknown-probability counterparts $b_1$ and $b_3$, again contradicting the subjective probability hypothesis[8] and reflecting the same type of ambiguity aversion as in the Three-Color Paradox. In the right-hand example, suggested to Ellsberg by Kenneth Arrow and known as the *Four-Color Ellsberg Paradox*,[9] the typical rankings $c_1 \succ c_2$ and $c_3 \prec c_4$ imply prob(green) > prob(black) and prob(green) < prob(black) respectively, and reveal the same type of ambiguity aversion as in the previous examples. Ellsberg observed that such examples can be viewed as providing systematic violations of Savage's *Sure-Thing Principle*.[10] As mentioned, such examples have received a great deal of experimental confirmation (see Note 6).

TWO-URN ELLSBERG PARADOX

| | URN I | | URN II | | |
|---|---|---|---|---|---|
| | 100 balls | | 50 balls | 50 balls | |
| | red | green | black | yellow | |
| $b_1$ | \$100 | \$0 | | | $\prec$ |
| $b_2$ | | | \$100 | \$0 | |
| $b_3$ | \$0 | \$100 | | | $\prec$ |
| $b_4$ | | | \$0 | \$100 | |

FOUR-COLOR ELLSBERG PARADOX

| | (single urn) | | | | |
|---|---|---|---|---|---|
| | 100 balls | | 50 balls | 50 balls | |
| | red | green | black | yellow | |
| $c_1$ | \$100 | \$100 | \$0 | \$0 | $\succ$ |
| $c_2$ | \$100 | \$0 | \$100 | \$0 | |
| $c_3$ | \$0 | \$100 | \$0 | \$100 | $\prec$ |
| $c_4$ | \$0 | \$0 | \$100 | \$100 | |

7 Ellsberg (1961, pp.650-651,653).

8 $b_1 \prec b_2$ would reveal prob(red) < prob(black), but $b_3 \prec b_4$ would reveal prob(green) < prob(yellow), violating the requirement that these probabilities satisfy prob(red) + prob (green) = prob(black) + prob (yellow) = 1.

9 Ellsberg (1961, p.651, note 1).

10 Savage (1954, p.23, Postulate P2). These three examples also violate Axiom P4* (*Strong Comparative Probability*) of Machina and Schmeidler (1992, p.761).

### *B. Major Models of Ambiguity Aversion*

Ellsberg's examples have spurred the development of models which generalize and/or weaken the classic Subjective Expected Utility Model to allow for ambiguity aversion. In the finite-outcome case, the objects of choice consist of *purely objective lotteries* $\mathbf{P}$ = $(x_1, p_1;\ldots;x_n, p_n)$ yielding $x_j$ with probability $p_j$ for some (say, monetary) outcome set $\mathcal{X} = \{x\}$, *purely subjective acts* $f(\cdot) = [x_1 \text{ on } E_1 ; \ldots ; x_n \text{ on } E_n]$ yielding $x_i$ on event $E_i$ for some partition $\{E_1, \ldots, E_n\}$ of a subjective state space $S = \{\ldots,s,\ldots\}$ or $S \subseteq R^N$, and *mixed objective/subjective bets*[11] $[\mathbf{P}_1 \text{ on } E_1;\ldots;\mathbf{P}_n \text{ on } E_n]$, which are subjective bets whose "outcomes" consist of objective lotteries $\mathbf{P}_i = (\ldots; x_{ij}, p_{ij}; \ldots)$. The family of mixed objective/subjective bets is seen to include the family of purely objective lotteries and the family of purely subjective acts.

Classical *Subjective Expected Utility* preferences over such prospects can be represented by a preference function which takes the form

(1) $$W_{SEU}(\ldots; x_i \text{ on } E_i ; \ldots) \;=\; \sum_i U(x_i)\cdot\pi(E_i)$$

over purely subjective acts, and more generally, the form

(2) $$W_{SEU}(\ldots;(\ldots; x_{ij}, p_{ij};\ldots) \text{ on } E_i ;\ldots) \;=\; \sum_i \Big[\sum_j U(x_{ij})\cdot p_{ij}\Big]\cdot\pi(E_i)$$

over mixed objective/subjective bets, for some increasing cardinal *utility function* $U(\cdot)$ over outcomes and additive *subjective probability measure* $\pi(\cdot)$ over events.

One of the major models of ambiguity aversion over subjective or mixed objective/subjective bets is the *Rank-Dependent* (or *Choquet*) *Model* of Schmeidler (1989),[12] which takes the form

(3) $$W_{RD}(\ldots; x_i \text{ on } E_i;\ldots) \;=\; \sum_i U(x_i)\cdot\Big(C(\cup_{k=1}^{i} E_k) - C(\cup_{k=1}^{i-1} E_k)\Big)$$

over purely subjective acts, and more generally

(4) $$W_{RD}(\ldots;(\ldots; x_{ij}, p_{ij};\ldots) \text{ on } E_i;\ldots) = \sum_i \Big[\sum_j U(x_{ij})\cdot p_{ij}\Big]\cdot\Big(C(\cup_{k=1}^{i} E_k) - C(\cup_{k=1}^{i-1} E_k)\Big)$$

[11] Such bets are also known as *Anscombe-Aumann acts* (Anscombe and Aumann (1963)). To demonstrate that the issues discussed in this paper are not due to two-stage resolution of uncertainty, we assume that, as in the Bent Coin problem, all prospects in the paper involve simultaneous resolution of all sources of uncertainty (e.g. Machina (2004,Sect.4.1), Wakker (2010,Sect.4.9)).

[12] See also Gilboa (1987), which derives from an earlier version of Schmeidler (1989).

over mixed objective/subjective bets, for some nonadditive measure $C(\cdot)$ termed a *capacity*, and where in (3) the outcomes $x_i$ and their corresponding events $E_i$ are labeled so that $x_1 \succcurlyeq \ldots \succcurlyeq x_n$, and in (4) the conditional lotteries $(\ldots;x_{ij},p_{ij};\ldots)$ and their corresponding events $E_i$ are labeled so that $\sum_j U(x_{1j})\cdot p_{1j} \succcurlyeq \ldots \succcurlyeq \sum_j U(x_{nj})\cdot p_{nj}$.[13] The intuition behind the use of a nonadditive measure is that the union of two ambiguous events (such as black and yellow in the Three-Color Urn) could well be purely objective, and it requires a nonadditive measure over events to capture this. The event $\cup_{k=1}^{i} E_k$ on which a payoff of at least $x_i$ is received is sometimes referred to as the bet's *good news event* for the outcome level $x_i$.

A second model, formalized by Gilboa and Schmeidler (1989) and termed the *Expected Utility with Multiple Priors* (or simply *Multiple Priors*) *Model*, captures ambiguity aversion by means of the form

(5) $$W_{MP}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}_0} \sum_i U(x_i)\cdot\pi(E_i)$$

over purely subjective acts, and more generally

(6) $$W_{MP}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}_0} \sum_i \big[\sum_j U(x_{ij})\cdot p_{ij}\big]\cdot\pi(E_i)$$

for some increasing cardinal $U(\cdot)$ and some family $\mathcal{P}_0$ of subjective probability measures $\pi(\cdot)$ over events.[14] The intuition behind this form is that an ambiguity averter evaluates each subjective or mixed bet in the most pessimistic way, given the family of measures $\mathcal{P}_0$.[15]

A more recently proposed model is the *Smooth Ambiguity Model* of Klibanoff, Marinacci and Mukerji (2005),[16] developed in part to eliminate the "kinks at certainty" properties of the Rank-Dependent and Multiple Priors forms. This model takes the form

(7) $$W_{SM}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \int_{\pi(\cdot)\in\mathcal{P}} \phi\Big(\sum_i U(x_i)\cdot\pi(E_i)\Big)\cdot d\mu(\pi(\cdot))$$

over purely subjective acts, and more generally

(8) $$W_{SM}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \int_{\pi(\cdot)\in\mathcal{P}} \phi\Big(\sum_i \big[\sum_j U(x_{ij})\cdot p_{ij}\big]\cdot\pi(E_i)\Big)\cdot d\mu(\pi(\cdot))$$

13 Schmeidler (1989,Theorem (pp.578-579)).

14 In this paper we restrict consideration to families $\mathcal{P}_0$ with uniformly bounded and uniformly continuous densities.

15 Schmeidler (1986,Prop.3;1989,pp.582-584) provides conditions under which the Multiple Priors Model contains the Rank-Dependent Model as a special case.

16 See also the earlier analysis of Segal (1987,1990).

for some increasing cardinal functions $U(\cdot)$ and $\phi(\cdot)$, the family $\mathcal{P}$ of *all* subjective probability measures $\pi(\cdot)$ over events, and subjective probability measure $\mu(\cdot)$ over $\mathcal{P}$. For each $\pi(\cdot)$, the expected utility of the mixed objective/subjective prospect $(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$ would be $\sum_i[\sum_j U(x_{ij})\cdot p_{ij}]\cdot\pi(E_i)$, and the individual is averse to the uncertainty in these expected utility levels which results from their subjective uncertainty about $\pi(\cdot)$ as represented by $\mu(\cdot)$. Risk aversion over objective uncertainty is captured by concavity of the utility function $U(\cdot)$, and ambiguity aversion captured by concavity of $\phi(\cdot)$.

The fourth major model of ambiguity aversion is the *Variational Preferences Model* of Maccheroni, Marinacci and Rustichini (2006), which takes the form

(9) $$W_{VP}(\ldots;x_i \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}}\Big(\sum_i U(x_i)\cdot\pi(E_i) + \eta\big(\pi(\cdot)\big)\Big)$$

over purely subjective acts, and more generally

(10) $$W_{VP}(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots) \;=\; \min_{\pi(\cdot)\in\mathcal{P}}\Big(\sum_i\big[\sum_j U(x_{ij})\cdot p_{ij}\big]\cdot\pi(E_i) + \eta\big(\pi(\cdot)\big)\Big)$$

for some increasing cardinal function $U(\cdot)$, the family $\mathcal{P}$ of all subjective probability measures $\pi(\cdot)$ over events, and convex function $\eta(\cdot)$ over $\mathcal{P}$. For each $\pi(\cdot)$, the expected utility of the mixed objective/subjective bet $(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$, namely $\sum_i(\sum_j U(x_{ij})\cdot p_{ij})\cdot\pi(E_i)$, is supplemented by a value $\eta(\pi(\cdot))$ representing the individual's attitudes toward ambiguity, and the combined value then minimized over the family $\mathcal{P}$. The above authors have shown how this model includes the Multiple Priors Model and the *Multiplier Preferences Model* of Hansen and Sargent (2001) as special cases.

It is important to note how these models are applied to mixed objective/subjective prospects. In order to fully separate and represent their objective and subjective uncertainty, such prospects are expressed in the Anscombe-Aumann form $[\mathbf{P}_1 \text{ on } E_1;\ldots;\mathbf{P}_n \text{ on } E_n]$, and then evaluated as in equations (4), (6), (8) and (10). Thus, for the Three-Color Urn, the appropriate state space on which to apply the models is not the *mixed* objective/subjective space {red,black,yellow}, but rather, the underlying *purely subjective* space $\{s_0,\ldots,s_{60}\}$ = {0 black balls,…,60 black balls}, whose uncertainty is, after all, the underlying source of the urn's ambiguity.[17] Expressed in this manner, Ellsberg's Three-Color bets take the following form, where in each case $i$ runs from 0 to 60:

[17] See the Appendix for a further discussion of this issue.

$$(11)\qquad \begin{array}{ll} a_1 = \left[\ldots;(\$0,\tfrac{60}{90};\$100,\tfrac{30}{90}) \text{ if } s_i;\ldots\right] & a_2 = \left[\ldots;(\$0,\tfrac{90-i}{90};\$100,\tfrac{i}{90}) \text{ if } s_i;\ldots\right] \\ a_3 = \left[\ldots;(\$0,\tfrac{i}{90};\$100,\tfrac{90-i}{90}) \text{ if } s_i;\ldots\right] & a_4 = \left[\ldots;(\$0,\tfrac{30}{90};\$100,\tfrac{60}{90}) \text{ if } s_i;\ldots\right] \end{array}$$

Each of the above models has been shown to be consistent with standard Ellsberg-type preferences in the above and similar examples, each has been formally axiomatized by the above researchers, and each has seen applications in economics.[18]

### III. Allais-Type Problems Under Purely Subjective Uncertainty

Our first observation is trivial: since the mixed-uncertainty forms (4), (6), (8) and (10) of the above models posit expected utility preferences over the subset of purely objective lotteries,[19] they are subject to purely objective phenomena such as the Allais Paradox, Common Consequence Effect and Common Ratio Effect.[20]

Why did the developers of these models embed expected utility objective risk preferences into their forms? In the two-outcome world of the classic Ellsberg Paradoxes, the issue is of little consequence – as already noted, if the only possible outcomes are {\$0,\$100}, it would be impossible to violate expected utility in choice over any menu of purely objective prospects.

Even if three or more possible outcomes are allowed, one could still argue that such models were not developed to address the "Allais-type" phenomenon of *nonlinearity in probabilities*, but rather, the "Ellsberg-type" phenomenon of *nonseparability across events*, and it is only their ability to model the latter phenomena which should matter. But it turns out that in a world of three or more outcomes, even preferences over *purely subjective* prospects – specifically, the purely subjective formulas (5), (7) and (9) of the Multiple Priors, Smooth Ambiguity and Variational Preferences models – can be subject to standard *Allais-type* phenomena.[21]

The reason why Allais-type problems can extend to purely subjective uncertainty is that some purely subjective events can be said to be "more objective" than others. Take a continuum state space $S$ = [0,1], partition it into $m$ equal intervals

[18] E.g. Chamberlain (2001), Epstein (2001), Hansen and Sargent (2001), Mukerji and Tallon (2004), Klibanoff, Marinacci and Mukerji (2005), Epstein and Schneider (2010), the papers in Gilboa (2004) and the 2011 Symposium Issue of *Economic Theory* in honor of the 50th anniversary of the Ellsberg Paradox (Ellsberg, et al. (2011)).

[19] This follows from the bracket term in each equation.

[20] See MacCrimmon and Larsson (1979), Schoemaker (1982) or Machina (1987) for reviews of these effects.

[21] The purely subjective Allais-type examples of this section are thus distinct from the purely subjective Ellsberg-type examples of event-nonseparability examined by Tversky and Kahneman (1992), Fennema and Wakker (1996) and Wu and Gonzalez (1999). (See the latter for an especially thorough experimental treatment, as well as the early approach of MacCrimmon and Larsson (1979,pp.364-365).)

$\{[0,1/m),\ldots,[i/m,(i+1)/m),\ldots,[(m-1)/m,1)\}$,[22] and for each $\alpha \in [0,1]$ define $[0,\alpha)\underset{m}{\times}S$ as the union of the left $\alpha$ portions of these intervals, so that $[0,\alpha)\underset{m}{\times}S = \cup_{i=0}^{m-1}[i/m,(i+\alpha)/m)$. As shown by Poincaré (1912) and others,[23] such events will satisfy $\lim_{m\to\infty} \pi([0,\alpha)\underset{m}{\times}S) = \alpha$ for any measure $\pi(\cdot)$ over [0,1] with a sufficiently regular density.

More generally, for any set $\wp \subseteq [0,1)$ consisting of a finite union of intervals, and any positive integer $m$, define the event $\wp\underset{m}{\times}S$ by

$$(12) \qquad \wp\underset{m}{\times}S \;=\; \bigcup_{i=0}^{m-1}\left\{ (i+\omega)/m \,\middle|\, \omega \in \wp \right\}$$

that is, as the union of the natural images of $\wp$ into each of $S$'s equal-length intervals. Events with this type of periodic structure are termed *almost-objective events*, and satisfy $\lim_{m\to\infty} \pi(\wp\underset{m}{\times}S) = \lambda(\wp)$ where $\lambda(\cdot)$ is the uniform Lebesgue measure over [0,1]. In the limit, agents with "event-smooth" preference functions will treat these events in much the same way as objective events. As $m\to\infty$ such agents will exhibit common outcome-invariant revealed likelihoods of almost-objective events, independence of almost-objective and fixed subjective events, probabilistically sophisticated preferences over almost-objective bets, and for subjective expected utility maximizers, linearity in almost-objective likelihoods and mixtures.[24]

We accordingly posit that as $m$ grows large, individuals will converge to indifference between an almost-objective bet $[x_1 \text{ on } \wp_1\underset{m}{\times}S;\ldots;x_n \text{ on } \wp_n\underset{m}{\times}S]$ and its corresponding purely objective lottery $(x_1,\lambda(\wp_1);\ldots;x_n,\lambda(\wp_n))$. Indeed, most so-called "objective" randomizing devices actually generate almost-objective uncertainty: the events {heads,tails} for a 50:50 coin are each periodic events in the (subjective) force of the flip, and each slot on a roulette wheel is periodic in the subjective force of the spin, yet they are viewed by both decision makers and decision theorists as purely objective.

Given an individual with the typical[25] preference rankings $\mathbf{P}_1 \prec \mathbf{P}_2$ and $\mathbf{P}_3 \succ \mathbf{P}_4$ over the Common Ratio Effect lotteries

[22] For simplicity, we ignore the rightmost state $s = 1$ in this and subsequent almost-objective partitions and bets.

[23] See Machina (2004, p.9).

[24] Machina (2004,Thms. 1, 2 & 5, 3, 6). The definition of "event-smooth" is that of Machina (2004, pp. 34-36).

[25] Kahneman and Tversky (1979, p.267) reported that 86% of their experimental subjects preferred $\mathbf{P}_2$ over $\mathbf{P}_1$, and 73% preferred $\mathbf{P}_3$ over $\mathbf{P}_4$.

$$(13)\qquad \begin{array}{lll} \mathbf{P}_1 = (\$6{,}000,.45\,;\,\$0,.55) & \prec & \mathbf{P}_2 = (\$3{,}000,.90\,;\,\$0,.10) \\ \mathbf{P}_3 = (\$6{,}000,.001\,;\,\$0,.999) & \succ & \mathbf{P}_4 = (\$3{,}000,.002\,;\,\$0,.998) \end{array}$$

pick $\varepsilon > 0$ small enough so that both $\mathbf{P}_1 \prec \mathbf{P}_2^{\varepsilon} = (\$3{,}000-\varepsilon,.90;\$0,.10)$ and $\mathbf{P}_3^{\varepsilon} = (\$6{,}000-\varepsilon,.001;\$0,.999) \succ \mathbf{P}_4$. Thus, in the Thermometer Problem, they would presumably exhibit the rankings

$$(14)\qquad \begin{array}{l} f_{1,m}(\cdot) = \big[\$6\text{K on }[0,.45)\underset{m}{\times}S;\$0\text{ on }[.45,1)\underset{m}{\times}S\big] \prec \\ \qquad\qquad \big[\$3\text{K}-\varepsilon\text{ on }[0,.90)\underset{m}{\times}S;\$0\text{ on }[.90,1)\underset{m}{\times}S\big] = f^{\varepsilon}_{2,m}(\cdot) \\ f^{\varepsilon}_{3,m}(\cdot) = \big[\$6\text{K}-\varepsilon\text{ on }[0,.001)\underset{m}{\times}S;\$0\text{ on }[.001,1)\underset{m}{\times}S\big] \succ \\ \qquad\qquad \big[\$3\text{K on }[0,.002)\underset{m}{\times}S;\$0\text{ on }[.002,1)\underset{m}{\times}S\big] = f_{4.m}(\cdot) \end{array}$$

for all $m$ greater than some $m_0$. Since all such $m$ are finite, all such acts are purely subjective.

To see that such preferences over these purely subjective acts are incompatible with the Multiple Priors form (5), consider a family $\mathcal{P}_0$ of priors $\pi(\cdot)$ over a state space $S = [0,1]$, with uniformly bounded and uniformly continuous densities. By Machina (2004,Thm.0), for each finite interval union $\wp \subseteq [0,1)$, the convergence of $\pi(\wp\underset{m}{\times}S)$ to $\lambda(\wp)$ will be uniform over $\mathcal{P}_0$. We thus have

$$(15)\qquad \begin{array}{l} \lim\limits_{m\to\infty} W_{MP}\big(x_1\text{ on }\wp_1\underset{m}{\times}S;\ldots;x_n\text{ on }\wp_n\underset{m}{\times}S\big) = \lim\limits_{m\to\infty}\min\limits_{\pi(\cdot)\in\mathcal{P}_0}\sum_{i=1}^{n}U(x_i)\cdot\pi(\wp_i\underset{m}{\times}S) = \\ \min\limits_{\pi(\cdot)\in\mathcal{P}_0}\lim\limits_{m\to\infty}\sum_{i=1}^{n}U(x_i)\cdot\pi(\wp_i\underset{m}{\times}S) = \min\limits_{\pi(\cdot)\in\mathcal{P}_0}\sum_{i=1}^{n}U(x_i)\cdot\lambda(\wp_i) = \sum_{i=1}^{n}U(x_i)\cdot\lambda(\wp_i) \end{array}$$

That is to say, as $m \to \infty$, Multiple Priors preferences over almost-objective bets converge to expected utility. Setting $U(\$0) = 0$, (14) and (15) then yield the incompatible inequalities

$$(16)\qquad \begin{array}{l} .45\cdot U(\$6\text{K}) = \lim\limits_{m\to\infty} W_{MP}(f_{1,m}(\cdot)) \le \lim\limits_{m\to\infty} W_{MP}(f^{\varepsilon}_{2,m}(\cdot)) < .90\cdot U(\$3\text{K}) \\ .001\cdot U(\$6\text{K}) > \lim\limits_{m\to\infty} W_{MP}(f^{\varepsilon}_{3,m}(\cdot)) \ge \lim\limits_{m\to\infty} W_{MP}(f_{4,m}(\cdot)) = .002\cdot U(\$3\text{K}) \end{array}$$

To see that such preferences are also incompatible with the Smooth Ambiguity form (7), assume enough regularity so that the limit can be moved inside both the integral sign and $\phi(\cdot)$, so that

$$(17)\quad \begin{aligned} &\lim_{m\to\infty} W_{SM}\left(x_1 \text{ on } \wp_1 \underset{m}{\times} S;\ldots;x_n \text{ on } \wp_n \underset{m}{\times} S\right) \\ &= \lim_{m\to\infty}\int_{\pi(\cdot)\in\mathcal{P}} \phi\left(\textstyle\sum_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)\right)\cdot d\mu(\pi(\cdot)) \\ &= \int_{\pi(\cdot)\in\mathcal{P}} \phi\left(\lim_{m\to\infty}\textstyle\sum_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)\right)\cdot d\mu(\pi(\cdot)) \\ &= \int_{\pi(\cdot)\in\mathcal{P}} \phi\left(\textstyle\sum_{i=1}^n U(x_i)\cdot\lambda(\wp_i)\right)\cdot d\mu(\pi(\cdot)) \quad = \quad \phi\left(\textstyle\sum_{i=1}^n U(x_i)\cdot\lambda(\wp_i)\right) \end{aligned}$$

That is to say, as $m \to \infty$, this model's preferences over almost-objective bets also converge to expected utility.[26] Defining $f_{1,m}(\cdot)$, $f^{\varepsilon}_{2,m}(\cdot)$, $f^{\varepsilon}_{3,m}(\cdot)$ and $f_{4,m}(\cdot)$ as in (14) and setting $U(\$0) = 0$ again yields incompatible inequalities:

$$(18)\quad \begin{aligned} &\phi\left(.45\cdot U(\$6\text{K})\right) = \lim_{m\to\infty} W_{SM}(f_{1,m}(\cdot)) \le \lim_{m\to\infty} W_{SM}(f^{\varepsilon}_{2,m}(\cdot)) < \phi\left(.90\cdot U(\$3\text{K})\right) \\ &\phi\left(.001\cdot U(\$6\text{K})\right) > \lim_{m\to\infty} W_{SM}(f^{\varepsilon}_{3,m}(\cdot)) \ge \lim_{m\to\infty} W_{SM}(f_{4,m}(\cdot)) = \phi\left(.002\cdot U(\$3\text{K})\right) \end{aligned}$$

To see that such preferences are also incompatible with the Variational Preferences form (9), assume enough regularity so that the limit can be moved inside the minimum function, to get

$$(19)\quad \begin{aligned} &\lim_{m\to\infty} W_{VP}\left(x_1 \text{ on } \wp_1 \underset{m}{\times} S;\ldots;x_n \text{ on } \wp_n \underset{m}{\times} S\right) \\ &= \lim_{m\to\infty}\min_{\pi(\cdot)\in\mathcal{P}}\left(\textstyle\sum_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S) + \eta\left(\pi(\cdot)\right)\right) \\ &= \min_{\pi(\cdot)\in\mathcal{P}}\left(\lim_{m\to\infty}\textstyle\sum_{i=1}^n U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S) + \eta\left(\pi(\cdot)\right)\right) \\ &= \min_{\pi(\cdot)\in\mathcal{P}}\left(\textstyle\sum_{i=1}^n U(x_i)\cdot\lambda(\wp_i) + \eta\left(\pi(\cdot)\right)\right) \\ &= \textstyle\sum_{i=1}^n U(x_i)\cdot\lambda(\wp_i) \;+\; \min_{\pi(\cdot)\in\mathcal{P}}\eta\left(\pi(\cdot)\right) \end{aligned}$$

Since the "min" term in the final expression is a constant independent of both the $x_i$'s and $\wp_i$'s of an almost-objective prospect, such preferences again converge to expected utility over such prospects, and are thus incompatible with the Common Ratio Effect preferences (14).

A similar argument can establish that the Multiple Priors, Smooth Ambiguity and Variational preferences forms (5), (7) and (9) are incompatible with the classic Allais Paradox and other Common Consequence Effect preferences.

It is important to observe that the above incompatibilities *do not* arise from how the models evaluate purely objective (or even mixed) uncertainty – that is, they do not arise from the square bracketed terms in equations (6), (8) and (10). Rather, they arise from the "conditional

[26] This has also been observed by Klibanoff, Marinacci and Mukerji (2005, p.1855, n.3).

*subjective* expected utility" structure of the purely subjective forms (5), (7) and (9) – that is, from the purely subjective expression $\sum_i U(x_i)\cdot\pi(E_i)$ as the subjective probability measure $\pi(\cdot)$ on the state space $\mathcal{S}$ ranges over some family of measures $\mathcal{P}_0$ or $\mathcal{P}$. This implies that merely replacing the bracketed *objective* expected utility terms in equations (6), (8) and (10) with non-expected utility functions will not eliminate the purely subjective Allais-type problems which can arise once three or more outcomes are considered. It also explains why the purely subjective Rank-Dependent form (3), which in general does not have the conditionally subjective expected utility form, is not necessarily subject to Allais-type effects.[27]

## IV. Attitudes Toward Different Sources (and Amounts) of Uncertainty

As noted, the subjective Rank-Dependent form (3) is not subject to the Allais-type problems of the previous section. However, it is subject to a different type of difficulty.

In bets that involve only two outcomes, such as the classic Ellsberg examples, the outcome values are necessarily "adjacent" to each other. However, adding a third possible outcome to an Ellsberg-type bet allows for possible interactions between outcome values which are not adjacent, such as the outcomes +\$8,000 and –\$8,000 in the Slightly-Bent Coin Problem.

The Slightly-Bent Coin Problem differs from a typical Allais- or Ellsberg-type problem in that it does not include any purely objective prospect, or even any purely objective event. In the author's view, the ambiguity-averse choice would be for Bet I, which spreads the uncertainty of receiving +\$8,000 versus –\$8,000 across the less ambiguous coin rather than the more ambiguous ball. Others have told me they would prefer Bet II. A strict preference in either direction would violate the Subjective Expected Utility Model: since informational symmetry would imply $\mu(BH) = \mu(BT) = \mu(WH) = \mu(WT) = ¼$, both bets would have a common subjective expected utility of $¼\cdot U(+\$8{,}000) + ½\cdot U(\$0) + ¼\cdot U(–\$8{,}000)$.

Although the ambiguity-averse ranking Bet I $\succ$ Bet II can be shown to be consistent with the Multiple Priors, Smooth Ambiguity and Variational Preferences models, neither it nor the reverse ranking Bet I $\prec$ Bet II is consistent with the Rank-Dependent Model, which implies that the two bets *must be indifferent* to each other. Although their good news events *BH* and *WT* for the payoff +\$8,000 are not the same, they are informationally-symmetric, which

[27] For example, define the capacity $C(\cdot)$ by $C(E) = c(\pi(E))$ for some probability measure $\pi(\cdot)$ to obtain $\lim_{m\to\infty} W_{RD}(\ldots; x_i \text{ on } \wp_i \overset{\times}{_m} S;\ldots) = V_{RD}(\ldots;x_i,\lambda(\wp_i);\ldots)$ for some Quiggin (1982) type *anticipated utility* preference function $V_{RD}(\ldots;x_i,p_i;\ldots) = \sum_i U(x_i)\cdot(c(\sum_{k=1}^{i} p_k) - c(\sum_{k=1}^{i-1} p_k))$, which with appropriate choice of cumulative probability transformation function $c(\cdot)$ is compatible with Allais-type preferences. We say that the Rank-Dependent form (3) is not *necessarily* subject to Allais-type effects since it includes subjective expected utility itself as a special case. A referee has observed that (3) includes other special cases which will be subject to Allais-type effects.

would presumably imply $C(BH) = C(WT)$. The two bets *do* have the same good news event for the payoff \$0, namely $BH \cup WH \cup WT$. Together, this implies that the formula (3) values

$$
\begin{aligned}
W_{RD}(\text{Bet 1}) &= U(+8K) \cdot C(BH) + U(0) \cdot [C(BH \cup WH \cup WT) - C(BH)] \\
&\qquad + U(-8K) \cdot [1 - C(BH \cup WH \cup WT)] \\
W_{RD}(\text{Bet 2}) &= U(+8K) \cdot C(WT) + U(0) \cdot [C(BH \cup WH \cup WT) - C(WT)] \\
&\qquad + U(-8K) \cdot [1 - C(BH \cup WH \cup WT)]
\end{aligned}
\tag{20}
$$

must be equal, so that the Rank-Dependent model implies indifference between the two (completely subjective) bets. In other words, the Rank-Dependent Model cannot represent a preference, in either direction, for one of these sources of ambiguity (coin versus urn) over the other. This is presumably relevant if real-world decisions involve different sources, with different amounts, of ambiguity.[28]

To see that this difficulty arises from the triple of outcome values {+\$8,000, \$0, –\$8,000}, replace the \$0's by +\$8,000 and –\$8,000 to obtain the adjacent-outcome bets Bet I* and Bet II* below. An ambiguity averter would presumably still prefer that the ±\$8,000 uncertainty be based on the less ambiguous coin than on the more ambiguous ball color, and prefer Bet I* over Bet II*. Since the key feature of the Slightly-Bent Coin Problem is that the partition $\{BH \cup WH, BT \cup WT\}$ is less ambiguous than $\{BH \cup BT, WH \cup WT\}$, the Rank-Dependent model can capture ambiguity-averse preferences over these two-outcome bets by simply positing $C(BH \cup WH) > C(WH \cup WT)$ for the less ambiguous good news event $BH \cup WH$ of Bet I* and the more ambiguous good new event $WH \cup WT$ of Bet II*.

| | Bet I* | | | | Bet II* | |
|---|---|---|---|---|---|---|
| | black | white | | | black | white |
| heads | +\$8,000 | +\$8,000 | vs. | heads | –\$8,000 | +\$8,000 |
| tails | –\$8,000 | –\$8,000 | | tails | –\$8,000 | +\$8,000 |

But as seen, the Rank-Dependent Model may not be able to capture attitudes toward different sources or amounts of ambiguity when it is across nonadjacent outcome values. Choice between the original Bets I and II is presumably still derives from the fact that the partition $\{BH \cup WH, BT \cup WT\}$ is less ambiguous than $\{BH \cup BT, WH \cup WT\}$. But since none of the events in these partitions is the "good news" event for any outcome value, none of the key

[28] Why a *bent* coin? Why not use an objective 50:50 coin to make the same point? As noted by Lo (2008,n.1), such a version appears in a manuscript version of Machina (2009). (Lo applies this version to the model of Klibanoff (2001), and Blavatskyy (2013) applies it to other models.) We use a bent (and therefore purely subjective) coin here in order to show that the difficulties with the Rank-Dependent Model appear even in a setting of purely subjective uncertainty – that is, directly from equation (3), and not solely in the mixed uncertainty domain of equation (4).

discriminating values $C(BH \cup WH)$, $C(WH \cup WT)$, $C(BH \cup WH)$ or $C(WH \cup WT)$ appear in the formulas (20), allowing the equality $C(BH) = C(WT)$ to imply $W_{RD}$(Bet I) = $W_{RD}$(Bet II).

In fact, the difficulties raised by the two-source Slightly-Bent Coin Problem extend to the case of just a single source of subjective uncertainty. Say you've learned that a meteor of unknown size or speed has been spotted, is predicted to strike the earth sometime the day after tomorrow, and you are betting on a single subjective variable, namely the longitude $\ell$ of its strike. Since you know nothing more, from your point of view the circular state space $S$ = $[0^{\text{o}}, 360^{\text{o}})$ is informationally-symmetric. The following bets are based on whether $\ell$ lands in the interval $[0^{\text{o}}, 180^{\text{o}})$ or $[180^{\text{o}}, 360^{\text{o}})$, or whether its seventh decimal is even or odd. This is seen to be a single-source analog of the Slightly-Bent Coin Problem, where the highly subjective ball is replaced by the highly subjective partition $\{[0^{\text{o}}, 180^{\text{o}}), [180^{\text{o}}, 360^{\text{o}})\}$, and the slightly bent coin is replaced by the almost-objective partition $\{7^{\text{th}}$ decimal even, $7^{\text{th}}$ decimal odd$\}$.[29] Presumably, an ambiguity-averse decision maker would prefer to spread the ±\$8,000 uncertainty across the almost-objective 7th decimal than across the much more subjective hemispheres, and prefer Bet I over Bet II. However, the good news events for the payoff \$8,000 are informationally-symmetric for the two bets, and their good news events for \$0 are again identical, so the Rank-Dependent Model cannot allow a preference for one of these sources of ambiguity over the other.

METEORITE PROBLEM ($\ell$ = LONGITUDE OF FIRST STRIKE)

| | BET I | | | | BET II | |
|---|---|---|---|---|---|---|
| | $\ell \in [0^{\text{o}}, 180^{\text{o}})$ | $\ell \in [180^{\text{o}}, 360^{\text{o}})$ | | | $\ell \in [0^{\text{o}}, 180^{\text{o}})$ | $\ell \in [180^{\text{o}}, 360^{\text{o}})$ |
| 7th dec. even | +\$8,000 | \$0 | vs. | 7th dec. even | \$0 | \$0 |
| 7th dec. odd | –\$8,000 | \$0 | | 7th dec. odd | –\$8,000 | +\$8,000 |

## V. Attitudes Toward Ambiguity at Different Outcome Levels

A third aspect of ambiguity aversion which can arise in a world of three or more outcomes is that, as with objective risk, ambiguity can occur at different final wealth levels or different gain/loss levels in a prospect, and individuals may exhibit different amounts of ambiguity aversion at these different outcome levels. This is not seen in the three-outcome bets of the Slightly-Bent Coin Problem, where in each bet, its upper and lower outcomes +\$8,000 and –

[29] Since it is based on a single source of purely subjective uncertainty, this example differs from the examples in Machina (2009), each of which involves both an objective and two separate sources of subjective uncertainty.

\$8,000 enter with equal ambiguity. But in general, prospects with three or more outcomes can exhibit more ambiguity at or about some outcome levels than others, which can allow this feature of ambiguity aversion to reveal itself.

This can be seen from the Three-Color Ambiguity at Low vs. High Outcomes Problem at the start of the paper. Recall that each urn's bet involves the same triple of outcomes \$0, \$$c$ and \$100, where \$$c$ is the decision maker's (or your own) certainty equivalent of the objective lottery (\$0,½;\$100,½).[30] But as noted, the ambiguity in Urn I is across the middle and lower outcomes \$$c$ and \$0, whereas in Urn II it is across the middle and upper outcomes \$$c$ and \$100. If ambiguity aversion somehow involves "pessimism," mightn't an ambiguity averter have a strict preference for Urn II over Urn I?

Maybe yes, maybe no. But whether or not an ambiguity averter *should* prefer one bet over the other, none of the models we are considering *allow* this to happen in either direction. To see this, express these mixed bets in a manner which separates their objective from their subjective uncertainty – that is, as Anscombe-Aumann acts – so they can be evaluated by the models' formulas (4), (6), (8) and (10). The following table presents these mappings from states to objective lotteries, where the underlying subjective uncertainty of each urn is given by its state space $\{BB,BW,WB,WW\}$ denoting the respective colors of its two unknown-color balls. For example, when both the unknown-color balls in Urn I are black, it yields a 2/3 chance of paying \$0 and a 1/3 chance of paying \$100; if both are white, it yields a 2/3 chance of \$$c$ and a 1/3 chance of \$100, etc. Under the objective expected utility hypothesis adopted by each of the four models (the bracketed terms in equations (4), (6), (8) and (10)), we can assign utility values of {0,½,1} to the outcomes {\$0,\$$c$,\$100} to obtain

AMBIGUITY AT LOW VS. HIGH OUTCOMES PROBLEM: MAPPING FROM STATES TO OBJECTIVE LOTTERIES AND EXPECTED UTILITY VALUES

| | *BB* | *BW* | *WB* | *WW* |
|---|---|---|---|---|
| URN I | (\$0,⅔;\$100,⅓) | (\$0,⅓;\$$c$,⅓;\$100,⅓) | (\$0,⅓;\$$c$,⅓;\$100,⅓) | (\$$c$,⅔;\$100,⅓) |
| URN II | (\$0,⅓;\$$c$,⅔) | (\$0,⅓;\$$c$,⅓;\$100,⅓) | (\$0,⅓;\$$c$,⅓;\$100,⅓) | (\$0,⅓;\$100,⅔) |
| exp. utility | ⅓ | ½ | ½ | ⅔ |

and equations (4), (6), (8) and (10) imply

[30] Lest the value \$$c$ be interpreted as coming from backwards induction of some two-stage prospects, we assume that this monetary amount has been independently and previously elicited from preferences over single-stage, purely objective lotteries.

*Rank-Dependent Model* (*equation* (4)):

$$
(21)\qquad \begin{aligned} W_{RD}(\text{URN I}) \;=\; & \tfrac{2}{3}\cdot C(WW) + \tfrac{1}{2}\cdot[C(WW\cup BW\cup WB) - C(WW)] \\ & + \tfrac{1}{3}\cdot[1 - C(WW\cup BW\cup WB)] \;=\; W_{RD}(\text{URN II}) \end{aligned}
$$

*Multiple Priors Model* (*equation* (6)):

$$
(22)\qquad W_{MP}(\text{URN I}) = \min_{(p_{BB},p_{BW},p_{WB},p_{WW})\in\mathcal{P}_0}\left[\tfrac{1}{3}\cdot p_{BB}+\tfrac{1}{2}\cdot p_{BW}+\tfrac{1}{2}\cdot p_{WB}+\tfrac{2}{3}\cdot p_{WW}\right] = W_{MP}(\text{URN II})
$$

*Smooth Ambiguity Model* (*equation* (8)):

$$
(23)\qquad \begin{aligned} W_{SM}(\text{URN I}) \;=\; & \int \phi\left(\tfrac{1}{3}\cdot p_{BB}+\tfrac{1}{2}\cdot p_{BW}+\tfrac{1}{2}\cdot p_{WB}+\tfrac{2}{3}\cdot p_{WW}\right)\cdot d\mu(p_{BB},p_{BW},p_{WB},p_{WW}) \\ =\; & W_{SM}(\text{URN II}) \end{aligned}
$$

*Variational Preferences Model* (*equation* (10)):

$$
(24)\qquad \begin{aligned} & W_{VP}(\text{URN I}) \;=\; \\ & \min_{(p_{BB},p_{BW},p_{WB},p_{WW})\in\mathcal{P}}\left[\tfrac{1}{3}\cdot p_{BB}+\tfrac{1}{2}\cdot p_{BW}+\tfrac{1}{2}\cdot p_{WB}+\tfrac{2}{3}\cdot p_{WW}+\eta(p_{BB},p_{BW},p_{WB},p_{WW})\right] \\ & \qquad\qquad = \; W_{VP}(\text{URN II}) \end{aligned}
$$

In other words, none of the four models allow the decision maker to exhibit an aversion to ambiguity in the lower outcome relative to ambiguity in the upper outcome, or vice versa, in this decision problem.

What is it in the structure of these bets that necessitates this indifference? As seen from the first two rows of the above table, the two bets have different mappings from states to objective lotteries. But as seen in the third row, they have the *same statewise assignment of objective expected utility values.* Any decision model which evaluates a bet solely on its mapping from states to these values – as do the Rank-Dependent, Multiple Priors, Smooth Ambiguity and Variational Preferences models[31] – must necessarily be indifferent between the two bets, in spite of the fact that their ambiguity occurs at different outcome levels.[32]

One might argue that preferences for Urn I versus Urn II in this example would not be a feature of an individual's ambiguity preferences at all, but rather, determined by their attitudes toward objective risk. But risk attitudes are completely incorporated into the statewise conditional expected utility levels. Preferences over the bets must be due to

[31] This follows since, for each event $E_i$, its conditional objective lottery $(\ldots;x_{ij},p_{ij};\ldots)$ only enters via the bracket term $[\sum_j U(x_{ij})\cdot p_{ij}]$ in each of equations (4), (6), (8) and (10).

[32] One might argue that the two bets do not have the same statewise assignments from states to expected utility levels, since their state spaces {*BB* in Urn I, *BW* in Urn I,…} and {*BB* in Urn II, *BW* in Urn II,…} are distinct. The use of separate urns in this example is for clarity of presentation – replacing Urn II's bet by the bet {\$$c$ if black; \$100 if white; \$0 if red} on Urn I yields an equivalent example over the common state space {*BB* in Urn I, *BW* in Urn I,…}. A similar argument holds for the following examples.

attitudes toward ambiguity at high versus low outcome levels – attitudes which cannot be captured by either subjective expected utility $W_{SEU}(\cdot)$ or any of the major models.

The use of four-outcome bets allows for a more explicit illustration of this phenomenon. Select monetary outcomes with equally spaced utility levels 1 through 4, and consider the following bets, expressed in terms of these utility levels. Urn I differs from the purely objective bet $b_0$ by introducing ambiguity across its lower values 1 and 2, whereas Urn II differs by introducing ambiguity over its higher values 3 and 4.

AMBIGUITY AT LOW VS. HIGH OUTCOMES PROBLEM – FOUR-COLOR VERSION

$$b_0 \qquad \overbrace{\;1\;}^{\text{1 ball}} \qquad \overbrace{\;2\;}^{\text{1 ball}} \qquad \overbrace{\;3\;}^{\text{1 ball}} \qquad \overbrace{\;4\;}^{\text{1 ball}}$$

$$\begin{array}{c c}
\text{URN I} & \text{URN II} \\
\overbrace{\begin{array}{cc}\text{black} & \text{white} \\ 1 & 2\end{array}}^{\text{2 balls}}
\;\overbrace{\begin{array}{c}\text{red} \\ 3\end{array}}^{\text{1 ball}}
\;\overbrace{\begin{array}{c}\text{green} \\ 4\end{array}}^{\text{1 ball}}
\quad \text{vs.} \quad &
\overbrace{\begin{array}{c}\text{red} \\ 1\end{array}}^{\text{1 ball}}
\;\overbrace{\begin{array}{c}\text{green} \\ 2\end{array}}^{\text{1 ball}}
\;\overbrace{\begin{array}{cc}\text{black} & \text{white} \\ 3 & 4\end{array}}^{\text{1 ball}}
\end{array}$$

| | *BB* | *BW* | *WB* | *WW* |
|---|---|---|---|---|
| URN I | (1,½;3,¼;4,¼) | (1,¼;2,¼;3,¼;4,¼) | (1,¼;2,¼;3,¼;4,¼) | (2,½;3,¼;4,¼) |
| URN II | (1,¼;2,¼;3,½) | (1,¼;2,¼;3,¼;4,¼) | (1,¼;2,¼;3,¼;4,¼) | (1,¼;2,¼;4,½) |
| exp. utility | 2¼ | 2½ | 2½ | 2¾ |

Again, informational symmetry implies that a subjective expected utility maximizer would have equal subjective probabilities over the states $\{BB,BW,WB,WW\}$, and accordingly be indifferent between the bets. But if ambiguity at lower outcome levels matters differently than ambiguity at higher outcome levels, an ambiguity averter may well have a strict preference for one bet over the other. But since the two urns again have equivalent mappings from states to expected utility levels, the four models must again rank them as indifferent – none can allow for a preference (or aversion) toward ambiguity about some outcome values rather than others.

The following example, which we choose to express in terms of gains and losses, has a different structure. Starting from a purely objective 50:50 bet over utility levels {–2,+2} ambiguity is introduced via an informationally-symmetric spread about its loss level –2 to obtain Urn I, and via an informationally-symmetric spread about its gain level +2 to obtain Urn II. Experimenters may find that individuals are more averse to ambiguity in losses than in gains (or vice versa). But since each urn has the same mapping $\{BR, BG, WR, WG\} \rightarrow \{-½, -½, +½, +½\}$ from states to expected utility levels, the four models must again rank the bets as indifferent.

AMBIGUITY IN LOSSES VERSUS GAINS

| URN I | | | | | URN II | | | |
|---|---|---|---|---|---|---|---|---|
| 1 ball | | 1 ball | | | 1 ball | | 1 ball | |
| black | white | red | green | vs. | red | green | black | white |
| –3 | –1 | +2 | +2 | | –2 | –2 | +1 | +3 |

In Section III we noted that the objective expected utility terms $\sum_j U(x_{ij})\cdot p_{ij}$ in formulas (4), (6), (8) and (10) left these models susceptible to Allais-type problems under purely objective uncertainty, and we have seen in this section how prospects with different ambiguity properties can nevertheless have identical mappings from states to expected utility levels. It might seem that both problems could be averted by replacing the models' objective expected utility terms by a Quiggin (1982) type *anticipated utility* term $\sum_j U(x_{ij})\cdot(c(\sum_{k=1}^{j} p_{ik}) - c(\sum_{k=1}^{j-1} p_{ik}))$ with appropriate choice of cumulative probability transformation function $c(\cdot)$.[33] While doing so does avert objective Allais-type problems, it cannot avert the types of difficulties presented in this section.

To see this, define $\hat{p}$ so that $c(\hat{p}) = ½$, which in turn implies $(1-c(\hat{p})) = ½$, and consider the following bets, each of which involves an objective $(1-\hat{p}){:}\hat{p}$ coin flip to determine from which of two urns a ball is to be drawn. Bet I differs from the objective lottery $(4, 1-\hat{p}; 8, \hat{p})$ by introducing informationally-symmetric ambiguity about its lower outcome level 4, whereas Bet II differs by introducing ambiguity about its higher outcome level 8. Below are the bets' statewise conditional objective lotteries with their conditional anticipated utility values:

[33] See, for example, Klibanoff, Marinacci and Mukerji (2005,p.1859-1860). Along the lines of equations (3) and (4), variables are labeled so that the terms $\sum_j U(x_{ij})\cdot\big(c(\sum_{k=1}^{j} p_{ik}) - c(\sum_{k=1}^{j-1} p_{ik})\big)$ are decreasing in $i$, and for each $i$, the values $x_{ij}$ are decreasing in $j$. The function $c(\cdot)$ defined over objective probabilities is to be distinguished from the capacity $C(\cdot)$ defined over events.

AMBIGUITY AT LOW VS. HIGH OUTCOMES – ANTICIPATED UTILITY VERSION

| BET I | | | | | BET II | | | |
|---|---|---|---|---|---|---|---|---|
| prob $(1–\hat{p})$ | | prob $\hat{p}$ | | | prob $(1–\hat{p})$ | | prob $\hat{p}$ | |
| URN I (1 ball) | | URN II (1 ball) | | | URN I (1 ball) | | URN II (1 ball) | |
| black | white | red | green | vs. | red | green | black | white |
| 3 | 5 | 8 | 8 | | 4 | 4 | 7 | 9 |

| | *BR* | *BG* | *WR* | *WG* |
|---|---|---|---|---|
| BET I | $(3,1–\hat{p};8,\hat{p})$ | $(3,1–\hat{p};8,\hat{p})$ | $(5,1–\hat{p};8,\hat{p})$ | $(5,1–\hat{p};8,\hat{p})$ |
| BET II | $(4,1–\hat{p};7,\hat{p})$ | $(4,1–\hat{p};7,\hat{p})$ | $(4,1–\hat{p};9,\hat{p})$ | $(4,1–\hat{p};9,\hat{p})$ |
| anticip. util. | 5½ | 5½ | 6½ | 6½ |

In this case, the bets have equivalent mappings from states to conditional anticipated utility levels, so forms which evaluate prospects solely on the basis of these mappings must again rank the bets as indifferent, and again cannot admit a preference for ambiguity in high versus low outcome values (or similarly, for ambiguity in gains versus losses).

It is worth contrasting a preference for ambiguity at high rather than low outcome values with what might be called "decreasing absolute ambiguity aversion."[34] This distinction can be illustrated by analogy with the case of objective risk preferences. Decreasing absolute risk aversion in the standard objective sense involves starting with a pair of prospects, adding some common $\Delta x$ to all their monetary outcomes, and comparing the ranking over the newly created second pair with the ranking over the original pair. In contrast, a preference for risk about higher rather than lower outcomes starts with a single prospect, and compares whether the individual would prefer a given mean-preserving spread about a higher rather than a lower outcome value.[35] These two properties of preferences are not equivalent, even under objective expected utility: under expected utility, the second is equivalent to $-U''(x)$ (rather than $-U''(x)/U'(x)$) decreasing in $x$. This property of preferences is strictly weaker than decreasing absolute risk aversion, in that it is exhibited by every decreasing *or constant* absolute risk averse von Neumann-Morgenstern utility function.

The distinction in the case of ambiguity preferences is analogous. Decreasing absolute ambiguity aversion would presumably involve adding some $\Delta U$ to all the utility outcomes of

[34] See the discussions of Klibanoff, Marinacci and Mukerji (2005,pp.1866,1876), Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011, Sect.5.2.4) and Grant and Polak (2012).

[35] In this sense it is intuitively similar to a preference for positive versus negative skewness.

a pair of prospects, and comparing the ranking over the second pair with the ranking over the first. But as portrayed in the examples of this section, a preference for ambiguity about higher rather than lower outcomes starts with a single purely objective prospect, and compares whether the individual would prefer introducing ambiguity in higher rather than lower utility levels. Although decreasing absolute ambiguity aversion cannot be exhibited by the Rank-Dependent, Multiple Priors or Variational Preferences forms (4), (6) or (10),[36] they can be easily adapted by putting their square bracketed terms inside an appropriate nonlinear function (or in the case of the Smooth Ambiguity model (8), by simply making the appropriate curvature assumptions on $\phi(\cdot)$). However the examples of this section imply that none of the four models can exhibit a preference for ambiguity about higher rather than lower outcome values (or vice versa), regardless of the shape of their $U(\cdot)$ functions, or whether they undergo the above adaptations.

Our final example illustrates the case of a consumer who, by paying ½ util to insure against a low-likelihood ambiguous loss of 4 utils, and another ½ util to buy a low-likelihood ambiguous lottery ticket with a prize of 4 utils, can go from Bet I to Bet II. Since neither transaction would affect the mapping from (the 16) states to their conditional expected utility values,[37] the subjective expected utility model (which is ambiguity-neutral) would be exactly indifferent to making either (or both) transactions. But for that same reason, so would each of the four major models, which means that none can model ambiguity attitudes which depart from ambiguity-neutrality in a Friedman-Savage (1948) type direction of the simultaneous purchase of insurance against low-likelihood ambiguous disastrous losses with holdings in low-likelihood ambiguous high-stakes business ventures.

FRIEDMAN-SAVAGE TYPE EXAMPLE

| | 1 ball | | 1 ball | | 1 ball | | 1 ball | |
|---|---|---|---|---|---|---|---|---|
| | black | white | red | green | blue | yellow | purple | amber |
| BET I | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| BET II | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 7 |

36 For each model, increasing $U(x_{ij})$ by $\Delta U$ for all $i, j$ implies that $W(\cdot)$ will increase by $\Delta U$.

37 After the informationally-symmetric reordering of the payoffs {3,3,3,3,3,3,3,7} to {3,7,3,3,3,3,3,3} in Bet II.

## VI. Conclusion

The examples of this paper show that a world of three or more possible outcome values allows for aspects of ambiguity and ambiguity aversion which cannot arise in the classic two-outcome Ellsberg examples, and are not fully captured by models developed in response to these original examples. Three such aspects have been presented in this paper:

- The Thermometer Problem of Section III arises from the fact that for all $\pi(\cdot)$ in a family $\mathcal{P}_0$ or $\mathcal{P}$ of smooth subjective probability measures, the purely subjective expression $\sum_i U(x_i)\cdot\pi(\wp_i \underset{m}{\times} S)$ will converge to its purely objective counterpart $\sum_i U(x_i)\cdot\lambda(\wp_i)$. Since the purely subjective Multiple Priors, Smooth Ambiguity and Variational forms (5), (7) and (9) evaluate almost-objective acts solely through these subjective expressions, preferences over these acts will, for large enough $m$, inherit the same types of Allais-type violations of objective expected utility that arise in a world of three or more outcomes.
- The Slightly-Bent Coin and Meteorite Problems of Section IV arise from the fact that in a world of three or more outcomes, subjective prospects with different ambiguity structures might still have informationally-symmetric good news events $\cup_{k=1}^{i} E_k$. Since the Rank-Dependent form (3) evaluates subjective prospects solely through these events, it cannot capture attitudes toward these different ambiguity structures.
- The Ambiguity at Low vs. High Outcomes Problems of Section V arise from the fact that in a world of three or more outcomes, prospects whose ambiguity occurs at differ outcome levels might still have the same statewise conditional expected utility values $\sum_j U(x_{ij})\cdot p_{ij}$. Since forms (4), (6), (8) and (10) of the four major models evaluate conditional objective uncertainty solely through these statewise conditional values, they cannot capture attitudes toward ambiguity at different outcome levels.

The major models of ambiguity aversion considered in this paper generalize the subjective expected utility specification of a unique probability measure over states by means of: a nonadditive measure over states (rank-dependent), a family of probability measures over states (multiple priors), the family of all probability measures over states (variational) or a prior on the set of all probability measures over states (smooth). None of these forms is able to accommodate all of the examples of this paper. There do, however, exist functional forms general enough to accommodate these and other examples, such as the two-stage recursive model of Segal (1987).[38] As always, generalizations of a model move us along the tradeoff between the ability to accommodate specific paradoxes/examples and predictive power for other economic decisions. The proper point on this tradeoff will, as always, depend on both the findings of experimenters and the applications of theorists.

An alternative approach would be to drop reliance on specific functional forms altogether, and try to address these phenomena via *curvature conditions* on general preference functions of the form $W(\ldots;x_i \text{ on } E_i;\ldots)$ or $W(\ldots;(\ldots;x_{ij},p_{ij};\ldots) \text{ on } E_i;\ldots)$. By "curvature conditions" I mean directions in which ambiguity-averse preferences "bend away" from classical subjective expected utility, much like risk aversion over objective lotteries has been modeled by von Neumann-Morgenstern utility functions which bend away from linearity in the direction of concavity.[39] After all, some of the most fundamental results in economics – the Slutsky equation, efficiency of competitive markets, existence of general equilibrium – are not based on functional forms at all, merely curvature conditions such as quasiconcavity, convexity, etc.

## Appendix: What's the Proper State Space for an Ellsberg Urn?

As noted above, there is more than one way to specify the underlying state space of an Ellsberg urn.[40] In our discussion of the classic Three-Color Paradox with its 60 balls of unknown color, we specified a 61-element space, with each state corresponding to the number of black balls. In our Section V discussion of the Three-Color Ambiguity at Low vs. High Outcomes Problem, we specified a 4-element space of the form $\{BB,BW,WB,WW\}$. Had we instead defined states by the number of black balls, we would have had a 3-element state space {0 black balls, 1 black ball, 2 black balls}. The relationship between the two specifications is that the latter space is slightly coarser that the former, with its state "1 black ball" being the union of the states *BW* and *WB*.

Does the choice of approach matter? Not always – given informational symmetry, the bets in the Two-Urn and Four-Color Ellsberg Paradoxes are such that the lessons of these examples would be the same under either specification.

However, the two approaches can yield different predictions over other types of bets, even under informational symmetry. Consider, as does Epstein (2010, p.2088), bets on the actual composition of an Ellsberg urn – say Urn I in the three-color version of the Ambiguity at

[38] See Dillenberger and Segal (2012).

[39] Or for a general preference function $V(x_1,p_1;\ldots;x_n,p_n)$ over objective lotteries, by the curvature condition that $\partial V(x_1,p_1;\ldots;x_n,p_n)/\partial p_i$ is concave in $x_i$ (Machina (1982,Thm.2;1987,pp.133-134)).

[40] See also the discussions in Wakker (2010, Sects. 4.9 and 10.7) and Machina (2011).

Low vs. High Outcomes Problem. Informational symmetry of the state space $\{BB,BW,WB,WW\}$ would lead to indifference between staking a prize on the two balls being the same color versus the two balls being of different colors. On the other hand, informational symmetry of the space {0 black balls, 1 black ball, 2 black balls} would lead to a strict preference for the first bet. This distinction is magnified when applied to the 90-ball Three-Color Urn.

Whether individuals treat the states $\{BB,BW,WB,WW\}$ or the states {0 black balls, 1 black ball, 2 black balls} as informationally-symmetric is ultimately an empirical question. It turns out that the distinction does not matter for our analysis of this example or any of our other examples. As mentioned, we explicitly adopted the state space $\{BB,BW,WB,WW\}$ in our analysis of this example. However, if we instead adopt the space {0 black balls, 1 black ball, 2 black balls}, the table preceding equations (21) through (24) would take the form

| | 0 black balls | 1 black ball | 2 black balls |
|---|---|---|---|
| URN I | $(\$c,⅔;\$100,⅓)$ | $(\$0,⅓;\$c,⅓;\$100,⅓)$ | $(\$0,⅔;\$100,⅓)$ |
| URN II | $(\$0,⅓;\$100,⅔)$ | $(\$0,⅓;\$c,⅓;\$100,⅓)$ | $(\$0,⅓;\$c,⅔)$ |
| expected utility | ⅔ | ½ | ⅓ |

and the equations themselves would take the forms

*Rank-Dependent Model:*

$$\begin{aligned} W_{RD}(\text{URN I}) &= \tfrac{2}{3}\cdot C(0\text{ black}) + \tfrac{1}{2}\cdot[C(0\text{ black}\cup 1\text{ black}) - C(0\text{ black})] \\ &\quad + \tfrac{1}{3}\cdot[1 - C(0\text{ black}\cup 1\text{ black})] \;=\; W_{RD}(\text{URN II}) \end{aligned} \tag{21*}$$

*Multiple Priors Model:*

$$\begin{aligned} W_{MP}(\text{URN I}) &= \min_{(p_{0\text{ black}},\,p_{1\text{ black}},\,p_{2\text{ black}})\in\mathcal{P}_0} \left[\tfrac{2}{3}\cdot p_{0\text{ black}} + \tfrac{1}{2}\cdot p_{1\text{ black}} + \tfrac{1}{3}\cdot p_{2\text{ black}}\right] \\ &= W_{MP}(\text{URN II}) \end{aligned} \tag{22*}$$

*Smooth Ambiguity Model:*

$$\begin{aligned} W_{SM}(\text{URN I}) &= \int \phi\left(\tfrac{2}{3}\cdot p_{0\text{ black}} + \tfrac{1}{2}\cdot p_{1\text{ black}} + \tfrac{1}{3}\cdot p_{2\text{ black}}\right)\cdot d\mu(p_{0\text{ black}}, p_{1\text{ black}}, p_{2\text{ black}}) \\ &= W_{SM}(\text{URN II}) \end{aligned} \tag{23*}$$

*Variational Preferences Model:*

$$
\begin{aligned}
&W_{VP}(\text{U}\text{RN I}) \quad = \\
(24^*) \qquad &\min_{(p_{0\,\text{black}},\, p_{1\,\text{black}},\, p_{2\,\text{black}}) \in \mathcal{P}} \left[ \tfrac{2}{3} \cdot p_{0\,\text{black}} + \tfrac{1}{2} \cdot p_{1\,\text{black}} + \tfrac{1}{3} \cdot p_{2\,\text{black}} + \eta(p_{0\,\text{black}}, p_{1\,\text{black}}, p_{2\,\text{black}}) \right] \\
&= \quad W_{VP}(\text{U}\text{RN II})
\end{aligned}
$$

In other words, the two urns will continue to have the same mapping from states to expected utility levels, and the four models will continue to imply indifference.

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