Solving Nonlinear Rational Expectations Models by Approximating the Stochastic Equilibrium System

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The Way Ahead

Motivation

- **Non-linearities** and **stochastic** environments are the central attributes of modern dynamic macroeconomic modeling.

- Predominant solution method for DSGE models: **Linearization**

- Major **Drawback**: **Cannot** account for **uncertainty** and ignores non-linearities and risk-sensitivity (e.g. precautionary savings).

- **Reason:** Conventionally computed from the **deterministic model** where shocks are simply neglected.

The AES-Approach

This paper proposes:

- Approximate the original DSGE model by a kth-order Taylor series expansion in the exogenous shocks about the deterministic model.

- Compute the linear solution to the kth-order approximation equilibrium system (**AES**).

- The **conventional** approach takes **0th-order** AES to compute the linear solution!

- The AES-approach is in straight analogy to Samuelson's Fundamental Approximation Theorem to portfolio choice model (REStud 1970).

Theoretical Implications of the AES-approach:

1. The approximated system of equilibrium conditions AES
   i) is non-stochastic,
   ii) it preserves the nonlinearity in the endogenous variables, and
   iii) it is linear in the first k moments of the exogenous disturbances.

2. The linear solution (steady state and 1st-order Taylor coefficients) to the AES captures the equilibrium effects of risk up to the first k moments of the exogenous disturbances.

3. The approximation error of the linear solution is in the same order of magnitude as the approximation error of AES and the implied Euler error.

Practical Implications for Researchers:

1. Linear state-space representation of the solution process which allows to study the equilibrium implications of nonlinearities in stochastic environments:

    - AES-approach captures the effects of risk up to the first kth-moments on: i) existence, ii) determinacy, iii) equilibrium distribution, and iv) dynamics.

    - Standard linear econometric tools (VARs, likelihood based approaches with linear filtering,...)

    - Enables use of linear toolbox to study risk in Macro-models: welfare comparisons across different policies, asset pricing in DSGE models, Markov-switching models, models with recursive preferences, time-varying volatility, etc...

Practical Implications for Researchers:

2. Accuracy of linear solution increases in the order k of the Taylor series expansion to the equilibrium system.
   By implication:

   - Accuracy of the approximated key local properties i) existence, ii) determinacy, iii) equilibrium distribution, and iv) dynamics increases in the order k.

   - Accuracy of Maximum Likelihood inference and hence parameter estimates based on the linear solution increases in the order k (compare Ackerberg et al (ECTRA, 2009)) .

   - Accuracy of forecasts based on the linear solution to the AES also increases in the order k.

Practical Implications for Researchers:

3. The AES-approach requires the solution to be locally determined for the kth-order AES but not for the deterministic version of the model.

   - Applicable to models as for instance portfolio choice models.

Outline of the Presentation

- The General Formulation of the Equilibrium Framework
- The Solution Procedure
- Implementation
- Application: Asset Pricing Model
- Related Literature
- Concluding Remarks

General Formulation of the Equilibrium Framework *(1/2)*

General form of the system of dynamic stochastic equilibrium equations:

$$0 \;=\; E_t f(y_t, s_t, y_{t+1}, s_{t+1})$$

- $y_t \in \mathbb{R}^{n_y}$ is the vector of endogenous non-predetermined variables

- $s_t \in \mathbb{R}^{n_s}$ is the vector of the endogenous (x_t) and exogenous (z_t) predetermined variables, i.e.

$$s_t = \begin{bmatrix} x_t \\ z_t \end{bmatrix}$$

- $z_t \in \mathbb{R}^{n_z}$ is described by $0 = M(z_{t+1}, z_t, \sigma\epsilon_{t+1})$.

- $\epsilon_t \in \mathbb{R}^{n_\epsilon}$ is the vector of i.i.d. exogenous disturbances.

General Formulation of the Equilibrium Framework *(2/2)*

- The solution process evolves according to:

$$y_t = g(x_t, z_t) \qquad \text{and} \qquad s_{t+1} = q(s_t, \sigma\epsilon_{t+1}) = \begin{pmatrix} h(x_t, z_t) \\ m(z_t, \sigma\epsilon_{t+1}) \end{pmatrix}$$

- The system of nonlinear stochastic equilibrium equations in terms of the solution:

$$\begin{aligned} 0 &= E_t f(g(x_t, z_t), x_t, z_t, g(h(x_t, z_t), m(z_t, \sigma\epsilon_{t+1})), h(x_t, z_t), m(z_t, \sigma\epsilon_{t+1})) \\ &\equiv E_t F(s_t, \sigma\epsilon_{t+1}; g(\cdot), q(\cdot)). \end{aligned}$$

- Regularity conditions are assumed to be satisfied for the order of approximation.

Approximating the Stochastic Equilibrium System *(1/4)*

At a given state s_t, the k-th order Taylor expansion of the stochastic equilibrium system $E_t F(s_t, \sigma\epsilon_{t+1}; g(\cdot), h(\cdot))$ in the perturbation parameter about the deterministic model when $\sigma = 0$ yields

$$\begin{aligned}
E_t F(s_t, \sigma\epsilon_{t+1}; g(\cdot), q(\cdot)) &= F(s_t, 0; g^0(\cdot), q^0(\cdot)) \\
&+ F_\epsilon(s_t, 0; g^0(\cdot), q^0(\cdot))\sigma E_t \epsilon_{t+1} \\
&+ \frac{1}{2} F_{\epsilon\epsilon}(s_t, 0; g^0(\cdot), q^0(\cdot))\sigma^2 E_t \left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \\
&+ \frac{1}{3!} F_{\epsilon\epsilon\epsilon}(s_t, 0; g^0(\cdot), q^0(\cdot))\sigma^3 E_t \left(\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \epsilon_{t+1}\right) \\
&+ ... + O(\sigma^{k+1}).
\end{aligned}$$

The Taylor series expansion is non-stochastic and non-linear in the state s_t, and it is linear in the first k moments $E_t\epsilon_{t+1}, E_t\left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right), ...$ of the exogenous shock distribution.

Approximating the Stochastic Equilibrium System *(2/4)*

The approximation to the equilibrium system in Proposition 1 can be stated explicitly as

$$
\begin{aligned}
E_t f(y_t, s_t, y_{t+1}, s_{t+1}) = {} & f\left(y_t, s_t, y_{t+1}, s_{t+1}\right) \\
& + f_{w_{t+1}}(\cdot) \begin{bmatrix} q_\epsilon^0 \\ g_s^0 q_\epsilon^0 \end{bmatrix} \sigma E_t \epsilon_{t+1} \\
& + \frac{1}{2} f_{w_{t+1} w_{t+1}}(\cdot) \left(\begin{bmatrix} q_\epsilon^0 \\ g_s^0 q_\epsilon^0 \end{bmatrix} \otimes \begin{bmatrix} q_\epsilon^0 \\ g_s^0 q_\epsilon^0 \end{bmatrix} \right) \sigma^2 E_t(\epsilon_{t+1} \otimes \epsilon_{t+1}) \\
& + \frac{1}{2} f_{w_{t+1}}(\cdot) \begin{bmatrix} q_{\epsilon\epsilon}^0 \\ g_{ss}^0 (q_\epsilon^0 \otimes q_\epsilon^0) + g_s^0 q_{\epsilon\epsilon}^0 \end{bmatrix} \sigma^2 E_t(\epsilon_{t+1} \otimes \epsilon_{t+1}) \\
& + \ldots + O(\sigma^{k+1}) \\
& \equiv f^\sigma(y_t, s_t, y_{t+1}, s_{t+1}) + O(\sigma^{k+1}),
\end{aligned}
$$

where $w_{t+1} = [y_{t+1}; s_{t+1}]$ for conciseness.

Approximating the Stochastic Equilibrium System *(3/4)*

The associated actual law of motion for z_t is appropriately approximated by the k-th order Taylor expansion in the perturbation parameter about the deterministic law,

$$
\begin{aligned}
z_{t+1} = {} & m(z_t, 0) \\
& + m_\epsilon(z_t, 0)\sigma\epsilon_{t+1} \\
& + \frac{1}{2!} m_{\epsilon\epsilon}(z_t, 0)\sigma^2 \left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \\
& + \frac{1}{3!} m_{\epsilon\epsilon\epsilon}(z_t, 0)\sigma^3 \left(\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \epsilon_{t+1}\right) \\
& + ... + O(\sigma^{k+1}) \\
& \equiv m^\sigma(z_t, \epsilon_{t+1}) + O(\sigma^{k+1}).
\end{aligned}
$$

Approximating the Stochastic Equilibrium System *(4/4)*

The k-th order Taylor expansion of the general model in the perturbation parameter σ about the deterministic model when $\sigma = 0$ can be stated as

$$k^{\sigma}(y_t, s_t, y_{t+1}, s_{t+1}, \epsilon_{t+1}) = \begin{bmatrix} f^{\sigma}(y_t, s_t, y_{t+1}, s_{t+1}) \\ z_{t+1} - m^{\sigma}(z_t, \epsilon_{t+1}) \end{bmatrix} = 0$$

The solution to the approximated equilibrium system AES is denoted by

$$s_{t+1} = q^{\sigma}(s_t, \epsilon_{t+1}) = \begin{bmatrix} h^{\sigma}(x_t, z_t) \\ m^{\sigma}(z_t, \epsilon_{t+1}) \end{bmatrix} \quad \text{and} \quad y_t = g^{\sigma}(s_t)$$

1. The steady state $(\bar{y}^\sigma, \bar{s}^\sigma)$ of the approximated equilibrium system AES satisfies

$$\bar{y}^\sigma = g(\bar{s}^\sigma), \quad \bar{x}^\sigma = h(\bar{s}^\sigma), \quad \text{and} \quad \bar{z}^\sigma = \bar{z} = m(\bar{z}, 0),$$

where the shock realization is $\epsilon = 0$,

and $\bar{y}^\sigma$ and $\bar{s}^\sigma$ solve

$$k^\sigma(\bar{y}^\sigma, \bar{s}^\sigma, \bar{y}^\sigma, \bar{s}^\sigma, 0) = 0.$$

2. The first-order Taylor approximation to $g^\sigma(s_t)$ and $h^\sigma(s_t)$ about the steady state $(\bar{y}^\sigma, \bar{s}^\sigma)$, i.e.

$$g^\sigma(\bar{s}^\sigma + \delta) = \bar{y}^\sigma + g_s^\sigma(\bar{s}^\sigma)\delta$$

$$h^\sigma(\bar{s}^\sigma + \delta) = \bar{x}^\sigma + h_s^\sigma(\bar{s}^\sigma)\delta.$$

The linear coefficients are computed from solving

$$k^\sigma_{w_{t+1}}(\bar{y}^\sigma, \bar{s}^\sigma, \bar{y}^\sigma, \bar{s}^\sigma, 0)\begin{pmatrix} q_s^\sigma \\ g_s^\sigma q_s^\sigma \end{pmatrix} + k^\sigma_{w_t}(\bar{y}^\sigma, \bar{s}^\sigma, \bar{y}^\sigma, \bar{s}^\sigma, 0)\begin{pmatrix} I_{n_s} \\ g_s^\sigma \end{pmatrix} = 0$$

(Here it goes to the **details**)

Implementation

The AES-approach contains three steps:

1. For a given initial state s_t (e.g. the deterministic steady state), find the respective derivative for the deterministic solution $g^0(s_t)$ and $q^0(s_t, 0)$ (! numerical terms).

2. Solve for the steady state to the approximated equilibrium system searching for the fix point in the symbolic expressions $f^\sigma(\cdot)$, namely $f\cdot, f_{w_{t+1}}\cdot, f_{w_{t+1}w_{t+1}}\cdot, \ldots$ (can be done by numerical solver).

3. Evaluate the terms $k^\sigma_{w_{t+1}}(\cdot)$ and $k^\sigma_{w_t}(\cdot)$ and compute the linear coefficients to the solution (standard approach, use e.g. P. Klein's code).

Example: Asset Pricing Model

The **asset pricing model** in Burnside (1998, JEDC)) which has a closed-form solution:

- Allows to compute the linear part of the exact (true) solution and to compute the true equilibrium distribution;
- Exact solution can be used as the data generating process for estimating the parameters of the model computing Maximum Likelihood.

Example: Asset Pricing Model *(2/1)*

- Purpose of the numerical exercise: assess the performance of the *AES-approach* and compare it to the conventional perturbation approach.

- Comparison is based on the 2nd-order AES and it considers four distinct criteria:

  (A) The Taylor coefficients and the implied equilibrium distribution.

  (B) Existence and uniqueness of the solution.

  (C) Maximum likelihood estimation.

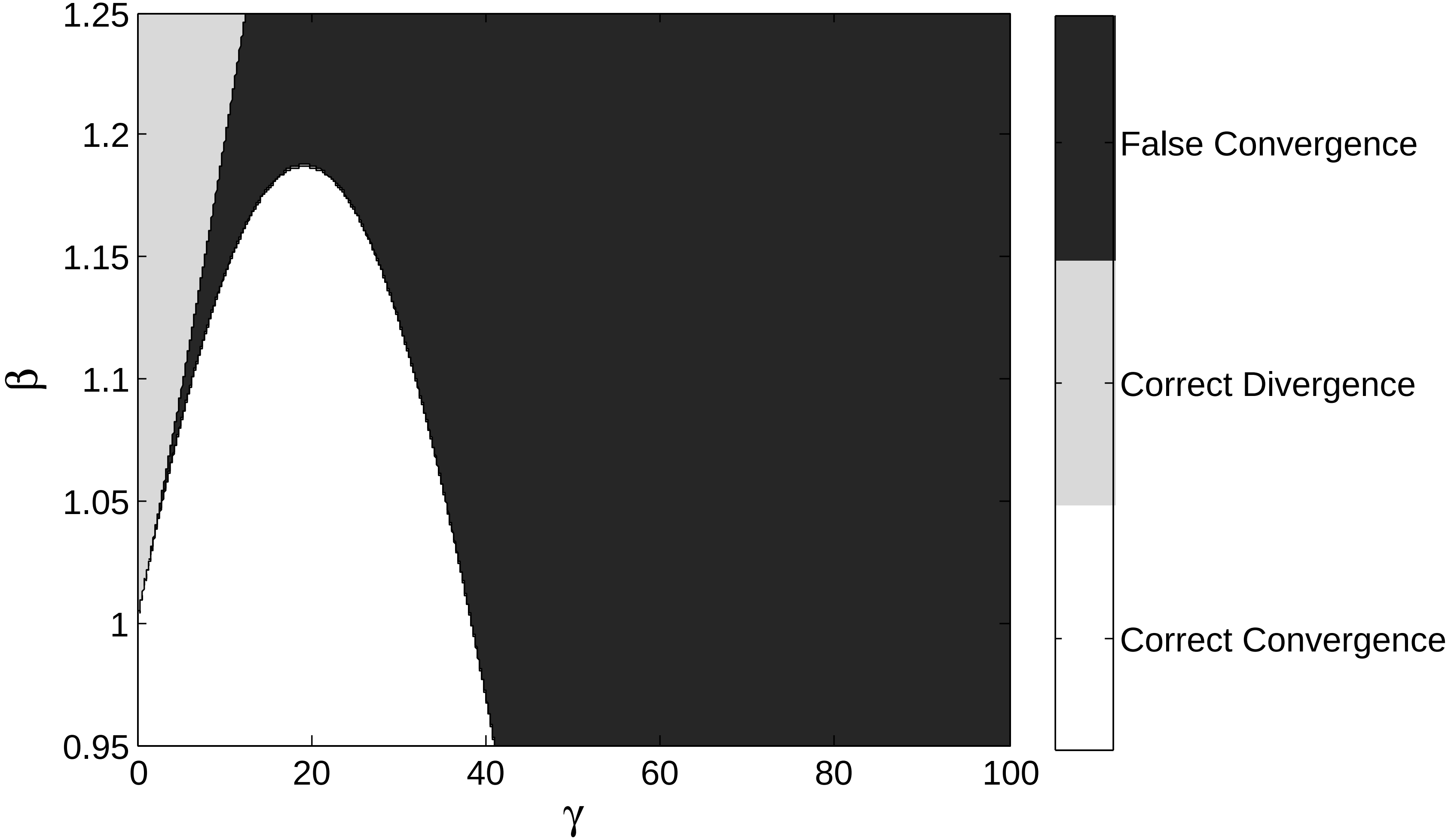
  (D) Comparison to the conventional 2nd-order approximation.

(A) Linear solution to the price-dividend ratio and implied equilibrium distribution in the asset pricing model.

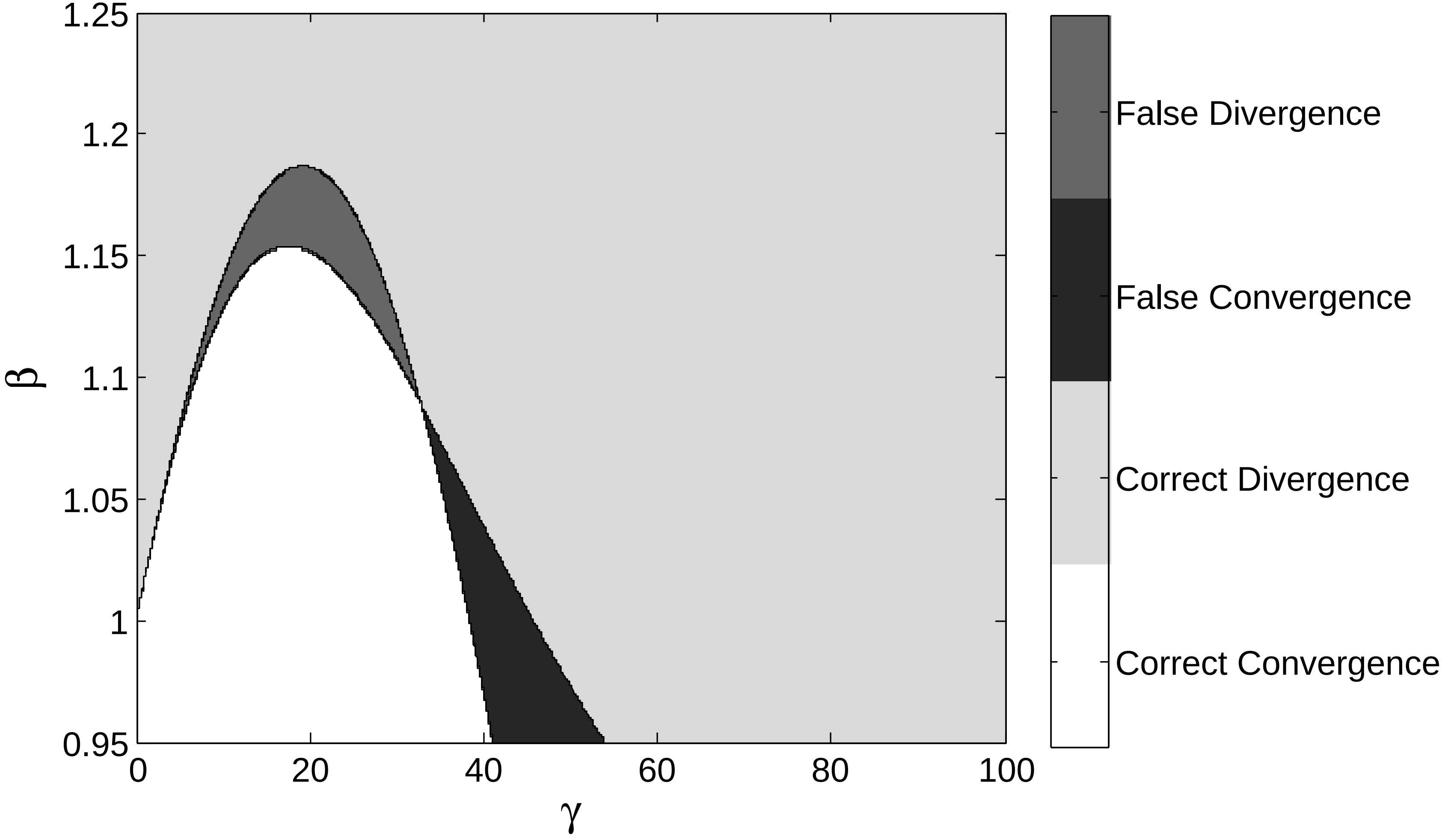
Case	Taylor Coefficients		Equilibrium Distribution	
Method	$\bar{g}$	g_z	Mean	Std. Dev.
<u>Benchmark:</u>				
True	12.481	2.306	12.482	0.081
AES	12.482 (0.00)	2.306 (0.00)	12.482 (0.00)	0.081 (0.00)
Conventional	12.304 (1.42)	2.273 (1.41)	12.304 (1.43)	0.080 (1.42)
<u>High Curvature: $\theta = -10$</u>				
True	5.024	6.260	5.029	0.220
AES	5.069 (0.89)	6.314 (0.87)	5.069 (0.79)	0.222 (0.72)
Conventional	3.862 (23.14)	4.834 (22.78)	3.862 (23.21)	0.170 (22.89)
<u>High Persistence: $\rho = .9$</u>				
True	14.879	-131.43	15.748	5.071
AES	14.144 (4.98)	-121.646 (7.45)	14.138 (10.22)	4.214 (16.90)
Conventional	12.304 (17.31)	-99.073 (24.62)	12.304 (21.87)	3.432 (32.32)

The benchmark calibration is $\beta = 0.95$, $\theta = -1.5$, $\rho = -0.139$, $\bar{z} = 0.0179$, and $\eta = 0.0348$.

(B.1) Uniqueness (Convergence) of the conventional perturbation approach.



(B.2) Uniqueness (Convergence) of linear solution to 2nd-order AES.

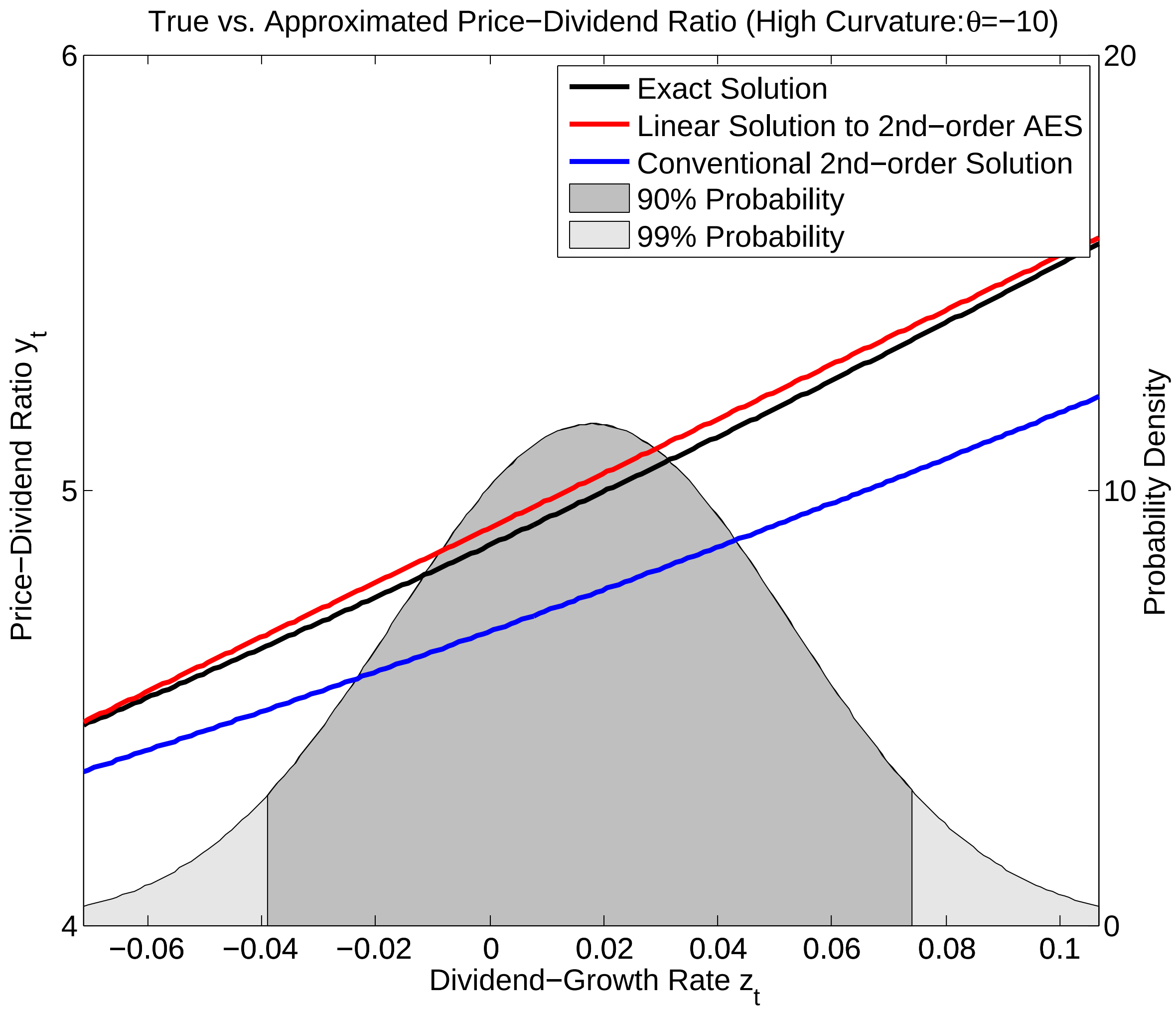


(C) Maximum Likelihood estimates of the linear solution in the asset pricing model.

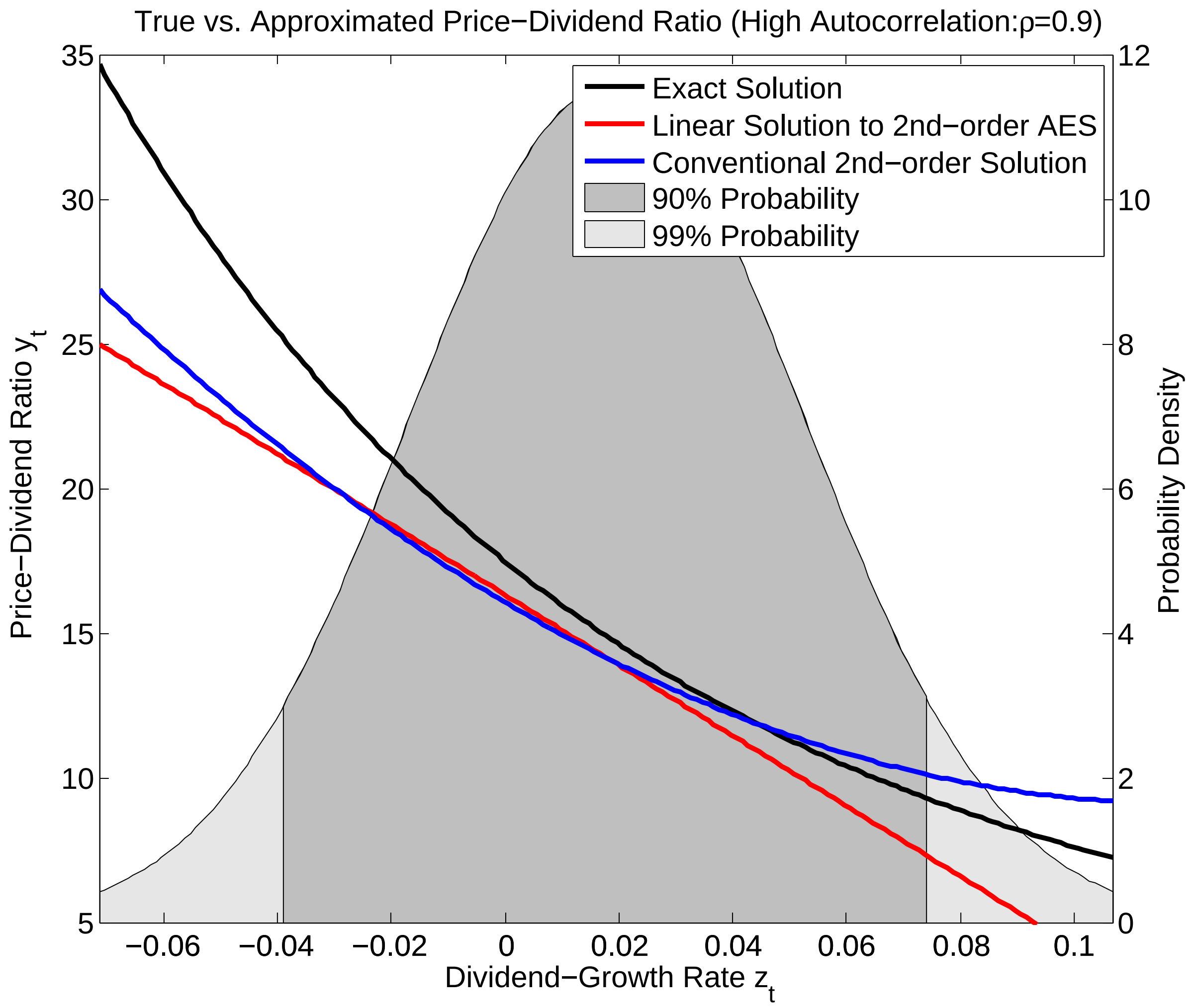
Case	Parameters		Shocks	
Method	ρ	θ	η	ς
Benchmark:				
True	-0.139	-1.5	0.0348	0.01
AES	-0.139 (0.06)	-1.500 (0.00)	0.0348 (0.05)	0.010 (0.29)
Conventional	-0,146 (4,67)	-1,440 (4,00)	0,0348 (0,05)	0,010 (0,29)
High Curvature:				
True	-0.139	-10	0.0348	0.01
AES	-0.137 (1.76)	-10.162 (1.62)	0.0347 (0.19)	0.012 (19.92)
Conventional	-0,200 (43,93)	-7,266 (27,34)	0,0348 (0,09)	0,012 (19,92)
High Persistence:				
True	0.9	-1.5	0.0151	0.01
AES	0.934 (3.82)	-1.129 (24.76)	0.0153 (1.20)	1.343 (13334,50)
Conventional	0,995 (10,53)	-0,580 (61,37)	0,0155 (2,39)	1,344 (13335,76)

The benchmark calibration is $\beta = 0.95$, $\theta = -1.5$, $\rho = -0.139$, $\bar{z} = 0.0179$, and $\eta = 0.0348$.

(D.1) Approximate solutions versus exact solution to the price-dividend ratio for high curvature ($\theta = -10$).



(D.2) Approximate solutions versus exact solution to the price-dividend ratio for high autocorrelation ($\rho = 0.9$).



Example: Asset Pricing Model *(9/1)*

(D.3) Standard and dynamic Euler Errors: The linear solution of the 2nd-order approximated equilibrium system (AES) vs. the conventional 2nd-order approximation to the solution (Conventional 2nd-order).

Case	Standard Euler Error		Dynamic Euler Error		Equilibrium Distribution	
Method	*Max*	*Avg*	*Max*	*Avg*	Mean	Std. Dev.
Benchmark:						
AES	0.026	0.003	1.480	0.342	12.481	0.081
	(0.005)	(0.000)	(0.153)	(0.009)	(0.002)	(0.002)
Convent'l 2nd-Order	0.048	0.019	1.444	1.434	12.479	0.080
	(0.003)	(0.000)	(0.002)	(0.000)	(0.002)	(0.002)
High Curvature: $\theta = -10$						
AES	1.538	0.500	11.340	2.612	5.004	0.217
	(0.228)	(0.004)	(1.147)	(0.077)	(0.006)	(0.005)
Convent'l 2nd-Order	7.153	4.714	30.209	29.574	4.790	0.170
	(0.250)	(0.022)	(0.152)	(0.004)	(0.005)	(0.004)
High Persistence: $\rho = .9$						
AES	95.442	9.201	195.410	29.596	14.434	4.621
	(32.189)	(1.126)	(52.564)	(2.794)	(0.652)	(0.312)
Convent'l 2nd-Order	43.363	7.159	7549.940	54.107	14.731	3.509
	(16.819)	(0.537)	(44118.842)	(46.678)	(0.483)	(0.312)

Related Literature *(1/3)*

1. Conventional **2nd or higher-order approximations** to the solution (Judd (1998), Sims (2000), Schmitt-Grohé & Uribe (2004) and Kim et al. (2008)).

   (a) Compute 2nd-order Taylor expansions to the solution about the deterministic steady state **jointly** with respect to

      i. the exogenous and endogenous state variables and

      ii. a **perturbation parameter** that scales the exogenous shocks.

   (b) Key implication: coefficients linear in the state vector are still independent of uncertainty ("Certainty Equivalence" !?)

   (c) Issues: i) higher-order terms in the state vector lead to explosive solution paths (pruning), ii) non-linear statistical method (e.g.particle filtering) are necessary for estimating the rational expectations model

Related Literature *(2/3)*

2. Proposals to approximate solution about the *Stochastic* or *Risky* Steady State (Juillard, Kaminek, Kliem and Uhlig, .. and others)

   - Compute 2nd or higher-order Taylor series to expectational equations in all variables - 2nd order terms are conditional variance of end. and exog. variables.

   - Compute moments using the linear solution to deterministic model.

   - Update moments iteratively using solution to the Taylor series to exp. equations.

   - Issues: Robustness/Convergence (iterative procedure); implementability into estimation routines.

Related Literature *(3/3)*

3. Solving for linear solution about risky steady state using indeterminate coefficients (Coerdacier et al. (AER PP, 2010)).

   - Compute 2nd order Taylor series to expectational equations in all variables - 2nd order terms are conditional variance of end. and exog. variables.

   - Postulate linear solution and solve numerically for indeterminate coefficients (steady state and linear coefficient) iteratively.

   - Issues: Robustness/Convergence; implementability into estimation routines.

Concluding Remarks *(1/2)*

- This paper demonstrates how to compute a linear solution to a DSGE model accounting for the interaction between non-linearities and the stochastic environment.

- The solution procedure makes linear toolbox applicable to many interesting problems where the focus is on the equilibrium interaction between uncertainty and non-linearities.

- The fundamental concept of the solution procedure is to disentangle the account of the stochastic environment of the original set of equilibrium conditions from computing the functional form of the solution processes.

- Computing the functional form of the solution processes itself is not restricted to perturbation methods at all but any local or global method is applicable.

Concluding Remarks *(2/2)*

End.

Example II: Precautionary Savings *(1/6)*

The risk averse household solves:

$$\max_{c_s} E_t \sum_{s=0}^{\infty} \beta^{t+s} u(c_{t+s})$$

s.t.

$$a_{t+s+1} + c_{t+s} = (1+r)a_{t+s} + w_{t+s},$$
$$w_{t+s+1} = (1-\rho)\bar{w} + \rho w_{t+s} + \epsilon_{t+s+1},$$

where $\epsilon_{t+s+1} \sim iid(0,1)$.

Given initial levels (a_t, w_t) and the transversality condition, the optimal consumption path satisfies the Euler equation

$$u'(c_t) - \beta(1+r)E_t u'(c_{t+1}) = 0.$$

The equilibrium conditions can be stated as

$$E_t f(c_t, a_t, w_t, c_{t+1}, a_{t+1}, w_{t+1}) = \begin{bmatrix} u'(c_t) - \beta(1+r)E_t u'(c_{t+1}) \\ a_{t+1} + c_t - (1+r)a_t - w_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and $$M(w_{t+1}, w_t, \sigma\epsilon_{t+1}) = w_{t+1} - (1-\rho)\bar{w} - \rho w_t - \sigma\epsilon_{t+1} = 0$$

Example II: Precautionary Savings *(3/6)*

In terms of the solution, the model can be stated as the stochastic equilibrium system

$$F(a_t, w_t, \sigma\epsilon_{t+1}; g(\cdot), h(\cdot))$$

$$= \begin{bmatrix} u'(g(a_t, w_t)) - \beta(1+r)E_t u'(g(h(a_t, w_t), m(w_t, \sigma\epsilon_{t+1}))) \\ h(a_t, w_t) + g(a_t, w_t) - (1+r)a_t - w_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the exogenous process

$$m(z_t, \sigma\epsilon_{t+1}) = (1-\rho)\bar{w} + \rho z_t + \sigma\epsilon_{t+1}.$$

Note: consumption is the non-predetermined variable: $c_t = g(a_t, w_t)$.

Example II: Precautionary Savings *(4/6)*

The second order Taylor series expansion to the stochastic equilibrium system yields

$$f^{\sigma}(c_t, a_t, w_t, c_{t+1}, a_{t+1}, m(z_t, 0))$$

$$= \begin{bmatrix} u'(c_t) - \beta(1+r)u'(c_{t+1}) \\ -a_{t+1} - c_t + (1+r)a_t + w_t \end{bmatrix}$$

$$+ \ \frac{1}{2}\begin{bmatrix} -\frac{\sigma^2}{2}\beta(1+r)\left(u'''(c_{t+1}){g_w^0}^2 + u''(c_{t+1})g_{w,w}^0\right) \\ 0 \end{bmatrix}\sigma^2 E_t\epsilon_{t+1}^2,$$

where $g^0(\cdot)$ denotes the solution to the deterministic problem and g_w^0 solves $F_w(a_t, w_t, 0) = 0$ and g_{ww}^0 solves $F_{ww}(a_t, w_t, 0) = 0$.

Example II: Precautionary Savings *(5/6)*

The approximated Euler equation can be rearranged to state the equilibrium relationship between marginal rate of substitution and the riskless return on savings as

$$\frac{u'(c_t)}{(1+r)\beta u'(c_{t+1})} = 1 - \underbrace{\Psi_{t+1}\left(g^0_{w,w} + \Phi_{t+1}{g^0_w}^2\right)\frac{\sigma^2}{2}}_{\text{Precautionary Savings}}.$$

- $\Psi(c_{t+1}) = -\frac{u''(c_{t+1})}{u'(c_{t+1})} > 0$ denotes the measure of absolute risk aversion, and

- $\Phi(c_{t+1}) = -\frac{u'''(c_{t+1})}{u''(c_{t+1})}$ denotes the measure of absolute prudence.

Special Case: Constant Absolute Risk Aversion

$$u(c) = -\frac{1}{\alpha}\exp(-\alpha c).$$

Then $g_w = \frac{r}{1+r-\rho}$ and $g_{ww} = 0$. Setting $\beta(1+r) = 1$ and omitting the approximation error yields

$$f^{\sigma}(c_t, a_t, w_t, c_{t+1}, a_{t+1}, m(z_t, 0))$$

$$= \begin{bmatrix} c_t - c_{t+1} - \frac{\alpha}{2}\frac{r^2}{(1+r+\rho)^2}\sigma^2 \\ -a_{t+1} - c_t + (1+r)a_t + w_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{-2}$$

The Matrix objects to compute linear coefficients to a 2nd-order AES in Detail *(1/3)*

$$
\begin{aligned}
f^{\sigma}_{w_{t+1}}(\cdot) &= f_{w_{t+1}}(\bar{y}^{\sigma}, \bar{s}^{\sigma}, \bar{y}^{\sigma}, \bar{s}^{\sigma}) \\
&\quad + f_{w_{t+1},w_{t+1}}(\cdot)\left(\begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix} E_t \epsilon_{t+1} \otimes I_{2n}\right) \\
&\quad \frac{1}{2} f_{w_{t+1}w_{t+1}w_{t+1}}(\cdot)\left[\left(\begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix} \otimes \begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix}\right) E_t\left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \otimes I_{2n}\right] \\
&\quad + \frac{1}{2} f_{w_{t+1}w_{t+1}}(\cdot)\left[\begin{pmatrix} q^0_{\epsilon\epsilon} \\ g^0_{ss}\left(q^0_{\epsilon} \otimes q^0_{\epsilon}\right) + g^0_s q^0_{\epsilon\epsilon} \end{pmatrix} E_t\left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \otimes I_{2n}\right]
\end{aligned}
$$

The Matrix objects to compute linear coefficients to a 2nd-order AES in Detail *(2/3)*

$$
\begin{aligned}
f^{\sigma}_{w_{t+1}}(\cdot) &= f_{w_t}(\bar{y}^{\sigma}, \bar{s}^{\sigma}, \bar{y}^{\sigma}, \bar{s}^{\sigma}) \\
&+ f_{w_{t+1},w_t}(\cdot)\left(\begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix} E_t \epsilon_{t+1} \otimes I_{2n}\right) \\
&+ \begin{bmatrix} \Gamma_{\epsilon} & 0_{n\times n_y} \end{bmatrix} \\
&+ \frac{1}{2} f_{w_{t+1}w_{t+1}w_t}(\cdot)\left[\left(\begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix} \otimes \begin{pmatrix} q^0_{\epsilon} \\ g^0_s q^0_{\epsilon} \end{pmatrix}\right) E_t\left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \otimes I_{2n}\right] \\
&+ \frac{1}{2} f_{w_{t+1}w_t}(\cdot)\left[\begin{pmatrix} q^0_{\epsilon\epsilon} \\ g^0_{ss}\left(q^0_{\epsilon} \otimes q^0_{\epsilon}\right) + g^0_s q^0_{\epsilon\epsilon} \end{pmatrix} E_t\left(\epsilon_{t+1} \otimes \epsilon_{t+1}\right) \otimes I_{2n}\right] \\
&+ \frac{1}{2}\begin{bmatrix} \Gamma_{\epsilon\epsilon} & 0_{n\times n_y} \end{bmatrix}
\end{aligned}
$$

The Matrix objects to compute linear coefficients to a 2nd-order AES in Detail *(3/3)*

where

$$\Gamma_\epsilon = f_{w_{t+1}}(\cdot)\begin{pmatrix} q^0_{\epsilon s}(E_t\epsilon_{t+1}\otimes I_{n_s}) \\ g^0_{ss}(q^0_\epsilon E_t\epsilon_{t+1}\otimes q^0_s) + g^0_s q^0_{\epsilon s}(E_t\epsilon_{t+1}\otimes I_{n_s}) \end{pmatrix}$$

and

$$\Gamma_{\epsilon\epsilon} = 2f_{w_{t+1}w_{t+1}}(\cdot)\left(\begin{pmatrix} q^0_{\epsilon s}(I_{n_\epsilon}\otimes I_{n_s}) \\ g^0_{ss}(q^0_\epsilon\otimes q^0_s) + g^0_s q^0_{\epsilon s}(I_{n_\epsilon}\otimes I_{n_s}) \end{pmatrix}\otimes\begin{pmatrix} q^0_\epsilon \\ g^0_s q^0_\epsilon \end{pmatrix}\right)E_t(\epsilon_{t+1}\otimes\epsilon_{t+1})$$
$$+ f_{w_{t+1}}(\cdot)\begin{pmatrix} q^0_{\epsilon\epsilon s}(I_{n^2_\epsilon}\otimes I_{n_s}) \\ g^0_{sss}(q^0_\epsilon\otimes q^0_\epsilon\otimes q^0_s) + 2g^0_{ss}\left(q^0_{\epsilon s}(I_{n_\epsilon}\otimes I_{n_s})\otimes q^0_\epsilon\right) \\ +g^0_{ss}(q^0_{\epsilon\epsilon}\otimes q^0_s) + g^0_s q^0_{\epsilon\epsilon s}(I_{n^2_\epsilon}\otimes I_n) \end{pmatrix}E_t(\epsilon_{t+1}\otimes\epsilon_{t+1})$$

summarize the changes in the risk adjusted equilibrium system when the initial state varies.

The Conventional Procedure *(1/7)*

Perturbing the solution in σ: The second order Taylor approximation to the solution $g(s_t;\sigma)$ and $h(s_t;\sigma)$ computed about the deterministic steady state is

$$
\begin{aligned}
g(\bar{s}^0+\delta;\sigma) &= g(\bar{s}^0;0)+g_{s^j}(\bar{s}^0;0)\delta^j+g_\sigma(\bar{s}^0;0)\sigma \\
&+\frac{1}{2}g_{s^j s^l}(\bar{s}^0;0)\delta^j\delta^l+g_{s^j\sigma}(\bar{s}^0;0)\sigma\delta^j+\frac{1}{2}g_{\sigma\sigma}(\bar{s}^0;0)\sigma^2,
\end{aligned}
$$

and

$$
\begin{aligned}
h(\bar{s}^0+\delta;\sigma) &= h(\bar{s}^0;0)+h_{s^j}(\bar{s}^0;0)\delta^j+h_\sigma(\bar{s}^0;0)\sigma \\
&\frac{1}{2}h_{s^j s^l}(\bar{s}^0;0)\delta^j\delta^l+h_{s^j\sigma}(\bar{s}^0;0)\sigma\delta^j+\frac{1}{2}h_{\sigma\sigma}(\bar{s}^0;0)\sigma^2,
\end{aligned}
$$

- The deterministic steady state is defined as

  $\bar{y}^0 = g(\bar{s}^0; 0)$, $\bar{x}^0 = h(\bar{s}^0; 0)$, and $\bar{z}^0 = m(\bar{z}^0, 0)$,

  and it solves the deterministic system of equilibrium conditions

$$F(\bar{s}^0, 0; 0) = 0.$$

The Conventional Procedure *(3/7)*

- The first order Taylor coefficients in σ:

$g_\sigma(\bar{s}^0; 0)$ and $h_\sigma(\bar{s}^0; 0)$ solve

$$F_\epsilon(\bar{s}^0, 0; 0) E_t \epsilon_{t+1} + F_\sigma(\bar{s}^0, 0; 0) = 0,$$

where

$$F_\sigma(\bar{s}^0, 0; 0) = f_{y_t} g_\sigma + f_{y_{t+1}}(g_x h_\sigma + g_\sigma) + f_{x_{t+1}} h_\sigma,$$

Assuming $E_t \epsilon_{t+1} = 0$: $g_\sigma(\bar{s}^0; 0) = 0$ and $h_\sigma(\bar{s}^0; 0) = 0$.

- The cross terms $g_{s\sigma}(\bar{s}^0;0)$ and $h_{s\sigma}(\bar{s}^0;0)$ are computed from solving

$$F_{s\epsilon}(\bar{s}^0,0;0)E_t\epsilon_{t+1} + F_{s\sigma}(\bar{s}^0,0;0) = F_{s\sigma}(\bar{s}^0,0;0) = 0,$$

which implies $g_{s\sigma}(\bar{s}^0;0) = 0$ and $h_{s\sigma}(\bar{s}^0;0) = 0$.

- The second order coefficients in σ: $g_{\sigma\sigma}(\bar{s}^0;0)$ and $h_{\sigma\sigma}(\bar{s}^0;0)$ solve

$$F_{\epsilon^j,\epsilon^l}(\bar{s}^0,0;0)\Omega_{j,i} + F_{\sigma\sigma}(\bar{s}^0,0) = 0$$

where $\Omega = E_t\epsilon_{t+1}\epsilon_{t+1}^T$.

- The first and second order terms in the state vector are computed by solving

$$F_s(\bar{s}^0, 0; 0) = 0 \quad \text{and} \quad F_{ss}(\bar{s}^0, 0; 0) = 0.$$

respectively.

The Conventional Procedure *(6/7)*

As a result, the second order Taylor approximation to the solution $g(s_t;\sigma)$ and $h(s_t;\sigma)$ reduces to

$$g(\bar{s}^0+\delta;\sigma) = g(\bar{s}^0;0) + g_{s^j}(\bar{s}^0;0)\delta^j + \frac{1}{2}g_{s^j s^l}(\bar{s}^0;0)\delta^j\delta^l + \frac{1}{2}g_{\sigma\sigma}(\bar{s}^0;0)\sigma^2,$$

$$h(\bar{s}^0+\delta;\sigma) = h(\bar{s}^0;0) + h_{s^j}(\bar{s}^0;0)\delta^j + \frac{1}{2}h_{s^j s^l}(\bar{s}^0;0)\delta^j\delta^l + \frac{1}{2}h_{\sigma\sigma}(\bar{s}^0;0)\sigma^2,$$

In order to capture the effect of riskiness on the terms linear in the state vector, minimum is to compute the third order Taylor coefficients $g_{\sigma\sigma s}(\bar{s}^0;0)$ and $h_{\sigma\sigma s}(\bar{s}^0;0)$ which solve

$$F_{\epsilon^j\epsilon^l s}(\bar{s}^0,0;0)\Omega_{j,i} + F_{\sigma\sigma s}(\bar{s}^0,0) = 0.$$

The difference between the two approaches: The solution has been parameterized in the perturbation coefficient σ.

The drawback of this approach is that the additional Taylor coefficients in the perturbation parameter have to be determined, too.

This necessarily increases the number of identifying restrictions by the number of additional Taylor coefficients:

$g_{\sigma\sigma}(\bar{s}^0;0)$ and $h_{\sigma\sigma}(\bar{s}^0;0)$, one has to solve $F_{\epsilon^j,\epsilon^l}(\bar{s}^0,0;0)\Omega_{j,i}+F_{\sigma\sigma}(\bar{s}^0,0)=0$.

Without parameterizing the solution in the perturbation parameter, the second order AES becomes

$$E_t F(\bar{s}^0, \sigma\epsilon_{t+1}) \;=\; F(\bar{s}^0, 0) + \frac{\sigma^2}{2} F_{\epsilon^j \epsilon_l}(\bar{s}^0, 0)\Omega_{j,i}.$$

The problem of a single agent is to choose consumption and equity holdings to maximize her expected discounted life-time utility

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{C_{t+\tau}^{\theta}}{\theta}$$

subject to the budget constraint

$$p_t e_{t+1} + c_t = (p_t + d_t) e_t.$$

β is the discount factor, c_t is period consumption, p_t denotes the price of the equity, e_t is household's equity holdings, and d_t are dividends on e_t.

Dividends d_t are assumed to grow at a rate x_t such that

$$d_t = \exp(x_t)d_{t-1}$$

The growth rate of dividends x_t follows the AR(1) process

$$x_t = (1-\rho)\tilde{x}^0 + \rho)x_{t-1} + \epsilon_t,$$

where ϵ_t is i.i.d. $\mathcal{N}(0, \sigma^2)$ with $|\rho| < 1$.

(Our $x_{t+1} = h(x_t) + \sigma\eta\epsilon_{t+1}$, where η equals σ in the model.)

The FOC is given by

$$p_t {c_t}^{\theta-1} = \beta E_t \left[{c_t}^{\theta-1} (p_{t+1} + d_{t+1}) \right].$$

Market clearing requires that $e_t = 1$ so that $c_t = d_t$.

Consequently,

$$p_t {c_t}^{\theta-1} = \beta E_t \left[\frac{d_{t+1}}{d_t}^{\theta-1} (p_{t+1} + d_{t+1}) \right].$$

Following Burnside, let y_t denote the price-dividend ratio $\frac{p_t}{d_t}$, ie.

$$y_t = \frac{p_t}{d_t}$$

.

As a result, the FOC reads

$$y_t = \beta E_t \left[\exp(\theta x_{t+1})(1 + y_{t+1}) \right].$$

(Our $E_t f(y_{t+1}, y_t, x_{t+1}, x_t) = 0$)

Closed form solution (Burnside (1998)):

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp\left(a_i + b_i(x_t - \tilde{x}^0)\right),$$

where

$$a_i = \theta \tilde{x}^0 i + \frac{\theta^2 \sigma^2}{2(1-\rho)^2}\left(i - \frac{2\rho(1-\rho^i)}{(1-\rho)} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right)$$

and

$$b_i = \frac{\theta\rho(1-\rho^i)}{1-\rho}.$$

The portfolio choice problem is to choose the portfolio weight w on a risky asset s.t.

$$\max_{w} EU\left((1-w) + w(a+1)\right).$$

The exogenous process for the excess return is simply

$$a = \sigma\epsilon,$$

where ϵ is a random shock and σ is the **perturbation parameter**.

The solution to the portfolio investment problem is characterized by the first order condition

$$EU'\left(w\sigma\epsilon + 1\right)\sigma\epsilon = 0$$

Step 1: Approximating the original stochastic equilibrium condition (computing AES)

The 2nd-order Taylor series expansion of the optimality condition in the perturbation parameter σ about $\sigma = 0$:

$$0 + U'(1)\sigma E\epsilon + U''(1)w\sigma^2 E\epsilon^2 + O(\sigma^3) = 0,$$

The remainder $O(\sigma^3)$ captures the approximation error of the second order Taylor expansion to the original condition.

Step 2: Computing the solution to AES

The solution to the portfolio fraction w of risky asset in the approximated equilibrium condition AES:

$$w = -\frac{\mu_a}{\sigma_a^2}\frac{U'(1)}{U''(1)} - O(\sigma^3).$$

where the mean and the variance are set as in Samuelson (RES, 1970), i.e. $E\epsilon = \sigma\mu_a$ and $E\epsilon^2 = \sigma_a^2$.

Samuelson's Portfolio Choice Problem *(4/4)*

The example highlights the key aspect of the proposed solution method:

1. The (2nd order) approximation AES is non-stochastic but captures uncertainty via the first and the second moments of the excess return.

2. The solution to the new equilibrium system therefore also depends on these moments.

3. The approximation error $O(\sigma^3)$ is simply the Euler error of the exact solution to the new equilibrium.

4. $O(\sigma^3)$ is also the approximation error to the true solution.

5. The procedure allows to solve problems where the solution is locally indeterminate in the deterministic version of the model.

Accuracy and the Euler Residuals *(1/7)*

Definition 1 (Euler Residuals)**.** Let $g^\sigma(x_t, z_t)$ and $h^\sigma(x_t, z_t)$ denote the solution process for the endogenous variables which solve the equilibrium system

$$F^\sigma(x_t, z_t; g^\sigma(\cdot), h^\sigma(\cdot)) = 0.$$

The corresponding Euler residual U^σ_{t+1} is defined as

$$U^\sigma_{t+1} = F(x_t, z_t, \sigma\epsilon_{t+1}; g^\sigma(\cdot), h^\sigma(\cdot)).$$

(compare e.g. den Haan & Marcet (RES, 1994), Santos (ECTRA, 2000)).

Proposition 1. *Let $g^\sigma(x_t, z_t)$ and $h^\sigma(x_t, z_t)$ denote the solution to the kth-order Taylor series approximation to the stochastic equilibrium system,*

$$F^\sigma(x_t, z_t; g^\sigma(\cdot), h^\sigma(\cdot)) = 0.$$

Then

$$\frac{1}{T}\sum_{s=1}^{T} U^\sigma_{t+s} \xrightarrow{a.s.} O(\sigma^{k+1}) \quad \text{as} \quad T \to \infty,$$

where $O(\sigma^{k+1})$ denotes the error of the Taylor approximation to the true stochastic equilibrium system.

Recall the discussion of Proposition 1:

$$E_t F(x_t, z_t, \sigma\epsilon_{t+1}; g(\cdot), h(\cdot)) \;=\; F^{\sigma}(x_t, z_t; g(\cdot), h(\cdot)) + O(\sigma^{k+1})$$

- $O(\sigma^{k+1})$ is thus a quantitative statement about the accuracy of the approximated equilibrium system.

- It is also a statement about the accuracy of the solution process $g^{\sigma}(\cdot)$ and $h^{\sigma}(\cdot)$ as the approximation to the true solution:

$$\begin{aligned} E_t F(x_t, z_t, \sigma\epsilon_{t+1}; g^{\sigma}(\cdot), h^{\sigma}(\cdot)) \;&=\; F^{\sigma}(x_t, z_t; g^{\sigma}(\cdot), h^{\sigma}(\cdot)) + O(\sigma^{k+1}) \\ &=\; O(\sigma^{k+1}). \end{aligned}$$

Proposition 2. *Let* $g(x_t, z_t)$ *and* $h(x_t, z_t)$ *denote the true solution and let* $g^\sigma(x_t, z_t)$ *and* $h^\sigma(x_t, z_t)$ *denote the solution to the* k*th-order Taylor series approximation to the stochastic equilibrium system*

$$F^\sigma(x_t, z_t; g^\sigma(\cdot), h^\sigma(\cdot)) = 0.$$

Then

$$g(x_t, z_t) - g^\sigma(x_t, z_t) = O(\sigma^{k+1}) \qquad \text{and} \qquad h(x_t, z_t) - h^\sigma(x_t, z_t) = O(\sigma^{k+1}),$$

where $O(\sigma^{k+1})$ *denotes a term in the order of magnitude of the error of the Taylor approximation to the true stochastic equilibrium system.*

Proposition 3. *Let* $(\bar{y}, \bar{x}, \bar{z})$ *denote the steady state of the true stochastic equilibrium system and* $(\bar{y}^\sigma, \bar{x}^\sigma, \bar{z})$ *the steady state of the approximated equilibrium system AES. Then*

$$\left[\begin{array}{c} \bar{y} - \bar{y}^\sigma \\ \bar{x} - \bar{x}^\sigma \end{array}\right] = O(\sigma^{k+1}) \quad \textit{and} \quad \left[\begin{array}{c} g_q(\bar{x}, \bar{z}) - g_q^\sigma(\bar{x}^\sigma, \bar{z}) \\ h_q(\bar{x}, \bar{z}) - h_q^\sigma(\bar{x}^\sigma, \bar{z}) \end{array}\right] = O(\sigma^{k+1}),$$

with $q = x, z$*.*

Accuracy and the Euler Residuals *(6/7)*

Moreover, let $\tilde{g}(x_t, z_t)$ *and* $\tilde{h}(x_t, z_t)$ *denote the linearized solution to the true stochastic equilibrium system computed about* $(\bar{y}, \bar{x}, \bar{z})$*, and let* $\tilde{g}^\sigma(x_t, z_t)$ *and let* $\tilde{h}^\sigma(x_t, z_t)$ *denote the linearized solution to the approximated equilibrium system AES computed about* $(\bar{y}^\sigma, \bar{x}^\sigma, \bar{z})$*. Then*

$$\tilde{g}(x_t, z_t) - \tilde{g}^\sigma(x_t, z_t) = O(\sigma^{k+1}) \quad \textit{and} \quad \tilde{h}(x_t, z_t) - \tilde{h}^\sigma(x_t, z_t) = O(\sigma^{k+1}).$$

Accuracy and the Euler Residuals *(7/7)*

Important Implications:

- Accuracy of the approximated key local properties i) existence, ii) determinacy, iii) equilibrium distribution, and iv) dynamics increases in the order k.

- Accuracy of Maximum likelihood inference and hence parameter estimates based on the linear solution (Compare Ackerberg et al (ECTRA, 2009)) increases in the order k.

- Accuracy of forecasts based on the linear solution to the AES also increases in the order k.

Solving Nonlinear Rational Expectations Models by Approximating the Stochastic Equilibrium System

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