

The Stability of Walrasian General Equilibrium under a replicator dynamic

Herbert Gintis and Antoine Mandel*

April 8, 2013

Abstract

We prove the stability of equilibrium in a completely decentralized Walrasian general equilibrium economy in which prices are fully controlled by economic agents, with production and trade occurring out of equilibrium.

Journal of Economic Literature Classifications:

C62—Existence and Stability Conditions of Equilibrium

D51—Exchange and Production Economies

D58—Computable and Other Applied General Equilibrium Economies

1 Introduction

Walras (1954 [1874]) developed a general model of competitive market exchange, but provided only an informal argument for the existence of a market-clearing equilibrium for this model. Wald (1951 [1936]) provided a proof of existence for a simplified version of Walras' model, and this proof was substantially generalized by Debreu (1952), Arrow & Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Negishi (1960), and others.

Walras was well aware that his arguments had to be backed by a theory of price adjustment that would ensure convergence and stability of the equilibrium. He considered that the key force leading to equilibrium was competition, which he saw materializing in the regular updating of prices by economic agents. In his own words:

*The authors contributed equally to this research. Gintis: Santa Fe Institute and Central European University; Mandel: Paris School of Economics University Paris 1 Panthéon-Sorbonne. The authors acknowledge the support of Institute for New Economic Thinking grant INO1200022.

The markets that are best organized from the competitive standpoint are those in which. . . the terms of every exchange are openly announced and an opportunity is given to sellers to lower their prices and to buyers to raise their bids (Walras 1954 [1874], paragraph 41).

However, he thought that a model where economic agents individually update their prices would be hard to track analytically and that a centralized model of price adjustment mimicking individual behavior (the auctioneer playing the role of a representative agent as far as price dynamics are concerned) would suffice to prove the stability of the equilibrium.

The stability of the Walrasian economy was a central research focus in the years following the existence proofs (Arrow and Hurwicz 1958, 1959, 1960; Arrow, Block and Hurwicz 1959; Nikaido 1959; McKenzie 1960; Nikaido and Uzawa 1960). Following Walras' tâtonnement process, these models assumed that there is no production or trade until equilibrium prices are attained, and out of equilibrium, there is a price profile shared by all agents, the time rate of change of which is a function of excess demand. These efforts at proving stability were successful only by assuming narrow and implausible conditions (Fisher 1983). Indeed, Scarf (1960) provided simple examples of unstable Walrasian equilibria under a tâtonnement dynamic.

Several researchers then explored the possibility that allowing trading out of equilibrium could sharpen stability theorems (Uzawa 1959, 1961, 1962; Negishi 1961; Hahn 1962; Hahn and Negishi 1962, Fisher 1970, 1972, 1973), but these efforts enjoyed only limited success. Moreover, Sonnenschein (1973), Mantel (1974, 1976), and Debreu (1974) showed that any continuous function, homogeneous of degree zero in prices, and satisfying Walras' Law, is the excess demand function for some Walrasian economy. These results showed that no general stability theorem could be obtained based on the tâtonnement process. Indeed, subsequent analysis showed that chaos in price movements is the generic case for the tâtonnement adjustment processes (Saari 1985, Bala & Majumdar 1992).

A novel approach to the dynamics of large-scale social systems, evolutionary game theory, was initiated by Maynard Smith & Price (1973), and adapted to dynamical systems theory in subsequent years (Taylor & Jonker 1978, Friedman 1991, Weibull 1995). The application of these models to economics involved the shift from biological reproduction to behavioral imitation as the mechanism for the replication of successful agents.

We apply this framework by treating the Walrasian economy as the stage game of an evolutionary process. We assume each agent is endowed in each period with goods he must trade to obtain the various goods he consumes. There are no inter-period exchanges. An agent's trade strategy consists of a set of *private prices*

for the good he produces and the goods he consumes, such that, according to the individual's private prices, a trade is acceptable if the value of goods received is at least as great as the value of the goods offered in exchange. The exchange process of the economy is hence defined as a multipopulation game where agents use private prices as strategies.

Competition materializes when a seller who slightly undercuts his competitors increases his sales and hence his income or when a buyer who overbids his competitors increases his purchases and hence his utility. In other words, when markets are in disequilibrium, there are always agents who have incentives to change their prices. More precisely, under rather mild conditions on the trading process, Walrasian equilibria are the only strict Nash equilibria of private prices games. Thus, if we assume that the strategies of traders are updated according to the replicator dynamic, the stability of equilibrium is guaranteed (see weibull (1995)).

Hence, the adjunction of private prices to the general equilibrium model allow on the one hand to propose a model closer to Walras original insights where prices are controlled by economic agents as in actual markets and on the other hand to revisit the issue of stability using game-theoretic and evolutionary insights to analyze the dynamic adjustment of prices to their equilibrium values.

The paper is organized as follows: section two introduces games with private prices in general exchange economies, section three reviews the stability properties of evolutionary dynamics and gives an introductory example of their application to games with private prices. Section four proves the stability of equilibrium in a stylized setting where the kinds of goods consumed and sold by each agent are independent of relative prices and section five extends this result to an arbitrary exchange economy. Section six discusses in more details the necessary condition for competition to bite on the price adjustment process in a setting with private prices. Section seven concludes.

2 The Walrasian Economy with Private Prices

We shall explore the stability of Walrasian equilibrium in an economy with a finite number of goods, indexed by $h = 1, \dots, l$, and a finite number of agents, indexed by $i = 1, \dots, m$ (see e.g Arrow & Debreu (1954)). Agent i has \mathbf{R}_+^l as consumption set, a utility function $u_i: \mathbf{R}_+^l \rightarrow \mathbf{R}_+$ and an initial endowment $e_i \in \mathbf{R}_+^l$. This economy is denoted by $\mathcal{E}(u, e)$.

In this setting an allocation $\bar{x} \in (\mathbf{R}_+^l)^m$ of goods is *feasible* if it belongs to the set $\mathcal{A}(e) = \{x \in (\mathbf{R}_+^l)^m \mid \sum_{i=1}^m x_i \leq \sum_{i=1}^m e_i\}$ and the demand of an agent i is the mapping $d_i: \mathbf{R}_+^l \rightarrow \mathbf{R}_+^l$ that associates to a price $p \in \mathbf{R}_+^l$ the utility-maximizing individual allocations satisfying the budget constraint, $d_i(p) :=$

$\operatorname{argmax}\{u(x_i) \mid x_i \in \mathbf{R}_+^l, p \cdot x_i \leq p \cdot e_i\}$. A feasible allocation $\bar{x} \in \mathcal{A}(e)$ then is an *equilibrium allocation* if there exists a price $\bar{p} \in S$ such that for all i , $\bar{x}_i \in d_i(\bar{p})$. We shall denote the set of such equilibrium prices by $\mathbf{E}(u, e)$.

The issue of stability is relevant only if equilibria actually exist. Let us therefore introduce sufficient conditions for the existence of an equilibrium. First, we assume about utility functions:

Assumption 1 (Utility) *For all $i = 1, \dots, m$, u_i is continuous, strictly concave, and locally non-satiated.*

This assumption ensures the existence of a quasi-equilibrium (see e.g. Florenzano 2005), that is an attainable allocation $x^* \in \mathcal{A}(e)$ and a price $p^* \in \mathbf{R}_+^l$ such that $u_i(x_i) > u_i(x_i^*)$ implies $p^* \cdot x_i > p^* \cdot x_i^*$. Note that the strict concavity is not necessary for the existence of a quasi-equilibrium but implies demand mappings are single-valued, which will prove useful below.

To ensure that every quasi-equilibrium is an equilibrium allocation, it suffices to assume that at a quasi-equilibrium the agents do not receive the minimal possible income (see Hammond 1993, Florenzano 2005):

Assumption 2 (Income) *For every quasi-equilibrium (p^*, x^*) , and for every $i = 1, \dots, m$, there exists $x_i \in \mathbf{R}_+^l$ such that $p^* \cdot x_i^* > p^* \cdot x_i$.*

This condition is satisfied under the survival assumption (i.e. when all initial endowments are in the interior of the consumption set) as well as in settings with corner endowments such as those investigated in Gintis (2007).

It is standard to show that under Assumptions (1) and (2) the economy $\mathcal{E}(u, e)$ has at least an equilibrium (see Florenzano 2005). We shall assume in the following they do hold and moreover restrict attention to the generic case where the economy has a finite set of equilibria (see Balasko 2009).

From Walras' perspective, such an equilibrium should be the outcome of free competition among economic agents. Walras himself considered free competition to materialize in actual market exchanges characterized by free entry and exit, the capacity of producers to choose their level of production and, most importantly for our purposes, the freedom to propose and to accept exchanges based upon their own assessment of the appropriate prices for the different goods (Dockes & Potier 2005). In Walras' words:

As buyers, traders make their demand by outbidding each other. As sellers, traders make their offers by underbidding each other... The markets that are best organized from the competitive standpoint are those in which... the terms of every exchange are openly announced

and an opportunity is given to sellers to lower their prices and to buyers to raise their bids (Walras 1954 [1874], paragraph 41).

Walras argues that, although in actual markets prices are controlled by economic agents themselves, he introduced the centralized tâtonnement in the belief that the stability of the price adjustment process would be thereby simplified. This belief was mistaken.

We show here that a model of price adjustment closer to Walras' vision of the actual functioning of competitive markets allows to prove convergence to equilibrium in a general setting. The key innovation of the model is allowing the prices to be controlled by economic agents themselves. Namely, we consider that each agent i is characterized by a private price $p_i \in \mathbf{R}_+^l$ whose coordinates represent the prices at which he is willing to sell the goods he supplies to the market and the maximum prices he is willing to pay for the goods he demands. Whatever structure one then considers for an exchange process, from a sequence of bilateral interactions as in Gintis (2007) to the simultaneous and multilateral exchanges implicitly considered in most general equilibrium models, the resulting allocation must be determined by the distribution of private prices. In other words, the exchange process will be modeled as a game where agents use private prices as strategies.

We shall associate with the economy $\mathcal{E}(u, e)$ the class of games $\mathcal{G}(u, e, \xi)$ where:

- Each agent has a finite set of prices $P \subset \mathbf{R}_+^l$ as strategy set.
- The game form is defined by an exchange mechanism $\xi: P^m \rightarrow \mathcal{A}(e)$ that associates to a profile of private prices $\pi = (p_1, \dots, p_m)$ an attainable allocation $\xi(\pi) = (\xi_1(\pi), \dots, \xi_m(\pi)) \in \mathcal{A}(e)$.
- The payoff $\phi_i: P^m \rightarrow \mathbf{R}_+$ of player i is evaluated according to the utility of the allocation it receives, that is $\phi_i(\pi) = u_i(\xi_i(\pi))$.

As strategic price set, we will mostly consider below that $P = K^{l-1} \times \{1\}$, where $K \subset \mathbf{R}_+$ is a finite set of commodity prices with minimum $p_{\min} > 0$ and maximum $p_{\max} > p_{\min}$ while good l is used as a numeraire and its price is fixed equal to 1. In any case, we will consider the price set P contains each of the finite number of equilibrium price profiles of the economy.

The exchange process ξ can be the outcome of a sequence of bilateral trades of the kind described in Gintis (2007,2012), where trades are proposed on the basis of demands computed at private prices and accepted or refused on the basis of values computed at private prices. One can also consider simpler exchange process where, as in the standard tâtonnement process, no trade takes place unless excess demand

vanishes, for example the exchange process ξ^0 such that

$$\xi_i^0(\pi) = \begin{cases} d_i(\bar{p}) & \text{if for all } i, \pi_i = \bar{p} \in \mathbf{E}(u, e) \\ e_i & \text{otherwise.} \end{cases} \quad (1)$$

The adjunction of such private prices games to the general equilibrium model allow on us the one hand to propose a model closer to Walras original insights where prices are controlled by economic agents as in actual markets and on the other hand to revisit the issue of stability using game-theoretic analysis for the dynamic adjustment of prices to their equilibrium values.

3 Evolutionary Stability of Equilibrium

We shall here focus on evolutionary dynamic, in particular on the replicator dynamic. This focus is motivated first by the observation that a major mechanism leading to convergence of economic behaviour is imitation in which poorly performing agents copy the behavior of, or are replaced by copies of, better-performing agents, second by the evidence of convergence of prices towards equilibrium obtained by simulation of stochastic evolutionary models in Gintis (2007) and third by the close relationships between stochastic evolutionary models and deterministic replicator dynamics (Helbing 1996, Benaim & Weibull 2003). More generally, replicator dynamics appears as the backbone of a wealth of adaptive processes ranging from reinforcement learning (Börgers & Sarin 1997) to bayesian updating (Shalizi 2009).

In the multipopulation game $\mathcal{G}(u, e, \xi)$, the replicator dynamics is defined as follows. First, the space of mixed strategies for player i is defined as

$$\Delta_i = \left\{ \sigma_i \in \mathbf{R}_+^P \mid \sum_p \sigma_{i,p} = 1 \right\}.$$

In an evolutionary context, an element $\sigma_i \in \Delta_i$ is interpreted as a population of players of type i whose share $\sigma_{i,p}$ uses the private price $p \in P$. The replicator dynamic then prescribes that the share of agents using price p in the population i should grow proportionally to the utility it is expected to yield in the exchange process under the assumption that trade partners are drawn uniformly in each of the populations. This yields the system of differential equations defined for all $i = 1, \dots, n$ and $p \in P$ by:

$$\frac{\partial \sigma_{i,p}}{\partial t} = \sigma_{i,p} (E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) - E_{\sigma}(u_i(\xi(\cdot)))) \quad (2)$$

where $E_{\sigma_{-i}}(u_i(\xi(p, \cdot)))$ represents the expected utility of the strategy p given the mixed strategy profile σ_{-i} , that is

$$E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) = \sum_{\rho \in P^{m-1}} \left(\prod_{j \neq i} \sigma_{j, \rho_j} \right) u_i(\xi(p, \rho)),$$

and $E_{\sigma}(u_i(\xi(\cdot)))$ represents the expected utility of the mixed strategy σ_i given the mixed strategy profile σ_{-i} ; that is

$$E_{\sigma}(u_i(\xi(\cdot))) = \sum_{\pi \in P^m} \left(\prod_{j=1}^m \sigma_{j, \pi_j} \right) u_i(\xi(\pi)).$$

For the replicator dynamic in a multi-population game, a strategy profile is asymptotically stable for the replicator dynamics if and only if it is a strict Nash equilibrium of the underlying game (see e.g. Weibull 1995 and Appendix A below). In the game $\mathcal{G}(u, e, \xi)$, a price profile $\pi \in P^m$ is a strict Nash equilibrium if and only if for all $i = 1 \dots m$, and all $p \neq \pi_i$, one has $u_i(\pi) > u_i(p, \pi_{-i})$ and one hence has following Weibull 1995¹:

Proposition 1 *A price profile $\pi \in P^m$ is asymptotically stable for the replicator dynamic if and only if for all $i = 1 \dots m$, and all $p \neq \pi_i$, one has $u_i(\pi) > u_i(p, \pi_{-i})$.*

Hence, private price games provide a consistent model of dynamic stability of equilibrium in an economy $\mathcal{E}(u, e)$ whenever these equilibria can be identified with strict Nash equilibria of the game $\mathcal{G}(u, e, \xi)$. Let us for example consider economies $\mathcal{E}(u, e)$ for which there are gains to trade in the sense of

Assumption 3 (Gains to trade) *For every $\bar{p} \in \mathbf{E}(u, e)$, we have for all $i = 1 \dots m$: $u_i(d_i(\bar{p})) > u_i(e_i)$.*

For every such economy, the exchange process ξ^0 introduced above is such that:

- At a strategy profile in which two or more agents use a non-equilibrium price, each agent is allocated his initial endowment and no unilateral deviation can modify this allocation, so that the profile is a non-strict Nash equilibrium
- At a strategy profile in which all agents but one use the same equilibrium price, each agent is allocated his initial endowment. Under Assumptions (3), the “non-equilibrium” agent makes everyone strictly better-off by deviating to the equilibrium price as each agent is then allocated his equilibrium demand. So, such a strategy profile is not a Nash equilibrium.

¹Similar results for a broader class of dynamics follow from the application of the results recalled in the appendix.

- At a strategy profile in which all agents use the same equilibrium price \bar{p} , each agent receives the equilibrium allocation $u_i(d_i(\bar{p}))$. Under Assumption 3, an agent deviating to a different price makes everyone strictly worse off as each agent is then allocated his initial endowment. Hence, the strategy profile is a strict Nash equilibrium.

So that one has:

Proposition 2 *Under Assumption 3, $\pi \in P^m$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi^0)$ if and only if there exists $\bar{p} \in \mathbf{E}(u, e)$ such that for all i , $\pi_i = \bar{p}$.*

As moreover, at a price $\bar{p} \in \mathbf{E}(u, e)$ each agent receives his equilibrium allocation, the strategy profile $(\bar{p}, \dots, \bar{p})$ can be unambiguously identified with the corresponding general equilibrium so that one can claim, using Proposition 1, that for the exchange process ξ^0 , general equilibria are the only asymptotically stable states in economies where there are gains to trade. That is:

Proposition 3 *Under Assumption (3), the only asymptotically stable strategy profiles for the replicator dynamic in the game $\mathcal{G}(u, e, \xi^0)$ are those in which each agent uses an equilibrium price $\bar{p} \in \mathcal{E}(u, e)$ and is allocated the corresponding equilibrium allocation $d_i(\bar{p})$.*

Proposition 3 hence illustrates that stability results can indeed be obtained for generic economies thanks to the introduction of private prices. It is worth pointing out in this respect that the process ξ^{nout} can be seen as a decentralized, but naive, version of the tatonnement process given that no trade takes place until equilibrium is reached, but convergence requires an extensive search of the state space.

4 Evolutionary Stability with Out of Equilibrium Trading

Considering that agents update their private prices on the basis of the results of out of equilibrium trading is much more in line with actual economic behavior and Walras' original insights. It also gives a much stronger sense of direction to the replicator dynamic. The analysis is however slightly more technical in this setting. In order to underline the key mechanisms at play, we will first restrict attention to a setting where for each agent the set of goods is partitioned between consumption and production goods, the agent being endowed with production goods only and deriving utility from consumption goods only (and only if it consumes some of each).

Assumption 4 (Goods partition) For $i = 1, \dots, m$, there exists a partition $\{P_i, C_i\}$ of $\{1, \dots, l\}$ such that:

1. for all $h \in P_i$, $e_{i,h} > 0$;
2. there exists $v_i : V_{C_i} \rightarrow \mathbf{R}_+$ such that²:
 - (a) for all $u_i(x) = v_i(\text{pr}_{V_{C_i}}(x))$
 - (b) $v_i(x) > 0 \Rightarrow \forall h \in C_i, x_h > 0$.

Out-of-equilibrium, there is no public price with regards to which agents can act as price takers. They must evaluate goods according to their private prices and determine their behavior accordingly. Namely, we shall consider that agents evaluate their demand by maximizing utility at private prices and trade only with peers having compatible prices. If one moreover assumes that there is competition both among buyers and sellers, so that lowest bidding sellers and highest bidding buyers have priority access to the market, then it is clear that when there is excess supply a seller who slightly undercuts his competitors increases his sales and hence his income while when there is excess demand, a buyer who slightly overbids his competitors increase his purchases and hence his utility.

Therefore with market disequilibrium, there are always some agents who have an interest to deviate. Conversely at equilibrium, an agent who deviates either prices himself out of the market or decreases his consumption and hence his utility. It therefore seems only Walrasian equilibria can be strict/stable Nash equilibria of the private price game. The key difference between ours and the standard tâtonnement process is that in our model, each agent is free to change his prices schedule at will in each period. In market equilibrium, of course, all prices for the same good are equal, and no agent has an incentive to change his price schedule.

We define the set of buyers of good h as $B_h := \{i \mid h \in C_i\}$, the set of sellers of good h as $S_h := \{i \mid h \in P_i\}$, the set of acceptable buyers as those whose prices is above the lowest buying price, that is $\mathcal{B}_h(\pi) := \{i \in B_h \mid \pi_{i,h} \geq \min_{j \in S_h} \pi_{j,h}\}$. the set of acceptable sellers as those whose prices is below the highest buying price, that is $\mathcal{S}_h(\pi) := \{i \in S_h \mid \pi_{j,h} \leq \max_{i \in B_h} \pi_{i,h}\}$ and the feasible income is $w_i(\pi) = \sum_{\{h \mid i \in S_h(\pi)\}} \pi_{i,h} e_h$. The set of price feasible allocations $\mathcal{A}'(e, \pi)$ is then defined as follows.

Definition 1 The set of price feasible allocations $\mathcal{A}'(e, \pi)$ is the subset of feasible allocations $\mathcal{A}(e, \pi)$ such that for every $x \in \mathcal{A}'(e, \pi)$ one has:

²For a subset $C \subset \{1, \dots, l\}$, we define the vector subspace V_C as $V_C := \{x \in \mathbf{R}^l \mid \forall h \notin C, x_h = 0\}$ and pr_{V_C} as the projection on V_C .

1. $x_i \leq d_i(\pi_i, w_i(\pi))$;
2. $\forall h \in C_i, x_{i,h} > 0 \Rightarrow i \in \mathcal{B}_h(\pi)$
3. $\sum_{i \in \mathcal{B}_h(\pi)} x_{i,h} \leq \sum_{j \in \mathcal{S}_h(\pi)} e_{j,h}$

Following the above discussion, condition 1 expresses the fact that demand are computed at private prices. Note that it is equivalent for our purposes to assume that each agent satisfies its private budget constraint, that is $\pi_i \cdot x_i \leq w_i(\pi)$. Condition 2 ensures that only buyers with acceptable prices have access to the market. Condition 3 ensures that only sellers with acceptable prices have access to the market. Stronger conditions, requiring for examples that agents only trade with peers using the same price, could be put forward but the present framework allows to distinguish between the delimitation of the trade space and the competitive advantages of buyers/sellers with high/low private prices, which is characterized below. In a sense, the definition of price-feasible allocations embed incentives for the agents to use common prices while assumption (6) below characterizes profitable deviations.

The set $\mathcal{A}'(e, \pi)$ can be seen as a bargaining set defined by the agents' private prices. The choice of an allocation within this bargaining set can be simulated numerically by an algorithmic implementation of the trading process as in Gintis (2007), but for our theoretical purposes, characterizing the exchange process ξ axiomatically as a solution of the bargaining problem $\mathcal{A}'(e, \pi)$ will prove more useful. We shall therefore assume that the exchange process is an efficient and monotonic solution to the bargaining problem $\mathcal{A}'(e, \pi)$. As far as efficiency is concerned, we shall only require it to hold when π is an equilibrium price profile, that is such that for all $i \in \{1, \dots, m\}$, $\pi_i = \bar{p}$ where $\bar{p} \in \mathbf{E}(u, e)$.

Assumption 5 (Bargaining Axioms) *The exchange process ξ is such that:*

1. (*Efficiency*) *If π is an equilibrium price profile, $\xi_i(\pi)$ is Pareto optimal in $\mathcal{A}'(e, \pi)$.*
2. (*Monotonicity*) *If π and π' are such that $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ then for all $i = 1 \dots n$, one has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.*

Our efficiency condition is weaker than what is standard in the bargaining literature and will allow us to identify equilibrium price profiles played in the game $\mathcal{G}(u, e, \xi)$ with the corresponding Walrasian equilibria of the economy $\mathcal{E}(u, e)$. The monotonicity condition is standard in the bargaining literature since the seminal paper by Kalai and Smorodinsky (1975).

Eventually, to prove stability we must ensure that competition is effective and that buyers have incentive to update their prices when there is excess demand. Walras' law implies that when the price profile π is uniform, that is when there exists $p \in P$ such that for all i , $\pi_i = p$, there is necessarily excess demand for one good unless the economy is in market equilibrium. As a matter of fact, our main results only require that agents have incentive to update their prices when the price profile is uniform.

Assumption 6 (Buyer's Incentive to Change) *If $\pi = (p, \dots, p) \in P^m$ is a uniform price profile and there exists an $h \in \{1, \dots, l\}$ such that*

$$\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h},$$

then there exists $p' \in P$ such that $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$.

Assumption (6) is the counterpart of Walras' description of buyers raising their bids to price out competitors. It could perhaps be taken as face value, as is the updating of prices by the auctioneer. Nevertheless, sufficient conditions for its satisfaction are extensively discussed below. One should yet note that it could symmetrically be assumed that sellers have incentives to decrease their prices when there is excess supply but the proof would be a little more involved. However, increasing their prices when there is excess demand is not a viable strategy for sellers as they price themselves out of the market if the buyers do not update their prices.

Now, Assumptions (4) through (6) imply that the only strict Nash equilibria of $\mathcal{G}(u, e, \xi)$ are the Walrasian equilibria of $\mathcal{E}(u, e)$. Namely:

Proposition 4 *Under Assumptions (4) – (6), $\pi \in \Pi$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if there exists $\bar{p} \in \mathbf{E}(u, e)$ such that for all $i \in \{1, \dots, m\}$, $\pi_i = \bar{p}$.*

Proof: *Suppose π is an equilibrium price profile, that is π is such that for all $i \in \{1, \dots, m\}$, $\pi_i = \bar{p}$ where $\bar{p} \in \mathbf{E}(u, e)$. Condition (1) in the definition of $\mathcal{A}'(\pi, e)$ then implies that for all i , $\xi_i(\pi) \leq d_i(\bar{p})$. As $\xi_i(\pi)$ is Pareto optimal in $\mathcal{A}'(e, \pi)$ according to assumption (5.1) and $(d_1(\bar{p}), \dots, d_m(\bar{p})) \in \mathcal{A}'(\pi, e)$, it can only be that for all i , $\xi_i(\pi) = d_i(\bar{p})$.*

Assume agent i deviates to price $p \neq \bar{p}$. Then:

- If there is $h \in P_i$ such that $p_h > \bar{p}_h$, one has $i \notin S_h(p', \pi_{-i})$ and agent i can no longer sale good h and does not rise any income on the good h market. If there is $h \in P_i$ such that $p_h < \bar{p}_h$, one clearly has $p_h e_h < \bar{p}_h e_h$. All in all, one clearly has $w_i(p', \pi_{-i}) \leq w_i(\pi)$, the inequality being strict whenever $p_h \neq \bar{p}_h$ for at least an $h \in P_i$.

Now, with regards to consumption,

- if there exists $h \in C_i$ such that $p_h < \bar{p}_h$, then $i \notin B_h(p', \pi_{-i})$ and $\xi_{i,h}(p', \pi_{-i}) = 0$, so that according to assumption (4.2b), $u_i(\xi_{i,h}(p', \pi_{-i})) = 0 < u_i(d_i(\bar{p}))$.
- Otherwise for all $h \in C_i$ one has $p_h \geq \bar{p}_h$ and either the inequality is strict for at least an h and $w_i(p', \pi_{-i}) \leq w_i(\pi)$ or none of the inequality is strict but according to the preceding $w_i(p', \pi_{-i}) < w_i(\pi)$. In any case, one has $u_i(d_i(p', w_i(p', \pi_{-i}))) < u_i(d_i(\bar{p}))$ so that $u_i(\xi_i(p', \pi_{-i})) \leq u_i(d_i(p', w_i(p', \pi_{-i}))) < u_i(d_i(\bar{p})) = u_i(\xi_i(\pi))$.

To sum up, if agent i deviates to price p' either his income or his choice set or both are strictly reduced: he cannot by definition obtain an allocation as good as $d_i(\bar{p})$ and hence is strictly worse off.

Suppose then π is not an equilibrium price profile.

- If π is a uniform price profile and there exists an $h \in \{1, \dots, l\}$ such that $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$, then assumption (6) implies π is not a (strict) Nash equilibrium.

Hence, from here on we can assume that π is not a uniform price profile.

- If there exists $h \in \{1, \dots, \ell\}$ and $i \in S_h/S_h(\pi)$, by setting $p'_h := \max_{i \in B_h} \pi_{i,h}$ and $p'_j := \pi_{i,j}$ for all $j \neq h$ one obtains a population $\pi' := (p', \pi_{-i})$ such that $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of \mathcal{A}' can only be relaxed. Therefore, $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$. According to assumption (5.2), it can not be that π is a strict Nash equilibrium. Similar arguments apply whenever there exists $h \in \{1, \dots, \ell\}$ and $i \in B_h/B_h(\pi)$.
- Otherwise, one must have $B_h = B_h(\pi)$, $S_h = S_h(\pi)$, and there must exist $i, i' \in \{1, \dots, m\}$ and $h \in \{1, \dots, l\}$ such that $\pi_{i,h} \neq \pi_{i',h}$, e.g $\pi_{i,h} < \pi_{i',h}$. One then has:
 - If $i, i' \in S_h$ by setting $\pi'_{i,h} = \pi_{i',h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i'$ or $k \neq h$, one has $w_i(\pi') \geq w_i(\pi)$ and hence $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption (5.2) The same holds true if $i \in S_h$ and $i' \in B_h$.

Hence, from here on we can assume that for all $i, i' \in S_h$, $\pi'_{i,h} = \pi_{i,h}$.
Then:

- If $i, i' \in B_h$ by setting $\pi'_{i',h} = \pi_{i,h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i$ or $k \neq h$, the private budget constraint of agent i' (condition 1 in definition of \mathcal{A}') is relaxed and one then has $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption (5.2)
- If $i \in B_h$ and $i' \in S_h$, it must be, given that $S_h = S_h(\pi)$, that there exist $i'' \in B_h$ such that $\pi_{i',h} \leq \pi_{i'',h}$ and the preceding argument applies to $i, i'' \in B_h$ such that $\pi_{i,h} < \pi_{i'',h}$.

Summing up, π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.

Through proposition 1, proposition 4 admits the following dynamical counterpart.

Proposition 5 *Under assumptions (4) through (6), the only asymptotically stable strategy profiles for the replicator dynamic in $\mathcal{G}(u, e, \xi)$ are those for which each agent uses an equilibrium price $\bar{p} \in \mathcal{E}(u, e)$ and agent i is allocated his equilibrium allocation $d_i(\bar{p})$.*

In other words, in every economy satisfying assumption (4) and for every exchange process based on private prices satisfying assumptions (5) and (6), Walrasian equilibria are the only asymptotic stable states for the replicator dynamics.

5 Extension to an arbitrary economy

Propositions 4 and 5 require that the goods an agent buys and sells be fixed independently of relative prices. This assumption can be relaxed by localizing the notions of acceptable buyers and sellers defined in the preceding section. In the following, we do not assume that assumption 4 holds and consequently adapt the definitions and proofs of the preceding section (incidentally, we slightly overload some of the notations).

Given a population π , we define the set of buyers of good h as $B_h(\pi) := \{i \mid d_{i,h}(\pi_i) > e_{i,h}\}$, the set of sellers of good h as $S_h(\pi) := \{i \mid d_{i,h}(\pi_i) \leq e_{i,h}\}$, the set of acceptable buyers as those agents whose prices is above the lowest selling price that is $\mathcal{B}_h(\pi) := \{i \in \{1, \dots, m\} \mid \pi_{i,h} \geq \min_{j \in S_h(\pi)/\{i\}} \pi_{j,h}\}$. the set of acceptable sellers as those agents whose prices is below the highest buying price, that is $\mathcal{S}_h(\pi) := \{i \in \{1, \dots, m\}/\{i\} \mid \pi_{j,h} \leq \max_{i \in B_h(\pi)} \pi_{i,h}\}$ and the

feasible income as $w_i(\pi) = \sum_{\{h|i \in \mathcal{S}_h(\pi)\}} \pi_{i,h} e_h$. The key differences with the preceding section is that here an acceptable buyer (resp. seller) is not necessarily a buyer (resp. seller). In particular, at an uniform price profile (i.e when all the agents use the same private price), each agent is both an acceptable buyer and an acceptable seller. However in order to avoid tautological definitions, an agent can't autonomously qualify himself as an acceptable buyer (resp. seller).

The set of price feasible allocations $\mathcal{A}''(e, \pi)$ is then defined as follows

Definition 2 *The set of price feasible allocations $\mathcal{A}''(e, \pi)$ is the subset of feasible allocations $\mathcal{A}(e, \pi)$ such that for every $x \in \mathcal{A}''(e, \pi)$ one has:*

1. $x_i \leq d_i(\pi_i, w_i(\pi))$;
2. $\forall h, \forall i \in B_h(\pi), x_{i,h} > e_{i,h} \Rightarrow i \in \mathcal{B}_h(\pi)$;
3. $\sum_{i \in \mathcal{B}_h(\pi) \cap B_h(\pi)} (x_{i,h} - e_{i,h}) \leq \sum_{j \in \mathcal{S}_h(\pi)} (e_{j,h} - x_{j,h})$

The key difference with definition (1) is that the conditions that applied to individual demand or supply there, apply here on individual excess demand or excess supply. Still, assumption (5) as an exact counterpart:

Assumption 7 (Bargaining Axioms Bis) *The exchange process ξ is such that:*

1. (Efficiency) *If π is an equilibrium price profile, $\xi_i(\pi)$ is Pareto optimal in $\mathcal{A}''(e, \pi)$.*
2. (Monotonicity) *If π and π' are such that $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ then for all $i = 1 \dots n$, one has $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$.*

Assumptions (6) and (7) then suffice to establish the counterpart of proposition (4) but for the two following caveats. First, the definition of acceptable sellers and buyers prevent an agent from being the sole buyer and seller of a given good. Therefore, we have to assume that (at least at equilibrium) there are at least a buyer and a distinct seller for every good. Second, in our framework the change of the private price of a commodity he neither consumes nor is endowed with has no effect whatsoever on an agent's utility, so that two strategies that differ only for such a good yield exactly the same utility. This might prevent the identification of Walrasian equilibria with strict equilibria. In order to avoid this failure, one shall assume that either each agent consumes or is endowed with each good or that an agent's strategy space is reduced to meaningful prices: these of commodities he consumes or sells.

Assuming both conditions fulfilled, we can proceed with the proof of the following proposition, which is very similar to that of proposition (4) and hence given in the appendix.

Proposition 6 *Under Assumptions (6) and (7), $\pi \in \Pi$ is a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if and only if there exists $\bar{p} \in \mathbf{E}(u, e)$ such that for all $i \in \{1, \dots, m\}$, $\pi_i = \bar{p}$.*

6 Sufficient Conditions for Competition

It remains to analyze how restrictive is assumption (6) about buyers' incentive to increase prices when there is excess demand. Walras' description of buyers raising their bids is the report of an empirical observation, the "natural" expression of competition. We shall investigate here how this idea of competition can be grounded in the exchange process ξ .

Let us first consider a basic example with two agents and two goods. The first agent derives utility from consumption of good 2 only, e.g his utility function is $u_1(x_1, x_2) = x_2$, and is endowed with a quarter unit of good 1, i.e his initial endowment is $e_1 = (\frac{1}{4}, 0)$. The second agent has Cobb-Douglas preferences $u_2(x_1, x_2) = x_1x_2$ and initial endowment $e_2 = (\frac{1}{4}, \frac{3}{4})$. Good 2 is the numéraire and its price is fixed equal to one. It is straightforward to check that the only equilibrium is such that the price equals $(1, 1)$, agent 1 is allocated $(0, \frac{1}{4})$ and agent 2 is allocated $(\frac{1}{2}, \frac{1}{2})$. It is also clear that if both agents adopt $(1, 1)$ as private price, the only efficient price feasible allocation is the equilibrium one. Yet, if both agents adopt $(p, 1)$ as private price, then the demands of agent 1 and 2 respectively are $d_1(p) = (0, \frac{p}{4})$ and $d_2(p) = (\frac{1}{8} + \frac{3}{8p}, \frac{p}{8} + \frac{3}{8})$. Whenever $p < 1$, there is excess demand for good 1 and the only efficient allocation is $(0, \frac{p}{4})$ to agent 1 and $(\frac{1}{2}, \frac{3-p}{4})$ to agent 2. Let us examine assumption (6) in this setting. Agent 1 who is not rationed would be worse off if he decreased his private price for good 1 and hence its income, he cannot increase it unilaterally as he would price himself out of the market. Agent 2 cannot decrease his private price for good 1 as he would price himself out of the market. One would expect that "competition" induces him to increase his price for good 1 but he has no incentive to do so as this would only decrease his purchasing power and hence his utility. The key issue is that agent 2 actually faces no competition on the good 1 market as there is no other buyer he could outbid by increasing his price.

Let us then consider a more competitive situation by “splitting in two” agent 2. That is we consider an economy with three agents and two goods. The first agent still has utility function $u_1(x_1, x_2) = x_2$, and initial endowment $e_1 = (\frac{1}{4}, 0)$. The second and third agent have Cobb-Douglas preferences $u_2(x_1, x_2) = u_3(x_1, x_2) = x_1x_2$ and initial endowment $e_2 = e_3 = (\frac{1}{8}, \frac{3}{8})$. As above, It is straightforward to check that the only equilibrium is such that the price equals $(1, 1)$, agent 1 is allocated $(0, \frac{1}{4})$ while agents 2 and 3 are allocated $(\frac{1}{4}, \frac{1}{4})$. It is also clear that if both agents adopt $(1, 1)$ as private price, the only efficient price feasible allocation is the equilibrium one. Now, if each agent adopts $(p, 1)$ as private price, then the demands respectively are $d_1(p) = (0, \frac{p}{4})$ and $d_2(p) = d_3(p) = (\frac{1}{16} + \frac{3}{16p}, \frac{p}{16} + \frac{3}{16})$. Whenever $p < 1$, there is excess demand for good 1. It seems natural to assume that the exchange process would then allocate its demand $d_1(p) = (0, \frac{p}{4})$ to agent 1 who is not rationed. As far as agents 2 and 3 are concerned, the allocation is efficient provided they are allocated no more than their demand $\frac{1}{16} + \frac{3}{16p}$ in good 1. It seems sensible to consider that the allocation is symmetric and hence that both agents are allocated $(\frac{1}{4}, \frac{3-p}{8})$ and so are rationed in good 1. As before agent 1 has no incentive to change his private price for good 1 and neither agents 2 nor 3 can further decrease their private price for good 1. However, if the exchange process implements a form of competition between buyers by fulfilling in priority the demand of the agent offering the highest price for the good, then agents 2 and 3 have an incentive to increase their private prices for good 1. Indeed assume that agent 2 increases his private price for good 1, so $q > p$. He will then be allocated his demand $(\frac{1}{16} + \frac{3}{16q}, \frac{q}{16} + \frac{3}{16})$ and be better off than at price p provided that $(\frac{1}{16} + \frac{3}{16q})(\frac{q}{16} + \frac{3}{16}) > \frac{1}{4} \cdot \frac{3-p}{8}$. It is straightforward to check that this equation holds for $q = p$ (whenever $p < 1$) and hence by continuity in a neighborhood of p . In particular, there exists $q > p$ such that agent 2 is better off adopting q as private price for good 1.

The above examples show that necessary conditions for assumption (6) to hold are potential and actual competition among buyers, respectively *via* the presence of more than one buyers for every good and the priority given to highest bidding buyers in the trading process. It turns out these two conditions are in fact sufficient.

Since Edgeworth (1881), notably in (Debreu & Scarf 1963), the seminal way to ensure competition is effective in a general equilibrium economy is to consider that the economy consists of sufficiently many replicates of a given set of primitive

types. For our purposes, it suffices to assume that the economy $\mathcal{E}(u, e)$ is a 2-fold replicate of some underlying simple economy.

Assumption 8 (Replicates) *For every $i \in \{1 \cdots m\}$ there exists $i' \in \{1 \cdots m\} \setminus \{i\}$ such that $u_i = u_{i'}$ and $e_i = e_{i'}$. Types i and i' are called replicates.*

In our framework, a companion assumption is to consider that the exchange process is symmetric with respect to replicates using the same private price. That is:

Assumption 9 (Symmetry) *For every $\pi \in P^m$, if $i, i' \in \{1 \cdots m\}$ are replicates such that $\pi_i = \pi_{i'}$, then $\xi_i(\pi) = \xi_{i'}(\pi)$.*

Then, to ground in the exchange process ξ actual effects of competition, we shall assume that if there is excess demand for one good the highest bidding seller has priority access to the market. That is a buyer who deviates upwards from a uniform price profile has its demand fulfilled in priority.

Assumption 10 (High Bidders Priority) *Let $\pi = (p, \dots, p) \in P^m$ be a uniform price profile, $i, i' \in \{1, \dots, m\}$ be replicates and π' a price profile such that $\pi'_{i,h} > p_h$ for every $h \in C_i$ and $\pi'_{j,k} = p_k$ otherwise. Then defining*

$$X_i(\pi') = \{x_i \in \mathbf{R}_+^l \mid x_i \leq \xi_i(\pi) + \xi_i(\pi') \text{ and } \pi'_i x_i \leq \pi'_i e_i\},$$

we have

$$u_i(\xi_i(\pi')) \geq \max_{x_i \in X_i(\pi')} u_i(x_i).$$

That is, given that agent i gains priority over his replicate by decreasing its price, everything goes as if he could pick any allocation satisfying his private budget constraint in the pool formed by adding the allocations he and his replicates were formerly allocated.

As announced, the latter conditions suffice to ensure that competition holds in the sense of assumption (6). Namely, one has

Proposition 7 *If assumptions (4), (5), (8), (9), and (10) hold, then assumption (6) holds.*

Proof: *Let $\pi = (p, \dots, p) \in P^m$ be a uniform price profile such that for some $h \in \{1, \dots, l\}$, one has $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$. We shall prove that there exists $p' \in P$ such that $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$. If there exists i such that for some $h \in C_i$, $\xi_{i,h}(\pi) = 0$, the proof is straightforward according to assumption (4). Otherwise, let us then consider an agent i such that $\xi_{i,h}(\pi) < d_{i,h}(\pi_i, w_i(\pi))$. It is then clear, given condition 1 in the definition of \mathcal{A}' , that $u_i(\xi_i(\pi)) < u_i(d_{i,h}(\pi_i, w_i(\pi)))$. Hence it must either be that:*

- the private budget constraint of agent i is not binding, that is one has $p \cdot \xi_i(\pi) < p \cdot e_i$,
- or there are budget neutral utility improving shifts in consumption, that is there exists $v \in \mathbf{R}_+^l$ such that $v_h = 0$ for all $h \notin C_i$ and $p \cdot v = 0$ such that for all sufficiently small $t > 0$, $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$.

In the case where agent i private budget constraint is not binding it is clear, given assumption 10, that agent i can still afford and obtain $\xi_i(\pi)$ if he shifts to a price $p' \in P$ such that $p'_h > p_h$ for every $h \in C_i$, $p'_h \geq p_h$ otherwise and p' is sufficiently close to p . In the latter case where for all sufficiently small $t > 0$, one has $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$, denoting by i' , the replicate of i (who receives the same allocation according to assumption (9)) one has for $t > 0$ sufficiently small: $(\xi_i(\pi) + tv) \leq \xi_i(\pi) + \xi_{i'}(\pi)$. Then for any p' such that $p'_h > p_h$ for every $h \in C_i$, and $p'_h \geq p_h$, let us set $x_i(p') = \frac{p' \cdot e_i}{p \cdot e_i}(\xi_i(\pi) + tv)$. One clearly has $x_i(p') \leq \xi_i(\pi) + \xi_{i'}(\pi)$ and $p' \cdot x_i(p') \leq p' \cdot e_i$. This implies according to assumption (10) that $\xi_i(\pi') \geq u_i(x_i(p'))$ (where p' is defined as in assumption (10)). Moreover, for p' sufficiently close to p , one has using the continuity of the utility that $u_i(x_i(p')) > u_i(\xi_i(\pi))$. This ends the proof.

7 Conclusion

We have shown that the equilibrium of a Walrasian market system is a strict Nash equilibrium of an exchange game in which the requirements of the exchange process are quite mild and easily satisfied. Assuming producers update their private price profiles periodically by adopting the strategies of more successful peers, we have a multipopulation game in which strict Nash equilibria are asymptotically stable in the replicator dynamic. Conversely, all stable equilibria of the replicator dynamic are strict Nash equilibria of the exchange process and hence Walrasian equilibria of the underlying economy.

The major innovation of our model is the use of private prices, one set for each agent, in place of the standard assumption of a uniform public price faced by all agents, and the replacement of the tâtonnement process with a replicator dynamic. The traditional public price assumption would not have been useful even had a plausible stability theorem been available using such prices. This is because there is no mechanism for prices to change in a system of public prices—no agent can alter the price schedules faced by the large number of agents with whom any one agent has virtually no contact.

The private price assumption is the only plausible assumption for a fully decentralized market system not in equilibrium, because there is in fact no natural way

to define a common price system except in equilibrium. With private prices, each individual is free to alter his price profile at will, market conditions alone ensuring that something approximating a uniform system of prices will prevail in the long run.

There are many general equilibrium models with private prices in the literature, based for the most part on strategic market games (Shapley & Shubik 1977, Sahi & Yao 1989, Giraud 2003) in which equilibrium prices are set on a market-by-market basis to equate supply and demand, and it is shown that under appropriate conditions the Nash equilibria of the model are Walrasian equilibria. These are equilibrium models, however, without known dynamical properties, and unlike our approach they depend on an extra-market mechanism to balance demand and supply.

The equations of our dynamical system are too many and too complex to solve analytically or to estimate numerically. However, it is possible to construct a discrete version of the system as a finite Markov process. The link between stochastic Markov process models and deterministic replicator dynamics is well documented in the literature. Helbing (1996) shows, in a fairly general setting, that mean-field approximations of stochastic population processes based on imitation and mutation lead to the replicator dynamic. Moreover, Benaim & Weibull (2003) show that large population Markov process implementations of the stage game have approximately the same behavior as the deterministic dynamical system implementations based on the replicator dynamic. This allows us to study the behavior of the dynamical market economy for particular parameter values. For sufficiently large population size, the discrete Markov process captures the dynamics of the Walrasian economy extremely well with near certainty (Benaim & Weibull 2003). While analytical solutions for the discrete system exist (Kemeny & Snell 1969, Gintis 2009), they also cannot be practically implemented. However, the dynamics of the Markov process model can be studied for various parameter values by computer simulation (Gintis 2007, 2012).

Macroeconomic models have been especially handicapped by the lack of a general stability model for competitive exchange. The proof of stability of course does not shed light on the fragility of equilibrium in the sense of its susceptibility to exogenous shocks and its reactions to endogenous stochasticity. These issues can be studied directly through Markov process simulations, and may allow future macroeconomists to develop analytical microfoundations for the control of excessive market volatility.

Appendix A: Asymptotic stability and replicator dynamics

Let G be an n -player game with finite strategy sets $\{S_i | i = 1, \dots, n\}$, the cardinal of which is denoted by $k_i = |S_i|$, with strategies indexed by $h = 1, \dots, k_i$ and payoff functions $\{\pi_i | i = 1, \dots, n\}$. Let $\Delta_i = \{\sigma_i \in \mathbf{R}^{k_i} | \forall h, \sigma_{i,h} \geq 0 \text{ and } \sum_{h=1}^{k_i} \sigma_{i,h} = 1\}$ the which is the mixed strategy space of agent i , and let $\Delta = \prod_{i=1}^m \Delta_i$. In an evolutionary game setting, an element $\sigma_i \in \Delta_i$ represents a population of players i with a share $\sigma_{i,h}$ of the population playing strategy $h \in S_i$.

Dynamics for such population of players $(\sigma_1, \dots, \sigma_N) \in \Delta$ are defined by specifying, a growth rate function $g : \Delta \rightarrow \mathbf{R}^{\sum_{i=1}^m k_i}$, for all $i = 1, \dots, n$ and $h = 1, \dots, k_i$:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} g_{i,h}(\sigma) \quad (3)$$

We shall restrict attention to growth-rate functions that satisfy a regularity condition and maps Δ into itself (Weibull 1995).

Definition 3 *A regular growth-rate function is a Lipschitz continuous function g defined in a neighborhood of Δ such that for all $\sigma \in \Delta$ and all $i = 1, \dots, n$ we have $g_i(\sigma) \cdot \sigma_i \neq 0$.*

The dynamics of interest in a game-theoretic setting are those that satisfy minimal properties of monotonicity with respect to payoffs. Strategies of player i in $B_i(\sigma) := \{s \in S_i | u_i(s, \sigma_{-i}) > u_i(\sigma)\}$ that have above average payoffs against σ_{-i} , have a positive growth-rate in the following sense:

Definition 4 *A regular growth-rate function g is weakly payoff-positive if for all $\sigma \in \Delta$ and $i = 1, \dots, n$,*

$$B_i(\sigma) \neq \emptyset \Rightarrow g_{i,h} > 0 \text{ for some } s_{i,h} \in B_i(\sigma), \quad (4)$$

where $s_{i,h}$ denotes the h^{th} pure strategy of player i .

Among the class of weakly-payoff positive dynamics, the replicator dynamic is by far the most commonly used to represent the interplay between population dynamics and strategic interactions. It corresponds to the system of differential equations defined for all $i = 1, \dots, n$ and $h = 1, \dots, |S_i|$ by:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} (\pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)). \quad (5)$$

That is thus the system of differential equation corresponding to the growth rate function $g_{i,h}(\sigma) = \pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)$.

It is standard to show that the system of differential equations (3) associated with a regular and weakly-payoff monotonic growth function has a unique solution defined at all times for every initial condition in Δ . We will generically denote the solution mapping by $\psi : \mathbf{R}_+ \times \Delta \rightarrow \Delta$, so $\psi(t, \sigma_0)$ gives the value at time t of the solution to (3) with initial condition $\sigma(0) = \sigma_0$. Stability properties of (3), are then defined in terms of this solution mapping:

Definition 5 A strategy profile $\sigma^* \in \Delta$ is called Lyapunov stable if every neighborhood V of σ^* contains a neighborhood W of σ^* such that $\psi(t, \sigma) \in V$ for all $\sigma \in W \cap \Delta$.

Definition 6 A strategy profile $\sigma^* \in \Delta$ is called asymptotically stable if it is Lyapunov stable and there exists a neighborhood V of σ^* such that for all $\sigma \in V \cap \Delta$:

$$\lim_{t \rightarrow +\infty} \psi(t, \sigma) = \sigma^*.$$

Appendix B: Proof of proposition 6.

The proof of proposition 6 proceeds as follows.

Proof: Suppose π is an equilibrium price profile, that is π is such that for all $i \in \{1, \dots, m\}$, $\pi_i = \bar{p}$ where $\bar{p} \in \mathbf{E}(u, e)$. Condition (1) in the definition of $\mathcal{A}''(\pi, e)$ then implies that for all i , $\xi_i(\pi) \leq d_i(\bar{p})$. As $\xi_i(\pi)$ is Pareto optimal in $\mathcal{A}''(e, \pi)$ according to assumption (5.1) and $(d_1(\bar{p}), \dots, d_m(\bar{p})) \in \mathcal{A}''(\pi, e)$, it can only be that for all i , $\xi_i(\pi) = d_i(\bar{p})$.

Assume agent i deviates to price $p \neq \bar{p}$ and let $\pi' = (p, \pi_{-i})$. Then:

- If there is h such that $p_h > \bar{p}_h$, one has $i \notin \mathcal{S}_h(\pi')$ and agent i can no longer sale good h and does not rise any income on the good h market. If there is h such that $p_h < \bar{p}_h$, one clearly has $p_h e_{i,h} < \bar{p}_h e_{i,h}$. Given that at the equilibrium price profile π , every agent is an acceptable seller for every good so that $w_i(\pi) = \sum_{h=1}^{\ell} \pi_{i,h} e_{i,h}$, it follows that $w_i(\pi') \leq w_i(\pi)$, the inequality being strict unless $e_{i,h} = 0$ for every h such that $p_h \neq \bar{p}_h$.
- If the latter condition holds, there must be h such that $p_h \neq \bar{p}_h$ and $e_{i,h} = 0$ (so that for any price profile ρ , $i \in \mathcal{B}_h(\rho)$). For any h such that $p_h < \bar{p}_h$, then $i \notin \mathcal{B}_h(\pi')$ and one necessarily has $\xi_{i,h}(p', \pi_{-i}) = 0$ according to the definition of \mathcal{A} (and given that $e_{i,h} = 0$).
- Hence agent i can only consume goods h such that $p_h \geq \bar{p}_h$.

- To sum up, if agent i deviates to price p' his income and his choice set are both reduced and one of them is strictly reduced. He cannot by definition obtain an allocation as good as $d_i(\bar{p})$ and hence is strictly worse off.

Suppose then π is not an equilibrium price profile.

- If π is a uniform price profile and there exists an $h \in \{1, \dots, l\}$ such that $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$, then assumption (6) implies π is not a (strict) Nash equilibrium.

Hence, from here on we can assume that π is not a uniform price profile. Then:

- If there exists $h \in \{1, \dots, l\}$ and $i \in S_h(\pi)/\mathcal{S}_h(\pi)$, by setting $p'_h := \max_{i \in B_h(\pi)/\{i\}} \pi_{i,h}$ and $p'_j := \pi_{i,j}$ for all $j \neq h$ one obtains a population $\pi' := (p', \pi_{-i})$ such that $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of \mathcal{A}'' can only be relaxed. Therefore $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$. According to assumption (5.2), it can not be that π is a strict Nash equilibrium. Similar arguments apply whenever there exists $h \in \{1, \dots, l\}$ and $i \in B_h(\pi)/\mathcal{B}_h(\pi)$.
- Otherwise, one must have $B_h(\pi) \subset \mathcal{B}_h(\pi)$, $S_h(\pi) \subset \mathcal{S}_h(\pi)$, and there must exist $i, i' \in \{1, \dots, m\}$ and $h \in \{1, \dots, l\}$ such that $\pi_{i,h} \neq \pi_{i',h}$, e.g. $\pi_{i,h} < \pi_{i',h}$. One then has:
 - If $i, i' \in S_h(\pi)$ by setting $\pi'_{i,h} = \pi_{i',h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i$ or $k \neq h$, one has $w_i(\pi') \geq w_i(\pi)$ while the other constraints in the definition of \mathcal{A}'' can only be relaxed. Therefore $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption (5.2) The same holds true if $i \in S_h(\pi)$ and $i' \in B_h(\pi)$.

Hence, from here on we can assume that for all $i, i' \in S_h$, $\pi'_{i,h} = \pi_{i',h}$. Then:

- If $i, i' \in B_h(\pi)$ by setting $\pi'_{i,h} = \pi_{i,h}$ and $\pi'_{j,k} = \pi_{j,k}$ for $j \neq i$ or $k \neq h$, the private budget constraint of agent i' (condition 1 in definition of \mathcal{A}'') is relaxed and one then has $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ so that π can not be a strict Nash equilibrium according to assumption (5.2)
- If $i \in B_h(\pi)$ and $i' \in S_h(\pi)$, it must be, given that $S_h = \mathcal{S}_h(\pi)$, that there exist $i'' \in B_h$ such that $\pi_{i',h} \leq \pi_{i'',h}$ and the preceding argument applies to $i, i'' \in B_h(\pi)$ such that $\pi_{i,h} < \pi_{i'',h}$.

Summing up, π cannot be a strict Nash equilibrium of $\mathcal{G}(u, e, \xi)$ if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.

References

- Arrow, K., Block, H. D. & Hurwicz, L. (1959), 'On the stability of the competitive equilibrium, ii', *Econometrica* **27**, 82–109.
- Arrow, K. & Hurwicz, L. (1958), 'On the stability of the competitive equilibrium, i', *Econometrica* **26**, 522–552.
- Arrow, K. & Hurwicz, L. (1959), Competitive stability under weak gross substitutability: Nonlinear price adjustment and adaptive expectations, Technical report, Office of Naval Research Contract Nonr-255, Department of Economics, Stanford University. Technical Report No. 78.
- Arrow, K. & Hurwicz, L. (1960), 'Some remarks on the equilibria of economics systems', *Econometrica* **28**, 640–646.
- Arrow, K. J. & Debreu, G. (1954), 'Existence of an equilibrium for a competitive economy', *Econometrica* **22**(3), 265–290.
- Bala, V. & Majumdar, M. (1992), 'Chaotic tatonnement', *Economic Theory* **2**, 437–445.
- Balasko, Y. (2009), *The Equilibrium Manifold. Postmodern Developments in the Theory of General Economic Equilibrium*, MIT Press, Cambridge.
- Benaim, M. & Weibull, J. W. (2003), 'Deterministic approximation of stochastic evolution in games', *Econometrica* **71**(3), 873–903.
- Börgers, T. & Sarin, R. (1997), 'Learning through reinforcement and replicator dynamics', *Journal of Economic Theory* **77**(1), 1 – 14.
URL: <http://www.sciencedirect.com/science/article/pii/S002205319792319X>
- Debreu, G. (1952), 'A social equilibrium existence theorem', *Proceedings of the National Academy of Sciences* **38**, 886–893.
- Debreu, G. (1974), 'Excess demand function', *Journal of Mathematical Economics* **1**, 15–23.
- Debreu, G. & Scarf, H. (1963), 'A limit theorem on the core of an economy', *International Economic Review* **4**(3), pp. 235–246.
URL: <http://www.jstor.org/stable/2525306>

- Dockes, P. & Potier, J.-P. (2005), Léon walras et le statut de la concurrence : une étude partir des éléments d' économie politique pure, *in* G. Bensimon, ed., 'Histoire des représentations du marché: Xe Colloque de l'Association Charles Gide pour l'étude de la pensée économique', Michel Houdiard, Paris.
- Edgeworth, F. Y. (1881), *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*, Kegan Paul, London.
- Fisher, F. M. (1970), 'Quasi-competitive price adjustment by individual firms: A preliminary paper', *Journal of Economic Theory* **2**, 195–206.
- Fisher, F. M. (1972), 'On price adjustments without an auctioneer', *Review of Economic Studies* **39**, 1–15.
- Fisher, F. M. (1973), 'Stability and competitive equilibrium in two models of search and individual price adjustment', *Journal of Economic Theory* **6**, 446–470.
- Fisher, F. M. (1983), *Disequilibrium Foundations of Equilibrium Economics*, Cambridge University Press, Cambridge.
- Florenzano, M. (2005), *General Equilibrium Analysis: Existence and Optimality Properties of Equilibria*, Springer, Berlin.
- Friedman, D. (1991), 'Evolutionary games in economics', *Econometrica* **59**(3), 637–666.
- Gale, D. (1955), 'The law of supply and demand', *Math. Scand.* **30**, 155–169.
- Gintis, H. (2007), 'The dynamics of general equilibrium', *Economic Journal* **117**, 1289–1309.
- Gintis, H. (2009), *Game Theory Evolving* Second Edition, Princeton University Press, Princeton.
- Gintis, H. (2012), The dynamics of pure market exchange, *in* M. Aoki, K. Binmore, S. Deakin & H. Gintis, eds, 'Complexity and Institutions: Norms and Corporations', Palgrave, London.
- Giraud, G. (2003), 'Strategic market games: an introduction', *Journal of Mathematical Economics* **39**(5-6), 355–375.
- Hahn, F. (1962), 'A stable adjustment process for a competitive economy', *Review of Economic Studies* **29**, 62–65.

- Hahn, F. & Negishi, T. (1962), 'A theorem on non-tâtonnement stability', *Econometrica* **30**, 463–469.
- Hammond, P. J. (1993), Irreducibility, resource relatedness and survival in equilibrium with individual nonconvexities in general equilibrium, *in* R. Becker, M. Boldrin, R. Jones & W. Thomson, eds, 'Growth and Trade', Academic Press, San Diego, pp. 73–115.
- Helbing, D. (1996), 'A stochastic behavioral model and a 'microscopic' foundation of evolutionary game theory', *Theory and Decision* **40**, 149–179.
- Kalai, E. & Smorodinsky, M. (1975), 'Other solutions to nash's bargaining problem', *Econometrica* **43**(3), 513–18.
- Kemeny, J. & Snell, J. (1969), *Finite Markov chains*, University series in undergraduate mathematics, repr edn, VanNostrand.
- Mantel, R. (1974), 'On the characterization of aggregate excess demand', *Journal of Economic Theory* **7**, 348–53.
- Mantel, R. (1976), 'Homothetic preferences and community excess demand functions', *Journal of Economic Theory* **12**, 197–201.
- Maynard Smith, J. & Price, G. R. (1973), 'The logic of animal conflict', *Nature* **246**, 15–18.
- McKenzie, L. W. (1959), 'On the existence of a general equilibrium for a competitive market', *Econometrica* **28**, 54–71.
- McKenzie, L. W. (1960), 'Stability of equilibrium and value of positive excess demand', *Econometrica* **28**, 606–617.
- Negishi, T. (1960), 'Welfare economics and the existence of an equilibrium for a competitive economy', *Metroeconomica* **12**, 92–97.
- Negishi, T. (1961), 'On the formation of prices', *International Economic Review* **2**, 122–126.
- Nikaido, H. (1956), 'On the classical multilateral exchange problem', *MetroEconomica* **8**, 135–145.
- Nikaido, H. (1959), 'Stability of equilibrium by the brown-von neumann differential equation', *MetroEconomica* **27**, 645–671.

- Nikaido, H. & Uzawa, H. (1960), ‘Stability and nonnegativity in a walrasian tâtonnement process’, *International Economic Review* **1**, 50–59.
- Saari, D. G. (1985), ‘Iterative price mechanisms’, *Econometrica* **53**, 1117–1131.
- Sahi, S. & Yao, S. (1989), ‘The non-cooperative equilibria of a trading economy with complete markets and consistent prices’, *Journal of Mathematical Economics* **18**(4), 325–346.
- Scarf, H. (1960), ‘Some examples of global instability of competitive equilibrium’, *International Economic Review* **1**, 157–172.
- Shalizi, C. R. (2009), ‘Dynamics of Bayesian updating with dependent data and misspecified models’, *Electronic Journal of Statistics* **3**, 1039–1074.
- Shapley, L. S. & Shubik, M. (1977), ‘Trade using one commodity as a means of payment’, *Journal of Political Economy* **85**(5), 937–68.
- Sonnenschein, H. (1973), ‘Do walras’ identity and continuity characterize the class of community excess demand functions?’, *Journal of Economic Theory* **6**, 345–354.
- Taylor, P. & Jonker, L. (1978), ‘Evolutionarily stable strategies and game dynamics’, *Mathematical Biosciences* **40**, 145–156.
- Uzawa, H. (1959), ‘Walras’ tâtonnement in the theory of exchange’, *Review of Economic Studies* **27**, 182–194.
- Uzawa, H. (1961), ‘The stability of dynamic processes’, *Econometrica* **29**(4), 617–631.
- Uzawa, H. (1962), ‘On the stability of edgeworth’s barter process’, *International Economic Review* **3**(2), 218–231.
- Wald, A. (1951 [1936]), ‘On some systems of equations of mathematical economics’, *Econometrica* **19**(4), 368–403.
- Walras, L. (1954 [1874]), *Elements of Pure Economics*, George Allen and Unwin, London.
- Weibull, J. W. (1995), *Evolutionary Game Theory*, MIT Press, Cambridge, MA.