Blood Allocation with Replacement Donors: Theory and Application*

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Abstract

Blood donations by volunteer non-remunerated donors can only meet less than 50% of demand in 56 countries where blood banks had to adopt replacement donor programs to provide blood to patients in return for donations made by their donors. These programs are inefficient, as they limit the direct exchange of donors among patients. We introduce a novel framework and propose a general mechanism class accommodating efficiency objectives that embed current practices and other plausible fairness criteria. These mechanisms are incentive-compatible for patients to truthfully reveal their utility functions and replacement donors. Our framework also applies to other multi-unit exchange problems.

Keywords: Multi-unit exchange, compatibility-based preferences, blood transfusion, endogenous pricing, market design

JEL Codes: D47, C78, I19, D82, D78

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1 Introduction

Transfusions are commonly used to treat various medical conditions to replace lost blood or add inadequate blood components. Replacements of red blood cells and other blood components such as platelets, plasma, and clotting factors are essential for patients going through certain procedures such as surgery, chemotherapy, and childbirth, and for patients with trauma and blood diseases.[1] In the US, according to Pfuntner, Wier, and Stocks (2013), blood transfusion was the most common procedure performed during hospitalizations in 2011. Even though transfusion is an essential procedure in health care, many patients around the world do not have access to safe blood due to significant shortages.

Around the world, the collection and distribution of blood is organized through blood banks where donated blood is processed and stored. Unlike most solid human organs and tissues, blood replenishes after donation, and most blood products can be stored for a period of time. Different compatibility requirements apply for each blood component — see Appendix A for medical and institutional details of blood transfusion.

The most adequate and reliable supply of blood is through *volunteer non-remunerated donors* (*VNRDs*), who mostly donate blood, often repeatedly, through blood drives or other campaigns. These donors provide the safest supply of blood since the prevalence of blood-borne infections is lowest among this group of donors.[2] However, according to the World Health Organization (WHO), only 79 countries (38 high-income, 33 middle-income, and 8 low-income) collect more than 90% of their blood supply from VNRDs (WHO, 2020).

Although it seems relatively costless to donate blood, there are severe blood shortages in many developing countries, as well as seasonal shortages in developed countries (Gilcher and McCombs, 2005). Cultural and religious factors create frictions that deter VNRDs, especially in some developing countries. Furthermore, some blood components, such as platelets, have short shelf life, are in high demand, and are more difficult to collect than other components. This leads to shortages of such components even in the developed world.

[1] Since most patients require a specific blood component for treatment, whole blood is rarely used in modern transfusion medicine except in some low-income countries (WHO, 2020).

[2] Blood is forbidden to be exchanged using valuable remuneration in most countries. Nevertheless, it is reported that 16 countries collect blood through paid donations as of 2018 (WHO, 2020). The paid donors are considered to be the least safe, as they are usually in poorer health than VNRDs. Such donors may also have incentives to hide their health status, causing adverse selection problems.

In 56 countries worldwide (9 high-income, 37 middle-income, and 10 low-income), more than 50% of the blood supply is met by *replacement donors* and, in some rare cases, through paid donors (WHO, 2020). As an effective method to boost blood component reserves, blood banks in many places — including highly populated countries such as India, Brazil, Pakistan and China — employ official or unofficial *replacement donor programs*. Such a program requires each patient to nominate a number of willing donors, who are typically family members, to donate blood in order for the patient to receive transfusion. Notwithstanding the important role they play in addressing blood shortages, existing replacement donor programs suffer from two major shortcomings.

The first one is the loss of welfare due to the lack of optimization. As far as we know, there is no explicit optimized allocation method based on patients' needs and donor screening in current replacement donor programs, and most programs use allocation schemes that work as *first-come first-serve* (*FCFS*) procedures. Such procedures can be highly inefficient as patients are allowed to exchange donor blood only with the bank ruling out exchanges among patients. However, in many places, the blood bank cannot carry enough inventory to make intermediated exchanges close to efficient. In the face of chronic shortages, a natural policy objective can be to optimize the transfusion volume by organizing exchanges among the patients and the blood bank inventory properly. Indeed, Tagny (2012) notes that (especially in the context of programs in Sub-Saharan Africa):

> "Effective procedures of the management of family donors are not yet clearly worked out and reported. All the current international recommendations refer to only VNR donation, from donor recruitment to donor retention. The data from family blood donors, related to blood collection, processing and delivery including information on the potential recipient, cannot be managed by the current software. Actually, family replacement donation is mainly used in a decentralised model based on the hospital blood banks (Tagny et al., 2012). Thus, adapted or hybrid centralised models must be clearly defined to integrate an effective management of family replacement donation."

The second shortcoming is that replacement donor programs generally operate on exogenously fixed exchange rates between units (of blood) received by the patient and units supplied by the patient through her replacement donors, which can induce unfair, unethical, and inefficient outcomes. Certain patients may not be able to recruit the required number of donors that they are obligated to provide, making it difficult to receive transfusion. This gives rise to coercion and black markets through which third parties are paid to assume the role of replacement donors. In addition, the rules are sometimes bent in favor of some patients with few or no paired donors, inducing ad hoc and informal endogenous exchange rates.[3] Around the world, replacement donor programs appear to be highly non-transparent in their blood allocation operations, and it is difficult to find institutional guidelines (see Appendix A for more details). Even in the absence of these problems, a fixed exchange rate regime based exclusively on units of blood received or supplied without any regard to blood types limits the scope of admissible exchanges and allocations, leading to welfare loss.

In this paper, we model blood allocation with VNRDs and replacement donors, and design new allocation schemes to address these shortcomings. The problem is inherently discrete: as blood components are stored and allocated in packs, they are effectively treated as indivisible goods. Thus, we introduce a general theory on multi-unit discrete exchange with compatibility-based preferences, which can also be applied to other similar market design contexts.

In the model, each patient has a *maximum need* of compatible blood[4] that is usually determined by her medical condition. She has a (possibly empty) set of replacement donors as her endowment, which is private information. Each donor can donate one unit of blood. The blood bank also has an inventory of existing blood of each type, which can be interpreted as originating from VNRDs.

The replacement donor program is a centralized entity with the goal of assigning each patient a *schedule*, which designates the amount of blood she receives and the amount of blood she supplies (or the amount her donors donate). A patient has a separable utility function over schedules, defined as the difference of a concave blood valuation function and a convex cost function for blood supply.

We start off with the case of quasi-linear utility, where there is a common valuation function for all patients, and the cost function is linear with a privately known slope

[3]For example, such accounts were communicated through personal communication with the director of the Tucuman Blood Bank in Argentina, Dr. Felicitas Agote, on 07/07/2020.

[4]We explain the details of compatibility requirements in Appendix A.1. To give a concrete example, consider red blood cell transfusion. The set of blood types relevant for compatibility is $\mathcal{B} = \{O+, O-, A+, A-, B+, B-, AB+, AB-\}$. Signs $+$ and $-$ represent Rh D$+$ and Rh D$-$ types, respectively, and letters denote the ABO types. Two separate compatibility requirements are needed. First, Rh D$-$ red blood cell packs can be transfused to all while Rh D$+$ packs can only be transfused to Rh D$+$ patients. Second, there are different regional standards for ABO compatibility depending on how the packs are prepared, which usually requires *ABO-identical transfusion* or *ABO-cellular compatible transfusion*. In the former case, every pack can only be transfused to patients of its ABO type. In the latter case, type O packs can be transfused to all, type A (resp. type B) packs can be transfused to type A (resp. type B) patients and type AB patients, and type AB packs can only be transfused to type AB patients.

representing the patient's type. The type captures her marginal rate of substitution between receiving and supplying blood. In this case, the revelation problem is also simplified, making it more practical to implement. We show later that our results mostly carry over to the case of general utilities.

Incentivizing truthful revelation of both donor sets and utility functions (or types) is essential for efficiency and increasing the overall transfusion level, as well as obtaining a fair system that levels the playing field for patients (cf. Pathak and Sönmez, 2008). In order to achieve incentive compatible and efficient blood allocation while accommodating flexible and endogenous exchange rates, we introduce two interdependent policy levers.

The first one, a *feasible schedule menu*, is novel in market design as it makes it possible to deal with unrestricted multi-unit exchanges. In particular, many replacement donor programs operate with non-uniform exchanges, i.e., a patient does not necessarily supply one unit for each unit received. This cannot be addressed by the existing tools in the literature, which, as far as we know, all rely on the classical assumption of one-for-one exchange. The feasible schedule menu of each patient is an idiosyncratic function specifying the collection of schedules that she can possibly be assigned for any set of donors she provides. A profile of individual feasible schedule menus then constrain the allocations that a *mechanism* — our second policy lever — will choose from. More specifically, for any given feasible schedule menus, a mechanism chooses an allocation for each problem so that every patient's schedule in the allocation is feasible. Therefore, a patient's eventual exchange rate in her assigned schedule can be endogenously determined by the mechanism.[5]

The current blood allocation practices can be formally assessed using these tools. We map the existing exchange rate policies into feasible schedule menus, and start by showing that, while the FCFS mechanism in practice can often be incentive compatible under such feasible schedule menus, it is inefficient and can even be Pareto dominated by an incentive compatible mechanism under the most commonly used exogenous and fixed exchange rates (Theorem 1). This entails that current blood allo-

[5]The more traditional approach would be avoiding defining the concept of feasible schedule menus altogether, and defining the combination of a single feasible schedule menu profile and an allocation rule respecting this feasible schedule menu profile as a mechanism. We do not follow this approach to highlight limitations caused or possibilities created by these two levers separately as usually different feasible schedule menus are used together with the same allocation rule, FCFS, in most real-life practices. Thus, we effectively refer to a pair of allocation rule–feasible schedule menu profile as a mechanism.

cation practices can be improved upon leading to unambiguous gains for both patients and the bank. Our subsequent simulation results (in Section 5.2) further reveal that the size of improvement can be significantly large in real-life scenarios.

The Pareto-improving mechanism is still not necessarily Pareto efficient. A natural efficient solution that also minimally interferes with the current institutions is to replace FCFS with a *priority mechanism*, which essentially incorporates exchanges among patients into the sequential utility optimization scheme of FCFS.[6] Then, to embed more complex prioritization schemes or other objectives popular in similar medical exchange problems (such as maximizing total transplantations in kidney exchange), we propose a broader and intuitive class of new mechanisms. A *weighted utilitarian mechanism* is defined with respect to some positive weights assigned to individual utilities. Given a profile of feasible schedule menus, the mechanism chooses a feasible allocation that maximizes the weighted sum of utilities for each problem. The weighted utilitarian mechanisms nest both priority mechanisms and *maximal mechanisms*, which are the only previously studied incentive compatible mechanisms in similar allocation problems — albeit studied only under the one-for-one exchange rate (see Section 6 for more on this and the related literature).

The weighted utilitarian mechanisms together with properly designed feasible schedule menus overcome the two shortcomings of current replacement donor programs outlined above.

First, any weighted utilitarian mechanism is Pareto efficient for the given feasible schedule menus (Proposition 1). Moreover, in the simulations, we find that the counterpart to FCFS in our class, a priority mechanism, almost Pareto dominates FCFS. Besides efficiency, more egalitarian allocations can be achieved by choosing appropriate weights of patients. We show that, if the feasible schedule menus satisfy a discrete convexity notion, *L(attice)-convexity*, and the blood valuation function is strictly concave, then some weighted utilitarian mechanism approximately satisfies equal treatment of equals (Proposition 2).

Then, as a prelude to incentive properties of the weighted utilitarian mechanisms, we show that they satisfy a weak and basic incentive criterion of *donor monotonicity*, i.e., concealing some donors never enables a patient to receive more blood, when three natural properties are satisfied by the feasible schedule menus (Theorem 2): the menus

[6] Such a solution is in line with the recent *minimalist market design* paradigm, formally advocated by Sönmez (2023).

are L-convex;[7] if a patient can potentially receive more (or less) blood, then she can potentially receive this amount by supplying more (or less); and the feasible schedule set becomes weakly more favorable for a patient as her donor set expands. Achieving donor monotonicity is particularly important in this context as it helps align patients' individual incentives with the blood bank's objective of increasing transfusion.

Results on stronger incentive criteria are developed from the above theorem. After strengthening the third property of the feasible schedule menus, the weighted utilitarian mechanisms are incentive compatible in the particularly plausible scenario of lexicographic preferences (Theorem 3), where the transfusion amount is always of first-order importance to every patient. Without restricting preferences, if we take the schedule $(0, 0)$ as the outside option and further require the feasible schedule menus to exhibit a mild property of individual rationality, then the priority mechanisms, including the maximal mechanisms, remain incentive compatible (Theorem 5), while the whole class of weighted utilitarian mechanisms is at least incentive compatible with respect to the revelation of donors (Theorem 4).

Second, the innovation of feasible schedule menus allows for various exchange rates between units received and supplied, while weighted utilitarian mechanisms endogenously determine these exchange rates. This helps to rectify the shortcoming caused by a fixed exchange rate in current programs, as the feasible schedule menus can be tailored fairly for patients who can intrinsically recruit fewer donors or for different medical conditions, which may alleviate ethical and health problems associated with black markets. Our approach provides a framework to assess and improve the effectiveness of the existing replacement donor programs, and makes it possible to offer rigor and transparency to their organization. Further toward this goal, we provide concrete policy designs with feasible schedule menus that can help achieve equitable blood allocation, scarce blood type targeting, and approximating fixed exchange regimes with better incentives and welfare.

2 The Model

We focus on the design of blood markets and use the corresponding terminology, although the model introduced below applies to general multi-unit discrete exchange with compatibility-based preferences.

[7] Besides its important roles in fairness and incentives, L-convexity also guarantees that the outcome of a weighted utilitarian mechanism for certain patient utility functions can be found in polynomial time (see the earlier draft of our paper, Han, Kesten, and Ünver, 2021).

Consider the market for a single blood component, which we simply refer to as blood. Let I be a set of **patients** and $\mathcal{B}$ be the set of **blood types**. Each $X \in \mathcal{B}$ denotes a specific blood type used in compatibility requirements.[8]

- **Blood need.** A given patient $i \in I$ has type $\beta_i \in \mathcal{B}$ blood and **needs** a **maximum** of $\overline{n}_i$ units of compatible blood, which is a positive integer. Let $\beta_I = (\beta_i)_{i \in I}$ and $\overline{n} = (\overline{n}_i)_{i \in I}$. For each blood type X, $\mathcal{C}(X) \subseteq \mathcal{B}$, where $\mathcal{C}(X) \neq \emptyset$, is the set of blood types compatible with a type X patient. Compatibility requirements depend on the blood component and the medical context.[9]

- **Blood supply.** Each patient i has a (possibly empty) set of willing **replacement donors**[10] D_i such that each donor $d \in D_i$ can provide one unit of type $\beta_d \in \mathcal{B}$ blood.[11] Let $D = (D_i)_{i \in I}$, $\beta_D = (\beta_d)_{d \in \bigcup_{i \in I} D_i}$, and a positive integer δ be a commonly known **upper bound** on the number of donors any patient can bring forward. Moreover, the **blood bank**, denoted as b, has v_X units of type X blood in its **inventory** for each blood type X. Let $v = (v_X)_{X \in \mathcal{B}}$.

- **Patient preferences and utilities.** Each patient i has strict preferences over her blood allocation **schedules**, each of which is a non-negative integer vector (r, s), where r denotes the amount of compatible blood **received** and s denotes the amount of blood **supplied** through her replacement donors. Such preferences are represented by a one-to-one separable utility function u_i such that for every $(r, s) \in \{0, 1, \ldots, \overline{n}_i\} \times \{0, 1, \ldots, \delta\}$,

$$u_i[r, s] = \rho_i(r) - \sigma_i(s),$$

  where

[8] Our baseline model can be slightly extended to cover several blood components at the same time, using a more general specification of patient and donor types. Although in practice replacement donor programs function for each blood component independently, it is plausible that higher welfare gains can be achieved by integrating these markets. For instance, a patient requesting red blood cells can have her donors donate platelets to another patient, while the latter patient's donors donate red blood cells to the former patient.

[9] See Appendix A.1 for medical details of compatibility.

[10] Eligibility requirements for a replacement donor are regulated in many countries. For example, they are often called *family (replacement) donors* in the literature (see, for instance, Tagny, 2012, Allain and Sibinga, 2016, and Kyeyune-Byabazaire and Hume, 2019), as they need to have a familial relationship with the patient, to prevent black markets. Therefore, this donor set is usually well-defined at the onset of the problem.

[11] We normalize that each donor donates a single unit. For whole blood or red blood cell donations, which are the most common blood donations, each donor typically donates one single unit, and thus, there is no need for normalization.

- her blood valuation function $\rho_i : \{0, 1, \ldots, \overline{n}_i\} \to \mathbb{R}$ is strictly increasing and concave[12] — the patient prefers to receive more blood up to her maximum need,[13] and the marginal utility from blood is decreasing,
- her cost function for blood supply $\sigma_i : \{0, 1, \ldots, \delta\} \to \mathbb{R}$ is strictly increasing and convex — the patient prefers to have fewer of her donors to donate and the marginal cost of supplying blood is increasing, and
- for any $0 \leq r < \overline{n}_i$ and $0 \leq s < \delta$,

$$u_i[r+1, s+1] > u_i[r, s].$$

  That is, she always would like to supply one more unit to receive one more unit of blood. In other words, the *marginal rate of substitution* (*MRS*) of supply over receipt is always greater than 1.[14]

For the ease of exposition, and to focus more on the transfusions received by patients, we start with the slightly more restricted case of linear cost functions. Moreover, we assume all patients share the same and commonly known blood valuation function $\rho : \{0, 1, \ldots, \max\{\overline{n}_i : i \in I\}\} \to \mathbb{R}$ without loss of generality. That is, each patient i has the following **quasi-linear** utility function

$$u_i\big[(r, s),\ \theta_i\big] = \rho(r) - \theta_i \cdot s,$$

where $\theta_i > 0$ denotes the **type** of the patient i. Therefore, we are able to simply use a single parameter to capture patients' potential differences in MRS. We assume that there is a finite type space Θ, and let $\theta = (\theta_i)_{i \in I} \in \Theta^{|I|}$. In Section 4.4, we discuss that our design of efficient and incentive compatible blood allocation mechanisms mostly carries over to the general case where each patient reveals a convex cost function as well as her own valuation function.

- **Blood bank utility.** Finally, the blood bank cares about the remaining blood level of each type in its inventory, $(r_X)_{X \in \mathcal{B}}$, and possibly the total transfusions to the patients, r_t. We simply assume that it views the different types of blood as substitutes with different marginal rates of substitution, and its preferences are

[12]For a one-variable function f on a discrete and integer domain, concavity and convexity are defined in the standard way. That is, f is concave (or convex) if for any $x-1$, x and $x+1$ in the domain, $f(x) - f(x-1) \geq f(x+1) - f(x)$ (or $f(x) - f(x-1) \leq f(x+1) - f(x)$).

[13]The monotonic nature of preferences is motivated in Appendix A.2.

[14]Additionally, the hardbound of donor number at δ and satiation of need at $\overline{n}_i$ generate implicitly zero MRS for large receipt or supply levels.

represented by a linear utility function u_b such that

$$u_b\Big[(r_X)_{X\in\mathcal{B}},\ r_t\Big] = \sum_{X\in\mathcal{B}} (\lambda_X \cdot r_X) + \lambda_t \cdot r_t,$$

where $\lambda_X > 0$ for each $X \in \mathcal{B}$, $\lambda_t \geq 0$, and u_b is one-to-one with respect to $(r_X)_{X\in\mathcal{B}}$.[15]

A blood allocation **problem** is denoted by a list $\left[I, \beta_I, \overline{n}, D, \beta_D, \theta, v, u_b\right]$. We fix $I, \beta_I, \overline{n}$, v, and u_b. Then a problem is represented by a donor profile and type profile pair (D, θ).[16]

Allocations. Given a donor profile D, an **allocation** α is the vector of non-negative integers

- $\alpha_X(i)$ for each patient i and her compatible blood type $X \in \mathcal{C}(\beta_i)$ indicating the amount of type X blood received by i,
- $\alpha(d) \in \{0, 1\}$ for each donor d indicating whether the donor d is selected to donate blood or not, and
- $\alpha_X(b)$ for each blood type X indicating the amount of type X blood left in the inventory

such that:

1. for every patient i, $\quad \sum_{X\in\mathcal{C}(\beta_i)} \alpha_X(i) \ \leq \ \overline{n}_i$, and
2. for every blood type X, $\quad \left(\sum_{i\,:\,X\in\mathcal{C}(\beta_i)} \alpha_X(i)\right) + \alpha_X(b) = \left(\sum_{d\,:\,\beta_d = X} \alpha(d)\right) + v_X.$

The first condition requires that no patient receives more blood than her maximum need. The second one is a market clearing condition and makes sure that, for each blood type, the total volume allocated is equal to the sum of the existing blood at the blood bank and the blood collected from the patients' donors. Denote the set of all allocations for D as $\mathcal{A}(D)$.

[15] We generally employ the blood bank utility function as a tie-breaker among various allocations by keeping $\lambda_X > 0$ for each blood type $X \in \mathcal{B}$. On the other hand, $\lambda_t = 0$ is possible. Although the efficiency definition relies on this utility function, we can rather think of Pareto efficiency as a constrained efficiency axiom subject to certain health authority objectives. For example, when $\lambda_t > 0$ and is sufficiently larger than each λ_X, this utility represents a priority of a benevolent health authority to maximize total received amounts, a common objective in scarce medical resource allocation.

[16] Without loss of generality, we use this notation for brevity, assuming β_D is determined once D is given.

Only the schedules induced by an allocation will be economically relevant in the analysis. Thus, given an allocation $\alpha \in \mathcal{A}(D)$, for brevity we use $\alpha(i) = \big(\alpha_r(i), \alpha_s(i)\big)$, where

$$\alpha_r(i) = \sum_{X \in \mathcal{C}(\beta_i)} \alpha_X(i) \quad \text{and} \quad \alpha_s(i) = \sum_{d \in D_i} \alpha(d),$$

to denote the schedule of patient i, and $\alpha(b) = \big(\alpha_X(b)\big)_{X \in \mathcal{B}}$ to denote the eventual inventory schedule of the blood bank. Thus, allocation α induces the **schedule profile**

$$\Big(\big(\alpha(i)\big)_{i \in I}\,,\ \alpha(b)\Big).$$

Given a problem (D, θ), an allocation $\alpha \in \mathcal{A}$ is **Pareto efficient within $\mathcal{A}$** for a subset of allocations $\mathcal{A} \subseteq \mathcal{A}(D)$, if there is no allocation $\alpha' \in \mathcal{A}$ such that

- $u_i\big[\alpha'(i),\ \theta_i\big] \geq u_i\big[\alpha(i),\ \theta_i\big]$ for every patient i,
- $u_b\left[\alpha'(b),\ \sum\limits_{i \in I} \alpha'_r(i)\right] \geq u_b\left[\alpha(b),\ \sum\limits_{i \in I} \alpha_r(i)\right]$, and
- either $u_j\big[\alpha'(j),\ \theta_j\big] > u_j\big[\alpha(j),\ \theta_j\big]$ for some patient j or

$$u_b\left[\alpha'(b),\ \sum_{i \in I} \alpha'_r(i)\right] > u_b\left[\alpha(b),\ \sum_{i \in I} \alpha_r(i)\right].$$

2.1 Two Policy Levers: Feasible Schedule Menus and Mechanisms

In our current setup, a patient's schedule induced by an allocation is unconstrained, and might be normatively and practically unacceptable. In fact, blood banks around the world typically use various exchange regimes to control the rates at which blood received will be *traded* for blood supplied. On the other hand, the donor sets and preferences over schedules, or patient types, are private information and unknown to the blood bank. To incorporate these two important aspects of blood allocation in practice, we introduce two interdependent policy levers that the policy designer can use in conjunction to allocate blood for reported donors and types.

The first one is the concept of a *feasible schedule menu*, which allows us to formally incorporate (potentially endogenous) exchange rates between supplied and received units of blood, and the second one is an *allocation mechanism* which together with feasible schedule menus makes it possible to organize patient incentives while also ensuring efficient blood allocation. It is important to highlight that the two levers are inseparable in a multi-unit exchange setting such as ours. The feasible schedule menus alone are insufficient to pin down an allocation. They do not specify what patients get and which donors donate. Similarly, a mechanism needs to operate together with a suitable set of feasible schedule menus so that exchange regimes are respected and, more subtly, proper incentives are provided to patients.

Feasible Schedule Menus. The first policy lever constrains the possible schedules for the patients. The blood bank presents a *menu* to each patient, which shows the collection of schedules that are feasible to her, as a function of her reported donor set. Formally, for each patient i, a **feasible schedule menu** $\mathcal{F}_i$ is a function that assigns a non-empty set of schedules $\mathcal{F}_i(D_i)$ to each donor set D_i such that

$$\mathcal{F}_i(D_i) \subseteq \mathbb{S}_i(D_i) = \{0, 1, \ldots, \overline{n}_i\} \times \{0, \ldots, |D_i|\},$$

where $\mathbb{S}_i(D_i)$ denotes the set of all schedules that do not exceed the maximum need and donor capacity of the patient.

Let $\mathcal{F} = (\mathcal{F}_i)_{i \in I}$ denote a profile of feasible schedule menus. An arbitrary feasible schedule menu profile may not guarantee the existence of at least one feasible allocation, especially when there is not enough inventory in the bank. It is also common practice around the world that banks with enough inventories define a minimum guaranteed amount for a patient to receive when she brings forward some minimum number of donors. We explicitly define such guarantees for patients and tie them with feasible schedule menus so that the existence of an allocation that induces feasible schedules is ensured. We assume there exists a non-negative integer vector of **minimum guarantees** $\underline{g} = (\underline{g}_i)_{i \in I} \leq \overline{n}$ such that

1. for any non-empty subset of blood types $\mathcal{B}' \subseteq \mathcal{B}$,
$$\sum_{i \,:\, \beta_i \in \mathcal{B}'} \underline{g}_i \quad \leq \sum_{X \in \bigcup_{Y \in \mathcal{B}'} \mathcal{C}(Y)} v_X, \quad \text{and}$$
2. for each patient i, her feasible schedule menu $\mathcal{F}_i$ and donor set D_i,
$$\mathcal{F}_i(D_i) = \{(0,0)\} \qquad \text{or} \qquad \min\{r \,:\, (r,s) \in \mathcal{F}_i(D_i)\} = \underline{g}_i.$$

The minimum guarantee is what the bank promises for each patient to receive (the first condition ensures that the bank carries enough blood through Hall's Theorem, Hall, 1935) if the patient's donor set meets the basic requirement by the bank for participating in the replacement donor program (the second condition). The bank determines the minimum guarantees based on its inventory, which can be set to zero.

In going forward, we will only consider allocations in which each patient's induced schedule is feasible according to the given feasible schedule menus $\mathcal{F}$. That is, for any donor profile D, we restrict attention to allocations in the set

$$\mathcal{A}(\mathcal{F}, D) = \{\alpha \in \mathcal{A}(D) \,:\, \alpha(i) \in \mathcal{F}_i(D_i), \ \forall i \in I\},$$

which is non-empty by the definition of feasible schedule menus and the assumption regarding minimum guarantees.

Allocation Mechanisms. A **mechanism** is a procedure f that maps each problem (D, θ) to an allocation $f\Big(\mathcal{F},\, D,\, \theta\Big) \in \mathcal{A}(\mathcal{F}, D)$ under every feasible schedule menu profile $\mathcal{F}$. Defining mechanisms through feasible schedule menus gives a practical and formal method to formalize the policy outcomes if the blood bank pursues goals such as transparency, fairness, and other systematic normative criteria besides quantifiable optimization objectives. Moreover, if the blood bank updates the feasible schedule menu profile used, the mechanism outcome can be transparently traced through this general setup.

A mechanism f is **Pareto efficient** if for every problem (D, θ) and feasible schedule menu profile $\mathcal{F}$, its outcome is Pareto efficient within $\mathcal{A}(\mathcal{F}, D)$.

Incentive Compatibility. Patients' incentives to truthfully report donors and types depend on the mechanism as well as the feasible schedule menus. Thus, we say a mechanism f is **incentive compatible under $\boldsymbol{\mathcal{F}}$** if for any problem (D, θ), patient i, donor subset $D'_i \subseteq D_i$ and type θ'_i, we have

$$u_i\Big[f\Big(\mathcal{F},\, (D_i, D_{-i}),\, (\theta_i, \theta_{-i})\Big)(i),\ \theta_i\Big] \geq u_i\Big[f\Big(\mathcal{F},\, (D'_i, D_{-i}),\, (\theta'_i, \theta_{-i})\Big)(i),\ \theta_i\Big].$$

That is, given any problem, no patient can be strictly better-off by concealing some of her donors and misreporting her type.[17]

We will mostly separate the analysis on the two incentive considerations. The following weaker notion only requires that each patient cannot manipulate via under-reporting donors. A mechanism f is **incentive compatible with respect to donors under $\boldsymbol{\mathcal{F}}$** if for any problem (D, θ), patient i and donor subset $D'_i \subseteq D_i$, we have

$$u_i\Big[f\Big(\mathcal{F},\, (D_i, D_{-i}),\, \theta\Big)(i),\ \theta_i\Big] \geq u_i\Big[f\Big(\mathcal{F},\, (D'_i, D_{-i}),\, \theta\Big)(i),\ \theta_i\Big].$$

Donor Monotonicity. Blood donation is not nearly as costly as solid organ donation, leading to a much less invasive procedure and fast replenishment of blood. Therefore, the volume of blood received is often of first-order importance for a patient. In such cases a patient has *lexicographic preferences* over schedules, represented by the quasi-linear utility function with a sufficiently small type. Formally, we assume that there is

[17] For the report of donor set we focus on a patient's incentive to hide her donors. While situations in which a patient exaggerates her donors, i.e., reports a larger set of donors than she actually has, are theoretically conceivable, this type of manipulation is often practically infeasible, since donor registration requires legally verifiable donor identification information.

a type $\vartheta^L \in \Theta$ such that

$$\vartheta^L \cdot \delta < \rho(r+1) - \rho(r)$$

for all $0 \leq r < \max\{\overline{n}_i \,:\, i \in I\}$.

When it is known that all patients have lexicographic preferences, we assume that patient types are also known and $\Theta = \{\vartheta^L\}$. In this case the two incentive notions introduce above coincide. Moreover, as long as a patient does not receive less blood, providing more donors to the system may not be as undesirable for her. Based on this motivation, we introduce a weak and plausible incentive property in this context. A mechanism f is **donor monotonic under $\mathcal{F}$** if for any problem (D, θ), patient i and donor subset $D_i' \subseteq D_i$,

$$f_r\Big(\mathcal{F},\, (D_i, D_{-i}),\, \theta\Big)(i) \geq f_r\Big(\mathcal{F},\, (D_i', D_{-i}),\, \theta\Big)(i).$$

3 Blood Allocation Practices Around the World

To motivate our approach in this paper, we start off by discussing the current blood allocation systems and their deficiencies in various parts of the world. The tools we have introduced in the previous section enable us to formally describe these systems in terms of the mechanisms and the underlying feasible schedule menus.

We first give representative examples of exchange rate policies mapped to feasible schedule menus.

(P1) **One-for-one policy.** The most common exchange rate in the world is one-for-one, i.e., the units supplied must equal the units received. Formally, this leads to the following feasible schedule menu:

$$\mathcal{F}_i(D_i) = \big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s = r\big\}.$$

(P2) **Two-for-one policy.** In Cameroon, Congo, and Mexico, for each unit of blood received, two units of blood have to be supplied (Tagny, 2012; Thompson, 2020):

$$\mathcal{F}_i(D_i) = \big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s = 2r\big\}.$$

(P3) **Fixed donor policy.** Each patient has to register x donors regardless of the amount of blood she needs. This can be modeled as the following feasible schedule menu:

$$\mathcal{F}_i(D_i) = \begin{cases} \big\{(0,0)\big\} & \text{if } |D_i| < x \\ \big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s = x\big\} & \text{otherwise} \end{cases}.$$

For example, in Delhi, India, $x = 1$ (Delhi State Health Mission, 2016), and in

Turkey $x = 3$.[18]

(P4) **Forgiving policy.** In some places, a combination of a strict and a relaxed *forgiving policy* that is idiosyncratic for each patient is used. For example, in Tucuman, Argentina, a patient's replacement donors donate after the transfusion, and the rate is one-for-one. However, it is not as strictly enforced, and debts may sometimes be forgiven as mentioned in the Introduction. We can formalize such ad hoc policy as:

$$\mathcal{F}_i(D_i) = \big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s \leq r \leq |D_i|\big\}.$$

(P5) **Minimum guarantee policy.** Many replacement donor programs feature minimum guarantees based on the good samaritanship of the patients in the past. For instance, in Xi'an, China, a patient is guaranteed three units for each unit she has donated before, and the exchange rate is one-for-one beyond this guarantee (She, 2020). Let x_i be the amount of previous donations from a patient i. Then, with $\underline{g}_i = \min\{3x_i, \overline{n}_i\}$, her feasible schedule menu is formalized as:

$$\mathcal{F}_i(D_i) = \big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s = r - \underline{g}_i\big\}.$$

Aside from the ad hoc forgiving policy, in each case the exchange rates are exogenously fixed, in the sense that the amount of supply required for each possible amount of receipt is determined before blood is allocated.

As blood transfusion is one of the most common medical procedures, patients requesting blood can be highly heterogenous in terms of demographics and type of medical condition. In the presence of such heterogeneity, with the aim of improving fairness and equity in the delivery of care, patient prioritization tools (PPTs) are globally used for ordering patients for emergency or elective (i.e., medically necessary but non-emergency) services based on various criteria. By and large, patients are prioritized based on medical urgency and the order with which blood is requested.[19] Therefore, given that blood banks do not explicitly organize exchanges among patients and donors as far as we know, the current practice of allocating blood is similar to a

[18] Based on three separate personal communications with Haluk Ertan, Professor of Microbiology at UNSW, on 05/09/2022, Seda Hatice Gökler, Post-doctoral Fellow at Sakarya University, on 05/18/2022, and Ilhan Uyaner, Doctor at Ankara Oncology Hospital of University of Health Sciences, on 05/17/2022.

[19] For example, in Australia, patients with the most urgent need for surgery (Category 1) take precedence over those in semi-urgent need (Category 2) who take precedence over non-urgency patients (Category 3). Then, within each urgency category patients are treated in the same order in which they were added to the waiting list (CBHS, 2020).

first-come first-serve (FCFS) mechanism, where the patients are sequentially treated based on a predetermined order. This is also consistent with the current platelet allocation practice at a regional hospital in India as discussed by Ødegaard and Roy (2021).[20]

More formally, given any feasible schedule menu profile $\mathcal{F}$, FCFS with respect to a fixed order of patients is defined as follows. The bank first reserves blood units for each patient with a positive minimum guarantee. Then, given a problem (D, θ), in each step $k \geq 1$, the bank uses its current available inventory and the k^{th} patient's own donors to satisfy the patient as much as possible, or to achieve an optimal "individual allocation". That is, if the k^{th} patient, denoted as i, has a donor set D_i and type θ_i, and there are v_i units of compatible blood in inventory for her, then the bank selects her donors D_i' to donate and gives her r units of compatible blood so that they solve the following problem:

$$\max_{r \geq 0,\, D_i' \subseteq D_i} u_i\big[\big(r, |D_i'|\big),\ \theta_i\big]$$

$$\text{subject to}$$

$$r \leq \Big|\big\{d \in D_i' \,:\, \beta_d \in \mathcal{C}(\beta_i)\big\}\Big| + v_i \qquad \text{and} \qquad \big(r, |D_i'|\big) \in \mathcal{F}_i(D_i).$$

Then patient i leaves, and the bank serves the next patient with the updated inventory.

An FCFS mechanism works similarly to a *serial dictatorship* mechanism, and provides straightforward incentives to the patients. The mechanism is incentive compatible under the five policies listed above, P1-P5.[21] However, a major deficiency of the FCFS mechanism is that it can lead to large welfare loss.

Example 1 (FCFS is inefficient) *Suppose that the set of patients is $I = \{1, 2, 3, 4, 5\}$ and the set of relevant blood types is $\mathcal{B} = \{O, A, B, AB\}$. Assume ABO-identical transfusion, i.e., $\mathcal{C}(X) = \{X\}$ for each $X \in \mathcal{B}$. The blood bank has only one unit of type O blood in its inventory. Let $\overline{n}_i = 1$ for all $i \in I$. Each patient's blood type and donor set are given as follows.*

- *$\beta_1 = O$, and Patient 1 has one type AB donor and one type B donor.*
- *$\beta_2 = B$, and Patient 2 has one type A donor.*

[20] Ødegaard and Roy (2021) focus on platelet inventory management and model the platelet allocation policy at the hospital as FCFS. However, they omitted the replacement donor program with one-for-one exchange, which is unofficially implemented at the hospital, from their framework.

[21] This is based on the presumption that a patient's report of donors and types does not affect the assignments of those before her, which is satisfied when the bank treats the patients in the order of arrival.

- *$\beta_3 = B$, and Patient 3 has one type A donor.*
- *$\beta_4 = O$, and Patient 4 has one type B donor.*
- *$\beta_5 = A$, and Patient 5 has one type O donor.*

Suppose the exchange rate is one-for-one (policy P1) and the patients are treated in the order of $1-2-3-4-5$ under FCFS. Then Patient 1 first receives one unit of type O blood. Depending on which one of her donors is selected to donate, there are two possible outcomes:

- *If the type AB donor donates, none of the remaining patients receives any blood, and the bank ends up with one unit of type AB blood.*
- *If the type B donor donates, Patient 2 receives blood and her A donor donates. Patients 3 and 4 cannot receive any blood. Patient 5 receives one unit of type A blood, and the bank ends up with one unit of type O blood from her donor.*

Assume that the weights in the blood bank's utility function are such that λ_t is sufficiently large compared to λ_X for each blood type X, i.e., the bank values the total transfusions sufficiently more than its left-over blood. Then both allocations are inefficient, since there is an allocation α^* where $\alpha^*(i) = (1,1)$ for every $i \in I$ and $\alpha^*_A(b) = 1$. ∎

Example 1 illustrates two main sources of welfare loss under FCFS.

The first one is due to the arbitrariness of donor selection. Since FCFS does not have optimal donor screening based on the needs of patients, a donor whose blood type may later be in short supply can be sent home. For example, the first allocation in the example denies blood to all patients after Patient 1 since the "wrong donor", i.e., the type *AB* donor, is selected to donate.

The second source of welfare loss is due to the sequential nature of the procedure. If a patient requests blood at the "wrong time" when there is little or none available, it not only hurts her but also causes her donors to be not used, which subsequently hurts those who are served after her. For example, in the second allocation in Example 1, Patient 3 cannot receive blood despite that Patient 4 has a donor whose blood type matches Patient 3's. Note that Patient 4 can also use the blood from the donor of Patient 5. In other words, if the processing order of the patients were $1-2-5-4-3$, this coordination problem would have been solved. In general, of course, there is no way for the blood bank to foresee the correct processing order of patients.

The potentially significant welfare loss under FCFS, as will be confirmed by our simulation results in Section 5.2, provides the main rationale for our approach in this paper. It turns out that an FCFS mechanism is not only inefficient, but can also be

Pareto dominated by another incentive compatible mechanism under the existing and common policies P1 and P5, when the bank's choices satisfy a mild *consistency* assumption.

Note that, under P1 and P5, for a problem (D, θ), at any step k of the mechanism there can be multiple solutions in finding an optimal individual allocation to the k^{th} patient, i, and the solutions are independent of the type of patient i. We then assume the bank's decision only depends on the relevant variables at this step: for a different problem (D', θ'), if $D'_i = D_i$ and the bank has the same available blood units in its inventory at step k, then the bank gives the same compatible blood units to patient i and selects the same donors to donate.[22]

We summarize the main findings on FCFS as follows.

Theorem 1 *An FCFS mechanism may not be Pareto efficient. Moreover, there exists a mechanism f such that, under feasible schedule menus P1 and P5, f is incentive compatible, for any environment the utilities of every patient and the bank are weakly higher under f, and for some environment the outcome of f Pareto dominates that of the FCFS mechanism.*[23]

As we will explore in the next section, among the real-life polices introduced above, only P1 and P5 satisfy the sufficient conditions for obtaining incentive compatible and Pareto efficient mechanisms in the general class we introduce.

Theorem 1 shows that it is possible to improve upon the *status quo allocation* of FCFS without giving up incentive compatibility. The recent literature on assignment problems has underscored the difficulty of obtaining strategy-proof Pareto improvements over strategy-proof inefficient mechanisms (see, for example, Kesten, 2010, Abdulkadiroğlu, Pathak, and Roth, 2009, Erdil, 2014, and Alva and Manjunath, 2019). Indeed, in a number of models, Pareto improvement upon a strategy-proof mechanism often leads to a loss of straightforward incentives, or provides rather limited welfare gains by allocating wasted resources. Our context stands out as an interesting exception to this strand of literature. Furthermore, the simulation results show that the welfare gains over FCFS can be highly significant.

[22] This also ensures that FCFS is incentive compatible under P1 and P5.

[23] An environment refers to a collection of all possible elements defined in our blood allocation model except the two policy levers. Therefore, in this theorem we allow elements such as I and v to vary. Alternatively, if we fix an environment except the donor profile and the type profile (D, θ) (i.e., the problem), then it can be shown that under a small inventory v, a relatively large bound on number of donors δ, and the existing compatibility relations (see Footnote 4), there is always a problem for which the outcome of f Pareto dominates that of the FCFS mechanism.

We close this section with an informal description of the incentive compatible improvement over FCFS. The idea is based on a meticulously constructed two-stage mechanism:[24]

- In the first stage we calculate the FCFS allocation.
- In the second stage, patients are first divided into three groups:
  1. Patients who did not receive any blood from the inventory under FCFS (although they may have received blood from their own compatible donors).
  2. Patients who have the largest donor sets, i.e., δ donors, and received blood from the inventory under FCFS.
  3. All other patients.

  Then we apply one of the mechanisms introduced in the next section to the *original* allocation problem, under the updated feasible schedule sets for the last two groups: for any patient in Group 2, eliminate any feasible schedule worse than the one received under FCFS; for any patient in Group 3, the only feasible schedule is the one received under FCFS.

By organizing exchanges in the second stage through our mechanism, Groups 1 and 2 could be improved. Although Group 3 is not improved, including these patients in the improvement stage can help rectify the donor selection in FCFS. For instance, in Example 1 if the type AB donor of Patient 1 donates, then in the second stage improvement, she still receives the schedule $(1, 1)$ but her type B donor donates, which helps Patient 2 or Patient 3 receive blood, leading to the (unique) efficient allocation α^* in the example.

The observation that FCFS can lead to significant welfare loss raises a natural question: Are there any mechanisms that are both Pareto efficient and incentive compatible under some suitable set of feasible schedule menus? In the next section, we drop FCFS as our status quo benchmark and give a positive answer to this question by providing a large and intuitive class of such mechanisms.

4 A New Mechanism Class and Main Results

4.1 Weighted Utilitarian Mechanisms

We study a natural class of mechanisms that choose feasible allocations to maximize the weighted sum of patient and bank utilities. Given a collection of positive

[24] We give details of this mechanism in the proof of Theorem 1.

weights $w = (w_i)_{i \in I}$, define an objective function U such that for any feasible schedule menu profile $\mathcal{F}$, problem (D, θ) and allocation $\alpha \in \mathcal{A}(\mathcal{F}, D)$,

$$U(w,\theta,\alpha) = \sum_{i \in I} \Big(w_i \cdot u_i\big[\alpha(i),\ \theta_i\big] \Big) + u_b\big[\alpha(b),\ \sum_{i \in I} \alpha_r(i)\big].$$

Assume that if there is an allocation $\alpha' \in \mathcal{A}(\mathcal{F}, D)$ such that α and α' induce different schedule profiles, then $U(w,\theta,\alpha) \neq U(w,\theta,\alpha')$.[25] We say f is a **weighted utilitarian mechanism with respect to w** if

$$f\Big(\mathcal{F},\ D,\ \theta\Big) \in \underset{\alpha \in \mathcal{A}(\mathcal{F},D)}{\arg\max}\ U(w,\theta,\alpha)$$

for every feasible schedule menu profile $\mathcal{F}$ and problem (D, θ).

Since all the weights are positive, the welfare property of such mechanisms follows immediately from the construction:

Proposition 1 *Every weighted utilitarian mechanism is Pareto efficient.*

There are two familiar subclasses of weighted utilitarian mechanisms:

- *Priority mechanisms*. Given a priority order over the patients and the blood bank, the priority mechanism sequentially maximizes their utilities. That is, in the first step we find the allocations that maximize the utility of the first agent (a patient or the bank); in Step $k \in \{2, \ldots, |I| + 1\}$ we find the allocations that maximize the utility of the k^{th} agent among those obtained from the previous step.[26]
- *Maximal mechanisms*. When the bank sufficiently values the total transfusions, i.e., λ_t is sufficiently large compared to λ_X for each X, a priority mechanism with the bank having the first priority chooses an allocation that maximizes the total transfusions, and thus it is referred to as a maximal mechanism.

Priority mechanisms and maximal mechanisms are the most general forms of the mechanisms that were studied in the literature for compatibility-based allocation (see Section 6 for details), and these two principles each have been used in kidney exchange

[25]If U does not satisfy this assumption, the blood bank can always slightly adjust the weights (by arbitrarily small amounts) to break all the possible ties. Specifically, it can be shown that for any vector of positive numbers w and $\epsilon > 0$, there exists a positive weight vector w' with $\|w - w'\| < \epsilon$ such that for any $\mathcal{F}$, (D, θ), and $\alpha, \alpha' \in \mathcal{A}(\mathcal{F}, D)$, we have the following: (1) if α and α' induce different schedule profiles, then $U(w',\theta,\alpha) \neq U(w',\theta,\alpha')$, and (2) if $U(w,\theta,\alpha) > U(w,\theta,\alpha')$, then $U(w',\theta,\alpha) > U(w',\theta,\alpha')$.

[26]Such priority mechanism is a weighted utilitarian mechanism with properly chosen weights such that, for each k, the weight for the k^{th} agent is sufficiently larger than the sum of the weights of the agents with lower priorities.

programs, another application of human sourced allocation in real life.[27]

The idea behind priority mechanisms is similar to that behind FCFS used in practical replacement donor programs, as discussed in the previous section, with one important difference: a patient can only exchange with the blood bank in FCFS, while, in a priority mechanism, she can potentially exchange with patients who have either lower or higher priority than her, in addition to the blood bank. Moreover, when there is no simple and linear priority order for patients, the general weighted utilitarian mechanisms can achieve more egalitarian allocations than priority mechanisms, by assigning appropriate weights to patients.

Our cardinal framework and the concavity of the blood valuation function can also help achieve fair allocations in a more nuanced way: when the valuation function ρ is strictly concave,[28] a weighted utilitarian mechanism tends to assign similar amounts of blood to similar patients. For instance, if there are two identical patients with the same weights and one of them receives significantly less blood than the other, transferring one unit of blood from the latter to the former would increase the weighted sum of their utilities. However, the feasibility of such transfer hinges on the feasibility of schedules.[29] Therefore, we next introduce a discrete convexity condition on feasible schedule menus, which will also play a central role in the incentive compatibility of weighted utilitarian mechanisms.

Generally, a set $S \subseteq \mathbb{Z}^2_+$ is **L-convex** (where L stands for lattice) if for every $x, y \in S$, we have

$$\left\lfloor \frac{x+y}{2} \right\rfloor, \left\lceil \frac{x+y}{2} \right\rceil \in S.$$

L-convexity is one of the two widely used generalizations of convexity to discrete domains.[30]

Definition 1 *A feasible schedule menu profile $\mathcal{F}$ satisfies **L-convexity** if the feasible schedule set $\mathcal{F}_i(D_i)$ is L-convex for every patient i and her donor set D_i.*

Figure 1 provides a geometric illustration with four examples of L-convex feasi-

[27] For example, New England Program for Kidney Exchange used a priority-based allocation scheme while Alliance for Paired Donation has used a maximality-based allocation scheme (see Sönmez and Ünver, 2017).

[28] That is, for any $0 < r < \max\{\overline{n}_i : i \in I\}$, $\quad \rho(r+1) - \rho(r) < \rho(r) - \rho(r-1)$.

[29] In general, a transfer of supply may also be required. For instance, under one-for-one exchange, giving a patient one additional unit of blood requires her to supply one more unit.

[30] The other one is *M-convexity*, where M stands for matroid. See Murota (2013) for a general treatment of discrete convexity notions and discrete convex analysis.

ble schedule sets. A special case that satisfies L-convexity is the classical one-for-one exchange, as depicted in the last graph of Figure 1.

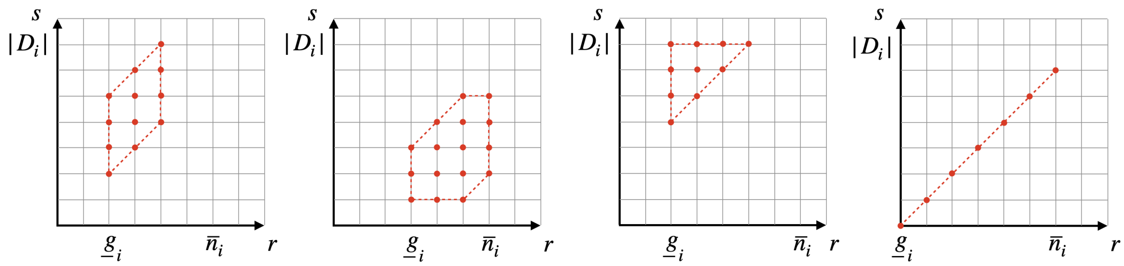

Figure 1: Illustration of L-convexity. The feasible schedule set $\mathcal{F}_i(D_i)$ is the integral points of a convex polygon with integral corners and at most six edges of slopes 1, 0, or ∞.

The next proposition shows that, under strict concavity of the blood valuation function and L-convexity of feasible schedule menus, we can construct a weighted utilitarian mechanism that satisfies a discrete and approximate version of "equal (medical) treatment of equals", i.e., the difference in two identical patients' transfusions can only be attributed to the indivisibility of blood packs.

Proposition 2 *Assume ρ is strictly concave. There exists a weighted utilitarian mechanism f such that for any L-convex feasible schedule menu profile $\mathcal{F}$, problem (D, θ) and two patients i and j, if $\beta_i = \beta_j$, $\overline{n}_i = \overline{n}_j$, $\left|\{d \in D_i : \beta_d = X\}\right| = \left|\{d \in D_j : \beta_d = X\}\right|$ for each $X \in \mathcal{B}$, $\mathcal{F}_i(D_i) = \mathcal{F}_j(D_j)$, and $\theta_i = \theta_j$, then*

$$\left| f_r\Big(\mathcal{F}, D, \theta\Big)(i) - f_r\Big(\mathcal{F}, D, \theta\Big)(j) \right| \leq 1.$$

In the proof, we show such weighted utilitarian mechanism is constructed by assigning sufficiently similar weights to patients.

4.2 Donor Monotonicity

We start to analyze the incentives under weighted utilitarian mechanisms by considering donor monotonicity. This will serve as the foundation for the incentive compatibility results later in Section 4.3. Moreover, as discussed before, donor monotonicity is also a plausible incentive requirement in the common real-life scenario where it is known that all patients have lexicographic preferences and are of the same type ϑ^L.

One may think that donor monotonicity is a straightforward requirement for a mechanism to satisfy given that it should be easy to construct mechanisms that give weakly more blood to patients when they bring forward more donors. However, this is not true under Pareto efficiency. Donor monotonicity is an elusive property that is not satisfied by even the most basic Pareto efficient mechanisms under general feasible schedule menus, such as priority mechanisms, a special case of weighted utilitarian mechanisms. As illustrated in detail below in Example 2, the difficulty stems from the fact that a patient can potentially under-report her donor set to alter the possible exchanges and thus the whole set of feasible allocations, from which the mechanism chooses its outcome. This is in contrast to an inefficient FCFS mechanism, which is more similar to an allocation mechanism rather than an exchange mechanism. Under FCFS, a patient's report of donors has no effect on the assignments of those before her, making donor monotonicity straightforward for the mechanism to satisfy. Thus, more direct exchanges among patients to enhance efficiency may, in general, cause a mechanism to violate donor monotonicity.

We will impose three regularity conditions on the feasible schedule menus that are satisfied by common real-life policies such as one-for-one exchange, to ensure donor monotonicity. All these conditions have natural interpretations.

L-convexity. The first one is L-convexity, introduced before for allocative fairness. The key role of this property in establishing donor monotonicity is that it prevents hidden complementarities between units of blood received, by ruling out *holes* in feasible schedule sets. In the following example we demonstrate such complementarities create incentives for misreporting donors under a priority mechanism. The example also shows that the concavity of blood valuation function plays a similar role as the L-convexity of feasible schedule sets does in incentives.

Example 2 (Violation of L-convexity) *Suppose that the set of patients is* $I = \{1, 2\}$ *and the set of relevant blood types is* $\mathcal{B} = \{O, A, B, AB\}$. *Each patient's blood type, maximum need and donor set are given as follows.*

- $\beta_1 = O$, $\overline{n}_1 = 3$, *and Patient 1 has three type B donors.*
- $\beta_2 = A$, $\overline{n}_2 = 2$, *and Patient 2 has one type O donor and two type B donors.*

The blood bank has two units of type O blood and one unit of type A blood in its inventory. Assume ABO-cellular compatible transfusion, where $\mathcal{C}(A) = \{A, O\}$ *and* $\mathcal{C}(O) = \{O\}$.

The exchange rate is one-for-one for both patients, except that $(2, 2)$ *is never a feasible schedule for Patient 1. Therefore, when she reports three donors, her feasible schedule set* $\{(0, 0), (1, 1), (3, 3)\}$ *is not L-convex.*

Consider a priority mechanism with the order $1 - 2 - b$, *and any type profile* θ. *When both patients truthfully report their donors, Patient 1 receives three units of type O blood, all of her donors donate, Patient 2 receives one unit of type A blood and her type O donor donates.*

However, if Patient 2 conceals her type O donor, Patient 1 can only receive one unit of type O blood due to the hole in her feasible schedule set. Then Patient 2 receives more compatible blood, i.e., one unit of type A blood and one unit of type O blood, and her utility is increased. ▪

Remark 1 *Decreasing marginal utility is not only a plausible assumption on the consumption of blood products, but also crucial for the incentives to report donors under general weighted utilitarian mechanisms, since a strictly convex valuation function ρ can also create complementarities as a non-L-convex feasible schedule set does. To see this, suppose the exchange rate is one-for-one for both patients in Example 2, $\theta_1 = \theta_2$, and $\rho(3) - \rho(2)$ is sufficiently larger than $\rho(2) - \rho(1)$, i.e., there is a similar complementarity between the second and the third units of compatible blood for Patient 1. Then Patient 1 receives three units of O blood under some weighted utilitarian mechanism where w_2 is slightly larger than w_1, when both patients truthfully report. However, if Patient 2 conceals her O donor, she receives one more unit of blood due to her higher weight, and her utility is increased.*

Feasibility of Positive Price. The second property generalizes the idea that each unit of blood has a positive "price." It says that when a patient can potentially receive more (or less) blood, there is a feasible schedule in which she receives this amount by also supplying more (or less) blood. Note that the patient does not necessarily supply more when she receives more: this property only requires that such a schedule is feasible.

Definition 2 *A feasible schedule menu profile $\mathcal{F}$ satisfies* ***feasibility of positive price*** *if for every patient i and her donor set D_i, the following holds:*

- *if $(r, s), (r', s') \in \mathcal{F}_i(D_i)$, $r' > r$ and $s < |D_i|$, then there exists $s'' > s$ such that $(r', s'') \in \mathcal{F}_i(D_i)$; and*
- *if $(r, s), (r', s') \in \mathcal{F}_i(D_i)$, $r' < r$ and $s > 0$, then there exists $s'' < s$ such that $(r', s'') \in \mathcal{F}_i(D_i)$.*

Geometrically, this property rules out a "flat top" or a "flat bottom" in the shape of a feasible schedule set except for the two extreme cases of $s = |D_i|$ and $s = 0$. The one-for-one exchange rate policy satisfies feasibility of positive price: each additional unit received costs exactly one unit supplied.

Both L-convexity and feasibility of positive price are novel in the market design literature, and the two properties are logically independent.[31] For example, the two-

[31] At a very high level, the role of the first two properties and concave valuation function ρ is similar to the role *same-side substitutes* and *cross-side complements* play in matching with trading networks for hav-

for-one exchange rate policy satisfies feasibility of positive price but not L-convexity;[32] the second feasible schedule set in Figure 1 violates feasibility of positive price as it has a "flat top" at $s = 5 < |D_i|$ and a "flat bottom" at $s = 1 > 0$, while it is L-convex. The other sets in this figure satisfy feasibility of positive price, although the third one has a "flat top." This is because it occurs at the maximum possible supply $s = |D_i|$.

As illustrated in Example A.3 in Appendix C, a flat top in a patient's feasible schedule set imposes a constant upper bound on her supply and thus can restrict her opportunity of exchanging donors with other patients, which may incentivize the patient to under-report her donor set, in order to reduce her current supply, or to reach a different feasible schedule set where she can be involved in more exchanges. On the other hand, a flat bottom can give the patient an advantage when she is competing for blood from the bank with other patients, since allocating more blood to her may not require more supply, leading to a larger increase in utility. Therefore, the patient has an incentive to hide donors if such feasible schedule set with a flat bottom can only be reached when she has less donors (see Example A.4 in Appendix C).

Non-diminishing Favorability in Donors. Until now both properties are imposed on the feasible schedule set for a given donor set. One can imagine that if the feasible schedule set becomes very unfavorable for the patient when more donors are presented to the blood bank, the patient would naturally hide these extra donors. Thus, we formalize this intuition through the last property.

Definition 3 *A feasible schedule menu profile $\mathcal{F}$ satisfies* ***non-diminishing favorability in donors*** *if for every patient i and pair of her donor sets D_i, D'_i such that $D'_i \subseteq D_i$, we have:*

- *if $(r, s) \in \mathcal{F}_i(D'_i)$ and $r \geq \underline{g}_i$, then there exists $s' \leq s$ such that $(r, s') \in \mathcal{F}_i(D_i)$; and*
- *if $(r, s) \in \mathcal{F}_i(D_i)$, $s \leq |D'_i|$ and $(r, s') \in \mathcal{F}_i(D'_i)$, then there exists $s'' \geq s$ such that $(r, s'') \in \mathcal{F}_i(D'_i)$.*

ing stable network structures (see Ostrovsky, 2008). Although these properties are all about preferences, while our first two properties are about feasibility, one can technically endogenize feasibility through preferences via acceptability of feasible allocations and unacceptability of infeasible allocations. Thus, these two properties regulate the feasible schedule sets so that received units (and supplied units) for a patient *do not* induce same-side complementarities among themselves and received-supplied units *do* induce cross-side complementarities. Observe that this analogy is extremely rough, as our tools are novel and have no exact precedence in matching or mechanism design theory. In our domain, each unit is a perfect substitute for another on the received side or the supplied side, unlike the trading networks. Moreover, our innovation is about incentives, while the literature on trading networks focuses on the existence of stable allocations.

[32] See Section 5.1 for a detailed discussion of this policy.

That is, when the donor set expands from D_i' to D_i, $\mathcal{F}_i(D_i)$ is weakly more favorable than $\mathcal{F}_i(D_i')$: (i) for any schedule in $\mathcal{F}_i(D_i')$ such that the amount received is at least the minimum guarantee, there is a schedule in $\mathcal{F}_i(D_i)$ where the patient receives the same amount by supplying weakly less blood; (ii) for any schedule in $\mathcal{F}_i(D_i)$ such that the amount supplied does not exceed the number of donors in D_i', whenever there is a schedule in $\mathcal{F}_i(D_i')$ where she receives the same amount of blood, there is a schedule in $\mathcal{F}_i(D_i')$ where she receives this amount by supplying weakly more blood.

Non-diminishing favorability in donors manifests itself geometrically as $\mathcal{F}_i(D_i)$ being an expansion of $\mathcal{F}_i(D_i')$ in the direction of receiving more blood, and/or a downward shift of $\mathcal{F}_i(D_i')$. The one-for-one exchange rate policy satisfies this property as well, since the feasible schedule set simply expands when the number of donors increases. In Figures 2 and 3, we give two examples involving endogenously determined exchange rates to further illustrate the implications of non-diminishing favorability in donors in conjunction with L-convexity and feasibility of positive price.

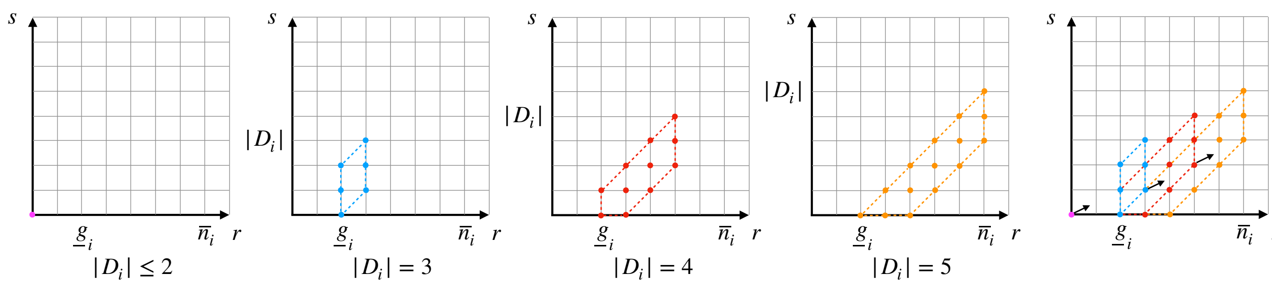

Figure 2: An illustration of a feasible schedule menu $\mathcal{F}_i$ satisfying L-convexity, feasibility of positive price, and non-diminishing favorability in donors. This particular policy relies only on the number of donors brought forward but not other specifics of the donor set. The first four graphs illustrate $\mathcal{F}_i(D_i)$ for $|D_i| = 0, \ldots, 5$, while the last graph shows how the feasible schedule set changes as the number of donors increases.

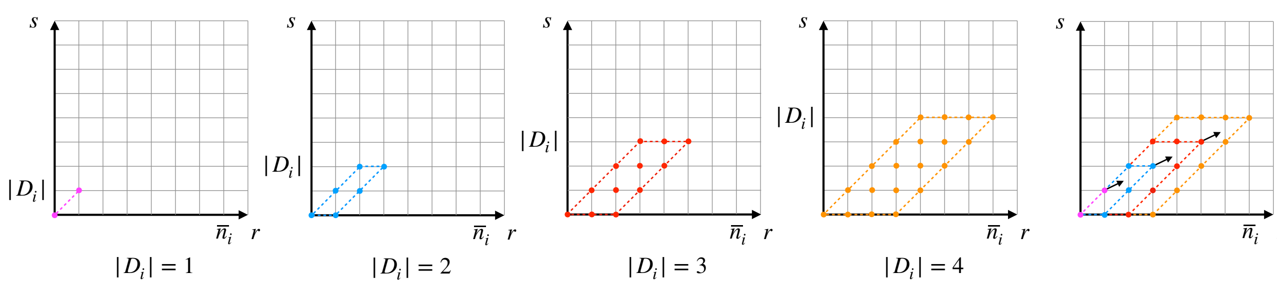

Figure 3: An illustration of a feasible schedule menu $\mathcal{F}_i$ satisfying L-convexity, feasibility of positive price, and non-diminishing favorability in donors. The first four graphs illustrate $\mathcal{F}_i(D_i)$ for $|D_i| = 1, \ldots, 4$. The last graph shows how the feasible schedule set changes as the number of donors increases.

The main result of this section is as follows:

Theorem 2 *If a feasible schedule menu profile $\mathcal{F}$ satisfies L-convexity, feasibility of positive price, and non-diminishing favorability in donors, then every weighted utilitarian mechanism is donor monotonic under $\mathcal{F}$.*[33]

A sketch of the proof of this theorem is given in Appendix B, before it is formally proved. Each of the three properties on feasible schedule menus is indispensable, which is established for the first two properties by Example 2, and Examples A.3 and A.4 in Appendix C. It is straightforward to show that non-diminishing favorability in donors cannot be dropped either. For example, for every patient i, $\mathcal{F}_i(\emptyset) = \{(1,0)\}$, and the blood bank has enough inventory to satisfy her minimum guarantee of one unit; if $D_i \neq \emptyset$, then $\mathcal{F}_i(D_i)$ shrinks to $\{(0,0)\}$. Such feasible schedule menu profile $\mathcal{F}$ violates non-diminishing favorability in donors, but satisfies L-convexity and feasibility of positive price. Under $\mathcal{F}$, no mechanism is donor monotonic.

It is also worth mentioning that even if we only want to ensure the donor monotonicity of priority mechanisms, each of the properties, except the absence of flat bottom in feasibility of positive price, is indispensable, as shown by the examples mentioned above.

In the end, as stressed before, the classical one-for-one exchange satisfies all the properties. Another interesting special case that satisfies these properties is when the feasible schedule menus are given by the universal consumption grids, i.e., for every patient i and her donor set D_i, $\mathcal{F}_i(D_i) = \mathbb{S}_i(D_i)$. Therefore, if we do not impose any feasibility restriction on schedules, the weighted utilitarian mechanisms are donor monotonic.

4.3 Incentive Compatibility

We first consider the plausible real-life scenario where it is known that every patient has lexicographic preferences, i.e., $\Theta = \{\vartheta^L\}$. Under a weighted utilitarian mechanism with feasible schedule menus that satisfy the properties in Theorem 2, a patient may be able to conceal some donors so that she receives the same amount of blood by supplying less. To prevent such manipulations, we need a stronger restriction on how feasible schedule sets can change when a patient reports different donor sets.

Definition 4 *A feasible schedule menu profile $\mathcal{F}$ satisfies* ***strong non-diminishing favorability in donors*** *if for every patient i and pair of donor sets D_i, D_i' such that $D_i' \subseteq D_i$, we*

[33] In fact, we prove a stronger version of this theorem: If the three properties are imposed on a patient i and only L-convexity is imposed on the other patients, then patient i cannot receive more blood by under-reporting her donor set.

have:

- *if* $(r,s) \in \mathcal{F}_i(D_i')$ *and* $r \geq \underline{g}_i$*, then there exists* s' *such that* $(r,s') \in \mathcal{F}_i(D_i)$*; and*
- *if* $(r,s) \in \mathcal{F}_i(D_i)$ *and* $(r,s') \in \mathcal{F}_i(D_i')$*, then* $s \leq s'$*.*

It is straightforward to see that strong non-diminishing favorability in donors implies non-diminishing favorability in donors. Therefore, under L-convexity, feasibility of positive price, and strong non-diminishing favorability in donors, the weighted utilitarian mechanisms are donor monotonic by Theorem 2. Moreover, in this case, if a patient reports a subset of her donors and still receives the same amount of blood, then the second condition in the above definition implies that her donors do not donate less blood. Hence, we have the following result.

Theorem 3 *Assume* $\Theta = \{\vartheta^L\}$*. If a feasible schedule menu profile* $\mathcal{F}$ *satisfies L-convexity, feasibility of positive price, and strong non-diminishing favorability in donors, then every weighted utilitarian mechanism is incentive compatible under* $\mathcal{F}$*.*

In Figure 4, we give an example of a feasible schedule menu that satisfies the above properties and involves endogenous exchange rates. When the feasible sched-

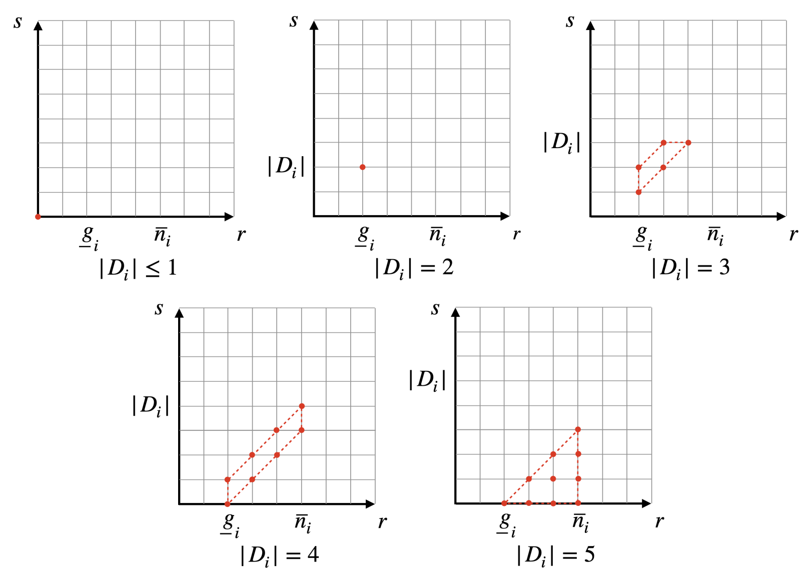

Figure 4: An illustration of a feasible schedule menu satisfying L-convexity, feasibility of positive price, and strong non-diminishing favorability in donors. The patient i has a maximum need of $\overline{n}_i = 5$, and $\delta = 5$.

ule menus feature *exogenous* exchange rates, i.e., for any patient i and donor set D_i there do not exist $(r,s) \in \mathcal{F}_i(D_i)$ and $(r,s') \in \mathcal{F}_i(D_i)$ such that $s \neq s'$, the strong and regular versions of non-diminishing favorability in donors are equivalent, and the above properties lead to an extension of one-for-one exchange.

Remark 2 *Suppose that the exchange rates are exogenous, and every patient i has the ability to bring forward some donors. Then L-convexity, feasibility of positive price, and non-diminishing favorability in donors pin down a particular class of feasible schedule menus for every patient i. Given a donor set D_i, if $\mathcal{F}_i(D_i) \neq \{(0,0)\}$, then there exist bounds $\underline{s}_i(D_i)$ and $\overline{r}_i(D_i)$ such that*

$$\mathcal{F}_i(D_i) = \Big\{(r,s) \in \mathbb{S}_i(D_i) \,:\, s - \underline{s}_i(D_i) = r - \underline{g}_i,\ \underline{g}_i \leq r \leq \overline{r}_i(D_i)\Big\}.$$

That is, she has to supply $\underline{s}_i(D_i)$ units to receive her minimum guarantee, and beyond this schedule, she has to supply one additional unit for each additional unit received, with the maximum amount received being restricted by $\overline{r}_i(D_i)$. Moreover, $\underline{s}_i(D_i)$ weakly decreases and $\overline{r}_i(D_i)$ weakly increases as her donor set expands. We refer to such feasible schedule menus as ***two-part tariffs****, which include the one-for-one exchange rate policy and the minimum guarantee Xi'an policy in Section 3 as special cases. Figure 5 provides a geometric illustration of a simple two-part tariff.*

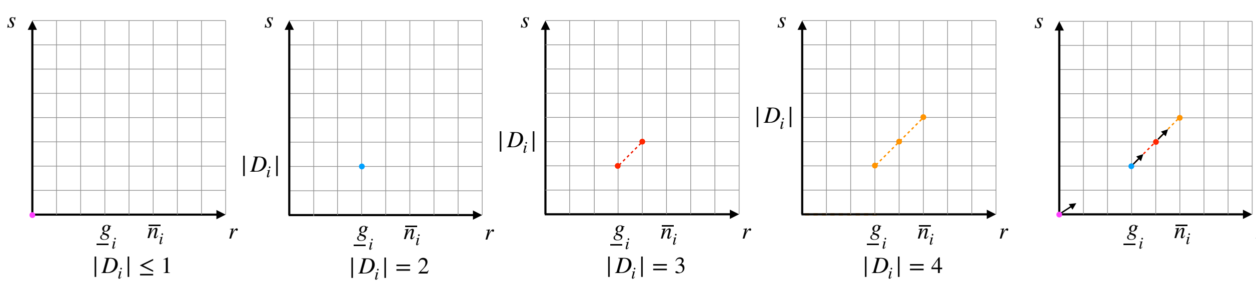

Figure 5: An illustration of a two-part tariff. The patient i has to supply two units to receive her minimum guarantee of $\underline{g}_i = 3$ units. The first four graphs illustrate $\mathcal{F}_i(D_i)$ for $|D_i| \in \{0, \ldots, 4\}$, while the last graph shows how the feasible schedule set changes as the number of donors increases.

In our model, the schedule $(0,0)$ can be interpreted as the outside option for each patient. Ensuring voluntary participation in the replacement donor program, or *individual rationality*, is straightforward under lexicographic preferences: we only need to make sure that any schedule $(0,s)$ with $s > 0$ is not in a feasible schedule set. We next turn to the case that patients have general quasi-linear utilities. Note that, as MRS is assumed to be greater than 1, for any patient i and θ_i, we have $u_i\big[(r,s),\theta_i\big] \geq u_i\big[(0,0),\theta_i\big]$ whenever $r \geq s$. We then directly impose this condition on feasible schedules to guarantee individual rationality:

Definition 5 *A feasible schedule menu profile $\mathcal{F}$ satisfies **individual rationality** if for every patient i and her donor set D_i, we have $r \geq s$ for any $(r, s) \in \mathcal{F}_i(D_i)$.*

In practice, it is often a common norm that a patient shall not supply more blood than what she receives. Moreover, the examples of feasible schedule menus we gave before in Figures 2, 3, 4 and 5 all satisfy individual rationality.

This property is also important for the incentives to truthfully report donors under general quasi-linear utilities. Built upon Theorems 2 and 3, we show that if individual rationality is further required, then patients cannot manipulate any weighted utilitarian mechanism via under-reporting donors:

Theorem 4 *If a feasible schedule menu profile $\mathcal{F}$ satisfies L-convexity, feasibility of positive price, strong non-diminishing favorability in donors and individual rationality, then every weighted utilitarian mechanism is incentive compatible with respect to donors under $\mathcal{F}$.*

Under a general weighted utilitarian mechanism defined through an objective function U, a patient can potentially misreport her type to alter the function U, leading to a better outcome for her. To prevent type manipulations, we are restricted to priority mechanisms, where the sequential utility maximization ensures truthful revelation of types. Then, in light of Theorem 4, under a priority mechanism any patient cannot be better-off by misreporting her type and donor set simultaneously, when the same restrictions are imposed on the feasible schedule menus.

Theorem 5 *If a feasible schedule menu profile $\mathcal{F}$ satisfies L-convexity, feasibility of positive price, strong non-diminishing favorability in donors and individual rationality, then all priority mechanisms, including the maximal mechanisms, are incentive compatible under $\mathcal{F}$.*

As for Theorem 2, each restriction on feasible schedule menus is in general indispensable (and strong non-diminishing favorability in donors cannot be weakened to non-diminishing favorability in donors) for Theorems 3, 4 and 5, with the exception that (only) the absence of flat bottom in feasibility of positive price is not needed for priority mechanisms. For L-convexity and feasibility of positive price, this is still established by the previously discussed examples, i.e., Examples 2, A.3 and A.4. In Appendix C, we give two additional examples for strong non-diminishing favorability in donors and individual rationality.

4.4 Beyond Quasi-linear Utilities

Suppose that each patient i has the following utility function

$$u_i[r,s] = \rho_i(r) - \sigma_i(s)$$

with ρ_i being concave as before and —now additionally— σ_i being convex. Then each problem is represented by a pair (D, u), where $u = (u_i)_{i \in I}$ is a profile of utility functions reported by the patients. The only limitation caused by this general formulation to our current theory is that Theorem 2 no longer holds for all weighted utilitarian mechanisms, while it still holds for priority mechanisms. When L-convexity, feasibility of positive price, and the weaker requirement of non-diminishing favorability in donors (instead of strong non-diminishing favorability in donors) are imposed, we can design more flexible feasible schedule menus such as those depicted in Figures 2 and 3. In this case, if some patient has a strictly convex cost function, then under some particular weighted utilitarian mechanisms, a large and increasing marginal cost of supply can restrict her supply in a similar way as a flat top in feasible schedules does, creating incentives for her to conceal donors. However, such chances of manipulations are eliminated, when strong non-diminishing favorability in donors is imposed. Therefore, Theorems 3, 4 and 5 are intact.

On the other hand, under the general utility functions we can extend Proposition 2 to supplies. It can be shown that, under L-convex feasible schedule menus, when the cost functions are strictly convex (or the blood valuation functions are strictly concave), there is a weighted utilitarian mechanism under which two identical patients' supplies (or receipts) differ by at most one unit.

Finally, it is worth mentioning that strictly concave cost functions can lead to incentives to under-report donors under weighted utilitarian mechanisms, even if the exchange rate is one-for-one for all patients, which satisfy all the properties on feasible schedule menus studied in this paper. As in Remark 1, this can also be illustrated using the problem in Example 2. Strictly decreasing marginal cost of supply creates complementarities between donors who do not donate, which are in the same vein as the complementarities between received units created by strictly increasing marginal utility of blood.

5 Policy Design for Blood Allocation

In this section, we first provide concrete designs of feasible schedule menus to illustrate how certain practical challenges in blood allocation can be addressed using our framework. Then, we assess the performance of the FCFS mechanism in practice and our mechanisms in simulated blood markets.

5.1 The Design of Feasible Schedule Menus

The feasible schedule menus can be designed to impose exchange rate policies and achieve more nuanced objectives regarding fairness, efficiency, and incentives.

Equitable Blood Allocation. An important flexibility of our proposal is that the exchange rates can be determined endogenously. This is especially useful when some patients may potentially have few or no paired donor candidates. We can design feasible schedule menus that accommodate for patients with and without donors as equitably as possible.

An example is provided in Figure 6. In this example, the patient i always receives the minimum guarantee of $\underline{g}_i = 1$ unit of blood, and she can receive up to her maximum need of $\overline{n}_i = 3$ units, even if she does not have any donor. The feasible schedule menu satisfies L-convexity, feasibility of positive price, and non-diminishing favorability in donors. Therefore, as she brings forward more donors, her chances of receiving more units of blood beyond $\underline{g}_i = 1$ under any weighted utilitarian mechanism weakly increase by donor monotonicity.

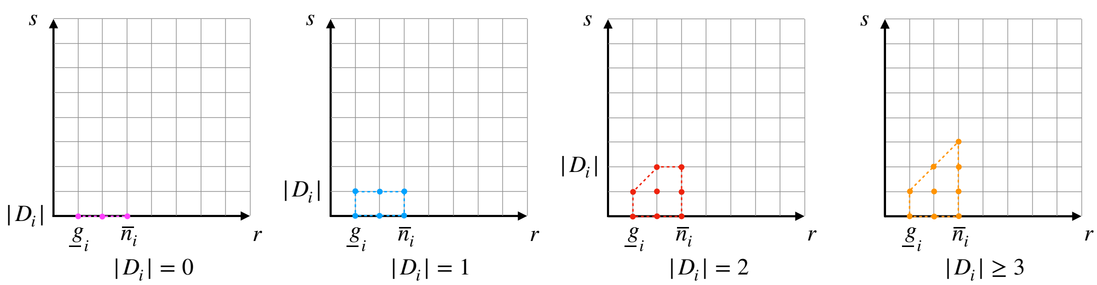

Figure 6: An equitable feasible schedule menu.

The proposal is also compatible with some existing equitable replacement donor policies. For example, in leading Chinese hospitals, patients who do not live in the city where the hospital is located are often not required to supply as many donors as local patients. The rationale behind this policy is that relatives of patients from other cities are usually not readily available to donate on behalf of the patients. Similarly, in Cambodia, replacement donor requirements are waived for a patient if she has no next-of-kin (Davies, 2004). Thus, the patient-specific nature of the feasible schedule menus can accommodate such fairness considerations as well.

A flexible policy with endogenous exchange rates can also help address some ethical concerns about replacement donor programs and enhance the overall efficiency of the system. Under a fixed exchange rate policy that is commonly observed around the world, a patient without enough donors may be forced to recruit illegal professional donors, leading to the issue of black markets for blood. On the other hand, if a fixed exchange rate, such as the one-for-one rate, is strictly enforced, then a patient without any donor cannot receive any blood even if the blood bank does have enough inventory for her, leading to obvious welfare loss.

In general, given the fairness, efficiency, and ethical issues of a fixed exchange rate policy, although rules may be bent in some way in practice (e.g., the ad hoc "forgiving policy" in Section 3), our design formalizes flexible and endogenous exchange rates, bringing rigor and transparency to the allocation system.

Blood Type Targeting. Blood banks occasionally fall short in blood components of certain blood types while others are aplenty. For example, the blood type distribution varies across different regions of the world, but AB Rh D$-$ is almost always the rarest type and components of this type are likely in short supply. On the other hand, although ABO-identical transfusion is required for certain blood components in some countries, this compatibility requirement is often relaxed in other cases. For instance, under ABO-cellular compatible red blood cell transfusion, blood type O Rh D$-$ is the universal donor, and under ABO-plasma compatible platelet transfusion, blood type AB is the universal donor. Therefore, it may be important for a blood bank to target the donation of certain types of blood. Since a patient's feasible schedule set depends on the observable characteristics of her donor set, this goal can be achieved by incentivizing the provision of donors of desired blood types through feasible schedule menu policies. In Figure 7, we give an example of a feasible schedule menu design that favors bringing forward more type O Rh D$-$ donors. In this case, a patient is able to receive the same amount of blood by supplying less if she has more donors of this type.

Approximating Fixed Exchange Beyond One-for-One. As mentioned before, Mexico and some countries in Africa (e.g., Congo and Cameroon), for various reasons, use two-for-one exchange rate: two units of blood need to be supplied for each unit received. The resulted feasible schedule menu violates L-convexity, and it can be shown that even a priority mechanism may not be donor monotonic under such exogenous exchange rate policy.

However, we can generate endogenous exchange rate policies that closely approximate the two-for-one exchange rate, such that under these policies the weighted util-

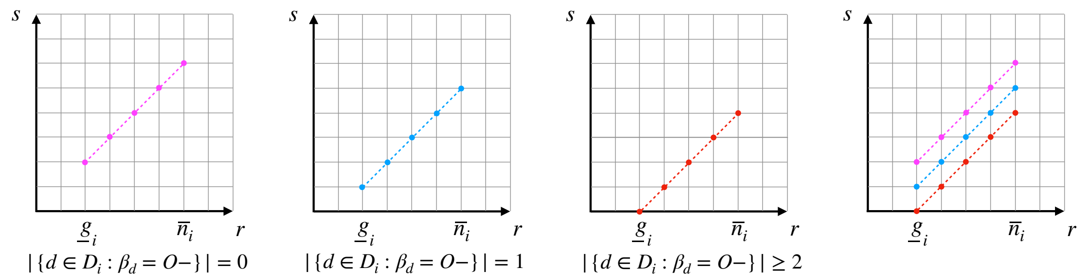

Figure 7: A feasible schedule menu that provides strong incentives to reveal type $O-$ donors. In each case, the feasible schedule set of patient i includes every schedule on the graph in which the amount supplied does not exceed her number of donors, and is $\{(0,0)\}$ if there is no such schedule. This feasible schedule menu satisfies L-convexity, feasibility of positive price, strong non-diminishing favorability in donors, and individual rationality. Assume that these properties are also satisfied for the other patients. Then, under any weighted utilitarian mechanism, if patient i has one or two type $O-$ donors, concealing a type $O-$ donor leads to a strictly lower utility.

itarian mechanisms are at least donor monotonic. See Figure 8 for an example. Such an approach can also be applied to approximate other exogenous exchange rates.

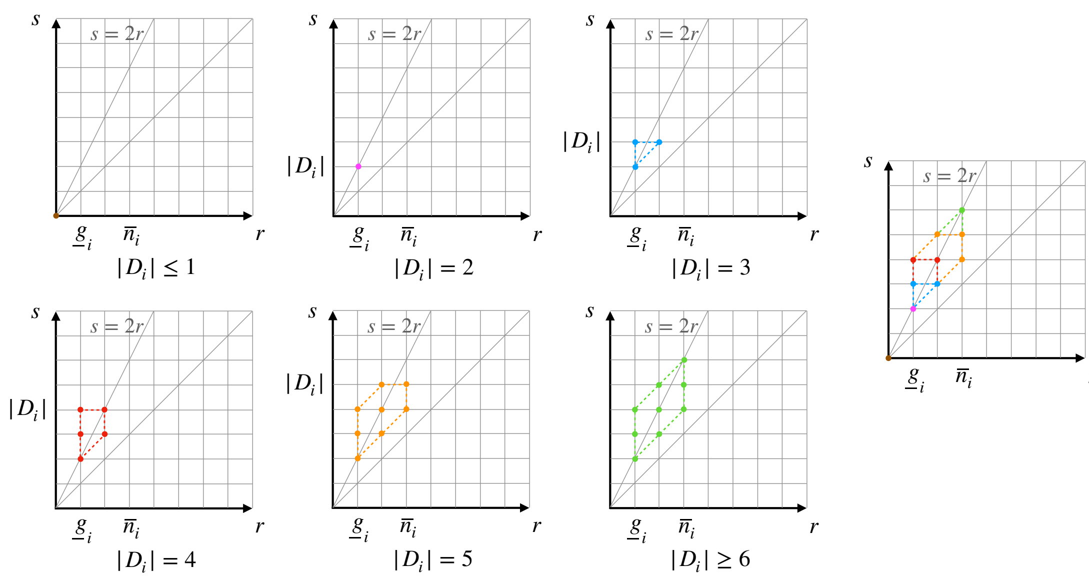

Figure 8: A feasible schedule menu designed to approximate the two-for-one exchange rate. The patient i is required to supply at least two units to receive her minimum guarantee. For any $s \in \{2, \ldots, 6\}$ such that $s \leq |D_i|$, $(\frac{1}{2}s, s)$ should be a feasible schedule when s is an even number, and we consider $(\lfloor \frac{s}{2} \rfloor, s)$ and $(\lceil \frac{s}{2} \rceil, s)$ feasible schedules when s is an odd number. Then the above graphs illustrate the feasible schedule menu that assigns the smallest set of schedules that include these feasible schedules in each case so that L-convexity, feasibility of positive price, and non-diminishing favorability in donors are satisfied for patient i.

5.2 Simulations on Mechanisms

As we have discussed earlier in Section 3, most blood banks use the FCFS mechanisms that do not require intricate implementation other than specifying the exchange rates in their policies. While FCFS is incentive compatible under most commonly used exchange rate policies, it is not Pareto efficient. Thus, one important question that connects our approach to practice is whether and under what conditions FCFS is approximately efficient, and when we have to use weighted utilitarian mechanisms to implement efficient outcomes.

These conditions include policy objectives of the blood bank, running levels of inventories, population density, and replacement donor recruitment ease, and countries that employ replacement donor programs are very diverse in these aspects. For example, rural Sub-Saharan African blood banks mostly operate with small inventories leading to wide-spread shortages (Tagny, 2012), while shortages are not regularly seen in some countries that also employ blood replacement programs, such as Turkey.[34]

For simplicity, in the rest of this section we compare different mechanisms only under the most common one-for-one exchange rate. Given an FCFS mechanism, the corresponding direct mechanism among the weighted utilitarian mechanisms is the priority mechanism with respect to the same order of patients, where the bank is ranked in the last place. The two mechanisms are in general not Pareto-comparable, but the priority mechanism Pareto dominates FCFS under zero-inventory. Thus, we expect that when the inventory level is low, the priority mechanism will induce substantially higher utility for most patients than FCFS.

On the other hand, if the blood bank has a sufficiently high inventory, then every patient can be fully satisfied under FCFS by simply trading with the blood bank. Thus, as the inventory level goes up, it is expected that FCFS becomes closer to the priority mechanism.

In addition, we consider the incentive compatible two-stage mechanism that improves upon FCFS (for the patients), introduced in Section 3, where in the first stage we run FCFS, and in the second stage we run the priority mechanism under the updated feasible schedule menus that are determined by the first stage.[35] We refer to this mechanism as *FCFS dominating mechanism*. It is equivalent to the priority mechanism under zero-inventory, as in this case all patients can be improved in the second stage. However, the significance of the improvement over FCFS is elusive under other

[34]Based on communications reported in Footnote 18.

[35]In the proof of Theorem 1 we used a slightly different priority mechanism in the second stage, where the bank has the first priority, to improve the utilities of both patients and the bank. We focus on the transfusions to patients in the simulations and thus give the bank the lowest priority. Nevertheless, the incentive compatibility of the current two-stage mechanism still follows from the same arguments in the proof.

inventory levels.

Below, we conduct simulations to give more precise evaluations of the three mechanisms.

Simulation Setup. The blood types of each patient, replacement donor, and unit in the blood bank inventory are drawn randomly and independently using the Indian blood-type distribution in Table 1.

$O+$	$A+$	$B+$	$AB+$	$O-$	$A-$	$B-$	$AB-$
27.85%	20.80%	38.14%	8.93%	1.43%	0.57%	1.79%	0.49%

Table 1: Blood-type frequencies in India (RhesusNegative.net, 2012-2019).

We simulate red blood cell transfusion that follows the commonly practiced ABO-identical and Rh-D-compatible protocol. We consider two patient set sizes, $|I| = 25$ and $|I| = 100$, representing small and large hospital systems and their blood banks, respectively. Each patient i is assumed to need a maximum of $\overline{n}_i$ units, determined by a random and independent draw from the uniform distribution with the support set $\{1, 2, \ldots, 6\}$ (so that the mean maximum need is 3.5, which is the reported number in Collins et al., 2015). Each patient i has a donor set D_i such that $|D_i|$ is determined by a random and independent draw from the uniform distribution with the support set $\{0, 1, \ldots, 6 - x\}$, where $x \in \{-1, 1\}$. That is, there is a slight shortage of replacement donors on average (when $x = 1$, the mean donor number is 2.5, and the mean individual maximum need is 3.5), or the mean donor number is equal to the mean maximum need (when $x = -1$). Finally, the number of units in the inventory is determined uniformly from the support set $\Big\{0, 1, \ldots, \langle 6\iota|I|\rangle\Big\}$, where $\iota \in \{0, 0.02, 0.04, 0.1, 0.2, 0.5, 1\}$, and $\langle x\rangle$ rounds x to the nearest integer. Therefore, ι is the ratio of the highest possible total inventory size to the highest possible total maximum need of the population (i.e., $6|I|$).

This design with 3 mechanisms (FCFS, priority mechanism, and FCFS dominating mechanism), 2 population sizes, 2 highest possible donor numbers, and 7 inventory ratios gives us 84 simulations.

Simulation Results and Policy Lessons. We randomly simulated 1,000 markets and summarize their average results through two figures. Figures 9 and 10 display the total transfusion amounts as percentages of the mean total maximum need $3.5|I|$ for $|I| = 25$ and $|I| = 100$, respectively (also see Table A.2 in Appendix D for standard errors and actual numbers in these figures).

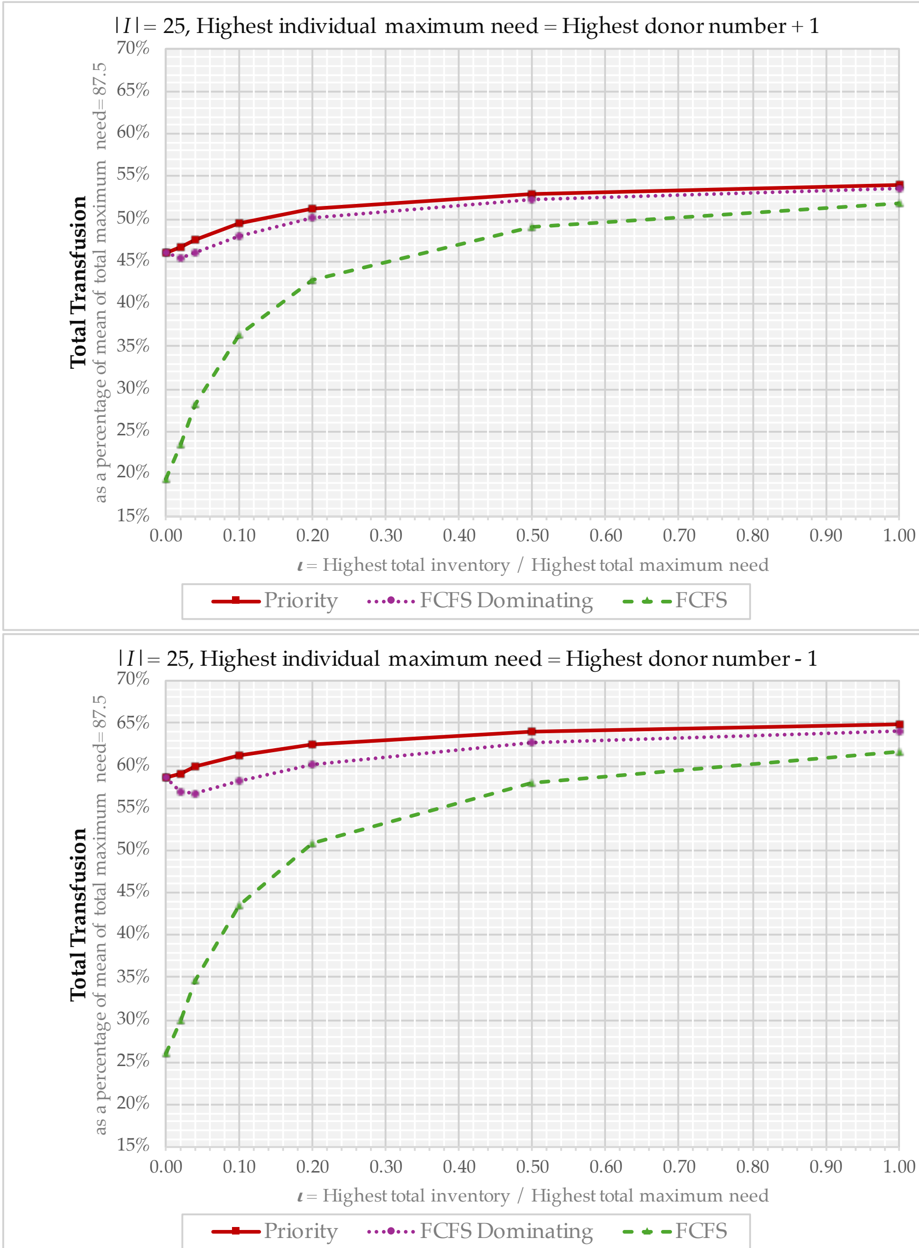

Figure 9: Blood transfused to the patients in the simulations for $|I| = 25$ as a function of ι.

We have five main observations where the first one concerns relatively low inventory, which is prevalent in many regions and one of the key reasons why replacement donor programs are employed in the first place.

Observation 1 *When the inventory rate $\iota \to 0^+$, a centralized implementation of the priority mechanism (or the FCFS dominating mechanism) is required to maximize the gains from a replacement donor program.*

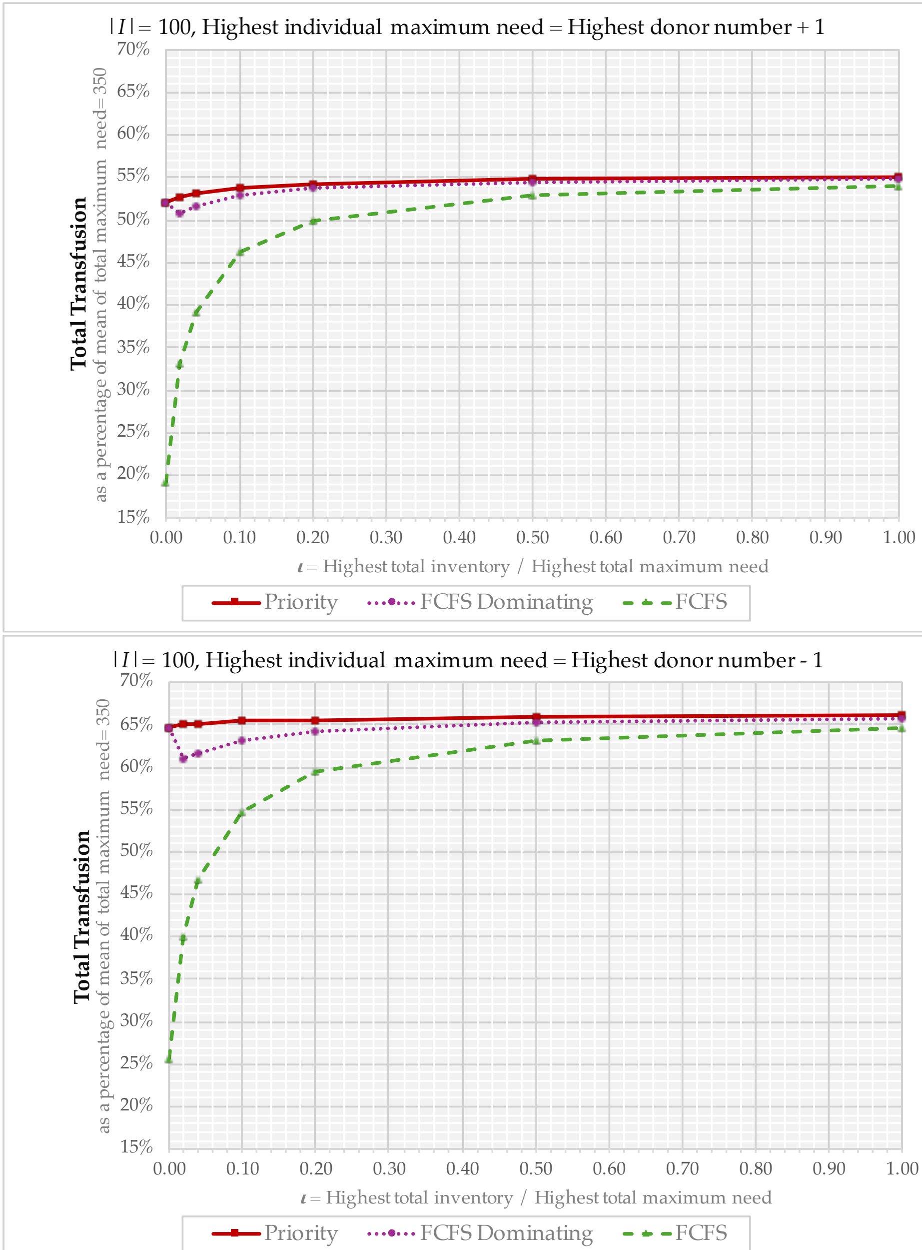

Figure 10: Blood transfused to the patients in the simulations for $|I| = 100$ as a function of ι.

At $\iota = 0$, FCFS achieves about $37 - 44\%$ of the total transfusions achieved by the other two mechanisms, which both generate the same transfusion level. Thus, in a place like sub-Saharan Africa, where inventory levels are historically low, running a centralized system with the priority mechanism for patients batched in small groups would be far superior to the current system.

On the other hand, when the highest initial inventory level approaches half of the highest total maximum need, the performance of FCFS substantially improves.

Observation 2 *When the inventory rate $\iota \to 0.5^-$, FCFS successfully approximates (from below) the priority mechanism and the FCFS dominating mechanism.*

For $\iota = 0.5$, FCFS achieves about 96% of the total transfusions achieved by the priority mechanism for $|I| = 100$, while this ratio is $91 - 92\%$ for $|I| = 25$. Moreover, when the population size is large, i.e., $|I| = 100$, FCFS performs well even for $\iota = 0.2$: it achieves about $91 - 92\%$ of the transfusions achieved by the priority mechanism.

We should note that when $\iota = 0.2$ and $|I| = 100$ the blood bank has on average 60 units of blood already in stock for 100 arriving patients (with a total maximum need of 350 units on average), which could be a substantial amount in many places that use replacement donor programs to mitigate blood shortages.

Our next result compares the FCFS dominating mechanism with the priority mechanism.

Observation 3 *The FCFS dominating mechanism and the priority mechanism are close, while the priority mechanism always achieves more transfusions on average. They most differ for small but non-zero values of the inventory rate ι.*

The FCFS dominating mechanism always achieves more than about 94% of the transfusions in the efficient outcome of the priority mechanism. Therefore, under small inventory levels, the current allocation system can in fact be significantly Pareto-improved for the patients in an incentive compatible way.

The next finding is about the sensitivity of the transfusions to changes in inventory, which can be important for robustness and fairness reasons regarding different blood banks with different inventory levels in a region.

Observation 4 *The priority mechanism is the least sensitive to the inventory rate ι, while FCFS is the most sensitive one.*

In the end, as the priority mechanism does not necessarily Pareto dominate FCFS or the FCFS dominating mechanism, we inspect the patients' preferences between pairs of mechanisms. The detailed comparison results are presented in Figures A.15 and A.16, and Table A.3 in Appendix D.

Observation 5 *The percentage of patients who prefer the priority mechanism over FCFS is $54 - 65\%$ when $\iota = 0$, and this percentage falls as ι increases. The percentage of patients who prefer FCFS over the priority mechanism never exceeds 1.9%, and is always less than about $\frac{1}{14}$ of the percentage of those who prefer the priority mechanism.*

Thus, the Pareto efficient and incentive compatible priority mechanism seems to be a perfect candidate to replace FCFS, as it almost Pareto dominates FCFS for the patients in the simulated markets.

6 Related Literature

Our paper contributes to two strands of literature in economics: market design with an emphasis on the economics of health care, and mechanism design with an emphasis on multi-unit discrete goods exchange without side payments.

On the market design front, an important predecessor is studies on living-donor kidney exchange spanned by Roth, Sönmez, and Ünver (2004, 2005), although most of this literature is about exchanging only one organ with the notable exception of Ergin, Sönmez, and Ünver (2017), which studies dual donor exchanges in lung, liver, and simultaneous liver-kidney transplantation with one-for-one exchange rate. The differences in institutional details between solid organ exchange applications and our main application are explained in Appendix A.4. Our donor monotonicity notion would reduce to the notion introduced in Roth, Sönmez, and Ünver (2005) if patients had unit demand and the exchange rate was one-for-one. Multi-tier priority mechanisms, special cases of our weighted utilitarian mechanisms, are proposed and applied in the context of kidney exchange (Andersson and Kratz, 2019 and Kratz, 2024).

The WHO guidelines suggest that blood should only come from VNRDs and economic incentives can adversely affect both blood safety and blood donation. The position of the WHO has been questioned based on recent evidence (Lacetera, Macis, and Slonim, 2013). In particular, Lacetera, Macis, and Slonim (2012) provide evidence from a natural field experiment showing that economic incentives have a positive effect on voluntary donation and can encourage pro-social behavior. Additionally, Slonim, Wang, and Garbarino (2014) also study blood donation from an economic perspective, and discuss methods to increase blood supply and improve the supply and demand balance without market prices. Pay-it-forward and pay-it-backward incentive schemes for encouraging COVID-19 convalescent plasma donation have recently been proposed by Kominers et al. (2020) in a market design context.[36]

[36]They propose issuing vouchers for the convalescent plasma donation of patients who recover from COVID-19 that can be used by these donors' family members who may become sick in the future to gain prioritized access to plasma therapy or for their own treatment, if they are still sick. Since one donor can donate plasma that can treat more than one patient, the system can collect enough plasma to treat all patients. Their paper inspects the steady-state analysis of a stylized large-market model, while ours is on mechanism design in a finite environment.

There are not many papers on market design for multi-unit exchange of indivisible goods without side payments, even under the restriction of one-for-one exchange using compatibility-based preferences. The existing models are isomorphic to a model where endowment provision costs do not matter or are simply equal to zero because of the one-for-one exchange rate assumption they carry, unlike our setting.

Besides Ergin, Sönmez, and Ünver (2017) mentioned above, two notable other papers are Manjunath and Westkamp (2021), who study shift exchanges for medical doctors, nurses, and other professionals as a market design problem, and Andersson et al. (2021), who consider the design of time banks or favor barter markets to be cleared by centralized clearinghouses.

In Manjunath and Westkamp (2021), for each agent there are three indifference classes of objects: desirable objects, undesirable objects that she is endowed with, and undesirable objects that she is not endowed with. This trichotomous preference domain is more general than our specification, and suits their application of shift exchange but not the blood allocation problem. They consider priority mechanisms and show they are individually rational, efficient, and strategy-proof. Similar to our study, Andersson et al. (2021) consider compatibility-based preferences, but their domain is more restrictive, since each agent is endowed with identical copies of an object. Compared to Manjunath and Westkamp (2021), they are able to achieve the stronger welfare requirement of maximality in a less general preference domain. They study maximal mechanisms with priority tie-breakers and show that they are individually rational and strategy-proof. The mechanisms in both studies are special cases of weighted utilitarian mechanisms with one-for-one exchange rate. One important difference between their models and ours is that we focus on the incomplete information regarding utilities and endowments of individuals in the mechanism design context, while they consider incomplete information regarding the compatibility relations of individuals. Our results can be generalized to the setting where the compatibility relations are private information as well (see an earlier draft of our paper, Han, Kesten, and Ünver, 2021), and subsume the latter paper's results.

Our paper as well as Andersson et al. (2021) substantially extends the priority mechanism introduced for bilateral kidney exchange, i.e., one-for-one donor exchange between two patients with unit demand, by Roth, Sönmez, and Ünver (2005).[37,38]

Price discovery and Pareto efficient allocation through endogenously determined exchange rates are the main features of competitive equilibrium. For the allocation of indivisible goods, this approach was pioneered by Hylland and Zeckhauser (1979) using pseudo-market equilibrium from equal "fake" monetary incomes. This approach fails to guarantee the existence of a competitive equilibrium with endowments and no monetary income—as in our model—even with single-unit demand under compatibility-based preferences and the possibility of probabilistic assignments (see Garg, Tröbst, and Vazirani, 2020 for an impossibility). Positive results are obtained with unit demand when some fake money is injected to the system (Echenique, Miralles, and Zhang, 2021). Moreover, competitive equilibrium as a mechanism is in general not incentive compatible in finite markets.[39]

Similar to our main insight in the blood allocation context, Agarwal et al. (2019) underline and calculate the welfare loss in the US kidney exchange due to inefficient mechanisms and agency problems. They argue that while the number of transplants that can be performed crucially depends on the marginal product of each patient-donor pair, current platform rules largely ignore this variation in the social value of submissions, much like the inefficiency caused by fixed exchange rates in blood allocation.

7 Concluding Remarks

We introduced a new market design problem and proposed a broad class of Pareto efficient and incentive-compatible mechanisms that have practical applications. We view our incentive compatibility notion as an important desideratum for a successful

[37]Matching models with unit demand and compatibility-based preferences have been studied in the context of graph theory. The incentive and fairness properties of mechanisms on such graphs were first analyzed by Bogomolnaia and Moulin (2004) in an economic model of two-sided matching. A recent related paper regarding matching and assignment with compatibility-based preferences is Nicolò, Sen, and Yadav (2019), who study the assignment of tasks to pairs of agents where each agent has separable compatibility-based preferences over her assigned partner and task. This paper focuses on finding core matchings in this domain. Another recent paper on the multi-unit exchange model with one-for-one exchange rate, Aziz (2019), derives a sufficient condition for the strategy-proofness of a mechanism.

[38]Multi-unit exchange with one-for-one exchange rate and non-compatibility-based preferences, such as strict *responsive* preferences, leads to the non-existence of any strategy-proof, individually rational, and efficient mechanism (Konishi, Quint, and Wako, 2001). On the other hand, positive results can be obtained if strategy-proofness is swapped with other incentive properties (for example, see Biró, Klijn, and Pápai, 2022).

[39]Also see Budish (2011) for the first model of competitive equilibrium with multi-unit demand and fake money, but without endowments.

field implementation.

The machinery needed to study blood allocation with replacement donors is also new and has not been developed before in the mechanism design literature to our knowledge. The concept of feasible schedule menus overcomes, using both positive and normative measures, the limitations put in place by the one-for-one exchange rate. Our mechanisms also substantially generalize well-known incentive compatible mechanisms in the context of compatibility-based preferences.

Unlike organ exchanges, dynamic analysis of the blood allocation problem is less of an issue as patients requesting blood transfusions do not typically have long waiting horizons. Instead, our mechanisms can be employed through batching (e.g., once in a few days). Notably, when the blood bank inventory is low, the full power of our approach is needed to boost blood transfusion volume, which can generate substantial welfare gains. On the other hand, when the inventory is sufficiently high, the currently used FCFS mechanism or other decentralized heuristics can be relatively successful.

In closing, we note that this is the first economics paper on a potentially important practical market design problem, as far as we know. It is our wish that unexplored features of this problem both on the more practical market design side and more abstract mechanism design framework will attract the attention of future researchers. We hope that, in addition to developing the theory for efficient blood allocation mechanisms with good incentive properties, our approach will be an important first step toward blueprints for transparent, equitable, and systematic replacement donor programs that are in line with the goals of the WHO. Relaxing the constraints imposed by fixed exchange rates, this approach can help to overcome important practical frictions such as coercion and emerging black markets in many places around the world where these programs are not adequately organized.

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009). "Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match." *American Economic Review*, 99, 1954–1978.

Agarwal, N., I. Ashlagi, E. Azevedo, C. R. Featherstone, and Ö. Karaduman (2019). "Market Failure in Kidney Exchange." *American Economic Review*, 109, 4026–70.

Allain, J.-P. and C. T. S. Sibinga (2016). "Family Donors Are Critical and Legitimate in Developing Countries." *Asian Journal of Transfusion Science*, 10 (1), 5–11.

Alva, S. and V. Manjunath (2019). "Strategy-proof Pareto-improvement." *Journal of Economic Theory*, 181, 121–142.

American Red Cross (2020). "Platelet Donation." Available at: https://www.redcrossblood.org/donate-blood/how-to-donate/types-of-blood-donations/platelet-donation.html, retrieved on 08/09/2020.

Andersson, T., A. Cseh, L. Ehlers, and A. Erlanson (2021). "Organizing Time Exchanges: Lessons from Matching Markets." *AEJ: Microeconomics*, 13 (1), 338–73.

Andersson, T. and J. Kratz (May 2019). "Pairwise Kidney Exchange over the Blood Group Barrier." *The Review of Economic Studies*, 87 (3), 1091–1133.

Aziz, H. (2019). "Strategyproof Multi-Item Exchange Under Single-Minded Dichotomous Preferences." *Autonomous Agents and Multi-Agent Systems*, 34 (1), 3.

Bates, I., G. Chapotera, S McKew, and N. Van Den Broek (2008). "Maternal Mortality in Sub-Saharan Africa: The Contribution of Ineffective Blood Transfusion Services." *BJOG: An International Journal of Obstetrics & Gynaecology*, 115 (11), 1331–1339.

Bates, I., G Manyasi, and A. M. Lara (2007). "Reducing Replacement Donors in Sub-Saharan Africa: Challenges and Affordability." *Transfusion Medicine*, 17, 434–442.

Biró, P., F. Klijn, and S. Pápai (2022). "Serial Rules in a Multi-Unit Shapley-Scarf Market." *Games and Economic Behavior*, 136, 428–453.

Bogomolnaia, A. and H. Moulin (2004). "Random Matching Under Dichotomous Preferences." *Econometrica*, 72, 257–279.

Budish, E. (2011). "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes." *Journal of Political Economy*, 119(6), 1061–1103.

CBHS (2020). "Understanding Elective Surgery in Australia." Available at: https://www.cbhs.com.au/mind-and-body/blog/understanding-elective-surgery-in-australia, retrieved on 10/16/2023.

Chen, Q. (2012). "Has Mutual Help Blood Donation Turned Sour?" China Youth Daily, 11 April 2012. Available at: http://zqb.cyol.com/html/2012-04/11/nw.D110000zgqnb_20120411_2-09.htm.

Cid, J., S. K. Harm, and M. H. Yazer (2013). "Platelet Transfusion-the Art and Science of Compromise." *Transfusion Medicine and Hemotherapy*, 40, 160–171.

Cockbain, A. J., T. Masudi, J. P. A. Lodge, G. J. Toogood, and K. R. Prasad (2010). "Predictors of Blood Transfusion Requirement in Elective Liver Resection." *HPB*, 12 (1), 50–55.

Collins, R., M. Wisniewski, J. Waters, D. Triulzi, L. Alarcon, et al. (2015). "Excessive Quantities of Red Blood Cells Are Issued to the Operating Room." *Transfusion Medicine*, 25, 374–379.

Davies, S. (2004). "Blood Banks No Longer in the Red." The Phnom Penh Post, February 13 2004. Available at: https://www.phnompenhpost.com/national/blood-banks-no-longer-red, retrieved on 08/27/2020.

Delhi State Health Mission (2016). "Standard Operating Procedures for Blood Bank." Available at: http://dshm.delhi.gov.in/pdf/QAC/SoPs/Blood%20bank.pdf, retrieved on 08/09/2020.

Dunbar, N. M., M. C. Katus, C. M. Freeman, and Z. M. Szczepiorkowski (2015). "Easier Said Than Done: ABO Compatibility and D Matching in Apheresis Platelet Transfusions." *Transfusion*, 55 (8), 1882–1888.

Echenique, F., A. Miralles, and J. Zhang (2021). "Constrained Pseudo-Market Equilibrium." *American Economic Review*, 111 (11), 3699–3732.

Eichbaum, Q., W. M. Smid, R. Crookes, N. Naim, A. Mendrone Jr, et al. (2015). "Apheresis in Developing Countries Around the World." *Journal of Clinical Apheresis*, 30 (4), 238–246.

Erdil, A. (2014). "Strategy-proof Stochastic Assignment." *Journal of Economic Theory*, 151, 146–162.

Ergin, H., T. Sönmez, and M. U. Ünver (2017). "Dual-Donor Organ Exchange." *Econometrica*, 85, 1645–1671.

Farrugia, A., J. Penrod, and J. M. Bult (2010). "Payment, Compensation and Replacement–the Ethics and Motivation of Blood and Plasma Donation." *Vox Sanguinis*, 99 (3), 202–211.

Garg, J., T. Tröbst, and V. V. Vazirani (2020). "An Arrow-Debreu Extension of the Hylland-Zeckhauser Scheme: Equilibrium Existence and Algorithms." arXiv:2009.10320.

Gilcher, R. O. and S. McCombs (2005). "Seasonal Blood Shortages Can Be Eliminated." *Current Opinion in Hematology*, 12 (6), 503–508.

Haddad, A., T. Bou Assi, and O. Garraud (2018). "How Can Eastern/Southern Mediterranean Countries Resolve Quality and Safety Issues in Transfusion Medicine?" *Frontiers in Medicine*, 5, 45.

Hall, P. (1935). "On Representatives of Subsets." *Journal of London Mathematical Society*, 10, 26–30.

Han, X., O. Kesten, and M. U. Ünver (2021). "Blood Allocation with Replacement Donors: A Theory of Multi-unit Exchange with Compatibility-based Preferences." WP.

Hylland, A. and R. Zeckhauser (1979). "The Efficient Allocation of Individuals to Positions." *Journal of Political Economy*, 87, 293–314.

Jones, J. M., M. R. Sapiano, A. A. Savinkina, K. A. Haass, M. L. Baker, et al. (2020). "Slowing Decline in Blood Collection and Transfusion in the United States–2017." *Transfusion*, 60, S1–S9.

Kesten, O. (2010). "School Choice with Consent." *The Quarterly Journal of Economics*, 125 (3), 1297–1348.

Kominers, S. D., P. A. Pathak, T. Sönmez, and M. U. Ünver (2020). "Paying It Backward and Forward: Expanding Access to Convalescent Plasma Therapy Through Market Design." NBER WP No. 27143.

Konishi, H., T. Quint, and J. Wako (2001). "On the Shapley-Scarf Economy: The Case of Multiple Types of Indivisible Goods." *Journal of Mathematical Economics*, 35, 1–15.

Kratz, J. (2024). "Conflicting Objectives in Kidney Exchange." *Journal of Economic Theory*, 217, 105803.

Kyeyune-Byabazaire, D. and H. A. Hume (2019). "Towards a Safe and Sufficient Blood Supply in Sub-Saharan Africa." *ISBT Science Series*, 14 (1), 104–113.

Lacetera, N., M. Macis, and R. Slonim (2012). "Will There Be Blood? Incentives and Displacement Effects in Pro-Social Behavior." *AEJ: Economic Policy*, 4(1), 186–223.

— (2013). "Economic Rewards to Motivate Blood Donation." *Science*, 340 (6135), 927–928.

Lozano, M., N. Heddle, L. M. Williamson, G. Wang, J. P. AuBuchon, et al. (2010). "Practices Associated With ABO-Incompatible Platelet Transfusions: A BEST Collaborative International Survey." *Transfusion*, 50 (8), 1743–1748.

Manjunath, V. and A. Westkamp (2021). "Strategy-Proof Exchange Under Trichotomous Preferences." *Journal of Economic Theory*, 193, 105197.

Marwaha, N. and R. Sharma (2009). "Consensus and Controversies in Platelet Transfusion." *Transfusion and Apheresis Science*, 41 (2), 127–133.

McCullough, J. (2010). "Overview of Platelet Transfusion." *Seminars in Hematology*, 47 (3), 235–242.

Murota, K. (2013). *Discrete Convex Analysis*. SIAM.

Nicolò, A., A. Sen, and S. Yadav (2019). "Matching With Partners and Projects." *Journal of Economic Theory*, 184, 104942.

Norfolk, D. (2013). *United Kingdom Blood Services: Handbook of Transfusion Medicine*. TSO, 5th Edition.

Norol, F., P. Bierling, F. Roudot-Thoraval, F. F. Le Coeur, C. Rieux, et al. (1998). "Platelet Transfusion: A Dose-Response Study." *Blood, The Journal of the American Society of Hematology*, 92 (4), 1448–1453.

Ødegaard, F. and S. N. Roy (2021). "Heuristic-Based Allocation of Supply Constrained Blood Platelets in Emerging Economies." *Journal of Heuristics*, 27 (5), 719–745.

Ostrovsky, M. (2008). "Stability in Supply Chain Networks." *American Economic Review*, 98 (3), 897–923.

Pathak, P. A. and T. Sönmez (2008). "Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism." *American Economic Review*, 98, 1636–1652.

Pfuntner, A., L. M. Wier, and C. Stocks (2013). "Most Frequent Procedures Performed in U.S. Hospitals, 2011: Brief #165." *Healthcare Cost and Utilization Project (HCUP) Statistical Briefs [Internet]*. Agency for Healthcare Research and Quality.

RhesusNegative.net (2012-2019). "Blood Type Frequencies by Country Including the Rh Factor." http://www.rhesusnegative.net/themission/bloodtypefrequencies/, retrieved on 01/24/2021.

Roth, A. E., T. Sönmez, and M. U. Ünver (2004). "Kidney Exchange." *Quarterly Journal of Economics*, 119 (2), 457–488.

— (2005). "Pairwise Kidney Exchange." *Journal of Economic Theory*, 125 (2), 151 –188.

She, X. (2020). "Behind the Mutual Help Blood Donation: Several Hospitals in Xi'an Said You Need to Donate Blood Before Using Blood." Chinese Business Gazette. Available at: http://news.hsw.cn/system/2020/0106/1144361.shtml.

Slonim, R., C. Wang, and E. Garbarino (2014). "The Market for Blood." *Journal of Economic Perspectives*, 28, 177–196.

Sönmez, T. (2023). "Minimalist Market Design: A Framework for Economists With Policy Aspirations." *mimeo*.

Sönmez, T. and M. U. Ünver (2017). "Market Design for Living-Donor Organ Exchanges: An Economic Policy Perspective." *Oxford Review of Economic Policy*, 33, 676–704.

Stroncek, D. F. and P. Rebulla (2007). "Platelet Transfusions." *Lancet*, 370, 427–438.

Tagny, C. T. (2012). "The Current Need for Family and Replacement Donation in Sub-Saharan Africa Should Not Hide the Difficulties of Its Management." *Transfusion Medicine*, 4, 298–299.

Tagny, C. T., M. D. Kouao, H. Touré, J. Gargouri, A. S. Fazul, et al. (2012). "Transfusion Safety in Francophone African Countries: An Analysis of Strategies for the Medical Selection of Blood Donors." *Transfusion*, 52 (1), 134–143.

Thompson, P. (2020). "Blood Donation Requirements in Puerto Vallarta." Available at: http://www.banderasnews.com/health-beauty/blooddonations.htm, retrieved on 08/13/2020.

WHO (2020). "Blood Safety and Availability." Available at: https://www.who.int/news-room/fact-sheets/detail/blood-safety-and-availability, retrieved on 08/03/2020.

For Online Publication: Appendices of "Blood Allocation with Replacement Donors: Theory and Application"

A Background for Blood Transfusion and Allocation

A.1 Main Blood Components and Compatibility

There are different transfusion protocols for different blood components, and the medical practices also vary across different regions of the world. We mainly focus on the three most-transfused blood components—red blood cells, platelets, and plasma—as well as whole blood, and provide a brief account starting with a general rule of thumb for compatibility requirements.

Blood-type compatibility plays an important role for the feasibility of transfusion. There are more than 300 human blood groups. Two of them are the most important in clinical practices. The first one, the ABO blood group system, is the most commonly known. A person's ABO blood type is determined by the presence of A or B antigens in her blood cells: her type may be O (if she has neither antigen), A (has only the A antigen), B (has only the B antigen), or AB (has both antigens). Each person has pre-formed antibodies in her plasma against every non-existent antigen. Antibodies against an antigen attack blood cells that carry this antigen, which can cause potentially fatal hemolysis.

Therefore, any transfusion including a significant amount of donor cells, by rule of thumb, should respect *ABO-cellular compatibility*: O blood-type cells can be donated to all, A blood-type cells can be donated to A and AB blood-type patients, B blood-type cells can be donated to B and AB blood-type patients, and AB blood-type cells can only be donated to AB blood-type patients.

On the other hand, any transfusion including a significant amount of donor plasma, which carries the donor's pre-formed antibodies, by rule of thumb, should respect *ABO-plasma compatibility*: AB blood-type plasma can be donated to all as it does not contain any antibodies, A blood-type plasma can be donated to A and O blood-type patients, B blood-type plasma can be donated to B and O blood-type patients, and O blood-type plasma can only be donated to O blood-type patients as it contains antibodies against both antigens.

The second crucial blood group system is Rh. The most clinically important Rh antigen is D. Its existence and non-existence correspond to Rh D+ type and Rh D−

type respectively. Antibodies to the Rh D antigen can only develop on an Rh D$-$ person after being exposed to Rh D$+$ red blood cells. Hence, the compatibility requirement is to avoid the transfusion of Rh D$+$ red blood cells to an Rh D$-$ patient, due to the risk of hemolytic reactions.

Most blood components are packed with others in solutions. Thus, depending on the amount of these components, different practices are followed for the compatibility of the pack with the patient.

Next, we turn our focus to specific blood components.

Red Blood Cells and Whole Blood. Red blood cells carry oxygen from the lungs to all parts of the body and are the most commonly transfused blood components. Red blood cell transfusion—the de-facto modern day replacement for the older whole blood transfusion therapy—is mostly used for patients with anemia due to cancer, blood diseases, and other causes, followed by surgical patients. Whole blood is still transfused in some low-income countries. For other countries, this is only occasionally performed in emergencies for patients with massive blood loss due to trauma, surgeries, etc. A person donates one unit (about a pint) of whole blood each time and she has to wait at least eight weeks between donations. Each unit of red blood cells is prepared from one unit of donated whole blood by removing plasma and adding preservative solutions, and can be stored for about 42 days.

ABO-identical and Rh D-compatible transfusion is generally practiced for whole blood transfusion.[40] For red blood cells, ABO-cellular compatible and Rh D-compatible transfusion is all that is needed in theory. However, as red blood cell packs usually carry some amount of donor plasma, ABO-identical (and Rh D-compatible) transfusion is often required.

Eight blood types are relevant for red blood cell or whole blood transfusion. However, in some populations, such as those in Asia, Rh D$-$ is so rare that there are effectively only four blood types.[41]

Platelets. These are tiny cells in the blood that form clots and stop bleeding. Platelet transfusions are mostly given to prevent or treat bleeding in patients with thrombocytopenia or abnormal platelet function, such as those undergoing chemotherapy or receiving a bone marrow transplant. McCullough (2010) states that the use of platelets

[40] An exception is that type O Rh D$-$ blood is often transfused in emergencies to patients with other or unknown blood types. For this reason it is also dubbed as the *global-donor* blood type.

[41] For example, in China, the Rh D antigen exists in more than 99% of the population.

has increased more than other blood components in the last 15 years. According to Red Cross of America, every 15 seconds someone needs platelets (American Red Cross, 2020). However, due to their storage requirement at room temperature, platelets have a much shorter shelf life than most other blood components: in most countries they can only be stored between four and seven days (Cid, Harm, and Yazer, 2013). As a result, platelets are in frequent shortages even in developed countries.

One unit of platelets can be prepared from 4-6 units of pooled whole blood, or obtained from a single donation through the technique of *apheresis*, which only takes platelets out of the donor's blood, leaving the other components in the blood stream. The whole process takes approximately three hours and a person can donate platelets in this way once a week, up to 24 times a year.[42] In addition to the efficiency in the production process, apheresis platelets are also safer to the patients due to the minimal donor exposure. Hence, it has become an increasingly common practice to give apheresis platelets, instead of whole-blood-derived platelets. In 2017, only 4.2% of the total transfused platelet units in the US were derived from whole blood (Jones et al., 2020).[43]

For platelets, the compatibility practices vary significantly among different institutions and countries. As platelets (weakly) express the ABO antigens and they are suspended in plasma in the platelet packs, ABO-identical transfusion is always preferred, although ABO incompatibility in platelet transfusion is generally not as risky as in whole blood or red blood cell transfusion. Given the frequent shortages, ABO-identical transfusion is often not possible. Both ABO-cellular compatible transfusion and ABO-plasma compatible transfusion (due to significant plasma amount in the packs) are commonly practiced, and there has been no consensus as to which is the better strategy (Dunbar et al., 2015; Lozano et al., 2010; Norfolk, 2013). Finally, as the Rh D antigen is not present on platelets, Rh D compatibility is usually not required (for example, see Cid, Harm, and Yazer, 2013).

Plasma. It is the non-cellular, protein- and antibody-rich liquid component of blood. The plasma used in everyday transfusion is usually *fresh frozen plasma*. Plasma transfusion is often utilized by patients with liver failure, heart surgery, severe infections, and serious burns. One unit of fresh frozen plasma can be prepared from one unit of whole blood after removing the blood cells. Alternatively, a person can donate up to three

[42] A donor usually donates one unit of platelets through apheresis, but double or triple-unit donation in a session may also be possible, depending on the health of the donor.

[43] The apheresis method has also become popular in developing countries (Eichbaum et al., 2015).

units through apheresis, which keeps other blood components in her blood stream and only extracts plasma. Fresh frozen plasma has the longest shelf life among the three main blood components: it can be stored for about a year. Its transfusion follows ABO-plasma compatibility, without regard to Rh D compatibility (as Rh D antibodies only form after exposure to the Rh D antigen and are not pre-formed).

Convalescent plasma, the antibody-rich plasma of a patient recovering from an infectious disease with no other known cure, such as Ebola and COVID-19, is commonly used to treat patients or to produce drugs against the disease. It can also be considered as a type of fresh frozen plasma.

In addition to plasma used for transfusion, plasma derivatives (such as albumin, coagulation factors, and immunoglobulins) manufactured from "source plasma" in fractionation centers are used in the treatment of various conditions. Unlike the blood used for transfusion, source plasma is commonly collected from paid donors in many countries.[44]

A.2 Blood Demand of a Patient

The amount of a blood component needed to treat each medical condition is idiosyncratic. For example, Collins et al. (2015) report that, at a tertiary referral center in the US, the average amount of red blood cell units used per surgery is close to 3.5 and this amount has a high variance due to different patient conditions. Besides the idiosyncratic demand, receiving more units is generally better under various outcome or preference metrics. We give three general examples of patient demand that have this common thread.

First, it is medically acceptable and feasible to transfuse various units to a patient with a particular condition such that more units lead to better outcomes. For example, platelets are often transfused prophylactically to prevent bleeding when a patient's platelet count is below a certain threshold. In such cases, both the strategy of higher doses in lower frequency and the strategy of lower doses in higher frequency are practiced (Stroncek and Rebulla, 2007). Norol et al. (1998) show that the high and very high dose treatments lead to significantly better platelet increment in the patients, compared to the medium dose treatment.

Second, the exact need of a patient can be ex-ante uncertain, with more blood leading to better outcomes on average. For example, a surgeon often orders significantly

[44]The US has a large source plasma industry that relies on paid donors, and it is responsible for 55% of the world's supply of plasma derivatives (Farrugia, Penrod, and Bult, 2010).

more blood than the patient ends up using during a surgery for cautionary reasons. Collins et al. (2015) report that 72% of the red blood cells ordered for surgeries go unused. The ratio of ordered to transfused red blood cells can be as high as 11 to 1 in elective liver resection surgical procedures (Cockbain et al., 2010). These ratios indicate that surgeons are quite risk averse, and ex-ante a surgeon has monotonic preferences over the amount of blood ordered.[45] Required operations are still conducted, but with less blood in hand, when there is a shortage.[46]

Third, blood components such as platelets and red blood cells are often transfused routinely to patients with chronic conditions and are administered in small doses over time. For example, Marwaha and Sharma (2009) state that patients undergoing chemotherapy require platelet transfusion once in at least every three days, and, when the bone marrow is adversely affected, every day. In such cases, more units are preferred to less to be administered through several transfusions in a time interval, although allocation can be done only once.

A.3 Replacement Donor Programs and Blood Bank Policies

Replacement donor programs are observed in all continents and are especially common in Africa, Latin America, and Central Asia (Allain and Sibinga, 2016). Populous countries such as Pakistan, Brazil, and Mexico collect their blood components almost entirely through replacement donor programs. On the other hand, countries such as India and China rely on these programs to meet the demand not met by VNRDs.

Within the medical community, there is an ongoing debate about the stance of the WHO regarding VNRDs being the safest blood supply. There has been considerable evidence suggesting that the blood collected through replacement donors is as safe as VNRDs. It is also argued that the motivations of the two types of donations are similarly altruistic, and the distinction between them from an ethical perspective is not clear cut. Allain and Sibinga (2016) provide an excellent survey of these views, empirical evidence, and references. In addition, there are significant economic and cultural reasons for the predominance of decentralized and often hospital-based replacement systems in many developing countries. Such a system is much less costly (Bates, Manyasi, and Lara, 2007), favors intra-group solidarity, and is culturally more

[45]Unused blood is usually discarded if it is out of blood bank storage for more than four hours or not kept in cold storage for more than thirty minutes.

[46]According to Bates et al. (2008), in Sub-Saharan Africa, where the blood supply heavily depends on replacement donor systems, about 26% of hemorrhage maternal deaths were due to lack of blood.

consistent with the presence of strong family or community bonds (Haddad, Bou Assi, and Garraud, 2018; Kyeyune-Byabazaire and Hume, 2019).

In a replacement donor program, a patient's donors can donate before or after the patient receives blood depending on the regional practice. Since direct donation from a donor to the patient (even if they are compatible) is not practiced in modern medicine due to health concerns (i.e., the donor blood needs to be tested and processed first), the blood bank is used as an intermediary.

Blood banks work with hospitals and blood centers. Hospitals relay the needs of patients to the blood banks, while the blood banks and blood centers collect donations from VNRDs and replacement donors. Hospitals are often required to maintain a small inventory of their own (for example, see Delhi State Health Mission, 2016).

Although replacement donor programs are very common and officially acknowledged in many countries, maybe surprisingly, it is difficult to find their exact institutional details. The most common practice in current replacement donor programs worldwide is that the blood bank announces, either officially or unofficially, a preset exchange rate between the units of blood received and supplied, often irrespective of the blood type sought or donated. Blood banks provide blood to patients exclusively based on these rates using schemes similar to first-come first-serve. Among these, the one-for-one exchange rate, i.e., one unit replacement per unit received, is most common around the world.

We also give some examples of other policies practiced. Although China banned the replacement donor programs in 2018, they are still used in several cities during shortages, especially for platelets (She, 2020). Different policies have been in place. In most cities, including Beijing, the exchange rate is one-for-one. As reported by She (2020), in Xi'an, during periods of shortages, a patient has the priority of receiving three units of blood for every unit she has donated before, and she has the priority of receiving one unit for every unit her replacement donors donate now. According to Chen (2012), in Guangzhou, there is not necessarily a fixed relation between the amount received and supplied. Moreover, in some regions there are restrictions on the blood types of replacement donations. As an extreme case, the blood type of a replacement donor must be identical to that of the patient in Jiangsu. While such a restriction is relatively rare for whole blood donations, it is not uncommon for replacement platelet donations throughout the country.

India has the second largest official replacement donor program in the world after

Pakistan. In Delhi, regardless of the amount of blood she needs, the patient is required to bring forward one replacement donor, unless the intervention needed is an emergency surgery (Delhi State Health Mission, 2016). In Turkey, a variation of the Delhi policy is used and a patient is required to bring forward three donors for any number of blood receipt.[47]

In Cameroon and Congo, the exchange rate has been two replacement units per unit received, as almost 25% of the donations are not suitable for transfusion due to infections (Tagny, 2012). The same exchange rate is also used in Puerto Vallarta, Mexico, for cost reasons (Thompson, 2020).

In Tucuman, Argentina, a patient's replacement donors donate after the transfusion. The exchange rate is fixed at one-for-one; however, it is not as strictly enforced.[48]

A.4 Comparisons to Solid Organ Exchanges

The feasibility of blood transfusion primarily depends on blood type compatibility. Therefore, replacement donor programs operate on the premise of the exchange of willing donors for compatible blood received by the paired patient. This is similar in principle to organ exchanges with the first-order difference that there is not yet an optimized central clearinghouse for replacement donors (for example, see Tagny, 2012). There are a number of other important institutional differences.

Unlike solid organ exchanges, replacement donor programs do not require the simultaneity of donation and transfusion, which gives the flexibility to schedule donations and transfusions separately. The donated blood must be tested and processed for safety reasons, which makes it unsuitable for immediate transfusion. It may take up to 24 hours to test and process donated blood. Thus, most exchanges are intermediated through the blood bank, i.e., patients receive from the inventory, and their donors donate to the inventory.

Moreover, other logistical constraints of blood donation are negligible compared to those in solid organ transplantations. The blood donation process takes at most a few hours and its effects wear off relatively quickly. On the other hand, organ transplantations carry risks and require careful planning before and after the operations. Once extracted, blood components can be stored for a certain period of time, which can facilitate the designer's choice of optimal timing of donation and transfusion. Many blood banks and hospitals often operate in coordination, making it possible to obtain the

[47] See Footnote 18.

[48] See Footnote 3.

necessary blood units from neighboring facilities. These lead to the observation that in blood allocation with replacement donors, the possibility of a donor reneging is not as much of a concern as in organ exchanges. The absence of logistical constraints together with the ability to store blood components make it possible to incorporate cycles and chains of arbitrary length into an allocation in our problem, unlike organ exchanges.

B Proofs

B.1 Proof of Proposition 2

Consider the vector w with $w_i = 1$ for all $i \in I$. As mentioned in Footnote 25, we can find a weighted utilitarian mechanism f with respect to weights w', where each w'_i is sufficiently close to 1, such that for any feasible schedule menu profile $\mathcal{F}$, problem (D, θ) and allocations $\alpha, \alpha' \in \mathcal{A}(\mathcal{F}, D)$, $U(w', \theta, \alpha) > U(w', \theta, \alpha')$ if $U(w, \theta, \alpha) > U(w, \theta, \alpha')$. Therefore, the outcome of f always maximizes the sum of patient and bank utilities.

Let $\mathcal{F}$ be any L-convex feasible schedule menu profile. We show that L-convexity implies the following assumption on $\mathcal{F}$, which will be useful in the proofs of other results as well.

Assumption 1. For every $i \in I$, D_i and (r, s), $(r', s') \in \mathcal{F}_i(D_i)$, we have

1. If $r' > r$ and $s' > s$, then
$$(r+1,\, s+1) \in \mathcal{F}_i(D_i) \quad \text{and} \quad (r'-1,\, s'-1) \in \mathcal{F}_i(D_i).$$
2. If $r' > r$ and $s' \leq s$, then
$$(r+1,\, s) \in \mathcal{F}_i(D_i) \quad \text{and} \quad (r'-1,\, s') \in \mathcal{F}_i(D_i).$$
3. If $s' > s$ and $r' \leq r$, then
$$(r,\, s+1) \in \mathcal{F}_i(D_i) \quad \text{and} \quad (r',\, s'-1) \in \mathcal{F}_i(D_i).$$

Lemma A.1 *The feasible schedule menu profile $\mathcal{F}$ satisfies Assumption 1.*

Proof of Lemma A.1. Consider any $i \in I$ and D_i. Let $\mathcal{F}_i(D_i) = S$. For any $x, y \in \mathbb{Z}^2_+$, where $x = (r, s)$ and $y = (r', s')$, denote $\text{dis}(x, y) = r' - r + s' - s$, and $y > x$ if $r' > r$ and $s' > s$. Suppose that $x = (r, s) \in S$, $y = (r', s') \in S$, and $y > x$. We want to first show that $(r+1, s+1) \in S$. It is true if $\text{dis}(x, y) = 2$. If $\text{dis}(x, y) > 2$, then consider

$z = \left\lceil \frac{x+y}{2} \right\rceil > x$. By L-convexity, $z \in S$. It follows from $\text{dis}(x, y) > 2$ that $\left\lceil \frac{r+r'}{2} \right\rceil < r'$ or $\left\lceil \frac{s+s'}{2} \right\rceil < s'$. Hence, $2 \le \text{dis}(x, z) < \text{dis}(x, y)$. If $\text{dis}(x, z) > 2$, we can repeat the argument and find $z' \in S$ such that $z' > x$ and $2 \le \text{dis}(x, z') < \text{dis}(x, z)$. Continuing in this fashion, in the end we must have $(r+1, s+1) \in S$. By symmetric arguments, it can be shown that $(r'-1, s'-1) \in S$. So Condition 1 in Assumption 1 is satisfied.

Next, we show Condition 2. Suppose that $x = (r, s) \in S$, $y = (r', s') \in S$, $r' > r$ and $s' \le s$. First, we argue that there exists $s'' \le s$ such that $(r+1, s'') \in S$. If $r' = r+1$, then it is true. If $r' > r+1$, then consider $\left\lceil \frac{x+y}{2} \right\rceil = (r_1, s_1)$. We have $r' > r_1 > r$ and $s_1 \le s$. By L-convexity, $(r_1, s_1) \in S$. If $r_1 > r+1$, we can repeat the argument and find $(r_2, s_2) \in S$ such that $r_1 > r_2 > r$ and $s_2 \le s$. Therefore, eventually we have $(r+1, s'') \in S$ for some $s'' \le s$. Denote $z = (r+1, s'')$. If $s'' < s$, consider $\left\lceil \frac{x+z}{2} \right\rceil = (r+1, s_3)$. Then $s'' < s_3 \le s$. By L-convexity, $(r+1, s_3) \in S$. If $s_3 < s$, we can repeat the argument and find some s_4 such that $(r+1, s_4) \in S$ and $s_3 < s_4 \le s$. Therefore, we must have $(r+1, s) \in S$. By symmetric arguments, it can be shown that $(r'-1, s') \in S$. Finally, Condition 3 in Assumption 1 can be shown using arguments similar to those in the proof of Condition 2. ■

Assume ρ is strictly concave. To prove the proposition by contradiction, suppose that there exist a problem (D, θ) and patients $i, j \in I$ such that $\beta_i = \beta_j$, $\overline{n}_i = \overline{n}_j$, $\left|\{d \in D_i : \beta_d = X\}\right| = \left|\{d \in D_j : \beta_d = X\}\right|$ for each $X \in \mathcal{B}$, $\mathcal{F}_i(D_i) = \mathcal{F}_j(D_j)$, $\theta_i = \theta_j$, and

$$f_r\Big(\mathcal{F}, D, \theta\Big)(j) - f_r\Big(\mathcal{F}, D, \theta\Big)(i) > 1.$$

For simplicity, let $f\Big(\mathcal{F}, D, \theta\Big) = \alpha$, $\alpha(i) = (r, s)$ and $\alpha(j) = (r', s')$.

We first consider the case that $s' > s$. By Condition 1 in Assumption 1, $(r+1, s+1) \in \mathcal{F}_i(D_i)$ and $(r'-1, s'-1) \in \mathcal{F}_j(D_j) = \mathcal{F}_i(D_i)$. Moreover, $s' > s$ indicates that there exist $d \in D_i$ and $d' \in D_j$ such that $\beta_d = \beta_{d'}$, $\alpha(d) = 0$ and $\alpha(d') = 1$. Therefore, we can construct another allocation $\alpha' \in \mathcal{A}(\mathcal{F}, D)$ based on α such that $\alpha'(i) = (r+1, s+1)$, $\alpha'(j) = (r'-1, s'-1)$, $\alpha'(k) = \alpha(k)$ for all $k \in I \setminus \{i, j\}$ and $\alpha'(b) = \alpha(b)$, by transferring one unit of compatible blood from j to i, and letting d replace the donation of d'. Then, by the strict concavity of ρ,

$$\rho(r+1) - \rho(r) > \rho(r') - \rho(r'-1),$$

which implies

$$u_i\big[\alpha'(i),\, \theta_i\big] - u_i\big[\alpha(i),\, \theta_i\big] > u_j\big[\alpha(j),\, \theta_j\big] - u_j\big[\alpha'(j),\, \theta_j\big]$$

since $\theta_i = \theta_j$. Therefore, the sum of utilities is larger under α' than under α, contradicting to the construction of f.

On the other hand, suppose $s' \leq s$. By Condition 2 in Assumption 1, $(r+1, s) \in \mathcal{F}_i(D_i)$ and $(r'-1, s') \in \mathcal{F}_j(D_j)$. Then we can simply construct another allocation $\alpha'' \in \mathcal{A}(\mathcal{F}, D)$ based on α by transferring one unit of compatible blood from j to i. As before, the strict concavity of ρ implies the sum of utilities is larger under α'' than under α, and a contradiction is reached.

B.2 Proof of Theorem 2

As the proof is involved, we first give a sketch of it.

Sketch of the Proof. Fix a weighted utilitarian mechanism f, a feasible schedule menu profile that satisfies the properties in the theorem, and a type profile. We first define an auxiliary matching market that is isomorphic to the original problem, which we refer to as an *extended problem*. In this market, the blood bank is represented as a pseudo-patient and its inventory is represented by pseudo-donors paired with it. For each blood type, we also introduce a dummy patient paired with dummy donors so that, without loss of generality, we can focus on the simple case where any (real or dummy) patient cannot receive blood from her own compatible donors.

In an extended problem, a *matching* specifies which donors are matched with each patient. A patient is not only matched with the donors who donate to her, but also those of her own donors who do not donate to anyone. Hence, this is a pure exchange economy. The analogue of a mechanism for extended problems is a *rule*, which assigns a matching to each extended problem. We then define a rule F that is isomorphic to the weighted utilitarian mechanism f, which chooses a matching by maximizing the weighted sum of the utilities of the real patients and the bank. In Lemma A.3, we show that f and F are welfare equivalent (for the real patients and the bank). Hence, to prove the theorem, it is sufficient to show that the rule F is donor monotonic. The rest of the proof consists of two lemmata.

The first one, Lemma A.4, is the most crucial result in the proof. This lemma essentially gives a general necessary condition for profitable manipulation under any rule. Consider two extended problems: the original one, denoted as $\widehat{D}$, and the one induced by some real patient i concealing exactly one of her donors, denoted as $\widehat{D'}$. Let M be a matching for $\widehat{D}$ and M' be a matching for $\widehat{D'}$ such that i receives more blood under M'. Then, Lemma A.4 says that there exists a particular graph theoretical structure, a *cycle* or a *chain*, relating these two matchings.

A cycle C from the matching M to the matching M' is a list of patients and donors in which each patient j points to a donor that is matched with j under M' but not under M, and each donor d points to the patient that is matched with d under M. We can "add" the cycle C to the matching M to make it closer to M': starting from M, we remove each donor d in the cycle from the match of the patient that is pointed by d, and add it to the match of the patient that points to d. Due to L-convexity, feasibility of positive price, and non-diminishing favorability in donors, the definition of a cycle is carefully tailored to ensure that these exchanges lead to a well-defined matching for $\widehat{D}$, denoted as $M + C$. We can also "remove" the cycle from M': starting from M', we remove each donor d in the cycle from the match of the patient that points to d, and add it to the match of the patient that is pointed by d. This results in a matching for $\widehat{D'}$, denoted as $M' - C$. On the other hand, a chain is similar to a cycle. The only differences are that the head patient in the chain does not point to any donor, and the tail patient in the chain is not pointed by any donor. Chain addition and removal operations are similarly defined and also lead to new matchings for the two extended problems.

Finally, Lemma A.5 states that the rule F is donor monotonic. We proceed by contradiction. Let $\widehat{D}$ be an extended problem, and $\widehat{D'}$ be the extended problem induced by a real patient i concealing a donor. Suppose that patient i receives more blood under the matching $F(\widehat{D'})$ than under the matching $F(\widehat{D})$. By Lemma A.4, there is a cycle or a chain C from $F(\widehat{D})$ to $F(\widehat{D'})$. Then, $F(\widehat{D}) + C$ is a matching for $\widehat{D}$ and $F(\widehat{D'}) - C$ is a matching for $\widehat{D'}$. We want to show that $F(\widehat{D})$ and $F(\widehat{D}) + C$ are welfare equivalent. Assume that this is not true. Then they do not give the same weighted sum of utilities, which, by the construction of F, implies the weighted sum of utilities must be higher under $F(\widehat{D})$ than under $F(\widehat{D}) + C$.

In the cycle or chain operations, a real patient who receives one more (or less) unit of blood under $F(\widehat{D}) + C$ than under $F(\widehat{D})$ must (1) receive one more (or less) unit of blood under $F(\widehat{D'})$ than under $F(\widehat{D'}) - C$, and (2) receive strictly more (or less) blood under $F(\widehat{D'})$ than under $F(\widehat{D})$. Hence, by the concavity of the blood valuation function, it can be shown that the weighted sum of utilities under $F(\widehat{D'}) - C$ is also higher than that under $F(\widehat{D'})$, which is a contradiction. Therefore, $F(\widehat{D})$ and $F(\widehat{D}) + C$ are welfare equivalent. Then, as patient i still receives more blood under $F(\widehat{D'})$ than under $F(\widehat{D}) + C$, we can apply Lemma A.4 again to show that there is a cycle or a chain C' from $F(\widehat{D}) + C$ to $F(\widehat{D'})$. By similar arguments as before, $F(\widehat{D}) + C$ and $(F(\widehat{D}) +$

$C) + C'$ are welfare equivalent. This process can be continued infinitely, which leads to a contradiction since each cycle or chain addition generates a new matching that is closer to $F(\widehat{D'})$.

Proof. Consider a weighted utilitarian mechanism f with respect to w. Fix a type profile θ, and a feasible schedule menu profile $\mathcal{F}$ that satisfies L-convexity, feasibility of positive price, and non-diminishing favorability in donors. For simplicity, we will drop θ and $\mathcal{F}$ from the relevant arguments (such as those of problems, feasible allocation sets, the mechanism, etc.) throughout the proof. Then each problem under consideration is simply represented by a donor profile D, and we want to show that for any D, $i \in I$ and $D_i' \subseteq D_i$,

$$f_r\Big(D\Big)(i) \geq f_r\Big(D_i', D_{-i}\Big)(i).$$

By Lemma A.1 in the proof of Proposition 2, L-convexity implies Assumption 1 on the feasible schedule menus. We first show that L-convexity, feasibility of positive price, and non-diminishing favorability in donors together imply the following additional assumption on $\mathcal{F}$, which will be useful in the proof.

Assumption 2. For every $i \in I$, D_i, D_i' with $D_i' \subseteq D_i$, $(r,s) \in \mathcal{F}_i(D_i)$ and $(r',s') \in \mathcal{F}_i(D_i')$, we have

1. If $r' > r$, $s' > 0$, and $s < \big|D_i\big|$, then
$$(r+1,\, s+1) \in \mathcal{F}_i(D_i) \quad \text{and} \quad (r'-1,\, s'-1) \in \mathcal{F}_i(D_i').$$
2. If $r' > r$ and $s' \leq s$, then
$$(r+1,\, s) \in \mathcal{F}_i(D_i) \quad \text{and} \quad (r'-1,\, s') \in \mathcal{F}_i(D_i').$$

Lemma A.2 *The feasible schedule menu profile $\mathcal{F}$ satisfies Assumption 2.*

Proof of Lemma A.2. Consider any $i \in I$, D_i, D_i' with $D_i' \subseteq D_i$, $(r,s) \in \mathcal{F}_i(D_i)$, and $(r',s') \in \mathcal{F}_i(D_i')$. Suppose that $r' > r, s' > 0$ and $s < \big|D_i\big|$. Since $r' > 0$, we have $\mathcal{F}_i(D_i') \neq \{(0,0)\}$ and $r' \geq \underline{g}_i$. Then by non-diminishing favorability in donors, there exists s_1 such that $(r', s_1) \in \mathcal{F}_i(D_i)$. Since $r' > r$ and $s < \big|D_i\big|$, by feasibility of positive price, there exists $s_2 > s$ such that $(r', s_2) \in \mathcal{F}_i(D_i)$. Then, given that $(r', s_2) > (r,s)$, it follows from Condition 1 in Assumption 1 that $(r+1, s+1) \in \mathcal{F}_i(D_i)$. This also implies that $\mathcal{F}_i(D_i) \neq \{(0,0)\}$, and hence $r \geq \underline{g}_i$. Recall that $\mathcal{F}_i(D_i') \neq \{(0,0)\}$. So there exists s_3 such that $(\underline{g}_i, s_3) \in \mathcal{F}_i(D_i')$. Since $r' > r \geq \underline{g}_i$ and $s' > 0$, by feasibility of positive price, there exists $s_4 < s'$ such that $(\underline{g}_i, s_4) \in \mathcal{F}_i(D'_i)$. Then, given that $(r', s') > (\underline{g}_i, s_4)$, it follows from Condition 1 in Assumption 1 that $(r'-1, s'-1) \in \mathcal{F}_i(D'_i)$.

On the other hand, to show Condition 2 in Assumption 2, suppose that $r' > r$ and $s' \leq s$. Then $r' \geq \underline{g}_i$. By non-diminishing favorability in donors, there exists $s_1 \leq s' \leq s$ such that $(r', s_1) \in \mathcal{F}_i(D_i)$. It follows from Condition 2 in Assumption 1 that $(r+1, s) \in \mathcal{F}_i(D_i)$. Then, we argue that $(r, s') \in \mathcal{F}_i(D_i)$. This is true if $s' = s$. Suppose that $s' < s$. Then consider $(r', s_1) \in \mathcal{F}_i(D_i)$ and $(r, s) \in \mathcal{F}_i(D_i)$, where $r' > r$ and $s_1 \leq s' < s$. By repeated applications of Condition 3 in Assumption 1, we have $(r, s') \in \mathcal{F}_i(D_i)$. Finally, since $\mathcal{F}_i(D_i) \neq \{(0,0)\}$, $r \geq \underline{g}_i$. Given that $r' > r \geq \underline{g}_i$, $(r', s') \in \mathcal{F}_i(D'_i)$ and $(\underline{g}_i, s_2) \in \mathcal{F}_i(D'_i)$ for some s_2, it is straightforward to see that, by L-convexity, there exists s_3 such that $(r, s_3) \in \mathcal{F}_i(D'_i)$. Since $(r, s') \in \mathcal{F}_i(D_i)$ and $s' \leq |D'_i|$, by non-diminishing favorability in donors, there exists $s_4 \geq s'$ such that $(r, s_4) \in \mathcal{F}_i(D'_i)$. As $(r', s') \in \mathcal{F}_i(D'_i)$, $r' > r$ and $s' \leq s_4$, it follows from Condition 2 in Assumption 1 that $(r'-1, s') \in \mathcal{F}_i(D'_i)$. ■

We introduce new machinery to prove the donor monotonicity of f. In particular, we will construct *extended problems* in which the blood bank inventory is represented as a donor set, each blood type has a replica and there are some new dummy agents. Such a construction serves two purposes: it helps us represent allocations as *matchings*, which specify the donors that each patient receives blood from; it allows us to focus on the simple case where no patient receives blood from her own compatible donors.

First, we treat the blood bank b as a *pseudo patient* and introduce a donor set for it. It has a set of (volunteer non-remunerated) donors D_b, where for each blood type $X \in \mathcal{B}$ the blood bank has v_X donors. That is, for each $X \in \mathcal{B}$, $\left|\{d \in D_b : \beta_d = X\}\right| = v_X$.

Then, for each blood type $X \in \mathcal{B}$, we construct a *dummy blood type* $\widehat{X}$. Let $\widehat{\mathcal{B}} = \mathcal{B} \cup \{\widehat{X} : X \in \mathcal{B}\}$. Define a compatibility relation $\widehat{\mathcal{C}}$ as follows: for each $X \in \mathcal{B}$,

$$\widehat{\mathcal{C}}(X) = \mathcal{C}(X) \cup \{\widehat{Y} : Y \in \mathcal{C}(X)\} \quad \text{and} \quad \widehat{\mathcal{C}}(\widehat{X}) = \{X\}.$$

For each $X \in \mathcal{B}$, we construct a *dummy patient* $i_{\widehat{X}}$ and her (fixed) set of *dummy donors* $D_{i_{\widehat{X}}}$ such that

$$\begin{aligned} \beta_{i_{\widehat{X}}} &= \beta_d = \widehat{X} \quad \text{for every } d \in D_{i_{\widehat{X}}}, \text{ and} \\ \overline{n}_{i_{\widehat{X}}} &= \left|D_{i_{\widehat{X}}}\right| = \sum_{i \in I} \overline{n}_i. \end{aligned}$$

Moreover, let $\underline{g}_{i_{\widehat{X}}} = 0$, and her feasible schedule set be

$$\mathcal{F}_{i_{\widehat{X}}}(D_{i_{\widehat{X}}}) = \{(r, s) \in \mathbb{Z}^2_+ : s = r \leq \overline{n}_{i_{\widehat{X}}}\}.$$

For any problem D (under $\mathcal{B}$ and $\mathcal{C}$), we use $\widehat{D}$ to represent the corresponding **extended problem** (under $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$), after we treat the blood bank as a pseudo patient and its inventory as a donor set, and add the dummy patients and the dummy donors to the problem D.

Given an extended problem $\widehat{D}$, let $\widehat{I} = I \cup \{b\} \cup \{i_{\widehat{X}}\}_{X \in \mathcal{B}}$ and $\widehat{\mathbf{D}} = \bigcup_{i \in \widehat{I}} D_i$. From now on in this proof, we refer to each $i \in \widehat{I}$ as a **patient** (in reality it can be a real patient, a dummy patient, or the blood bank) and each $d \in \widehat{\mathbf{D}}$ as a **donor** (it can be a real donor, a dummy donor, or a unit of blood in the bank's inventory). A(n) **(auxiliary) matching** is a function $M : \widehat{I} \to 2^{\widehat{\mathbf{D}}}$, where the **match** of every patient $i \in \widehat{I}$, $M(i)$, is denoted as M_i by a slight abuse of notation, such that

1. $M_i \cap M_j = \emptyset$ for every $i, j \in \widehat{I}$ with $i \neq j$, and $\bigcup_{i \in \widehat{I}} M_i = \widehat{\mathbf{D}}$,
2. for every $i \in \widehat{I} \setminus \{b\}$ and $d \in M_i \setminus D_i$, $\quad \beta_d \in \widehat{\mathcal{C}}(\beta_i)$, and
3. for every $i \in \widehat{I} \setminus \{b\}$, $\quad (|M_i \setminus D_i|, |D_i \setminus M_i|) \in \mathcal{F}_i(D_i)$.

Let $\mathcal{M}(\widehat{D})$ be the set of all the matchings for $\widehat{D}$. Every (feasible) allocation $\alpha \in \mathcal{A}(D)$ in the problem D is associated with a matching $M \in \mathcal{M}(\widehat{D})$ in its extended problem $\widehat{D}$ and vice versa, as shown in the proof of Lemma A.3 below.[49] In particular, the match of a patient $i \in \widehat{I} \setminus \{b\}$ consists of two parts:

- The first part $M_i \setminus D_i$ is the set of donors that she receives blood from. These donors necessarily belong to other patients, and the blood types of these donors are compatible with patient i (Condition 2 in the definition of a matching).
- The second part $M_i \cap D_i$ is the set of her own donors who end up not donating. They are matched back with patient i.[50]

Therefore, patient i never receives blood from her own donors in a matching. Although this may not be the case in an allocation, we introduced the dummy patients and their dummy donors to account for this possibility. If in an allocation a patient $i \in I$ receives blood from one of her own donors, this is represented in a matching in the following manner:[51]

- this donor $d \in D_i$ is matched with the dummy patient induced by her blood type, $i_{\widehat{\beta_d}}$,

[49] As mentioned earlier, since $\mathcal{F}$ is fixed, we drop it from the arguments of the set of allocations (and the set of matchings as well). That is, we write $\mathcal{A}(D)$ instead of $\mathcal{A}(\mathcal{F}, D)$, with a slight abuse of notation.

[50] Similarly, the blood bank b receives donations from the donors $M_b \setminus D_b$, while keeping the donors $M_b \cap D_b$.

[51] See the proof of Lemma A.3 for the details of this construction.

- patient i is matched with one of the dummy donors of this dummy patient, i.e., with some $d' \in D_{i_{\widehat{\beta_d}}}$, in return.

As a result, the set of donors of any patient $i \in \widehat{I} \setminus \{b\}$ who actually donate in a matching M is $D_i \setminus M_i$. Therefore, $(|M_i \setminus D_i|, |D_i \setminus M_i|)$ has to be in the feasible schedule set $\mathcal{F}_i(D_i)$ (Condition 3 in the definition of a matching).

The analogue of a mechanism in the extended problems is a **rule**, which is a function F that maps each extended problem $\widehat{D}$ to a matching $F(\widehat{D}) \in \mathcal{M}(\widehat{D})$. A rule F is **donor monotonic** if for any D, D' and $i \in I$ such that $D'_i \subseteq D_i$ and $D'_j = D_j$ for every $j \in I \setminus \{i\}$, we have

$$|F_i(\widehat{D}) \setminus D_i| \geq |F_i(\widehat{D'}) \setminus D'_i|.$$[52]

We define a rule that is the counterpart of the weighted utilitarian mechanism f with respect to w. For each extended problem $\widehat{D}$ and matching $M \in \mathcal{M}(\widehat{D})$, define

$$\begin{aligned}\widehat{U}(M) = &\sum_{i \in I} \Big(w_i \cdot u_i \Big[|M_i \setminus D_i|, |D_i \setminus M_i| \Big] \Big) \\ &+ u_b \Big[\Big(\big| \{ d \in M_b \,:\, \beta_d \in \{X, \widehat{X}\} \} \big| \Big)_{X \in \mathcal{B}}, \sum_{i \in I} |M_i \setminus D_i| \Big]\end{aligned}$$

which is the weighted sum of the utilities of $I \cup \{b\}$ under M and w, after the type $\widehat{X}$ blood received by b is counted as type X blood, for each $X \in \mathcal{B}$. Then let F be a rule that maximizes such weighted sum of utilities, i.e., for every extended problem $\widehat{D}$,

$$F(\widehat{D}) \in \underset{M \in \mathcal{M}(\widehat{D})}{\arg\max}\, \widehat{U}(M).$$

Given a problem D, let $\phi(\alpha)$ denote the schedule profile induced by an allocation $\alpha \in \mathcal{A}(D)$, and $\phi(M)$ denote the schedule profile (for $I \cup \{b\}$) induced by a matching $M \in \mathcal{M}(\widehat{D})$, i.e.,

$$\phi(M) = \Big(\big(|M_i \setminus D_i|, |D_i \setminus M_i| \big)_{i \in I}, \Big(\big| \{ d \in M_b \,:\, \beta_d \in \{X, \widehat{X}\} \} \big| \Big)_{X \in \mathcal{B}} \Big).$$

Then, the following result implies that to finish the proof of Theorem 2, it is sufficient to show that the rule F is donor monotonic.

Lemma A.3 *For every problem D, $\phi\big(f(D)\big) = \phi\big(F(\widehat{D})\big)$.*

Proof of Lemma A.3. Consider any problem D. By the construction of f and F, to prove $\phi\big(f(D)\big) = \phi\big(F(\widehat{D})\big)$, we only need to show that (1) for every $\alpha \in \mathcal{A}(D)$, there is $M \in \mathcal{M}(\widehat{D})$ such that $\phi(\alpha) = \phi(M)$, and (2) for every $M \in \mathcal{M}(\widehat{D})$, there is $\alpha \in \mathcal{A}(D)$

[52] Note that we do not consider manipulations by the dummy patients as their donor sets are fixed.

such that $\phi(M) = \phi(\alpha)$. We show this in the following two parts.

Part 1. Let $\alpha \in \mathcal{A}(D)$. Consider the extended problem $\widehat{D}$, and any blood type $X \in \mathcal{B}$. Since $|D_{i_{\widehat{X}}}| = \sum_{j\in I} \overline{n}_j$, there exists a collection of disjoint donor sets $\{M_j^{\widehat{X}}\}_{j\in I\,:\,X\in\mathcal{C}(\beta_j)}$ such that for every $j \in I$ with $X \in \mathcal{C}(\beta_j)$, $M_j^{\widehat{X}} \subseteq D_{i_{\widehat{X}}}$ and $|M_j^{\widehat{X}}| = \alpha_X(j)$.

Moreover, since

$$\sum_{j\in I\,:\,X\in\mathcal{C}(\beta_j)} \alpha_X(j) + \alpha_X(b) = v_X + \sum_{d\in\bigcup_{j\in I} D_j\,:\,\beta_d = X} \alpha(d),$$

the donors

$$\{d \in D_b \,:\, \beta_d = X\} \cup \{d \in \bigcup_{j\in I} D_j \,:\, \beta_d = X,\ \alpha(d) = 1\}$$

can be put into two disjoint sets $M_{i_{\widehat{X}}}^X$ and M_b^X such that $|M_{i_{\widehat{X}}}^X| = \sum_{j\in I\,:\,X\in\mathcal{C}(\beta_j)} |M_j^{\widehat{X}}|$, and $|M_b^X| = \alpha_X(b)$.

Then we construct a matching M for $\widehat{D}$ as follows:

- for each $j \in I$, $\quad M_j = \left(\bigcup_{X\in\mathcal{C}(\beta_j)} M_j^{\widehat{X}}\right) \cup \{d \in D_j \,:\, \alpha(d) = 0\}$,
- for each $X \in \mathcal{B}$, $M_{i_{\widehat{X}}} = M_{i_{\widehat{X}}}^X \cup \Big(D_{i_{\widehat{X}}} \setminus \left(\bigcup_{j\in I\,:\,X\in\mathcal{C}(\beta_j)} M_j^{\widehat{X}}\right)\Big)$, and
- $M_b = \bigcup_{X\in\mathcal{B}} M_b^X$.

Therefore, each patient $j \in I$ is matched with $\alpha_X(j)$ dummy donors of type $\widehat{X}$ for every $X \in \mathcal{C}(\beta_j)$ (recall that for the extended problem, $\widehat{X} \in \widehat{\mathcal{C}}(\beta_j)$), and j's own donor d is matched with j if and only if $\alpha(d) = 0$. Moreover, for each dummy patient $i_{\widehat{X}}$, the number of X donors from $I \cup \{b\}$ matched with her is equal to the number of her $\widehat{X}$ donors that are not matched with her (recall that $\widehat{\mathcal{C}}(\widehat{X}) = \{X\}$). Finally, b is matched with $\alpha_X(b)$ donors of each type $X \in \mathcal{B}$. Hence, M is a well-defined matching for $\widehat{D}$ and $\phi(\alpha) = \phi(M)$.

Part 2. On the other hand, let $M \in \mathcal{M}(\widehat{D})$. Construct α as follows:

- for each $j \in I$ and $X \in \mathcal{C}(\beta_j)$, let $\alpha_X(j) = \left|\{d \in M_j \setminus D_j \,:\, \beta_d \in \{X, \widehat{X}\}\}\right|$,
- for each $j \in I$ and $d \in D_j$, let $\alpha(d) = 0$ if $d \in M_j$, and $\alpha(d) = 1$ if $d \notin M_j$, and
- for each $X \in \mathcal{B}$, let $\alpha_X(b) = \left|\{d \in M_b \,:\, \beta_d \in \{X, \widehat{X}\}\}\right|$.

If α is an allocation for D, then it is straightforward to see that $\phi(M) = \phi(\alpha)$. To show that α is a well-defined allocation, we only need to verify the market clearing conditions: for any blood type $X \in \mathcal{B}$,

$$
\begin{aligned}
&\sum_{j \in I \,:\, X \in \mathcal{C}(\beta_j)} \alpha_X(j) + \alpha_X(b) \\
&= \sum_{j \in I \,:\, X \in \mathcal{C}(\beta_j)} \left|\{d \in M_j \setminus D_j \,:\, \beta_d = X\}\right| + \left|\{d \in M_b \,:\, \beta_d = X\}\right| + \sum_{j \in I \,:\, X \in \mathcal{C}(\beta_j)} \left|M_j \cap D_{i_{\widehat{X}}}\right| \\
&\quad + \left|M_b \cap D_{i_{\widehat{X}}}\right| \\
&= \sum_{j \in I \,:\, X \in \mathcal{C}(\beta_j)} \left|\{d \in M_j \setminus D_j \,:\, \beta_d = X\}\right| + \left|\{d \in M_b \,:\, \beta_d = X\}\right| + \left|\{d \in M_{i_{\widehat{X}}} \,:\, \beta_d = X\}\right| \\
&= \sum_{j \in I} \left|\{d \in D_j \setminus M_j \,:\, \beta_d = X\}\right| + \left|\{d \in D_b \,:\, \beta_d = X\}\right| \\
&= \sum_{d \in \bigcup_{j \in I} D_j \,:\, \beta_d = X} \alpha(d) + v_X
\end{aligned}
$$

where the second equality follows from the construction of $\mathcal{F}_{i_{\widehat{X}}}(D_{i_{\widehat{X}}})$, as well as the fact that $\widehat{\mathcal{C}}(\widehat{X}) = \{X\}$. ■

The proof of the donor monotonicity of the rule F relies on comparing two matchings for two extended problems and constructing two new ones based on the differences between the matches of the patients, respectively. We introduce the following graph theoretical concepts that are central to the proof.

Let $\widehat{D}$ and $\widehat{D'}$ be two extended problems such that $D'_i \subseteq D_i$ for every $i \in I$. For ease of exposition we also write $D'_{i_{\widehat{X}}} = D_{i_{\widehat{X}}}$ for every $X \in \mathcal{B}$ and $D'_b = D_b$. Given a matching M for $\widehat{D}$ and a matching M' for $\widehat{D'}$, a **cycle from M to M'** is a directed graph of patients and donors in which each patient/donor points to the next donor/patient, and is denoted as a list $C = (i_1, d_1, \ldots, i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, \ldots, \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

1. $i_t \in \widehat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then
   $$(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1) \in \mathcal{F}_{i_t}(D_{i_t}) \quad \text{and} \quad (\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1) \in \mathcal{F}_{i_t}(D'_{i_t}).$$
3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
   $$(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1) \in \mathcal{F}_{i_t}(D_{i_t}) \quad \text{and} \quad (\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1) \in \mathcal{F}_{i_t}(D'_{i_t}).$$
4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then $i \neq b$, and either (i) $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or (ii) $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

In a cycle C from M to M', each patient points to a donor that she is matched with

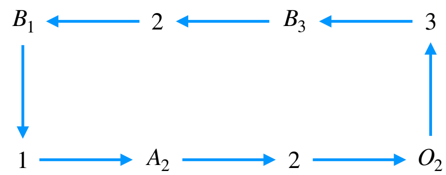

Figure A.11: Suppose that $I = \{1, 2, 3\}$, with $\beta_1 = A$, $\beta_2 = B$ and $\beta_3 = O$, $\widehat{D} = \widehat{D'}$, and the donor sets are given by $D_1 = \{B_1\}$, $D_2 = \{A_2, O_2\}$, $D_3 = \{B_3\}$ and $D_b = \emptyset$, where a type-X donor of a patient i is denoted as X_i. For simplicity, we omit the dummy patients and dummy blood types. For every $i \in I$, $\overline{n}_i = 1$, $\underline{g}_i = 0$ and the exchange rate is one-for-one. Assume ABO-identical transfusion. Consider the following two matchings M and M': $M_1 = \{B_1\}$, $M_2 = \{A_2, B_3\}$, $M_3 = \{O_2\}$ and $M_b = \emptyset$; $M'_1 = \{A_2\}$, $M'_2 = \{O_2, B_1\}$, $M'_3 = \{B_3\}$ and $M'_b = \emptyset$. The above graph gives a cycle C from M to M', and we have $M + C = M'$ and $M' - C = M$.

under M' but not under M, while each donor points to the patient that she is matched with under M. Note that each donor in the cycle must be in both extended problems, $\widehat{D}$ and $\widehat{D'}$. Starting from the base matching M, we can assign each patient in the cycle the donor she points to (who is one of her M' matches) instead of the donor she is pointed by (who is one of her M matches). That is, for each $t \in \{1, \ldots, \bar{t}\}$, add d_t to M_{i_t} and remove d_{t-1} from M_{i_t}. Condition 1 above guarantees that this leads to a well-defined function, which we denote as $M + C$ and satisfies Conditions 1 and 2 in the definition of a matching (for $\widehat{D}$). The patients involved in the cycle may not be distinct. But Condition 4 above says that if a patient $i \in \widehat{I}$ appears twice in the cycle, then $i \neq b$, and her schedule is not affected by the exchanges, i.e., the amount of blood received and the amount of blood supplied remain the same. Note that this condition also implies that any patient cannot appear more than twice in the cycle. Finally, if a patient $i \in \widehat{I} \setminus \{b\}$ is assigned a different schedule under $M + C$ than under M, then she appears only once in the cycle, and she either receives one more unit and supplies one more unit, or receives one less unit and supplies one less unit. Then Conditions 2 and 3 above imply Condition 3 in the definition of a matching. Therefore $M + C$ is a matching for $\widehat{D}$. Similarly, we can instead start from M' and assign each patient in the cycle the donor she is pointed by (who is one of her M matches) instead of the donor she points to (who is one of her M' matches). That is, for each $t \in \{1, \ldots, \bar{t}\}$, add d_{t-1} to M'_{i_t} and remove d_t from M'_{i_t}. These exchanges also lead to a well-defined matching for $\widehat{D'}$, denoted as $M' - C$. In Figure A.11, we give an example of a cycle and the construction of new matchings using this cycle.

It is wise to note that the cycle operations do not necessarily make all patients involved better off or worse off. Instead, they generate new matchings that are closer to each other in terms of the matches of the patients.

Another concept similar to a cycle is a chain. A **chain from M to M'** is a list $C = (i_1, d_1, \dots, i_{\bar{t}-1}, d_{\bar{t}-1}, i_{\bar{t}})$, $\bar{t} \geq 2$, such that

1. For every $t \in \{1, \dots, \bar{t}\}$, $i_t \in \widehat{I}$. Moreover, $i_1 \neq i_{\bar{t}}$, and $i_t = b$ implies $t \in \{1, \bar{t}\}$.
2. For every $t \in \{1, \dots, \bar{t}-1\}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
3. For every $t \in \{2, \dots, \bar{t}-1\}$, if $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then
$$\left(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1\right) \in \mathcal{F}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1\right) \in \mathcal{F}_{i_t}(D'_{i_t}).$$
4. For every $t \in \{2, \dots, \bar{t}-1\}$, if $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then
$$\left(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1\right) \in \mathcal{F}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1\right) \in \mathcal{F}_{i_t}(D'_{i_t}).$$
5. If $i_{\bar{t}} \neq b$, then
$$\left(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right| + 1\right) \in \mathcal{F}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad \left(\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right|, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| - 1\right) \in \mathcal{F}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$
when $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, and
$$\left(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right| - 1, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|\right) \in \mathcal{F}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad \left(\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| + 1, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right|\right) \in \mathcal{F}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$
when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$.
6. If $i_1 \neq b$, then
$$\left(\left|M_{i_1} \setminus D_{i_1}\right|, \left|D_{i_1} \setminus M_{i_1}\right| - 1\right) \in \mathcal{F}_{i_1}(D_{i_1}) \quad \text{and} \quad \left(\left|M'_{i_1} \setminus D'_{i_1}\right|, \left|D'_{i_1} \setminus M'_{i_1}\right| + 1\right) \in \mathcal{F}_{i_1}(D'_{i_1})$$
when $d_1 \in D_{i_1}$, and
$$\left(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right|\right) \in \mathcal{F}_{i_1}(D_{i_1}) \quad \text{and} \quad \left(\left|M'_{i_1} \setminus D'_{i_1}\right| - 1, \left|D'_{i_1} \setminus M'_{i_1}\right|\right) \in \mathcal{F}_{i_1}(D'_{i_1})$$
when $d_1 \notin D_{i_1}$.
7. If $i_t = i_{t'} = i$ for some t, t' such that $1 < t < t' < \bar{t}$, then either (i) $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or (ii) $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

   If $i_{\bar{t}} = i_t = i$ for some t such that $1 < t < \bar{t}$, then either (i) $d_t, d_{t-1} \in D_i$ and $d_{\bar{t}-1} \notin D_i$, or (ii) $d_t, d_{t-1} \notin D_i$ and $d_{\bar{t}-1} \in D_i$.

   If $i_1 = i_t = i$ for some t such that $1 < t < \bar{t}$, then either (i) $d_t, d_{t-1} \in D_i$ and $d_1 \notin D_i$, or (ii) $d_t, d_{t-1} \notin D_i$ and $d_1 \in D_i$.

A chain differs from a cycle as the last element of a chain is a patient and she does not point to any donor. We refer to this patient, $i_{\bar{t}}$, as the **head** of the chain. As a result there is no donor pointing back to i_1 whom we refer to as the **tail** of the chain. The head and the tail of the chain cannot be the same, and b can only appear as either the head or the tail (Condition 1).

$$1 \longrightarrow A_b \longrightarrow 2 \longrightarrow A_3 \longrightarrow 3$$

Figure A.12: Suppose that $I = \{1, 2, 3\}$ with $\beta_1 = \beta_2 = A$ and $\beta_3 = B$. The donor sets in two extended problems $\widehat{D}$ and $\widehat{D'}$ are given by $D_1 = \{B_1\}$, $D'_1 = \emptyset$, $D_2 = D'_2 = \emptyset$, $D_3 = D'_3 = \{A_3\}$ and $D_b = \{A_b, A'_b, B_b\}$, where X_i (or X'_i) denotes a type-X donor of i. For simplicity, we omit the dummy patients and dummy blood types. For every $i \in I$, $\overline{n}_i = 2$, $\underline{g}_i = 0$ and the feasible schedules are such that the amount supplied does not exceed the amount received. Assume ABO-identical transfusion. Consider a matching M for $\widehat{D}$, where $M_1 = \{A'_b\}$, $M_2 = \{A_b\}$, $M_3 = \{A_3, B_1, B_b\}$ and $M_b = \emptyset$, and a matching M' for $\widehat{D'}$, where $M'_1 = \{A_b, A'_b\}$, $M'_2 = \{A_3\}$, $M'_3 = \{B_b\}$ and $M'_b = \emptyset$. There does not exist a cycle from M to M', but the above graph gives a chain C from M to M'. Then $M + C$ is a matching for $\widehat{D}$, where $(M + C)_1 = \{A_b, A'_b\}$, $(M + C)_2 = \{A_3\}$, $(M + C)_3 = \{B_1, B_b\}$ and $(M + C)_b = \emptyset$. Moreover, $M' - C$ is a matching for $\widehat{D'}$, where $(M' - C)_1 = \{A'_b\}$, $(M' - C)_2 = \{A_b\}$, $(M' - C)_3 = \{A_3, B_b\}$ and $(M' - C)_b = \emptyset$.

Similar to the case of a cycle, given a chain C from M to M', we can construct a new matching, denoted as $M + C$, for $\widehat{D}$ as follows: starting from M, for each t such that $1 \le t \le \bar{t} - 1$, remove d_t from $M_{i_{t+1}}$ and add it to M_{i_t}. Condition 7 above implies that any patient cannot appear more than twice in a chain. Moreover, if a patient $i \in \widehat{I} \setminus \{b\}$ is assigned a different schedule under $M + C$ than under M, and she appears twice in the chain, then she must appear exactly once as the head or the tail, and only this appearance as the head or the tail affects her schedule. Then Conditions 3, 4, 5, and 6 ensure that the schedule of each patient $i \in \widehat{I} \setminus \{b\}$ under $M + C$ is indeed feasible. In particular, Conditions 3 and 4 are similar to those of a cycle, while Conditions 5 and 6 deal with special considerations for the head and tail patients. On the other hand, we can also construct a new matching, denoted as $M' - C$, for $\widehat{D'}$ as follows: starting from M', for each $1 \le t \le \bar{t} - 1$, remove d_t from M'_{i_t} and add it to $M'_{i_{t+1}}$. See Figure A.12 for an example of a chain and how new matchings are constructed using this chain.

Unlike in a cycle addition or removal, in the chain operations the number of donors that a patient is matched with only stays the same if she is neither the head nor the tail. Thus, the chain operations change the overall balance of the base matching, while cycle operations do not. The cycle operations would be all we needed if we were dealing with the one-for-one exogenous exchange rate. However, the chain operations play an important role in the general case with endogenously determined exchange rates.

The following observation is straightforward to show from the construction.

Observation 6 *Let C be a cycle or a chain from $M \in \mathcal{M}(\widehat{D})$ to $M' \in \mathcal{M}(\widehat{D'})$. For every $i \in \widehat{I} \setminus \{b\}$, we have*

$$\left|(M + C)_i \setminus D_i\right| - \left|M_i \setminus D_i\right| = \left|M'_i \setminus D'_i\right| - \left|(M' - C)_i \setminus D'_i\right| \in \{-1, 0, 1\},$$

and

$$\left|D_i \setminus (M+C)_i\right| - \left|D_i \setminus M_i\right| = \left|D'_i \setminus M'_i\right| - \left|D'_i \setminus (M'-C)_i\right| \in \{-1,0,1\}.$$

For every $X \in \widehat{\mathcal{B}}$,

$$\begin{aligned}&\left|\{d \in (M+C)_b \,:\, \beta_d = X\}\right| - \left|\{d \in M_b \,:\, \beta_d = X\}\right| \\ &= \left|\{d \in M'_b \,:\, \beta_d = X\}\right| - \left|\{d \in (M'-C)_b \,:\, \beta_d = X\}\right| \in \{-1,0,1\}.\end{aligned}$$

In the remaining of the proof of Theorem 2, we show two lemmata. The first one, Lemma A.4, is the most crucial result in the proof of the theorem. It gives a general necessary condition for any rule that is not donor monotonic. Using this result, we show that F is donor monotonic (Lemma A.5), which concludes the proof.

Lemma A.4 *Consider any D, D' and $i \in I$ such that $D'_i \subseteq D_i$, $\left|D_i \setminus D'_i\right| = 1$, and $D'_j = D_j$ for every $j \in I \setminus \{i\}$. If $M \in \mathcal{M}(\widehat{D})$, $M' \in \mathcal{M}(\widehat{D'})$, and $\left|M'_i \setminus D'_i\right| > \left|M_i \setminus D_i\right|$, then there exists a cycle or a chain C from M to M'. Moreover, for all $j \in \widehat{I} \setminus \{b\}$,*

$$\left|(M+C)_j \setminus D_j\right| > \left|M_j \setminus D_j\right| \textit{ implies } \left|M'_j \setminus D'_j\right| > \left|M_j \setminus D_j\right|, \textit{ and}$$
$$\left|(M+C)_j \setminus D_j\right| < \left|M_j \setminus D_j\right| \textit{ implies } \left|M'_j \setminus D'_j\right| < \left|M_j \setminus D_j\right|.$$

Proof of Lemma A.4. Consider two problems D, D' such that for some patient $i_1 \in I$, $D'_{i_1} \subseteq D_{i_1}$, $\left|D_{i_1} \setminus D'_{i_1}\right| = 1$, and $D'_i = D_i$ for every $i \in I \setminus \{i_1\}$. Suppose that $M \in \mathcal{M}(\widehat{D})$, $M' \in \mathcal{M}(\widehat{D'})$, and $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$. We first show the existence of a cycle or a chain from M to M'.

Since $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$, there exists a donor $d_1 \notin D_{i_1}$ such that $d_1 \in M'_{i_1} \setminus M_{i_1}$. We will iteratively construct a finite directed graph of patients and donors using the matchings M and M', which is denoted as $(i_1, d_1, i_2, d_2, \ldots)$. It starts with patient i_1, ends with either a patient or a donor, and each node in the list points to the next node.

We refer to this as the *pointing procedure from M to M'*:

Step 1: Let i_1 point to d_1, and d_1 point to $i_2 \in \widehat{I}$ such that $d_1 \in M_{i_2}$. If $i_2 = b$ then we stop at i_2 at Step 1, otherwise we continue with Step 2.

Step $t \geq 2$: At the end of Step $t-1$, patient $i_t \in \widehat{I} \setminus \{i_1, b\}$ is pointed by d_{t-1} where $d_{t-1} \in M_{i_t} \setminus M'_{i_t}$.

1. If $d_{t-1} \in D_{i_t}$: We have two cases:
   (a) If there exists $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point

to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.[53]

   (b) If there does not exist $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|D'_{i_t} \setminus M'_{i_t}\right| > \left|D_{i_t} \setminus M_{i_t}\right|$. We have two subcases:

      i. If $\left|M'_{i_t} \setminus D'_{i_t}\right| > \left|M_{i_t} \setminus D_{i_t}\right|$: Then there exists $d_t \notin D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.

      ii. If $\left|M'_{i_t} \setminus D'_{i_t}\right| \leq \left|M_{i_t} \setminus D_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.

2. If $d_{t-1} \notin D_{i_t}$: We have two cases:

   (a) If there exists $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.

   (b) If there does not exist $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|M'_{i_t} \setminus D'_{i_t}\right| < \left|M_{i_t} \setminus D_{i_t}\right|$. We have two subcases:

      i. If $\left|D'_{i_t} \setminus M'_{i_t}\right| < \left|D_{i_t} \setminus M_{i_t}\right|$: Then there exists $d_t \in D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.

      ii. If $\left|D'_{i_t} \setminus M'_{i_t}\right| \geq \left|D_{i_t} \setminus M_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.

If d_t is constructed, $i_t = i_{\underline{t}} \notin \{i_1, b\}$ for some $\underline{t} < t$, and neither

- $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, nor
- $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$

holds, then stop at donor d_t at Step t and remove i_{t+1} from the graph construction.

If d_t is constructed, the procedure does not stop at d_t, and $i_{t+1} \in \{i_1, b\}$, then stop at i_{t+1} at Step t.

Otherwise, continue with Step $t+1$.

Note that, according to the above construction, $i_t \neq i_{t+1}$ for any t. Moreover, the procedure stops under four circumstances:

- when some $i \notin \{i_1, b\}$ has appeared before, and the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,

[53]Generally for each $t \geq 1$, such i_{t+1} always exists, since $d_t \in \widehat{\mathbf{D}'} \subseteq \widehat{\mathbf{D}}$.

- when i_1 is pointed,
- when b is pointed,
- when some $i \notin \{i_1, b\}$ does not point.

The first circumstance implies that any patient can be pointed at most three times in the procedure. Hence, the procedure always stops in a finite number of steps.

We consider the following four cases based on these circumstances. Case 1 and Case 2 cover the first two circumstances in order and show the existence of a cycle in each case. Case 3 covers the third and the fourth circumstances together when i_1 does not supply more blood under M' than under M, and shows the existence of a chain. Finally, Case 4 is the most involved case. It covers the third and the fourth circumstances together when i_1 supplies more blood under M' than under M, and shows the existence of a cycle or a chain.

Case 1. The procedure stops at $d_{\bar{t}}$ at Step $\bar{t}$.

Then for some $\underline{t} < \bar{t}$, $i_{\underline{t}} = i_{\bar{t}} \notin \{i_1, b\}$ and neither of the following is true:

1. $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_{\underline{t}}}$.
2. $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \in D_{i_{\underline{t}}}$.

We show that $(i_{\underline{t}}, d_{\underline{t}}, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

First, for any t such that $\underline{t} < t \leq \bar{t} - 1$, $i_t \notin \{i_1, b\}$, since otherwise the procedure stops at i_t at Step $t-1$. It follows that $D_{i_t} = D'_{i_t}$ for every t such that $\underline{t} \leq t \leq \bar{t} - 1$. By the construction of the pointing procedure from M to M', Condition 1 in the definition of a cycle is satisfied. Next, we show Condition 2 and Condition 3.

First, consider any t such that $\underline{t} < t \leq \bar{t} - 1$. If $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then by the construction, we have $|M'_{i_t} \setminus D'_{i_t}| > |M_{i_t} \setminus D_{i_t}|$ and $|D'_{i_t} \setminus M'_{i_t}| > |D_{i_t} \setminus M_{i_t}|$. Since $(|M_{i_t} \setminus D_{i_t}|, |D_{i_t} \setminus M_{i_t}|) \in \mathcal{F}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \setminus D'_{i_t}|, |D'_{i_t} \setminus M'_{i_t}|) \in \mathcal{F}_{i_t}(D'_{i_t}) = \mathcal{F}_{i_t}(D_{i_t})$, it follows from Assumption 1 that

$(|M_{i_t} \setminus D_{i_t}| + 1, |D_{i_t} \setminus M_{i_t}| + 1) \in \mathcal{F}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \setminus D'_{i_t}| - 1, |D'_{i_t} \setminus M'_{i_t}| - 1) \in \mathcal{F}_{i_t}(D'_{i_t})$.

Similarly, if $d_{t-1} \notin D_{i_t}$ and $d_t \in D_{i_t}$, then by the construction we have $|M'_{i_t} \setminus D'_{i_t}| < |M_{i_t} \setminus D_{i_t}|$ and $|D'_{i_t} \setminus M'_{i_t}| < |D_{i_t} \setminus M_{i_t}|$. It follows from Assumption 1 that

$(|M_{i_t} \setminus D_{i_t}| - 1, |D_{i_t} \setminus M_{i_t}| - 1) \in \mathcal{F}_{i_t}(D_{i_t})$ and $(|M'_{i_t} \setminus D'_{i_t}| + 1, |D'_{i_t} \setminus M'_{i_t}| + 1) \in \mathcal{F}_{i_t}(D'_{i_t})$.

Second, consider $i_{\underline{t}}$. Suppose that $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$. Then either $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ or $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, as the procedure stops at the donor $d_{\bar{t}}$. Since we have either (i) $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$

and $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, or (ii) $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$, by the construction,

$$\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| > \left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| \text{ and } \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| > \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right|.$$

Then by Assumption 1,

$$(\left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| + 1, \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right| + 1) \in \mathcal{F}_{i_{\underline{t}}}(D_{i_{\underline{t}}}) \text{ and } (\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| - 1, \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| - 1) \in \mathcal{F}_{i_{\underline{t}}}(D'_{i_{\underline{t}}}).$$

That is, Condition 2 in the definition of a cycle is satisfied for $i_{\underline{t}}$. By similar arguments, it can be shown that Condition 3 is also satisfied for $i_{\underline{t}}$.

It remains to show Condition 4. If $i_t = i_{t'}$ and $\underline{t} < t < t' \leq \bar{t} - 1$, then either (i) $d_t, d_{t-1} \in D_{i_t}$ and $d_{t'}, d_{t'-1} \notin D_{i_t}$, or (ii) $d_t, d_{t-1} \notin D_{i_t}$ and $d_{t'}, d_{t'-1} \in D_{i_t}$, since otherwise the procedure stops at $d_{t'}$ at Step t'. Finally, suppose that $i_t = i_{\underline{t}}$ and $\underline{t} + 1 < t < \bar{t} - 1$. Since the procedure does not stop at d_t at Step t, we have either

(i) $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, or,

(ii) $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$.

Consider (i) first. Recall that $i_t = i_{\underline{t}} = i_{\bar{t}}$. If $d_{\bar{t}-1} \notin D_{i_t}$, then by the construction of the pointing procedure from M to M', $d_t \notin D_{i_t}$ implies that there exists a donor in $M'_{i_t} \setminus M_{i_t}$ that is not her own, and thus, she should again point to such a donor when she appears for the third time as $i_{\bar{t}}$: $d_{\bar{t}} \notin D_{i_t}$. So we have $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$, which contradicts to Case 1's assumption. Therefore, $d_{\underline{t}}, d_{\bar{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$. Similarly, if (ii) is true, then $d_{\bar{t}-1} \notin D_{i_t}$, since otherwise $d_t \in D_{i_t}$ implies $d_{\bar{t}} \in D_{i_t}$, leading to a contradiction. Hence, $d_{\underline{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$. This shows that Condition 4 holds, as well.

<u>Case 2.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t} - 1$ and $i_{\bar{t}} = i_1$.

To show that $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M', where $d_1 \notin D_{i_1}$, we verify Condition 2 in the definition of a cycle when $d_{\bar{t}-1} \in D_{i_1}$. Since $d_{\bar{t}-1} \in M_{i_1}$ and $d_{\bar{t}-1} \in M'_{i_{\bar{t}-1}}$, $\left|D_{i_1} \setminus M_{i_1}\right| < \left|D_{i_1}\right|$ and $\left|D'_{i_1} \setminus M'_{i_1}\right| > 0$. Then given that $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$, by Assumption 2, we have

$$(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right| + 1) \in \mathcal{F}_{i_1}(D_{i_1}) \text{ and } (\left|M'_{i_1} \setminus D'_{i_1}\right| - 1, \left|D'_{i_1} \setminus M'_{i_1}\right| - 1) \in \mathcal{F}_{i_1}(D'_{i_1}).$$

The other conditions on the cycle can be shown similarly as in Case 1.

<u>Case 3.</u> The procedure stops at $i_{\bar{t}}$ at Step $\bar{t} - 1$, $i_{\bar{t}} \neq i_1$, and $\left|D'_{i_1} \setminus M'_{i_1}\right| \leq \left|D_{i_1} \setminus M_{i_1}\right|$.

Then either $i_{\bar{t}} = b$ or in the procedure the patient $i_{\bar{t}} \in \widehat{I} \setminus \{i_1, b\}$ does not point. We show that $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. First, $i_t \neq b$ for any $t \in \{2, \ldots, \bar{t} - 1\}$ since otherwise the procedure stops at an earlier step. Second, we verify Condition 5 in the definition of a chain. Suppose that $i_{\bar{t}} \neq b$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then by the construction, $\left|D'_{i_{\bar t}} \setminus M'_{i_{\bar t}}\right| > \left|D_{i_{\bar t}} \setminus M_{i_{\bar t}}\right|$ and $\left|M'_{i_{\bar t}} \setminus D'_{i_{\bar t}}\right| \leq \left|M_{i_{\bar t}} \setminus D_{i_{\bar t}}\right|$. Given that $D_{i_{\bar t}} = D'_{i_{\bar t}}$, by Assumption 1,

$$(\left|M_{i_{\bar t}} \setminus D_{i_{\bar t}}\right|, \left|D_{i_{\bar t}} \setminus M_{i_{\bar t}}\right| + 1) \in \mathcal{F}_{i_{\bar t}}(D_{i_{\bar t}}) \text{ and } (\left|M'_{i_{\bar t}} \setminus D'_{i_{\bar t}}\right|, \left|D'_{i_{\bar t}} \setminus M'_{i_{\bar t}}\right| - 1) \in \mathcal{F}_{i_{\bar t}}(D'_{i_{\bar t}}).$$

The case that $d_{\bar t-1} \notin D_{i_{\bar t}}$ can be shown similarly. Next, Condition 6 follows from the fact that $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$ and $\left|D'_{i_1} \setminus M'_{i_1}\right| \leq \left|D_{i_1} \setminus M_{i_1}\right|$, as well as Assumption 2. Finally, we verify Condition 7 for i_1 and $i_{\bar t}$. For any $t \in \{2, \ldots, \bar t - 1\}$, $i_1 \neq i_t$, since otherwise the procedure stops at an earlier step. Suppose that $i_{\bar t} = i_t$ for some $t \in \{2, \ldots, \bar t - 1\}$. Then $i_{\bar t} = i_t \neq b$. First consider the case that $d_{\bar t-1} \in D_{i_t}$. If $d_t \in D_{i_t}$, then, given that $d_t \in M'_{i_t} \setminus M_{i_t}$, $i_{\bar t} = i_t$ should point to this donor (or some other donor of her own) at Step $\bar t$, which contradicts to the fact that the pointing procedure stops at $i_{\bar t}$. So $d_t \notin D_{i_t}$. Then $d_{t-1} \notin D_{i_t}$, since otherwise $i_{\bar t} = i_t$ should point to d_t (or some other donor that is not her own) at Step $\bar t$. In the case that $d_{\bar t-1} \notin D_{i_t}$, it can be similarly shown that $d_t, d_{t-1} \in D_{i_t}$. These are the crucial conditions to check; the other conditions can be shown similarly as in Case 1.

Case 4. The procedure stops at $i_{\bar t}$ at Step $\bar t - 1$, $i_{\bar t} \neq i_1$, and $\left|D'_{i_1} \setminus M'_{i_1}\right| > \left|D_{i_1} \setminus M_{i_1}\right|$.

In this case, we may not have $(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right|) \in \mathcal{F}_{i_1}(D_{i_1})$, and hence $(i_1, d_1, \ldots, d_{\bar t-1}, i_{\bar t})$ may not be a chain from M to M'.

Let $j_1 = i_1$. Since $\left|D'_{j_1} \setminus M'_{j_1}\right| > \left|D_{j_1} \setminus M_{j_1}\right|$, there exists a donor $c_1 \in D'_{j_1}$ such that $c_1 \in M_{j_1} \setminus M'_{j_1}$. To find a cycle or a chain, we consider the reverse of the previous construction and use the *pointing procedure from M' to M*. It starts with j_1 pointing to c_1. Then M and D in the pointing procedure from M to M' are replaced with M' and D' respectively, and M' and D' in the pointing procedure from M to M' are replaced with M and D respectively. This pointing procedure from M' to M constructs another directed graph of patients and donors, denoted as $(j_1, c_1, j_2, c_2, \ldots)$, and each node in the list points to the next node in the list. Compared to the previous procedure, there are two slight complications.

First, recall that $D'_{j_1} \subseteq D_{j_1}$ and $\left|D_{j_1} \setminus D'_{j_1}\right| = 1$. We refer to the donor in the set $D_{j_1} \setminus D'_{j_1}$ as the *concealed donor*. If the concealed donor is pointed by j_t at Step $t \geq 2$,[54] let this donor, c_t, point to $j_{t+1} = j_1$.

Second, there is an additional circumstance in which the procedure stops. At Step $t \geq 2$, if c_t is constructed, $j_t = i_{\underline{t}} \notin \{j_1, b\}$ for some $\underline{t} \in \{2, \ldots, \bar t - 1\}$, and neither

- $c_t, c_{t-1} \in D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{j_t}$, nor

[54] This can happen in Step t 1.(b)i and Step t 2.(a). Note that the concealed donor does not appear in the pointing procedure from M to M'.

- $c_t, c_{t-1} \notin D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{j_t}$

holds, then stop at donor c_t at Step t and remove j_{t+1} from the graph construction.

Then the pointing procedure from M' to M stops under five circumstances, instead of four:

- when some $j \notin \{j_1, b\}$ has appeared before in the pointing procedure from M' to M, and the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when some $j \notin \{j_1, b\}$ has appeared before in the pointing procedure from M to M', and in this previous appearance her role is not $i_{\bar{t}}$. Moreover, the following is not true: she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when b is pointed,
- when some $j \notin \{j_1, b\}$ does not point,
- when j_1 is pointed.

Due to the first circumstance, the pointing procedure from M' to M also stops in a finite number of steps. Since we are seeking a cycle or a chain from M to M', after the procedure stops we reverse the orientation of the constructed edges in $(j_1, c_1, j_2, c_2, \ldots)$.

We consider the following five subcases based on these circumstances. Subcase 4.1 and Subcase 4.2 cover the first two circumstances and show the existence of a cycle in each subcase. Subcase 4.3 covers the third and the fourth circumstances together and shows the existence of a cycle or a chain. Subcase 4.4 covers the fifth circumstance when j_1 is not pointed by the concealed donor, and shows the existence of a cycle. Finally, Subcase 4.5 covers the fifth circumstance when j_1 is pointed by the concealed donor and shows the existence of a cycle or a chain.

<u>Subcase 4.1.</u> The procedure stops at c_t at Step t, for some $\underline{t} < t$, $j_t = j_{\underline{t}} \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \in D_{j_t}$.

Then, after reversing the edges in the second directed graph, $(j_t, c_{t-1}, \ldots, j_{\underline{t}+1}, c_{\underline{t}})$ is a cycle from M to M'.

Subcase 4.2. The procedure stops at c_t at Step t, for some $\underline{t} \in \{2, \ldots, \bar{t}-1\}$, $i_{\underline{t}} = j_t \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{j_t}$.

We construct a cycle using the first directed graph given by the pointing procedure from M to M', $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$, and the second directed graph given by the pointing procedure from M' to M, $(j_1, c_1, \ldots, j_t, c_t)$. Recall that $j_1 = i_1$ and the orientation of the edges in the second graph should be reversed. Then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\underline{t}-1}, d_{\underline{t}-1})$ is a cycle from M to M'.

Subcase 4.3. The procedure stops at j_t at Step $t-1$, and $j_t \neq j_1$.

Then either $j_t = b$ or the patient j_t does not point.

If $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

If it is not true that $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. To see this, we verify $j_t \neq i_{\bar{t}}$ and Condition 6 in the definition of a chain. First, assume to the contrary, $j_t = i_{\bar{t}}$. Then $j_t = i_{\bar{t}} \in \widehat{I} \setminus \{j_1, b\}$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then $c_{t-1} \notin D_{i_{\bar{t}}}$, since otherwise in the pointing procedure from M' to M, j_t should point to $d_{\bar{t}-1}$ (or some other donor of her own) at Step t. However, by the construction of the two pointing procedures, $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$ implies $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| > \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$ and $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| \leq \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$, while $c_{t-1} \notin D_{i_{\bar{t}}}$ implies $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| > \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$ and $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| \leq \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$, contradiction. A similar contradiction can be reached when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$. Therefore, $j_t \neq i_{\bar{t}}$. Second, consider Condition 6. If $j_t \neq b$, and $c_{t-1} \in D_{j_t}$, then by the construction we have $\left|D'_{j_t} \setminus M'_{j_t}\right| < \left|D_{j_t} \setminus M_{j_t}\right|$ and $\left|M'_{j_t} \setminus D'_{j_t}\right| \geq \left|M_{j_t} \setminus D_{j_t}\right|$. It follows from Assumption 1 that

$$(\left|M_{j_t} \setminus D_{j_t}\right|, \left|D_{j_t} \setminus M_{j_t}\right| - 1) \in \mathcal{F}_{j_t}(D_{j_t}) \text{ and } (\left|M'_{j_t} \setminus D'_{j_t}\right|, \left|D'_{j_t} \setminus M'_{j_t}\right| + 1) \in \mathcal{F}_{j_t}(D'_{j_t}).$$

The case that $c_{t-1} \notin D_{j_t}$ can be shown similarly.

Subcase 4.4. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ and $c_{t-1} \notin D_{j_1} \setminus D'_{j_1}$.

Then $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is a cycle from M to M'.

Subcase 4.5. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ and $c_{t-1} \in D_{j_1} \setminus D'_{j_1}$.

Recall that $j_t = j_1 = i_1$ is the patient who concealed her donor, c_{t-1}. First, we have $j_{t'} \in \widehat{I} \setminus \{j_t, b\}$ for every $t' \in \{2, \ldots, t-1\}$, since otherwise the procedure stops at an earlier step. As j_t points to the concealed donor $c_{t-1} \notin M'_{j_t}$, $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is not a cycle from M to M'. However, we can still carry out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, starting from M: add c_{t-1} to M_{j_t} and remove c_{t-1} from $M_{j_{t-1}}, \ldots,$

add c_1 to M_{j_2} and remove c_1 from M_{j_t}. This leads to a well-defined matching M'' for $\widehat{D}$. Since $c_1, c_{t-1} \in D_{j_t}$, $|M''_{j_t} \setminus D_{j_t}| = |M_{j_t} \setminus D_{j_t}|$ and $|D_{j_t} \setminus M''_{j_t}| = |D_{j_t} \setminus M_{j_t}|$. That is, patient j_t receives and supplies the same amounts of blood under M'' and M.

Given that $|M''_{j_t} \setminus D_{j_t}| < |M'_{j_t} \setminus D'_{j_t}|$, we can repeat the previous analysis and identify a cycle or a chain from M'' to M', using the pointing procedure from M'' to M', and possibly the pointing procedure from M' to M''.

Note that the pointing procedure from M'' to M' starts with j_t pointing to some $d \notin D_{j_t}$ with $d \in M'_{j_t} \setminus M''_{j_t}$, and the pointing procedure from M' to M'' starts with j_t pointing to some $c \in D'_{j_t}$ with $c \in M''_{j_t} \setminus M'_{j_t}$. Since $c_{t-1} \notin M'_i$ for any $i \in \widehat{I}$, the concealed donor c_{t-1} is not pointed in the pointing procedure from M'' to M'. Moreover, $c_{t-1} \in M''_{j_t}$ implies that c_{t-1} is not pointed in the pointing procedure from M' to M''. Given that c_{t-1} does not appear in either procedure, this recursive Subcase 4.5 is never reached again, and hence a cycle or a chain C from M'' to M' can be found.

It remains to show that C is also a cycle or a chain from M to M'. We will only consider the case that C is a chain, since the proof for the case that C is a cycle is similar and simpler. Let $C = (\ell_1, a_1, \ldots, \ell_{\bar{x}-1}, a_{\bar{x}-1}, \ell_{\bar{x}})$, where $\bar{x} \geq 2$, $a_1, \ldots, a_{\bar{x}-1}$ are donors, and $\ell_1, \ldots, \ell_{\bar{x}}$ are patients. We verify the conditions in the definition of a chain from M to M'.

Since C is a chain from M'' to M', Condition 1 and Condition 7 are trivially satisfied for C to be a chain from M to M'. Consider any $x \in \{1, \ldots, \bar{x}-1\}$. We have $a_x \in M'_{\ell_x} \setminus M''_{\ell_x}$ and $a_x \in M''_{\ell_{x+1}} \setminus M'_{\ell_{x+1}}$. Given that M'' is obtained from M by carrying out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, we have $a_x \notin M_{\ell_x}$, since otherwise $a_x \in M_{\ell_x}$ and $a_x \notin M''_{\ell_x}$ imply that ℓ_x is pointed by a_x in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$ and hence, by the construction of the list, $a_x \notin M'_{\ell_x}$. Similarly, we have $a_x \in M_{\ell_{x+1}}$, since otherwise $a_x \notin M_{\ell_{x+1}}$ and $a_x \in M''_{\ell_{x+1}}$ imply that ℓ_{x+1} points to $a_x \neq c_{t-1}$ in the list and hence $a_x \in M'_{\ell_{x+1}}$. Therefore, Condition 2 is satisfied.

To show Conditions 3-6, we need the following result, which follows from the construction of (the reverse of) the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$ in the pointing procedure from M' to M. It essentially says that the schedule of every patient $i \neq b$ under M'' must be "between" her schedules under M and M'.

Observation 7 *For every $i \in \widehat{I} \setminus \{b\}$, if $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) \neq (|M_i \setminus D_i|, |D_i \setminus M_i|)$, then $i \neq j_t$, and either*

- $|M'_i \setminus D'_i| > |M_i \setminus D_i|$, $|D'_i \setminus M'_i| > |D_i \setminus M_i|$, *and* $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) = (|M_i \setminus D_i| + 1, |D_i \setminus M_i| + 1)$, *or*

- $|M'_i \setminus D'_i| < |M_i \setminus D_i|$, $|D'_i \setminus M'_i| < |D_i \setminus M_i|$, *and* $(|M''_i \setminus D_i|, |D_i \setminus M''_i|) = (|M_i \setminus D_i| - 1, |D_i \setminus M_i| - 1)$.

Consider any $x \in \{2, \ldots, \bar{x} - 1\}$ such that $a_{x-1} \in D_{\ell_x}$ and $a_x \notin D_{\ell_x}$. Condition 3 is clearly satisfied if $(|M''_{\ell_x} \setminus D_{\ell_x}|, |D_{\ell_x} \setminus M''_{\ell_x}|) = (|M_{\ell_x} \setminus D_{\ell_x}|, |D_{\ell_x} \setminus M_{\ell_x}|)$. Suppose that $(|M''_{\ell_x} \setminus D_{\ell_x}|, |D_{\ell_x} \setminus M''_{\ell_x}|) \neq (|M_{\ell_x} \setminus D_{\ell_x}|, |D_{\ell_x} \setminus M_{\ell_x}|)$. Then $\ell_x \neq j_t$. By the construction of the chain C from M'' to M', we have $|M'_{\ell_x} \setminus D'_{\ell_x}| > |M''_{\ell_x} \setminus D_{\ell_x}|$ and $|D'_{\ell_x} \setminus M'_{\ell_x}| > |D_{\ell_x} \setminus M''_{\ell_x}|$. Then by Observation 7, $|M'_{\ell_x} \setminus D'_{\ell_x}| > |M_{\ell_x} \setminus D_{\ell_x}|$ and $|D'_{\ell_x} \setminus M'_{\ell_x}| > |D_{\ell_x} \setminus M_{\ell_x}|$. Hence it follows from Assumption 1 that Condition 3 is satisfied. Condition 4 can be shown in a similar manner.

Next, consider Condition 5. Suppose that $\ell_{\bar{x}} \neq b$ and $a_{\bar{x}-1} \in D_{\ell_{\bar{x}}}$. For simplicity, denote

- $(|M_{\ell_{\bar{x}}} \setminus D_{\ell_{\bar{x}}}|, |D_{\ell_{\bar{x}}} \setminus M_{\ell_{\bar{x}}}|) = (r, s)$,
- $(|M''_{\ell_{\bar{x}}} \setminus D_{\ell_{\bar{x}}}|, |D_{\ell_{\bar{x}}} \setminus M''_{\ell_{\bar{x}}}|) = (r'', s'')$, and
- $(|M'_{\ell_{\bar{x}}} \setminus D'_{\ell_{\bar{x}}}|, |D'_{\ell_{\bar{x}}} \setminus M'_{\ell_{\bar{x}}}|) = (r', s')$.

Condition 5 is clearly satisfied if $(r, s) = (r'', s'')$. Suppose that $(r, s) \neq (r'', s'')$. Then $\ell_{\bar{x}} \neq j_t$. By the construction of the chain C from M'' to M', we have $s' > s''$ and $r' \leq r''$. Then by Observation 7, $r' > r$, $s' > s$ and $(r'', s'') = (r + 1, s + 1)$. Since $r' > r$ and $r' \leq r'' = r + 1$, we have $r' = r + 1$. By Assumption 1 and the fact that $r' > r$ and $s' > s$, $(r' - 1, s' - 1) = (r, s' - 1) \in \mathcal{F}_{\ell_{\bar{x}}}(D_{\ell_{\bar{x}}})$. Since $s' - 1 \geq s'' > s$ and $(r, s) \in \mathcal{F}_{\ell_{\bar{x}}}(D_{\ell_{\bar{x}}})$, by Assumption 1 again, we have $(r, s + 1) \in \mathcal{F}_{\ell_{\bar{x}}}(D_{\ell_{\bar{x}}})$. Finally, $(r', s' - 1) \in \mathcal{F}_{\ell_{\bar{x}}}(D'_{\ell_{\bar{x}}})$ since C is a chain from M'' to M'. The case that $a_{\bar{x}-1} \notin D_{\ell_{\bar{x}}}$ as well as Condition 6 can be shown similarly.

In the end, given the cycle or chain C from M to M' that is constructed in possibly any one of the above cases, we show the last statement in Lemma A.4 for any $i \in \widehat{I} \setminus \{b\}$, in the following two parts.

<u>Part 1.</u> Suppose that C is not constructed in Subcase 4.5. If $i \neq i_1$ and $|(M + C)_i \setminus D_i| > |M_i \setminus D_i|$, then either (1) i is pointed by her own donor and points to a donor that is not her own in C, or (2) C is a chain, i is the tail of the chain and points to a donor that is not her own. By the construction of the pointing procedures, $|M'_i \setminus D'_i| > |M_i \setminus D_i|$ and $|D'_i \setminus M'_i| > |D_i \setminus M_i|$ in the former case, and $|M'_i \setminus D'_i| > |M_i \setminus D_i|$ and $|D'_i \setminus M'_i| \leq |D_i \setminus M_i|$ in the latter case. Similarly, if $i \neq i_1$ and $|(M + C)_i \setminus D_i| < |M_i \setminus D_i|$, it can

be shown that $\left|M_i' \setminus D_i'\right| < \left|M_i \setminus D_i\right|$. On the other hand, if $i = i_1$, since i points to a donor that is not her own in the pointing procedure from M to M', and to her own donor in the pointing procedure from M' to M, it can be easily checked that i points to a donor that is not her own or she is pointed by her own donor, whenever she appears in C. Therefore, $\left|(M+C)_i \setminus D_i\right| \geq \left|M_i \setminus D_i\right|$. Then the statement holds for i since $\left|M_i' \setminus D_i'\right| > \left|M_i \setminus D_i\right|$.

Part 2. Suppose that C is constructed in Subcase 4.5, first as a cycle or a chain from M'' to M'. By arguments similar to those in Part 1,

$$\left|(M''+C)_i \setminus D_i\right| > \left|M_i'' \setminus D_i\right| \text{ implies } \left|M_i' \setminus D_i'\right| > \left|M_i'' \setminus D_i\right|, \text{ and}$$
$$\left|(M''+C)_i \setminus D_i\right| < \left|M_i'' \setminus D_i\right| \text{ implies } \left|M_i' \setminus D_i'\right| < \left|M_i'' \setminus D_i\right|.$$

Then, given that C is also a cycle or a chain from M to M', it follows from Observation 7 that the last statement in Lemma A.4 holds for i in this case. ■

Lemma A.5 *The rule F is donor monotonic.*

Proof of Lemma A.5. To prove that F is donor monotonic, it is sufficient to show that any $i \in I$ cannot receive more blood by concealing exactly one donor. Assume to the contrary, there exist D, D' and $i \in I$ such that $D_i' \subseteq D_i$, $\left|D_i \setminus D_i'\right| = 1$, $D_j' = D_j$ for every $j \in I \setminus \{i\}$, and $\left|F_i(\widehat{D'}) \setminus D_i'\right| > \left|F_i(\widehat{D}) \setminus D_i\right|$. For simplicity, denote $F(\widehat{D})$ and $F(\widehat{D'})$ as M and M' respectively. By Lemma A.4, there exists a cycle or a chain C from M to M'.

We first want to show that $\phi(M) = \phi(M+C)$. Suppose that it is not true. Then by the construction of F, $\widehat{U}(M) > \widehat{U}(M+C)$.[55] Consider any $j \in I$. By Observation 6 and the last statement of Lemma A.4, j receives the same amount of blood under M and $M+C$ if and only if she receives the same amount under M' and $M'-C$. Moreover, if she receives more (or less) under $M+C$ than under M, then (1) she receives only one more (or less) unit under $M+C$ than under M, (2) she receives one more (or less) unit under M' than under $M'-C$, and (3) she receives more (or less) under M' than under M. Then, the concavity of ρ implies

$$\rho\Big(\left|(M'-C)_j \setminus D_j'\right|\Big) - \rho\Big(\left|M_j' \setminus D_j'\right|\Big) \geq \rho\Big(\left|M_j \setminus D_j\right|\Big) - \rho\Big(\left|(M+C)_j \setminus D_j\right|\Big).$$

Since u_j is linear in supply, and u_b is linear, Observation 6 and the above inequality

[55]Note that due to the relation between schedules induced by allocations and schedules induced by matchings that is shown in the proof of Lemma A.3, for any $M, M' \in \mathcal{M}(\widehat{D})$, $\widehat{U}(M) \neq \widehat{U}(M')$ if $\phi(M) \neq \phi(M')$.

together imply

$$\widehat{U}(M'-C)-\widehat{U}(M')\geq\widehat{U}(M)-\widehat{U}(M+C)>0.$$

Then $\widehat{U}(M'-C)>\widehat{U}(M')$, contradicting to the fact that $M'=F(\widehat{D'})$ maximizes the weighted sum of utilities among $\mathcal{M}(\widehat{D'})$. Therefore, $\phi\big(F(\widehat{D})\big)=\phi\big(F(\widehat{D})+C\big)$.

Then by Lemma A.4 again, there exists a cycle or a chain C' from $F(\widehat{D})+C$ to $F(\widehat{D'})$. By similar arguments as before, it can be shown that $\phi\Big(\big(F(\widehat{D})+C\big)+C'\Big)=\phi\big(F(\widehat{D})+C\big)$. This process can be continued infinitely, which leads to a contradiction since each additional cycle or chain addition generates a new matching that is closer to $F(\widehat{D'})$. ∎

B.3 Proof of Theorem 4

Let f be a weighted utilitarian mechanism with respect to w, and $\mathcal{F}$ be a feasible schedule menu profile that satisfies L-convexity, feasibility of positive price, strong non-diminishing favorability in donors and individual rationality. Assume to the contrary, f is not incentive compatible with respect to donors under $\mathcal{F}$. Then there exist a problem (D,θ), patient i and donor subset $D'_i\subseteq D_i$ such that

$$u_i\big[(r',s'),\ \theta_i\big]>u_i\big[(r,s),\ \theta_i\big]$$

where $(r,s)=f\Big(\mathcal{F},\ D,\ \theta\Big)(i)$ and $(r',s')=f\Big(\mathcal{F},\ (D'_i,D_{-i}),\ \theta\Big)(i)$.

By Theorem 2, $r'\leq r$. By strong non-diminishing favorability in donors and the fact that u_i is strictly decreasing in supply, $u_i\big[(r',s'),\ \theta_i\big]>u_i\big[(r,s),\ \theta_i\big]$ implies $r'<r$. Then, given that u_i is strictly decreasing in supply, strictly increasing in receipt, and always exhibits a MRS greater than 1, it is straightforward to see that $s'<s$, and $s-s'>r-r'$. By individual rationality, $(r',s')\neq(0,0)$. Hence $r'\geq\underline{g}_i$. Then by strong non-diminishing favorability in donors, there exists $s''\leq s'$ such that $(r',s'')\in\mathcal{F}_i(D_i)$. Since $(r,s)\in\mathcal{F}_i(D_i)$, $r>r'$, $s>s''$ and $s-s''\geq s-s'>r-r'$, by (repeated applications of) Condition 1 in Assumption 1,

$$\Big(r-(r-r'),\ s-(r-r')\Big)=\Big(r',\ s-(r-r')\Big)\in\mathcal{F}_i(D_i).$$

However, this contradicts to strong non-diminishing favorability in donors, since $(r',s')\in\mathcal{F}_i(D'_i)$ and $s-(r-r')>s'$.

B.4 Proof of Theorem 5

Let f be a priority mechanism, and $\mathcal{F}$ be a feasible schedule menu profile that satisfies L-convexity, feasibility of positive price, strong non-diminishing favorability in donors and individual rationality. Consider any problem (D,θ), patient i, donor subset $D_i' \subseteq D_i$ and type θ_i'. Let

$$\alpha = f\Big(\mathcal{F},\, D,\, \theta\Big), \quad \alpha' = f\Big(\mathcal{F},\, (D_i', D_{-i}),\, \theta\Big), \quad \text{and} \quad \alpha'' = f\Big(\mathcal{F},\, (D_i', D_{-i}),\, (\theta_i', \theta_{-i})\Big).$$

By Theorem 4, $u_i\big[\alpha(i),\ \theta_i\big] \geq u_i\big[\alpha'(i),\ \theta_i\big]$. Then, since f always chooses an allocation by sequentially maximizing individual utilities, $u_i\big[\alpha'(i),\ \theta_i\big] \geq u_i\big[\alpha''(i),\ \theta_i\big]$. Therefore, we have $u_i\big[\alpha(i),\ \theta_i\big] \geq u_i\big[\alpha''(i),\ \theta_i\big]$, and f is incentive compatible under $\mathcal{F}$.

B.5 Proof of Theorem 1

Fix an arbitrary environment except D and θ. Let f be an FCFS mechanism. Suppose that each patient i has a minimum guarantee of $\underline{g}_i \geq 0$, and $\mathcal{F}$ is a feasible schedule menu profile where for every $i \in I$ and D_i,

$$\mathcal{F}_i(D_i) = \big\{(r,s) \in \mathbb{S}_i(D_i)\ :\ s = r - \underline{g}_i\big\}.$$

Note that this covers both P1 and P5.

For any problem (D, θ), let $G_1(D)$, $G_2(D)$ and $G_3(D)$ be the three groups of patients determined by the FCFS procedure as specified in Section 3, and $\mathcal{S}_i(D) \subseteq \mathcal{F}_i(D_i)$ be the updated feasible schedule set for each $i \in I$, i.e.,

1. $\mathcal{S}_i(D) = \mathcal{F}_i(D_i)$ if $i \in G_1(D)$.
2. $\mathcal{S}_i(D) = \Big\{(r,s) \in \mathcal{F}_i(D_i)\ :\ r \geq f_r\Big(\mathcal{F},\, D,\, \theta\Big)(i)\Big\}$ if $i \in G_2(D)$.
3. $\mathcal{S}_i(D) = \Big\{f\Big(\mathcal{F},\, D,\, \theta\Big)(i)\Big\}$ if $i \in G_3(D)$.[56]

Pick any priority mechanism in which the bank has the first priority. To improve upon f under $\mathcal{F}$, we apply this priority mechanism to each problem (D, θ) under the updated feasible schedule sets $\big\{\mathcal{S}_i(D)\big\}_{i \in I}$. That is, given a problem (D, θ), in choosing an allocation to sequentially maximize the bank and patient utilities, we only consider allocations in the set

$$\mathcal{A}' = \big\{\alpha \in \mathcal{A}(\mathcal{F}, D)\ :\ \alpha(i) \in \mathcal{S}_i(D),\ \forall i \in I\big\},$$

which is non-empty since $f\Big(\mathcal{F},\, D,\, \theta\Big) \in \mathcal{A}'$. Denote the resulting allocation as $h\Big(D,\, \theta\Big)$. By the construction of $\mathcal{S}_i(D)$, $u_i\Big[h\Big(D,\, \theta\Big)(i),\ \theta_i\Big] \geq u_i\Big[f\Big(\mathcal{F},\, D,\, \theta\Big)(i),\ \theta_i\Big]$ if $i \in G_2(D) \cup G_3(D)$. When $i \in G_1(D)$, i does not receive any blood from the inventory under the FCFS mechanism and thus $\underline{g}_i = 0$. Then

$$f_r\Big(\mathcal{F},\, D,\, \theta\Big)(i) = \min\Big\{\big|\big\{d \in D_i\ :\ \beta_d \in \mathcal{C}(\beta_i)\big\}\big|, \overline{n}_i\Big\}.$$

[56]Note that the above notions are independent of θ due to our consistency assumption on f.

By the efficiency of $h\Big(D, \theta\Big)$ within $\mathcal{A}'$,

$$h_r\Big(D, \theta\Big)(i) \geq \min\Big\{\big|\{d \in D_i \,:\, \beta_d \in \mathcal{C}(\beta_i)\}\big|, \overline{n}_i\Big\}.$$

Therefore, each patient in Group 1 is weakly better-off under $h\Big(D, \theta\Big)$ than under $f\Big(\mathcal{F}, D, \theta\Big)$. Furthermore, the bank also has weakly higher utility under $h\Big(D, \theta\Big)$, since $f\Big(\mathcal{F}, D, \theta\Big) \in \mathcal{A}'$ and it has the first priority in the priority mechanism.

To prove that h is incentive compatible under $\mathcal{F}$, we first show incentive compatibility with respect to donors, which is equivalent to donor monotonicity due to the specification of $\mathcal{F}$. Consider any problem (D, θ), patient $i \in I$ and donor subset $D'_i \subsetneq D_i$. For simplicity, let

- $r_f = f_r\Big(\mathcal{F}, D, \theta\Big)(i)$,
- $r'_f = f_r\Big(\mathcal{F}, (D'_i, D_{-i}), \theta\Big)(i)$,
- $r_h = h_r\Big(D, \theta\Big)(i)$, and
- $r'_h = h_r\Big((D'_i, D_{-i}), \theta\Big)(i)$.

Then $r_f \leq r_h$ and $r'_f \leq r'_h$. By the incentive compatibility of f under $\mathcal{F}$, $r'_f \leq r_f$. Assume $r_h < \overline{n}_i$, and we show $r'_h \leq r_h$ in the following three cases.

Case 1: $i \in G_3(D)$. If $i \in G_3(D'_i, D_{-i})$, then $r'_h = r'_f \leq r_f = r_h$. If $i \notin G_3(D'_i, D_{-i})$, then $i \in G_1(D'_i, D_{-i})$ and $\underline{g}_i = 0$. By the consistency assumption on f, when i reports D'_i, there is still compatible blood available for her in the inventory when she is served, while she does not receive any blood from the inventory. Then, given that $r'_f \leq r_f = r_h < \overline{n}_i$, D'_i does not include any donor that is incompatible with i. Therefore, $r'_h = |D'_i| \leq r_f = r_h$.

Case 2: $i \in G_2(D)$. Then $|D'_i| < \delta$. If $i \in G_3(D'_i, D_{-i})$, then $r'_h = r'_f \leq r_f \leq r_h$. If $i \in G_1(D'_i, D_{-i})$, then it can be shown as in Case 1 that $r'_h \leq r_h$.

Case 3: $i \in G_1(D)$. If $i \in G_3(D'_i, D_{-i})$, then $r'_h = r'_f \leq r_f \leq r_h$. Suppose that $i \in G_1(D'_i, D_{-i})$. Since $\underline{g}_i = 0$, and i does not receive any blood from the inventory or supply any blood to the inventory when she reports either D_i or D'_i, by the consistency assumption on f, $\mathcal{S}_j(D) = \mathcal{S}_j(D'_i, D_{-i})$ for all $j \in I \setminus \{i\}$. Then, since $\mathcal{S}_j(D)$ is L-convex for all $j \in I \setminus \{i\}$, $\mathcal{S}_i(D) = \mathcal{F}_i(D_i)$, $\mathcal{S}_i(D'_i, D_{-i}) = \mathcal{F}_i(D'_i)$, and $\mathcal{F}_i$ satisfies the properties of L-convexity, feasibility of positive price, and non-diminishing favorability in donors, by the arguments in the proof of Theorem 2, $r'_h \leq r_h$.[57]

In the end, consider any type θ'_i. It has been shown above that

$$u_i\Big[h\Big(D,\,\theta\Big)(i),\ \theta_i\Big] \geq u_i\Big[h\Big((D'_i, D_{-i}),\,\theta\Big)(i),\ \theta_i\Big].$$

Since the outcomes of both the mechanism f and the priority mechanism are independent of patients' types under $\mathcal{F}$, we have

$$h\Big((D'_i, D_{-i}),\,\theta\Big)(i) = h\Big((D'_i, D_{-i}),\,(\theta'_i, \theta_{-i})\Big)(i),$$

and thus

$$u_i\Big[h\Big(D,\,\theta\Big)(i),\ \theta_i\Big] \geq u_i\Big[h\Big((D'_i, D_{-i}),\,(\theta'_i, \theta_{-i})\Big)(i),\ \theta_i\Big].$$

Therefore, h is incentive compatible under $\mathcal{F}$.

Finally, it is straightforward to see that in an environment where $v_X = 0$ for all $X \in \mathcal{B}$, there are two patients without compatible donors, and each of them has a donor that is compatible with the other patient, the outcome of h Pareto dominates that of f.

C Examples Regarding Violations of Properties

Example A.3 (Violation of feasibility of positive price via flat top) *Suppose that the set of patients is $I = \{1, 2, 3, 4\}$ and the set of relevant blood types is $\mathcal{B} = \{O, A, B, AB\}$. Assume ABO-identical transfusion. Let $\overline{n}_2 = 2$, and $\overline{n}_i = 1$ for every other patient i. Each patient's blood type and donor set are given as follows.*

- *$\beta_1 = O$, and Patient 1 has one type AB donor.*
- *$\beta_2 = A$, and Patient 2 has one type O donor, one type B donor, and one type AB donor.*
- *$\beta_3 = A$, and Patient 3 has one type B donor.*
- *$\beta_4 = B$, and Patient 4 has one type A donor.*

In addition, the blood bank only has one unit of type A blood in its inventory. The exchange rate is one-for-one for every $i \in I \setminus \{2\}$. Let $\delta = 3$. Patient 2's feasible schedule set is given by Figure A.13 when $|D_2| = 3$, which has a flat top, and her exchange rate is one-for-one otherwise. Then the feasible schedule menu profile satisfies L-convexity, the requirement of "no flat bottom" in feasibility of positive price, strong non-diminishing favorability in donors, and individual rationality.

Let f be a priority mechanism with the order $1 - 2 - 3 - 4 - b$. Then, for any θ, f selects the following allocation when every patient truthfully reports her donor set:

[57] More specifically, see Footnote 33.

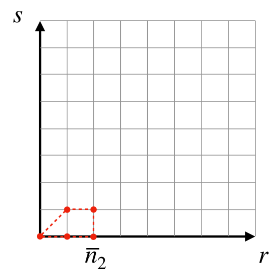

Figure A.13: Feasible schedule set for Patient 2 in Example A.3.

- *Each $i \in I$ receives one unit of type β_i blood.*
- *Patient 2's type O donor donates, and the donor of every other patient donates.*
- *The bank receives one unit of type AB blood.*

However, if Patient 2 conceals her O or AB donor, then she receives two units of A blood, one from the inventory and one from the exchange with Patient 4, and she also supplies two units.

Note that, when Patient 2 conceals her O donor, even if her feasible schedule set remains the same as in Figure A.13, i.e., starting from $(1,1)$ it is still not feasible for her to receive one more unit by supplying one more unit, she receives two units of A blood, while supplying one unit of B blood. ■

Example A.4 (Violation of feasibility of positive price via flat bottom) *Suppose that the set of patients is $I = \{1,2\}$ and the set of relevant blood types is $\mathcal{B} = \{O, A, B, AB\}$. Assume ABO-identical transfusion. The blood bank has two units of type A blood. Every patient $i \in I$ has type A blood, a maximum need of two, and two type B donors. For Patient 2 the exchange rate is one-for-one. On the other hand, when $D_1 \neq \emptyset$, $\mathcal{F}_1(D_1)$ is given by Figure A.14, where a flat bottom appears in the second graph. In addition, $\mathcal{F}_1(\emptyset) = \{(0,0)\}$. Then the feasible schedule menu profile satisfies L-convexity, the requirement of "no flat top" in feasibility of positive price, strong non-diminishing favorability in donors, and individual rationality.*

Let $\rho(r) = r$ and $\theta_1 = \theta_2 = \vartheta^L$. Consider a weighted utilitarian mechanism where the weights for the two patients are sufficiently larger than one, and

$$w_2 \cdot (1 - \vartheta^L) < w_1 < w_2.$$

When the patients truthfully report their donors, the bank gives one unit of A blood to Patient 1 to satisfy her minimum guarantee, and then exchanges with Patient 2 instead of Patient 1 since $w_2 > w_1$. However, if Patient 1 conceals one donor, then she supplies one unit of blood to receive her minimum guarantee, and then the bank also gives the other unit of A

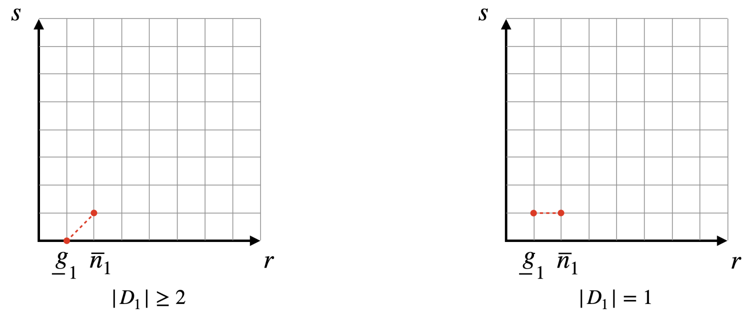

Figure A.14: Feasible schedule sets for Patient 1 in Example A.4.

blood to her without asking for more donations, instead of exchanging with Patient 2, since $w_2 \cdot (1 - \vartheta^L) < w_1$. *Therefore, by under-reporting her donor set, Patient 1's assigned schedule changes from* $(1, 0)$ *to* $(2, 1)$. ■

Example A.5 (Weakening strong non-diminishing favorability in donors) *Consider the setup in Example A.3. We only modify* $\mathcal{F}_2$ *so that for any* D_2*:*

$$\mathcal{F}_2(D_2) = \{(r, s) \in \mathbb{S}_2(D_2) \,:\, s \leq r\}.$$

Then the feasible schedule menu profile satisfies L-convexity, feasibility of positive price, non-diminishing favorability in donors, and individual rationality. Under the same priority mechanism, for any θ*, Patient 2 is assigned the schedule* $(2, 2)$ *if every patient truthfully reports her donors, and Patient 2 is assigned the schedule* $(2, 1)$ *if she hides her type O donor.* ■

Example A.6 (Violation of individual rationality) *Suppose that there is only one patient* i *with* $\overline{n}_i = \underline{g}_i = 1$ *and she has to supply two units to receive her minimum guarantee. That is, for any* D_i*,* $\mathcal{F}_i(D_i) = \{(1, 2)\}$ *if* $|D_i| \geq 2$*, and* $\mathcal{F}_i(D_i) = \{(0, 0)\}$ *otherwise. Then L-convexity, feasibility of positive price, and strong non-diminishing favorability in donors are satisfied, while individual rationality is violated. Assume* $\rho(r) = r$ *and* $\theta_i \in (\frac{1}{2}, 1)$*. It follows that* $u_i[(0, 0),\ \theta_i] > u_i[(1, 2),\ \theta_i]$*. Therefore, under any mechanism, if the patient has at least two donors, she is better-off by not reporting any donor.* ■

D Additional Simulation Results

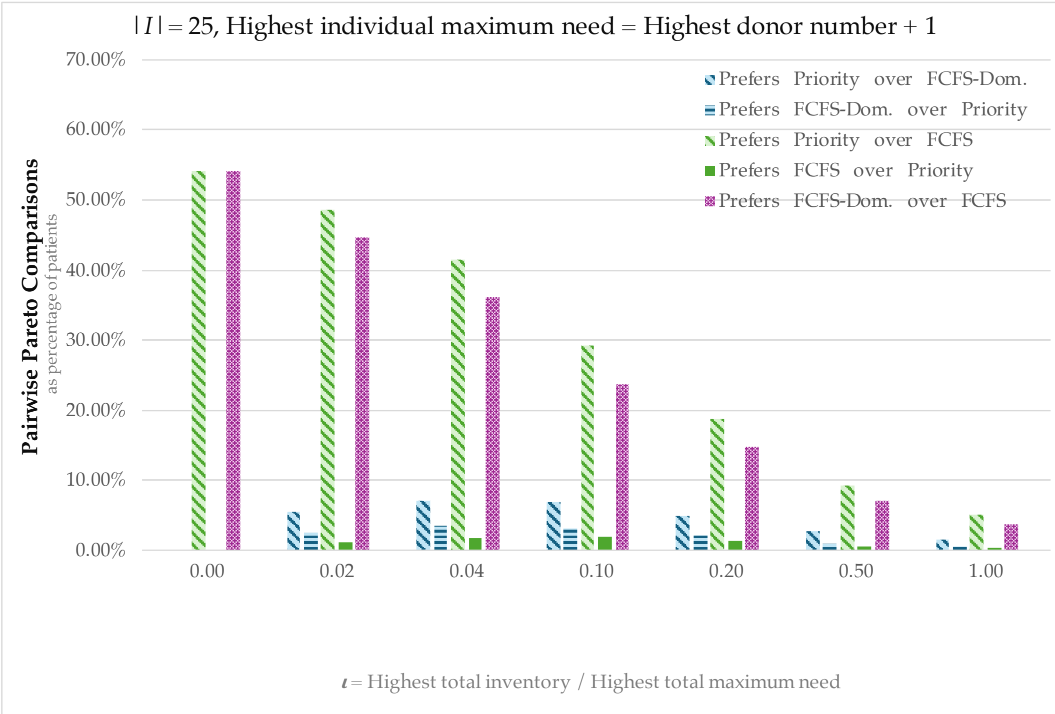

Figure A.15: Pareto comparisons between pairs of mechanisms when $|I| = 25$ as a function of the inventory ratio ι. No patient prefers FCFS to the FCFS dominating mechanism so this comparison percentage is always 0%.

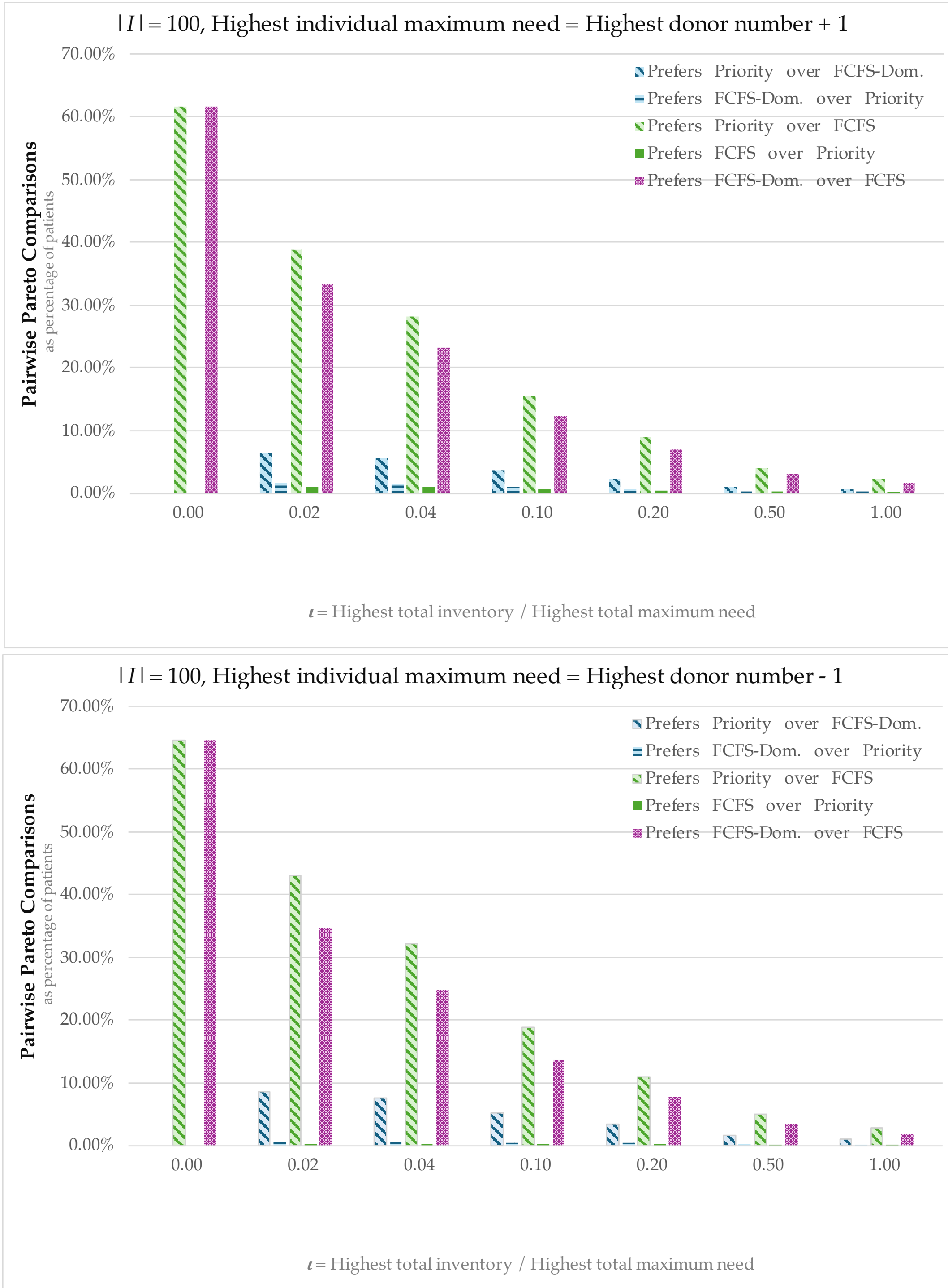

Figure A.16: Pareto comparisons between pairs of mechanisms when $|I| = 100$ as a function of the inventory ratio ι. No patient prefers FCFS to the FCFS dominating mechanism so this comparison percentage is always 0%.

Total Transfusion (as % of mean of the tot. max. need, $\mathbf{E}[\sum_i \overline{n}_i] = 87.5$ for $|I| = 25$ and $\mathbf{E}[\sum_i \overline{n}_i] = 350$ for $|I| = 100$)							
$\max \overline{n}_i - \delta$	ι	$|I| = 25$			$|I| = 100$		
		Priority	**FCFS Dom.**	**FCFS**	**Priority**	**FCFS Dom.**	**FCFS**
1	0	45.98% (8.95%)	45.98% (8.95%)	19.37% (4.72%)	52.09% (4.59%)	52.09% (4.59%)	19.16% (2.33%)
	0.02	46.68% (8.98%)	45.39% (8.72%)	23.40% (6.02%)	52.76% (4.52%)	50.79% (4.31%)	33.04% (7.86%)
	0.04	47.57% (8.95%)	45.97% (8.50%)	28.09% (7.87%)	53.22% (4.46%)	51.54% (4.30%)	39.16% (9.01%)
	0.1	49.50% (8.88%)	47.95% (8.56%)	36.27% (10.45%)	53.88% (4.35%)	52.82% (4.36%)	46.26% (8.75%)
	0.2	51.13% (8.81%)	50.03% (8.61%)	42.82% (11.45%)	54.31% (4.33%)	53.68% (4.37%)	49.95% (7.88%)
	0.5	52.99% (8.60%)	52.32% (8.65%)	48.94% (11.05%)	54.77% (4.27%)	54.48% (4.31%)	52.81% (6.30%)
	1	53.99% (8.60%)	53.58% (8.62%)	51.83% (10.27%)	55.04% (4.23%)	54.86% (4.27%)	54.02% (5.41%)
-1	0	58.58% (10.52%)	58.58% (10.52%)	25.97% (5.89%)	64.71% (5.02%)	64.71% (5.02%)	25.70% (2.78%)
	0.02	59.13% (10.48%)	56.87% (10.19%)	29.98% (7.03%)	65.00% (4.97%)	61.05% (4.84%)	40.08% (8.51%)
	0.04	59.85% (10.42%)	56.74% (9.71%)	34.75% (8.81%)	65.17% (4.93%)	61.68% (4.85%)	46.75% (9.91%)
	0.1	61.25% (10.19%)	58.09% (9.48%)	43.55% (11.61%)	65.41% (4.91%)	63.09% (4.98%)	54.82% (9.90%)
	0.2	62.51% (10.02%)	60.10% (9.65%)	50.79% (12.95%)	65.61% (4.91%)	64.21% (5.00%)	59.46% (9.10%)
	0.5	63.99% (9.82%)	62.61% (9.80%)	58.06% (12.71%)	65.91% (4.89%)	65.21% (4.98%)	63.10% (7.42%)
	1	64.86% (9.80%)	64.06% (9.90%)	61.63% (11.89%)	66.13% (4.87%)	65.72% (4.92%)	64.61% (6.36%)

Table A.2: The averages and the population standard errors in the simulations for the total transfusion as a percentage of the mean total maximum need as graphed in Figures 9 and 10. The standard errors of the averages can be found by dividing the population standard errors by $\sqrt{1000} \approx 31.62$, where 1000 is the number of random markets drawn for the simulations.

Pareto Comparisons Between Pairs of Mechanisms (as % of individuals strictly preferring the first mechanism to the second)											
		$\lvert I \rvert = 25$					$\lvert I \rvert = 100$				
$\max \overline{n}_i - \delta$	ι	**Priority ≻ FCFS Dom.**	**FCFS Dom. ≻ Priority**	**Priority ≻ FCFS**	**FCFS ≻ Priority**	**FCFS Dom. ≻ FCFS**	**Priority ≻ FCFS Dom.**	**FCFS Dom. ≻ Priority**	**Priority ≻ FCFS**	**FCFS ≻ Priority**	**FCFS Dom. ≻ FCFS**
1	0	0.00%	0.00%	54.16%	0.00%	54.16%	0.00%	0.00%	61.65%	0.00%	61.65%
		0.00%	0.00%	(11.62%)	0.00%	(11.62%)	0.00%	0.00%	(5.87%)	0.00%	(5.87%)
	0.02	5.42%	2.44%	48.68%	1.13%	44.69%	6.40%	1.70%	38.96%	1.03%	33.24%
		(6.03%)	(3.72%)	(12.42%)	(2.48%)	(13.67%)	(3.18%)	(2.02%)	(13.28%)	(1.60%)	(14.15%)
	0.04	7.13%	3.44%	41.43%	1.74%	36.10%	5.56%	1.53%	28.24%	1.00%	23.22%
		(6.03%)	(4.40%)	(13.48%)	(3.01%)	(14.63%)	(2.91%)	(1.96%)	(15.07%)	(1.62%)	(14.72%)
	0.1	6.91%	3.18%	29.18%	1.86%	23.78%	3.64%	1.03%	15.62%	0.69%	12.36%
		(5.52%)	(4.04%)	(15.42%)	(2.97%)	(15.48%)	(2.77%)	(1.56%)	(14.26%)	(1.27%)	(12.93%)
	0.2	4.92%	2.22%	18.68%	1.36%	14.76%	2.28%	0.70%	8.98%	0.46%	6.98%
		(4.87%)	(3.62%)	(15.62%)	(2.76%)	(14.67%)	(2.44%)	(1.29%)	(12.23%)	(1.03%)	(10.84%)
	0.5	2.67%	1.02%	9.31%	0.62%	7.13%	1.14%	0.37%	4.10%	0.23%	3.14%
		(4.20%)	(2.46%)	(13.64%)	(1.85%)	(11.87%)	(1.80%)	(0.92%)	(8.63%)	(0.70%)	(7.48%)
	1	1.57%	0.54%	5.08%	0.30%	3.80%	0.67%	0.20%	2.19%	0.13%	1.62%
		(3.37%)	(1.88%)	(10.82%)	(1.35%)	(9.38%)	(1.41%)	(0.73%)	(6.17%)	(0.59%)	(5.21%)
-1	0	0.00%	0.00%	58.29%	0.00%	58.29%	0.00%	0.00%	64.57%	0.00%	64.57%
		0.00%	0.00%	(10.92%)	0.00%	(10.92%)	0.00%	0.00%	(5.07%)	0.00%	(5.07%)
	0.02	6.32%	1.99%	53.03%	0.74%	47.86%	8.62%	0.66%	43.10%	0.29%	34.79%
		(6.58%)	(3.40%)	(11.75%)	(1.88%)	(13.60%)	(3.90%)	(1.25%)	(13.04%)	(0.79%)	(14.66%)
	0.04	8.82%	2.83%	46.44%	1.15%	39.01%	7.71%	0.66%	32.20%	0.33%	24.78%
		(6.80%)	(4.08%)	(12.97%)	(2.45%)	(14.91%)	(3.49%)	(1.20%)	(15.43%)	(0.82%)	(15.28%)
	0.1	8.98%	2.78%	33.51%	1.24%	25.78%	5.32%	0.58%	18.82%	0.32%	13.74%
		(6.17%)	(4.19%)	(15.82%)	(2.56%)	(16.04%)	(3.56%)	(1.19%)	(15.28%)	(0.85%)	(13.43%)
	0.2	6.93%	2.05%	22.72%	0.99%	16.72%	3.37%	0.50%	11.01%	0.27%	7.85%
		(5.63%)	(3.68%)	(16.75%)	(2.42%)	(15.59%)	(3.22%)	(1.07%)	(13.51%)	(0.77%)	(11.53%)
	0.5	3.84%	0.98%	11.64%	0.49%	8.26%	1.77%	0.32%	5.11%	0.17%	3.50%
		(5.15%)	(2.67%)	(15.03%)	(1.62%)	(12.83%)	(2.56%)	(0.77%)	(9.88%)	(0.53%)	(8.03%)
	1	2.31%	0.59%	6.50%	0.28%	4.50%	1.07%	0.19%	2.83%	0.10%	1.85%
		(4.16%)	(1.92%)	(12.18%)	(1.22%)	(10.22%)	(1.99%)	(0.59%)	(7.21%)	(0.41%)	(5.67%)

Table A.3: The averages and the population standard errors in the simulations for percentages of individuals who prefer one mechanism to the other in Figures A.15 and A.16. The standard errors of the averages can be found by dividing the population standard errors by $\sqrt{1000} \approx 31.62$, where 1000 is the number of random markets drawn for the simulations. The **FCFS dominating** mechanism Pareto dominates **FCFS** for the patients by construction, so no patient prefers FCFS to the FCFS dominating mechanism. Hence the column showing this comparison is omitted.