Selling Incremental Products Optimally*

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Abstract

This paper studies the optimal selling mechanisms for incremental products, with a particular emphasis on scenarios where consumers' private information profoundly influences their preferences across time periods. To mitigate their intertemporal information rent, the manufacturer encourages upgrades through refunds rather than payments, introducing inefficiencies but maximizing profits. We investigate how the manufacturer shapes consumer preferences by combining basic and novel upgrades in subsequent product iterations. The optimal strategy follows a bang-bang solution, prioritizing novel upgrades for consumers with lower valuations of the previous product iteration and basic upgrades for those with higher values. Additionally, we reveal that per-consumer information disclosure allows the manufacturer to tailor disclosures about novel functions strategically, enhancing incentives for truthful reporting and extracting more surplus from specific consumer segments.

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1 Introduction

In today's fiercely competitive business environment, firms are constantly seeking ways to enhance profitability and maintain market relevance. One promising avenue in this pursuit involves the strategic development and sales optimization of incremental products, which often take the form of new variants or novel features integrated into well-established offerings. A prime example of this concept is seen in the tech industry, where smartphone manufacturers iteratively introduce new models with enhanced features. The ability to self-iterate and adapt to consumer demands is a defining characteristic of these incremental products. Thus, this paper centers on addressing two research questions: What are the most effective selling mechanisms for firms' incremental products, enabling them to maximize profits? Moreover, with the ability to introduce new functions, how can they shape consumers' preferences to extract more surplus?

Consider a monopolistic manufacturer who introduces an incremental product, denoted as "P", to consumers across two periods. Initially, consumers privately evaluate the first generation, "P1". In the subsequent period, the manufacturer launches the second generation, "P2", and consumers re-evaluate it and decide whether to keep the old device or to upgrade to the newest version. The valuations for P2 may be correlated with their earlier assessments in period 1. Importantly, consumers can only use either P1 or P2, even if they own both.

Consumers' information rent from privately knowing the valuation of P1 (the part of consumer surplus that the manufacturer can not exact) extends beyond the initial period and persists into period 2. This enduring information rent is the key difference between incremental products and conventional products of which information rent typically does not carry over across time periods. It comes from two sources: i) the first source is obvious, consumers have the option to keep P1 and not to upgrade to P2, which means they still own the information rent gained initially; ii) in contrast, even if they upgrade, they can anticipate their valuation of P2 through P1 because P1 and P2 are not completely irrelevant products. In other words, the information rent embedded in the old device can partially transit to the new one.

Although consumers can anticipate the valuation of P2 through their knowledge of P1, they do not possess precise information about it. This gives the manufacturer chances to mitigate their information rent. In period 2, the optimal selling mechanism functions as the manufacturer incentivizes consumers to upgrade from P1 to P2 by offering refunds rather than requiring payment. This approach is distinctive, it encourages consumers to choose P2 over P1, even when owning P1 might offer a higher utility, which consequently brings additional inefficiencies.

In period 1, the payment for consumers to acquire P1 consists of three components: the price for P1; the expected additional surplus (or a deficit if the consumer chooses to upgrade even when P1 offers a higher utility), combined with the potential refund from replacement in period 2; and, through this replacement, the reduction in information rent which was compelled to be shared with the consumer. Notably, the second and third components together become the price of a “swap”. With this swap, consumers have the option to exchange P1 for P2, receiving a refund if their valuation of P2 surpasses a certain threshold.

At first glance, this mechanism may appear similar to a trade-in policy, which typically functions as if one is buying P2 at a reduced price when owning P1. However, in the optimal selling mechanism, the buying action for P2 is effectively concluded in period 1. In other words, the trade-in option is monetized in period 1.[1] This difference is crucial, as it enables the manufacturer to generate more profit, primarily because consumers cannot capitalize on the information rent derived from privately knowing the valuation of P2.

We then study how manufacturers can actively shape consumers’ preferences through different types of upgrades. Some upgrades are basic, like improving the phone’s chips to boost processing power or increasing battery capacity. These upgrades foster a stronger intertemporal dependency of consumers’ valuations. More novel upgrades, such as introducing AI modules or facial recognition, do not link as closely to the older device. We explore a scenario where the manufacturer decides to incorporate a customized mix of these upgrades. In period 2, based on the consumer’s valuation of P1, it commits to implementing the basic upgrades for some parts of the devices, and for the rest, the novel upgrades.

When this happens, the novel upgrades decrease the intertemporal dependency of consumers’ valuation of the old and new devices. This means that with the optimal selling mechanism, the manufacturer can reduce consumers’ information rent by privately knowing the valuation of the old devices, and extract more surplus.

[1] Apple’s iPhone upgrade program in the US, UK, and mainland China shares similarities with this optimal selling mechanism, but with less price discrimination. Under this program (US), consumers enter a 24-month installment loan for an eligible iPhone. Once they have paid the equivalent of at least twelve (12) installment payments, they are eligible to upgrade to a new eligible iPhone before the installment loan’s expiration date. If they decide to upgrade, they enroll in a new iPhone Upgrade Program. This includes applying for and entering into a new 24-month installment loan. Consequently, customers who upgrade after the release of a new iPhone are not required to pay the remaining 12-month installment loan, which is equivalent to a refund.

However, there's a downside brought by this approach; it might make the trade less efficient, which can hurt the overall revenue. We find that the consumer, nevertheless, bears all the efficiency loss that results from these novel upgrades. After eliminating these two opposing effects, the manufacturer faces a choice: maintain the virtual value from P1 in P2 or replace it with the expected value from novel functions. For those consumers with lower valuations of P1, prioritizing novel upgrades is optimal, while for those with higher values, basic upgrades prevail. Consequently, the manufacturer's optimal strategy follows a bang-bang solution.[2]

We further extend the study to the case where consumers may lack precise valuations for novel products before ownership. This allows the manufacturer to shape their perceptions through information disclosure about novel functions. With per-consumer disclosure, the manufacturer tailors disclosures to strengthen incentives for truthful reporting. Specifically, the value of information about novel functions becomes more significant for consumers with a higher valuation of P1 since their decision of whether to upgrade relies on whether P2 can bring more value. With a strategic decrease in the disclosure informativeness for those who report a low valuation of P1, like a trial before buying only offered to consumers reporting high valuations, the manufacturer can mitigate consumers' incentives of downward reporting, ultimately contributing to surplus extraction. This reveals the intricate interplay between information disclosure strategies and consumers' reporting behaviors in dynamic mechanism design.

The rest of the article is organized as follows: Section 2 presents the model. Section 3 introduces the optimal mechanism. Section 4 discusses how the manufacturer controls product design and information disclosure to generate maximum profit. Section 5 concludes the study and provides scope for further discussion. All proofs are in Appendix 6.2.

1.1 Related Literature

This study first contributes to the literature of dynamic mechanism design (Pavan et al. [2014], Bergemann and Välimäki [2019]). While the existing literature has explored various facets of optimal dynamic mechanisms (Baron and Besanko [1984], Riordan and Sappington [1987], Courty and Li [2000], Battaglini [2005], Eső and Szentes [2007], Board [2007], Krähmer and Strausz [2011], Boleslavsky

[2]Examples of this kind of design protocol can be found in the game industry. The gamer can either buy one game along with its remastered version and DLC. Or he can additionally subscribe a service like the Playstation plus premium, where he has access to new games every month.

and Said [2013]), our focus extends to situations where agents' private information influences their intertemporal preferences. In these contexts, the interplay between private information and preferences across periods adds an additional layer of complexity to the design of optimal selling mechanisms. Additionally, an ironing technique is introduced when dealing with the violation of monotonicity constraints.

This study contributes to several topics of industrial organization, including monopolistic pricing. Lu and Zhao [2023] study a related question, of how to sell two objects sequentially in different periods and consumers' utility is additive. They find that bundling is the best selling mechanism if the intertemporal values are negatively correlated. Our study focuses on a different case where the consumer has a unit demand and the best selling mechanism involves swaps or options. Doval and Skreta [2019] consider selling a durable good in multiple periods; they find that the best selling mechanism is fixing prices in each period. What we find is that the consumer has the option to upgrade or stay with the old products.

This study incorporates product design in the optimal dynamic selling mechanism. Schaefer [1999], Dahremöller and Fels [2015] and Veiga and Weyl [2016] focus on the product designed with multiple complementary components. Johnson and Myatt [2006] and Bar-Isaac et al. [2012] construct the product design with disperse value or demand rotation. In contrast to the previous studies, we follow a natural orthogonalization way of product design in which the value of the new device is composed of two parts. One is fully correlated with the old device, and the other is fully independent.

This study also incorporates information design (in the way of Bayesian persuasion, Kamenica and Gentzkow [2011]) in the optimal dynamic selling mechanism. Bergemann and Pesendorfer [2007] find that the information structures should be represented by monotone partition in monopolistic pricing. Bergemann et al. [2015] further study the situation where the pricing protocol is contingent on the information structure. Li and Shi [2017] find that per-consumer disclosure is weakly better than full disclosure in sequential screening problems. Bergemann et al. [2022] analyze a second-degree price discrimination problem with capacity constraint of quality where the firm also chooses the information that buyers have about their own value. We integrates the contributions to dynamic mechanism design, monopolistic pricing and strategic information disclosure, where the manufacturer tailors the information and influencing agents' incentive of truthfully reporting. Our study offers a richer understanding of how monopolists can shape consumer behavior across product cycles.

2 The Model

Consider a monopolistic manufacturer who sells an incremental product (P) to consumers over two periods. In period 1, the manufacturer introduces the first-generation product, referred to as "P1". Consumers observe their private valuations for P1, $v_1 > 0$, which follows some distribution F and the hazard rate $f/(1-F)$ is non-decreasing. In period 2, the manufacturer enhances the product and launches the second generation, labeled as "P2". Consumers observe their private valuations for P2, $v_2 > 0$, which can be correlated with the valuation in period 1, $v_2|v_1 \sim G$. The consumers' utility will be discussed later in the consumers' problem (section 2.1). We assume that both the manufacturer and consumers are fully patient, which means the discount factor is 1.

We assume that the manufacturer can commit to a selling mechanism that covers both periods. The parallel analysis of limited commitment is in Appendix 6.1. By the dynamic revelation principle (Sugaya and Wolitzky [2021]), the manufacturer can use a direct selling mechanism, where consumers report truthfully their private information in each period. In period 1, contingent on the consumer's report $\tilde{v}_1$, the allocation probability of P1 is $x_1(\tilde{v}_1) \in [0,1]$, accompanied by a corresponding monetary transfer of $t_1(\tilde{v}_1)$. In period 2, when the consumer owns P1, given his report $\tilde{v}_2$ in period 2 and $\tilde{v}_1$ in period 1, the allocation probability of P2 is $y_2^1(\tilde{v}_1, \tilde{v}_2) \in [0,1]$. Importantly, consumers get utility from only one product. Thus, there is no meaning to allocate both P1 and P2 in period 2. The two events are mutually exclusive. The probability of still owning P1 is $x_2^1(\tilde{v}_1, \tilde{v}_2) \in [0, 1 - y_2^1(\tilde{v}_1, \tilde{v}_2)]$, and the monetary transfer is $t_2^1(\tilde{v}_1, \tilde{v}_2)$. When consumers do not own P1, these variables are denoted as $y_2^0(\tilde{v}_1, \tilde{v}_2)$, $x_2^0(\tilde{v}_1, \tilde{v}_2)$ and $t_2^0(\tilde{v}_1, \tilde{v}_2)$ correspondingly.

2.1 Consumers' problem

We discuss the consumers' problem in a backward way.

Period 2

The consumer utility, given his true valuation and report, is represented as $u_2^i(v_1, \tilde{v}_1, v_2, \tilde{v}_2)$,

$$u_2^i(v_1, \tilde{v}_1, v_2, \tilde{v}_2) = \delta v_1 \cdot x_2^i(\tilde{v}_1, \tilde{v}_2) + v_2 \cdot y_2^i(\tilde{v}_1, \tilde{v}_2) - t_2^i(\tilde{v}_1, \tilde{v}_2), \; i \in \{0, 1\}$$

where δ represents the depreciation of P1 in period 2, and i is the indicator of whether P1 is allocated in period 1. Importantly, we can only guarantee truthful reporting on the equilibrium path. No IC restriction is put on those who do not report truthfully in period 1.

The Incentive Compatibility Condition, referred to as IC_2^i, states that **given the consumer reporting truthfully in period 1,** it is incentive compatible to also report truthfully in period 2.

Lemma 1. *IC_2^i is equivalent to*

1. *$y_2^i(v_1, v_2)$ is increasing in v_2 for any v_1.*

2. *For every combination of v_1 and v_2,*

$$t_2^i(v_1, v_2) = \underline{t}_2^i(v_1) + \delta v_1 \cdot x_2^i(v_1, v_2) + v_2 \cdot y_2^i(v_1, v_2) - \int_0^{v_2} y_2^i(v_1, t)dt$$

Then, we discuss consumers' strategic reporting off the equilibrium path. Denote $\hat{u}_2^i(v_1, \tilde{v}_1, v_2)$ the optimized utility in period 2 under any report in period 1,

$$\hat{u}_2^i(v_1, \tilde{v}_1, v_2) := u_2^i(v_1, \tilde{v}_1, v_2, \hat{v}_2^i), \;\; \hat{v}_2^i \in \arg\max_{\tilde{v}_2} u_2^i(v_1, \tilde{v}_1, v_2, \tilde{v}_2)$$

The First-Order Condition with respect to $\hat{v}_2^i$,

$$0 = \left.\frac{\partial u_2^i}{\partial \tilde{v}_2}\right|_{\tilde{v}_2 = \hat{v}_2^i} = \delta v_1 \frac{\partial x_2^i(\tilde{v}_1, \tilde{v}_2)}{\partial \tilde{v}_2} + v_2 \frac{\partial y_2^i(\tilde{v}_1, \tilde{v}_2)}{\partial \tilde{v}_2} - \left.\frac{\partial t_2^i(\tilde{v}_1, \tilde{v}_2)}{\partial \tilde{v}_2}\right|_{\tilde{v}_2 = \hat{v}_2^i}$$

Combining the expression of t_2^i, the FOC is equivalent to

$$\delta(v_1 - \tilde{v}_1)\frac{\partial x_2^i(\tilde{v}_1, \tilde{v}_2)}{\partial \tilde{v}_2} + (v_2 - \hat{v}_2^i)\frac{\partial y_2^i(\tilde{v}_1, \tilde{v}_2)}{\partial \tilde{v}_2} = 0$$

When the feasibility condition is binding, $y_2^i + x_2^i = 1$, we have

$$\hat{v}_2^i = v_2 + \delta(\tilde{v}_1 - v_1)$$

The Individual Rationality Condition, designated as IR_2^0, states that **given the consumer reporting truthfully in both periods,** $u_2^0(v_1, v_1, v_2, v_2) \geq 0, \; \forall v_1, v_2$. Given lemma 1,

$$u_2^0(v_1, v_1, v_2, v_2) = -\underline{t}_2^0(v_1) + \int_0^{v_2} y_2^0(v_1, t)dt \geq -\underline{t}_2^0(v_1) = u_2^0(v_1, v_1, 0, 0) \geq 0$$

Therefore IR_2^0 is equivalent to $\underline{t}_2^0(v_1) \leq 0, \; \forall v_1$.

Since the outside option for the consumer owning P1 is δv_1 in period 2, we define IR_2^1 as: **given the consumer reporting truthfully in both periods,** $u_2^1(v_1, v_1, v_2, v_2) \geq \delta v_1,\ \forall v_1, v_2$ which is equivalent to

$$u_2^1(v_1, v_1, 0, 0) = -\underline{t}_2^1(v_1) \geq \delta v_1,\ \forall v_1$$

Importantly, no IR restriction is put on those who do not report truthfully in period 1.

Period 1

Consumers' utility in period 1, given their true valuations and their reports, is

$$u_1(v_1, \tilde{v}_1) = v_1 \cdot x_1(\tilde{v}_1) - t_1(\tilde{v}_1) + (1 - x_1(\tilde{v}_1))\mathbb{E}_{v_2|v_1}[\hat{u}_2^0(v_1, \tilde{v}_1, v_2)] + x_1(\tilde{v}_1)\mathbb{E}_{v_2|v_1}[\hat{u}_2^1(v_1, \tilde{v}_1, v_2)]$$

The Incentive Compatibility Condition, IC_1, is defined as,

$$IC_1: \quad v_1 \in \arg\max_{\tilde{v}_1} u_1(v_1, \tilde{v}_1),\ \forall v_1$$

Lemma 2. *IC_1 is equivalent to*

1. $\forall v_1', v_1''$,

$$\int_{v_1'}^{v_1''} \left(\frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} - \frac{\partial u_1(v_1 = t, \tilde{v}_1 = v_1')}{\partial v_1} \right) dt \geq 0$$

2. *For every v_1,*

$$\begin{aligned} t_1(v_1) = \underline{t}_1 + v_1 \cdot x_1(v_1) + (1 - x_1(v_1))\mathbb{E}_{v_2}[\hat{u}_2^0(v_1, v_1, v_2)] \\ + x_1(v_1)\mathbb{E}_{v_2}[\hat{u}_2^1(v_1, v_1, v_2)] - \int_0^{v_1} \frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} dt \end{aligned}$$

The Individual Rationality Condition in period 1, IR_1, is $u_1(v_1, v_1) \geq 0, \forall v_1$, which is equivalent to

$$u_1(0, 0) = -\underline{t}_1 \geq 0$$

2.2 Manufacturer's problem

When considering a consumer's valuation in period 1, v_1, the revenue the manufacturer generates from this consumer is as follows:

$$
\begin{aligned}
R(v_1) &= t_1(v_1) + (1 - x_1(v_1))\mathbb{E}_{v_2|v_1}[t_2^0(v_1, v_2)] + x_1(v_1)\mathbb{E}_{v_2|v_1}[t_2^1(v_1, v_2)] \\
&= \underline{t}_1 + v_1 x_1(v_1) - \int_0^{v_1} \frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} dt \\
&\quad + (1 - x_1(v_1))\mathbb{E}_{v_2|v_1}[\delta v_1 \cdot x_2^0(v_1, v_2) + v_2 \cdot y_2^0(v_1, v_2)] \\
&\quad + x_1(v_1)\mathbb{E}_{v_2|v_1}[\delta v_1 \cdot x_2^1(v_1, v_2) + v_2 \cdot y_2^1(v_1, v_2)]
\end{aligned}
$$

The manufacturer's objective is to maximize its revenue

$$
\Pi(x_1, y_2^0, y_2^1, x_2^0, x_2^1) = \int_0^{+\infty} R(v_1) dF(v_1)
$$

with constraints

1. $y_2^i(v_1, v_2),\ i \in \{0, 1\}$ is increasing in v_2 for any v_1.
2. $\forall v_1', v_1''$,

$$
\int_{v_1'}^{v_1''} \left(\frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} - \frac{\partial u_1(v_1 = t, \tilde{v}_1 = v_1')}{\partial v_1} \right) dt \geq 0
$$

Apparently, $\underline{t}_1$ should be 0. Note that both $\underline{t}_2^0(v_1)$ and $\underline{t}_2^1(v_1)$ are offset in the expression of $R(v_1)$. Except for specific mention, we can fix $\underline{t}_2^0(v_1) = 0$ and $\underline{t}_2^1(v_1) = -\delta v_1,\ \forall v_1$, without loss of generality.

3 Optimal Mechanism

In this section, we solve the optimal selling mechanism. We first solve the extreme case where consumers' valuations are intertemporally uncorrelated. We find that the optimal mechanism brings inefficiency. Then, we present the general case.

3.1 Uncorrelated Valuation

We begin by considering the case where v_1 and v_2 are independent, i.e., $G(v_2|v_1) = G(v_2)$.

Proposition 1. *The allocation rule of the optimal mechanism is:*

1.

$$x_1(v_1) = \begin{cases} 0 & v_1 \le v_1^* \\ 1 & otherwise \end{cases}$$

where v_1^* is the Myersonian cutoff. ϕ is the Myersonian virtual value.

$$\phi(v_1^*) = v_1^* - \frac{1 - F(v_1^*)}{f(v_1^*)} = 0;$$

2.

$$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \le \delta\phi(v_1) \\ 1 & otherwise \end{cases}, \ and\ x_2^1(v_1, v_2) = 1 - y_2^1(v_1, v_2);$$

3.

$$y_2^0(v_1, v_2) = 1 \ and\ x_2^0(v_1, v_2) = 0.$$

Under this optimal selling mechanism, consumers might receive P2 in period 2, even if owning P1 would provide higher utility. This suggests that this mechanism introduces inefficiency, beyond the consumers with $v_1 < v_1^*$ not obtaining P1. This inefficiency arises because consumers benefit from privately knowing v_1 in both periods. When v_1 is high, it is less likely that consumers will receive P2 in period 2, which results in a higher information rent also in period 2. To mitigate this, the optimal mechanism increases the chances that consumers receive P2, even when it's not efficient to do so ($v_2 < \delta v_1$). Consumers accept this because they receive a refund for making this choice, which is reflected in the transfer in period 2 is

$$t_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \le \delta\phi(v_1) \\ -\delta\frac{1-F(v_1)}{f(v_1)} & otherwise \end{cases}$$

With

$$\begin{aligned} \left.\frac{\partial u_1}{\partial v_1}\right|_{\tilde{v}_1 = v_1} &= (1 - x_1(v_1))\delta\mathbb{E}_{v_2}[x_2^0(v_1, v_2)] + x_1(v_1)\left(1 + \delta\mathbb{E}_{v_2}[x_2^1(v_1, v_2)]\right) \\ &= x_1(v_1)\left(1 + \delta\mathbb{E}_{v_2}[1 - y_2^1(v_1, v_2)]\right), \end{aligned}$$

if $v_1 \le v_1^*$, the transfer in period 2 is 0, $t_2^0(v_1, v_2) = 0$, and

$$t_1(v_1) = \mathbb{E}_{v_2}[\hat{u}_2^0(v_1, v_1, v_2)] = \int_0^{+\infty} v_2 dG(v_2)$$

which is the expected surplus from P2.

If $v_1 > v_1^*$,

$$\begin{aligned} t_1(v_1) &= v_1 + \mathbb{E}_{v_2}[\hat{u}_2^1(v_1, v_1, v_2)] - \int_{v_1^*}^{v_1} \left(1 + \delta \mathbb{E}_{v_2}[1 - y_2^1(t, v_2)]\right) dt \\ &= (1+\delta)v_1^* + \mathbb{E}_{v_2}\left[(v_2 - \delta v_1) y_2^1(v_1, t) - t_2^1(v_1, t)\right] + \int_{v_1^*}^{v_1} \delta \mathbb{E}_{v_2}[y_2^1(t, v_2)] dt \\ &= (1+\delta)v_1^* + \int_{\delta\phi(v_1)}^{+\infty} (v_2 - \delta\phi(v_1)) dG(v_2) + \int_{v_1^*}^{v_1} \delta[1 - G(\delta\phi(t))] dt \end{aligned}$$

where the first item is the price for P1, the second item is the expected additional surplus (or deficit if $v_2 < \delta v_1$) plus refund from P2, and the remaining item measures the reduced information rent shared with the consumer. The last two items together constitute the value of a swap. With this swap, the consumer can choose to exchange P1 for P2 and get a refund, if his valuation is high enough. This mechanism looks similar but radically different from a trade-in policy. In a trade-in policy, customers typically buy P2 at a reduced price if they already own P1, while in the optimal selling mechanism, the purchase of P2 in period 2 is effectively completed in period 1. This approach appears to generate more profit than a traditional trade-in policy because consumers can't benefit from the information rent of privately knowing their v_2.

It is worth mentioning that

$$t_1(v_1^*) = (1+\delta)v_1^* + \int_0^{+\infty} v_2 dG(v_2)$$

and

$$t_2^1(v_1^*, v_2) = -\delta \frac{1 - F(v_1^*)}{f(v_1^*)} = -\delta v_1^*, \ \forall v_2$$

So, the surplus of those consumers with v_1^* is totally exacted.

Value of the Swap

$\phi'(v_1) > 1$, and

$$\begin{aligned} t_1'(v_1) = & -\delta\phi'(v_1)[1 - G(\delta\phi(v_1))] + \delta[1 - G(\delta\phi(v_1))] \\ = & \ \delta[\phi'(v_1) - 1][G(\delta\phi(v_1)) - 1] < 0 \end{aligned}$$

The value of the swap decreases as v_1 increases, since the consumer is unlikely to find P2 giving more additional value and thus does not execute this swap. Even if it is executed, the refund is low. Therefore, in period 1, consumers have no incentives to misreport upwards.

3.2 Correlated Valuation

Now assume that v_2 follows a distribution conditional on v_1, $G(v_2|v_1)$, and $G(v_2|v_1)$ first-order stochastic dominates $G(v_2|v_1')$ if $v_1 > v_1'$.

The expression for $\frac{\partial u_1}{\partial v_1}$, given $\tilde{v}_1 = v_1$, is different from the previous case. It takes the following form:

$$\begin{aligned}\left.\frac{\partial u_1}{\partial v_1}\right|_{\tilde{v}_1=v_1} &= x_1(v_1)\left(1+\delta\mathbb{E}_{v_2|v_1}[x_2^1(v_1,v_2)]\right) + (1-x_1(v_1))\delta\mathbb{E}_{v_2|v_1}[x_2^0(v_1,v_2)] \\ &\quad + x_1(v_1)\mathbb{E}_{v_2|v_1}[y_2^1(v_1,v_2)I_2(v_1,v_2)] + (1-x_1(v_1))\mathbb{E}_{v_2|v_1}[y_2^0(v_1,v_2)I_2(v_1,v_2)]\end{aligned}$$

where $I_2(v_1,v_2)$ is the impulse response function,

$$I_2(v_1,v_2) = -\frac{\frac{\partial G(v_2|v_1)}{\partial v_1}}{g(v_2|v_1)}$$

The value of I_2 indicates how much information v_1 provides about v_2. A higher value of I_2 signifies a stronger correlation.

Proposition 2. *Under the following assumption,*

1. $v_2 - \frac{1-F(v_1)}{f(v_1)}I_2(v_1,v_2)$ *is increasing in* v_2*,*
2. $\frac{1-F(v_1)}{f(v_1)}I_2(v_1,v_2)$ *is decreasing in* v_1*, and*
3. *Either* $I_2(v_1,v_2) \leq \delta,\ \forall v_1, v_2$ *and* $I_2(v_1,v_2)$ *is increasing in both* v_1 *and* v_2*, or* $I_2(v_1,v_2) \geq \delta,\ \forall v_1, v_2$ *and* $I_2(v_1,v_2)$ *is decreasing in both* v_1 *and* v_2*,*

the allocation rule of the optimal mechanism is as follows:

1. $$x_1(v_1) = \begin{cases} 0 & v_1 \leq v_1^* \\ 1 & \textit{otherwise} \end{cases}$$

2. $$y_2^1(v_1,v_2) = \begin{cases} 0 & v_2 \leq \bar{v}_2^1(v_1) \\ 1 & \textit{otherwise} \end{cases},$$

   where $\bar{v}_2^1(v_1)$ *is implicitly defined by the equation*

   $$\bar{v}_2^1(v_1) - \frac{1-F(v_1)}{f(v_1)}I_2(v_1,\bar{v}_2^1(v_1)) = \delta\phi(v_1),$$

   and

   $$x_2^1(v_1,v_2) = 1 - y_2^1(v_1,v_2);$$

3.
$$y_2^0(v_1,v_2)=\begin{cases}0 & v_2\le \bar v_2^0(v_1)\\ 1 & otherwise\end{cases},$$

where $\bar v_2^0(v_1)$ is implicitly defined by the equation

$$\bar v_2^0(v_1)-\frac{1-F(v_1)}{f(v_1)}I_2(v_1,\bar v_2^0(v_1))=0,$$

and

$$x_2^0(v_1,v_2)=0.$$

If $v_1<v_1^*$, in period 2, according to the allocation rule y_2^0, the consumer will receive P2 if their reported valuation is high enough. The transfer is

$$t_2^0(v_1,v_2)=\begin{cases}0 & v_2\le \bar v_2^0(v_1)\\ \bar v_2^0(v_1) & otherwise\end{cases}$$

In period 1, the consumer does not get P1 and his monetary transfer is

$$\begin{aligned}t_1(v_1)&=\mathbb{E}_{v_2|v_1}[\hat u_2^0(v_1,v_1,v_2)]-\int_0^{v_1}\mathbb{E}_{v_2|t}[\delta x_2^0(t,v_2)+y_2^0(t,v_2)I_2(t,v_2)]dt\\ &=\int_{\bar v_2^0(v_1)}^{+\infty}(v_2-\bar v_2^0(v_1))dG(v_2|v_1)-\int_0^{v_1}\mathbb{E}_{v_2|t}[y_2^0(t,v_2)I_2(t,v_2)]dt\end{aligned}$$

These two terms represent the value of a call option, which is the option to get P2 in period 2 and $\bar v_2^0(v_1)$ is the strike price. The first part is the additional value from P2 in period 2 and the second item is the reduction of consumer's information rent. The higher I_2 is, the lower the reduced amount is and the more consumer surplus the manufacturer has to share with the consumer.

Value of the Call Option

The derivative of the value of the call option is

$$\frac{\partial t_1(v_1)}{\partial v_1}=-\int_{\bar v_2^0(v_1)}^{+\infty}\frac{\partial \bar v_2^0(v_1)}{\partial v_1}dG(v_2|v_1)>0$$

The inequity comes from the strike price decreasing in v_1 derived from the implicit function theorem. The value of the call option increases in v_1 because the consumer with a relatively higher valuation for P1 is more likely to have a higher valuation for P2. Thus, the manufacturer sets a lower strike price to encourage trade in period 2 and extract more consumer surplus through this option.

If $v_1 > v_1^*$, the consumer receives P1 in period 1. With higher I_2, more information is embedded in v_2 regarding v_1. As a result, the manufacturer's strategy to reduce information rent through upgrading becomes less effective. One can observe this from the fact that the consumer owning P1 gets P2 less often. He gets P2 only if

$$v_2 > \bar{v}_2^1(v_1) = \delta v_1 - [\delta - I_2(v_1, \bar{v}_2^1(v_1))]\frac{1-F(v_1)}{f(v_1)}$$

where the right-hand side is increasing in I_2. The monetary transfer is

$$t_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \leq \bar{v}_2^1(v_1) \\ [I_2(v_1, \bar{v}_2^1(v_1)) - \delta]\,\frac{1-F(v_1)}{f(v_1)} & otherwise \end{cases}$$

Different from the uncorrelated case, if $I_2(v_1, v_2) \geq \delta$, $\forall v_1, v_2$, to get P2, consumers now need to pay rather than get a refund in period 2, $[I_2(v_1, \bar{v}_2^1(v_1)) - \delta]\frac{1-F(v_1)}{f(v_1)} > 0$.

The monetary transfer in period 1 is

$$\begin{aligned}
t_1(v_1) &= v_1 + \mathbb{E}_{v_2|v_1}[\hat{u}_2^1(v_1, v_1, v_2)] - \int_0^{v_1^*} \mathbb{E}_{v_2|t}[\delta x_2^0(t, v_2) + y_2^0(t, v_2)I_2(t, v_2)]dt \\
&\quad - \int_{v_1^*}^{v_1} \mathbb{E}_{v_2|t}[1 + \delta x_2^1(t, v_2) + y_2^1(t, v_2)I_2(t, v_2)]dt \\
&= (1+\delta)v_1^* + \int_{\bar{v}_2^1(v_1)}^{+\infty} (v_2 - \bar{v}_2^1(v_1))dG(v_2|v_1) - \int_0^{v_1^*} \mathbb{E}_{v_2|t}[y_2^0(t, v_2)I_2(t, v_2)]dt \\
&\quad + \int_{v_1^*}^{v_1} \mathbb{E}_{v_2|t}[(\delta - I_2(t, v_2))y_2^1(t, v_2)]dt
\end{aligned}$$

Similar to the uncorrelated case, the monetary transfer in period 1 consists of the price for P1 and a swap.

<u>*Value of the Swap*</u>

The derivative of the value of the swap is

$$\left(\delta - \frac{\partial \bar{v}_2^1}{\partial v_1}\right)(1 - G(\bar{v}_2^1|v_1))$$

Note that if $I_2(v_1, v_2) \leq \delta$, $\forall v_1, v_2$ and $I_2(v_1, v_2)$ is increasing in v_1, along with $\phi'(v_1) > 1$,

$$\frac{\partial \bar{v}_2^1}{\partial v_1} = \frac{I_2 + (\delta - I_2)\phi'(v_1) + \frac{1-F(v_1)}{f(v_1)}\frac{\partial I_2}{\partial v_1}}{1 - \frac{\partial I_2}{\partial v_2}\frac{1-F(v_1)}{f(v_1)}} > \delta$$

Similarly, as in the uncorrelated case, the value of the option decreases in v_1.

However, the opposite result can be found if $I_2(v_1, v_2) > \delta$, $\forall v_1, v_2$ and $I_2(v_1, v_2)$ is decreasing in both v_1 and v_2. This is because with more information about v_2 embedded in v_1, the consumer with high v_1 knows that v_2 is more likely to be high also, in which case the option becomes valuable.

3.3 Relaxing the Monotone Hazard Rate

We have assumed that the hazard rate $f/(1-F)$ is non-decreasing and in proposition 2, the impulse response function is monotone. In this section, we relax these assumptions and use the ironing technique (Myerson [1981]) to find the optimal mechanism. We first discuss the case where consumers' valuations are intertemporally uncorrelated. Then, we extend it to the general case. Moreover, without the monotone hazard rate, the manufacturer's revenue decreases compared to the relaxed problem. In section 4.3, we demonstrate how this discrepancy recovers with strategic information disclosure.

3.3.1 Uncorrelated Valuation

We need the necessary condition 1 in lemma 2 to guarantee the consumers' incentive compatibility condition in period 1. This condition holds if and only if $\frac{\partial u_1(v_1,\tilde{v}_1)}{\partial v_1} = x_1(\tilde{v}_1)\left(1 + \delta \mathbb{E}_{v_2}[x_2^1(\tilde{v}_1, \hat{v}_2)]\right) = x_1(\tilde{v}_1)\left(1 + \delta G\left(\delta v_1 - \delta \frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)}\right)\right)$ is increasing in $\tilde{v}_1$ for any v_1.[3] Without the non-decreasing hazard rate, however, it is not always non-decreasing in $\tilde{v}_1$. We discuss the optimal mechanism under this case, specifically for x_2^i and y_2^i, by ironing the hazard rate.

Proposition 3. *The allocation rule of the optimal mechanism is:*

1.

$$x_1(v_1) = \begin{cases} 0 & v_1 \le v_1^* \\ 1 & \textit{otherwise} \end{cases}$$

2.

$$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \le \bar{v}_2(v_1) := \delta v_1 + \kappa(v_1) \\ 1 & \textit{otherwise} \end{cases},$$

and

$$x_2^1(v_1, v_2) = 1 - y_2^1(v_1, v_2);$$

3.

$$y_2^0(v_1, v_2) = 1 \textit{ and } x_2^0(v_1, v_2) = 0.$$

[3] "If" part is obvious. "Only if" part comes from the separation of v_1 and $\tilde{v}_1$ in the expression.

Here, $\kappa(v_1)$ should satisfy the following properties,

1. *$\kappa(v_1)$ is non-decreasing;*

2. *there exists a partition of $[0, +\infty)$ into intervals such that $\kappa(v_1)$ is either equal to $-\delta\frac{1-F(v_1)}{f(v_1)}$ or a constant;*

3. *if $\kappa(v_1)$ is a constant on interval $(\underline{v}_1, \bar{v}_1)$, for any $v_1 \in (\underline{v}_1, \bar{v}_1)$,*

$$\int_{\underline{v}_1}^{v_1} \left(\kappa(v_1) + \delta\frac{1-F(v_1)}{f(v_1)}\right) g(\kappa(v_1) + \delta v_1) dF(v_1) \leq 0.$$

   with equality for $v_1 = \bar{v}_1$.

We next discuss how to find such a $\kappa(\cdot)$, in other words, the ironing procedure of a non-monotonic hazard rate. Panel 1a gives an example of a non-monotonic hazard rate. We first partition it into monotonic intervals, as shown in panel 1b. Then, we flatten ("iron") the decreasing part as shown in panel 1c. The value of the constant is determined by the equality in the third property of $\kappa(\cdot)$. Then, in panel 1d, we flatten the drop from the left. Again, the value of the constant is determined by the equality. In panel 1e, we flatten the right drop. We iteratively flattens all the other drops, as in panel 1f to 1g, and finally gets the $\kappa(\cdot)$. It is easy to verify that the $\kappa(\cdot)$ generated satisfies all the property in proposition 3.

3.3.2 Correlated Valuation

We relax the monotone hazard rate assumption and the assumption in proposition 2 and find the optimal mechanism by extending the ironing technique to the general case.

Proposition 4. *If $I_2(v_1, v_2) \leq \delta$, $\forall v_1, v_2$, the allocation rule of the optimal mechanism is:*

1. $$x_1(v_1) = \begin{cases} 0 & v_1 \leq v_1^* \\ 1 & otherwise \end{cases}$$

2. $$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \leq \bar{v}_2^1(v_1) := \delta v_1 + \kappa^1(v_1) \\ 1 & otherwise \end{cases},$$

   and

$$x_2^1(v_1, v_2) = 1 - y_2^1(v_1, v_2).$$

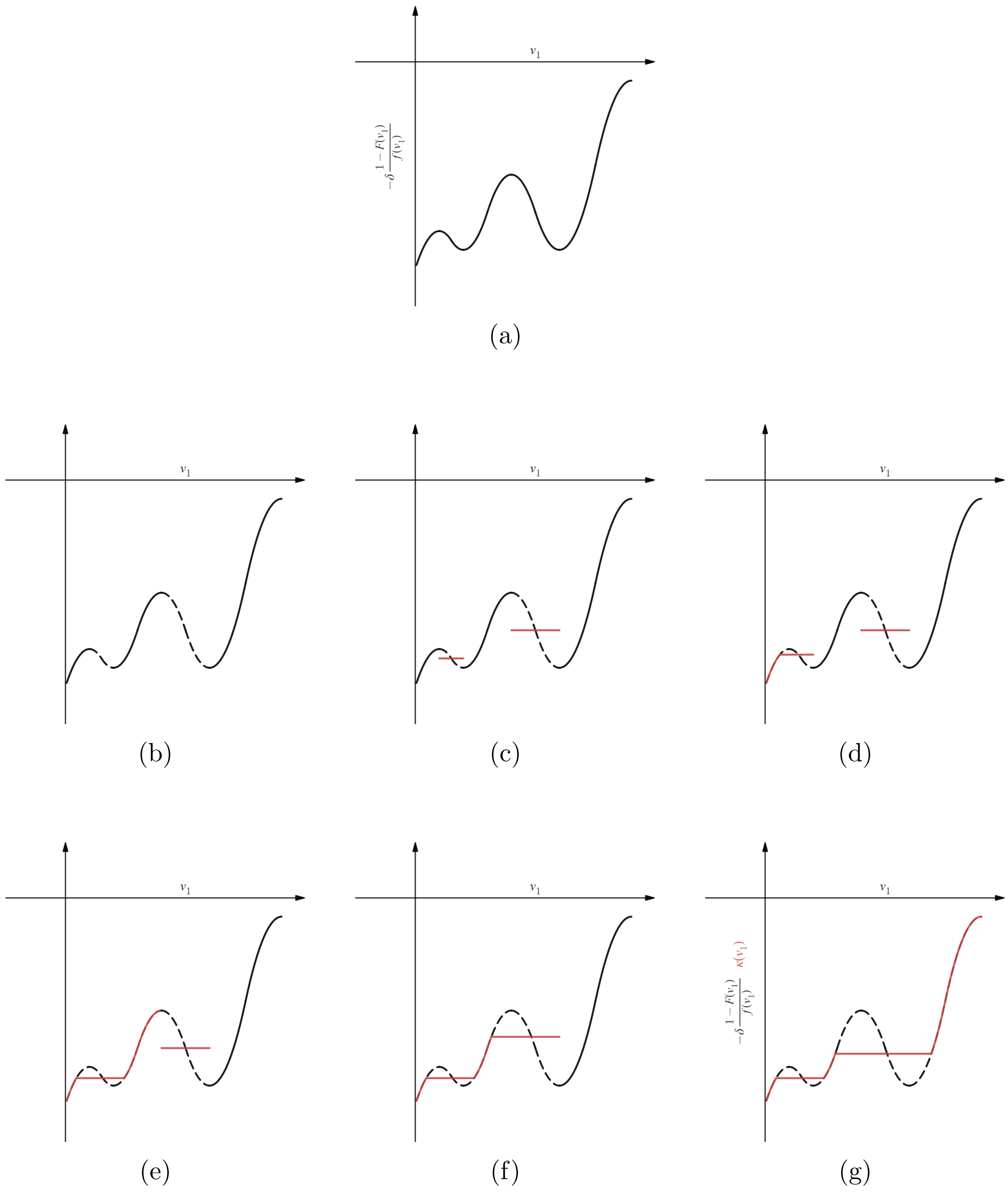

Figure 1: The ironing procedure of a non-monotonic hazard rate.

Here, $\kappa^1(v_1)$ should satisfy the following properties,

(a) $\kappa^1(v_1)$ is non-decreasing;

(b) there exists a partition of $[0, +\infty)$ into intervals such that $\kappa^1(v_1)$ is either equal to $[I_2(v_1, \kappa^1(v_1) + \delta v_1) - \delta]\frac{1-F(v_1)}{f(v_1)}$ or a constant;

(c) if $\kappa^1(v_1)$ is a constant on interval $(\underline{v}_1, \bar{v}_1)$, for any $v_1 \in (\underline{v}_1, \bar{v}_1)$,

$$\int_{\underline{v}_1}^{v_1} \left(\kappa^1(v_1) + [\delta - I_2(v_1, \kappa^1(v_1) + \delta v_1)]\frac{1-F(v_1)}{f(v_1)}\right) g(\kappa^1(v_1)+\delta v_1|v_1)dF(v_1) \leq 0.$$

with equality for $v_1 = \bar{v}_1$.

3.

$$y_2^0(v_1, v_2) = \begin{cases} 0 & v_2 \leq \bar{v}_2^0(v_1) := \kappa^0(v_1) \\ 1 & otherwise \end{cases},$$

and

$$x_2^0(v_1, v_2) = 0.$$

Here, $\kappa^0(v_1)$ should satisfy following properties,

(a) $\kappa^0(v_1)$ is non-increasing;

(b) there exists a partition of $[0, +\infty)$ into intervals such that $\kappa^0(v_1)$ is either equal to $I_2(v_1, \kappa^0(v_1))\frac{1-F(v_1)}{f(v_1)}$ or a constant;

(c) if $\kappa^0(v_1)$ is a constant on interval $(\underline{v}_1, \bar{v}_1)$, for any $v_1 \in (\underline{v}_1, \bar{v}_1)$,

$$\int_{\underline{v}_1}^{v_1} \left(\kappa^0(v_1) - I_2(v_1, \kappa^0(v_1))\frac{1-F(v_1)}{f(v_1)}\right) g(\kappa^0(v_1)|v_1)dF(v_1) \geq 0.$$

with equality for $v_1 = \bar{v}_1$.

To find such a $\kappa^1(\cdot)$, instead of ironing the non-monotonic hazard rate as in the uncorrelated valuation case, one needs to iron $\hat{\kappa}^1(v_1)$,

$$\hat{\kappa}^1(v_1) = \arg\max_k \int_k^{+\infty} \left(v_2 + (\delta - I_2(v_1, v_2 + \delta v_1))\frac{1-F(v_1)}{f(v_1)}\right) g(v_2 + \delta v_1|v_1)dv_2$$

And to find such a $\kappa^0(\cdot)$, instead of ironing the non-monotonic hazard rate as in the uncorrelated valuation case, one needs to iron $\hat{\kappa}^0(v_1)$,

$$\hat{\kappa}^0(v_1) = \arg\max_k \int_k^{+\infty} \left(v_2 - I_2(v_1, v_2)\frac{1-F(v_1)}{f(v_1)}\right) g(v_2|v_1)dv_2$$

3.4 More than 2 periods

The optimal selling mechanism can be extended to longer periods. In general, the virtual value of Pn in period t is

$$\Phi(n,t) = \delta^{t-n} \left(v_n - \frac{1-F(v_1)}{f(v_1)} \prod_{i=2}^{n} I_i(v_{i-1}, v_i) \right), \; n \le t$$

where

$$I_i(v_{i-1}, v_i) = -\frac{\frac{\partial G(v_i|v_{i-1})}{\partial v_{i-1}}}{g(v_i|v_{i-1})}$$

The consumer owning Pj upgrades to Pn in period n if and only if the virtual value of Pn is higher than that of Pj, $\Phi(n,n) > \Phi(j,n)$, and the monetary transfers are determined by the IC and IR conditions.

According to the expression of virtual values, one can find that consumers' information rent from privately knowing v_1 passes through I_i, the information about the new product embedded in the old one. Once one $I_i = 0$, consumers' information rent disappears in the sequential periods. The following section shows how the manufacturer benefits from this point.

4 Product Design and Disclosure

In the previous section, we assumed that the distribution of $G(v_2|v_1)$ is exogenous. However, the manufacturer has the capability to influence this distribution through various types of upgrades. Some of these improvements are **basic**, like upgrading chips, increasing battery capacity, or enhancing camera pixels, which intensify the intertemporal dependence of consumers' valuations. Conversely, other **novel** upgrades, such as Face ID, Apple Pay, or 3D Touch, result in consumer valuations that are less correlated with the older device.

4.1 Upgrade

In this section, we endogenize the distribution $G(v_2|v_1)$ by assuming that, with the consumer's report $\tilde{v}_1$ in period 1, the manufacturer commits to making basic upgrades in some proportion $\alpha(\tilde{v}_1) \in [\underline{\alpha}, 1]$[4] of the devices and replacing the rest with novel upgrades in period 2. This situation can be thought of as the consumer's fixed time spent with the device. To introduce a new function or technology to P2,

[4] $\underline{\alpha} < \delta$

the manufacturer must retire the old one. As a result, the relationship between v_1 and v_2 becomes,

$$v_2 = \alpha(\tilde{v}_1)v_1 + [1 - \alpha(\tilde{v}_1)]\omega$$

Here, $\omega \geq 0$ is a random variable independent of v_1 and follows a cumulative distribution function H (with corresponding probability density function h). In this case, we can determine that,

$$I_2(v_1, v_2; \tilde{v}_1) = -\frac{\frac{\partial G(v_2|v_1)}{\partial v_1}}{g(v_2|v_1)} = -\frac{\frac{\partial H(\frac{v_2-\alpha(\tilde{v}_1)v_1}{1-\alpha(\tilde{v}_1)})}{\partial v_1}}{\frac{1}{1-\alpha(\tilde{v}_1)}h(\frac{v_2-\alpha(\tilde{v}_1)v_1}{1-\alpha(\tilde{v}_1)})} = \alpha(\tilde{v}_1)$$

We consider the innovation cost as a sunk cost, not affecting the manufacturer's optimization problem. The manufacturer's objective function is defined as:

$$\Pi'(x_1, y_2^0, y_2^1, x_2^0, x_2^1, \alpha) = \int_0^{+\infty} R(v_1)dF(v_1)$$

According to the findings in Section 3, iterating the product with novel upgrades brings two opposite effects. First, a higher proportion of novel upgrades (lower α) reduces the intertemporal correlation between consumers' valuation. Thus, the manufacturer can reduce consumers' information rent from privately knowing v_1 under the optimal mechanism and extract more surplus. Second, it leads to an efficiency loss that, in turn, affects revenue negatively. These two effects offset each other.

One can observe that the marginal efficiency loss caused by novel upgrades is

$$\int_0^{+\infty} \frac{\partial \bar{v}_2(v_1)}{\partial \alpha(v_1)} \mathbb{E}_{v_2|v_1}[y_2^1(v_1, v_2)]dF(v_1) = \int_0^{+\infty} (1 - F(v_1))\mathbb{E}_{v_2|v_1}[y_2^1(v_1, v_2)]dv_1$$

which coincides with the marginal reduced information rent,

$$\int_0^{+\infty} \int_0^{v_1} \mathbb{E}_{v_2|t} \left[\frac{\partial I_2(t, v_2; t)}{\partial \alpha(t)} y_2^1(t, v_2)\right] dtdF(v_1) = \int_0^{+\infty} (1-F(v_1))\mathbb{E}_{v_2|v_1}[y_2^1(v_1, v_2)]dv_1$$

The equality comes from integrating by parts.

After eliminating these two effects, the manufacturer faces a trade-off between either keeping the virtual value $\phi(v_1)$ or replacing it with expected value brought by novel functions $\mathbb{E}[\omega]$. When v_1 is low, the virtual value is low, so introducing as many novel upgrades as possible is optimal. Otherwise, the manufacturer should just make basic upgrades. Therefore, the following proposition demonstrates that the manufacturer's optimal strategy α follows a bang-bang solution.

Proposition 5. *There is a unique cutoff $v_1^a > v_1^*$ which satisfies*

$$\int_{\frac{\delta-\underline{\alpha}}{1-\underline{\alpha}}\phi(v_1^a)}^{+\infty}(1-H(t))dt = \frac{1-\delta}{1-\underline{\alpha}}\phi(v_1^a).$$

The manufacturer chooses a level of basic upgrades $\alpha(\tilde{v}_1) = \underline{\alpha}$ if $\tilde{v}_1 < v_1^a$. Otherwise, it chooses $\alpha(\tilde{v}_1) = 1$.

4.2 Depreciation

In addition to novel upgrades, the manufacturer has the ability to adjust the depreciation level of old devices, either positively or negatively. This can be achieved through actions such as providing maintenance services or using non-replaceable components (e.g., non-replaceable batteries). The extent of depreciation is controlled by the parameter δ in the model.

A low depreciation level, corresponding to a large δ, has two significant effects. First, it increases the selling price of P1 because P1 retains a high value in period 2. Conversely, it reduces the likelihood of consumers upgrading from P1 to P2, resulting in a decrease in the sales of P2. The following proposition demonstrates that the first effect dominates. This is due to the fact that the first factor impacts all consumers purchasing P1, while the second effect only affects those consumers with sufficiently high valuations for P2.

Proposition 6. *Even without a commitment on the value of δ in period 1, The manufacturer chooses a depreciation level $\delta = 1$ with commitment in period 1, regardless of the distributions $F(v_1)$ or $H(\omega)$.*

Apparently, P1 with low depreciation discourages the consumers to upgrade for P2. Therefore, even without commitment in period 1, the manufacturer will keep the value of P1 to avoid paying a refund.

4.3 Information Disclosure

It is reasonable to expect that consumers do not have a precise valuation of P2 with novel functions before owning it. This gives the manufacturer areas to manipulate consumers' perception of ω by using different ways to disclose information. Specifically, it chooses a rule of information disclosure (signal), $s(\omega|\tilde{v}_1) : [0, +\infty) \to \Delta M$, where $s(\omega|\tilde{v}_1)$ is a signal observed by the consumer reporting $\tilde{v}_1$ in period 1 with valuation of novel functions ω and M is the signal space. Since the consumer's utility is linear in v_2, and thus linear in ω, $\mathbb{E}[\omega|s]$ is a sufficient statistic for determining their preference. Therefore, we only need to focus on the realization of $\mathbb{E}[\omega|s]$, of which the distribution is denoted as $\hat{H} \in \Delta([0,+\infty))$. $\hat{H}$ can be induced by a rule of information disclosure if and only if it is a mean-preserving contract of H,

$$\int_0^\omega \hat{H}(t|\tilde{v}_1)dt \leq \int_0^\omega H(t)dt,\ \forall \omega \in [0,+\infty)$$

and

$$\int_0^{+\infty} [1-\hat{H}(t|\tilde{v}_1)]dt \leq \int_0^{+\infty} [1-H(t)]dt$$

Then, we study the optimal per-consumer information disclosure. The following example shows how the manufacturer strengthens consumers' incentives to report truthfully by making customized disclosure of novel functions.

Example 1. *Suppose that $v_1 \in \{a,b\}$ with Probability $1/2$ each, $0 < a < b < 2a \leq 1$. $v_2 = \omega$ follows a uniform distribution H on $[0,1]$ which is independent of v_1. Denote t_a (t_b) the monetary transfer in period 1 if the consumer reports $\tilde{v}_1 = a$ ($\tilde{v}_1 = b$) in period 1. Denote r_a (r_b) the monetary transfer in period 2 if the consumer reports $\tilde{v}_1 = a$ ($\tilde{v}_1 = b$) in period 1. Assume $\delta = 1$. Then, the consumer will upgrade if $v_2 - r_{\tilde{v}_1} \geq v_1$, and*

$$u_1(v_1,\tilde{v}_1) = -t_{\tilde{v}_1} + v_1[1 + H(\max\{0, v_1 + r_{\tilde{v}_1}\})] + \int_{\max\{0,v_1+r_{\tilde{v}_1}\}}^1 (v_2 - r_{\tilde{v}_1})dH(v_2)$$

The IC_1 requires that $u_1(v_1,v_1) \geq u_1(v_1,\tilde{v}_1),\ \forall v_1, \tilde{v}_1$. The IR_1 requires that $u_1(v_1,v_1) \geq 0,\ \forall v_1$. The revenue of the manufacture is

$$\frac{1}{2}\sum_{v_1\in\{a,b\}} t_{v_1} + r_{v_1}[1 - H(\max\{0, v_1 + r_{v_1}\})]$$

It is easy to verify that the IC_1 for $v_1 = b$ and the IR_1 for $v_1 = a$ are binding. The solution is

$$r_a = a - b,\ r_b = 0$$

The complete solution is in Appendix 6.2.

Now, consider a mean-preserving contract of $H(v_2|\tilde{v}_1 = a)$ that it pools the interval $[2a-b, 1]$,

$$\hat{H}(v_2|\tilde{v}_1 = a) = \begin{cases} v_2 & v_2 \in [0, 2a-b) \\ 2a-b & v_2 \in [2a-b, \frac{1+2a-b}{2}) \\ 1 & v_2 \in [\frac{1+2a-b}{2}, 1] \end{cases}$$

It only decreases the value of $u_1(b, a)$.

The IC_1 *is not binding, which enables the manufacturer to extract more surplus from consumers with* $v_1 = b$ *through increasing* t_b. *Intuitively, the information about* v_2 *is more "valuable" to consumers with* $v_1 = b$ *since they only upgrade after knowing that* v_2 *is high enough. Thus, reducing the informativeness of* $\hat{H}(v_2|\tilde{v}_1 = a)$ *strengthens their incentive to report truthfully.*

We focus on the binary-partition signal structure in which for any report $\tilde{v}_1$ in period 1, it only reveals whether v_2 is above or below the threshold. The following proposition shows that when the monotone hazard rate assumption is violated, per-consumer information disclosure enables the manufacturer to generate more profit even if v_1 and v_2 are independent, $G(v_2|v_1) = H(v_2)$, which is different from the finding in Li and Shi [2017]. We simplify the discussion to $v_1 \in [v_1^*, \bar{v}]$ to guarantee full supply in period 1.

Proposition 7. *Assume that for any* v_1,

$$\mathbb{E}[v_2|v_2 < \delta\phi(v_1)] + \delta\frac{1 - F(v_1)}{f(v_1)} < \delta v_1^*;$$

and

$$\mathbb{E}[v_2|v_2 > \delta\phi(v_1)] + \delta\frac{1 - F(v_1)}{f(v_1)} > \delta\bar{v}.$$

The binary-partition disclosure, for any report $\tilde{v}_1$ *in period 1, only reveals whether* v_2 *is above or below the threshold* $\delta\phi(\tilde{v}_1)$, *enables the manufacturer to have a higher revenue than full disclosure when the hazard rate is not monotone.*

5 Conclusion

This paper emphasizes the unique nature of incremental products, where the enduring information rent arising from consumers' private knowledge of product valuations extends beyond the initial period. The manufacturer's challenge lies in reducing this rent by incentivizing consumers to upgrade to newer versions. This is achieved through a selling mechanism that provides refunds instead of requiring payments for upgrades. This distinction serves to extract additional surplus while introducing a level of inefficiency, thus defining incremental products.

This paper unveils the strategic implications of endogenizing the consumers' preference by incorporating consumer-reported valuations in dynamic mechanism design. By committing to a mix of basic upgrades and novel upgrades in period

2, the manufacturer influences consumer valuations, balancing the interplay between reduced information rent and efficiency loss. Despite the offsetting effects, the manufacturer's optimal strategy follows a bang-bang solution, maximizing profitability by selectively introducing either basic or novel upgrades based on consumers' reported valuations.

The manipulation of consumer perceptions through strategic information disclosure plays a crucial role in dynamic mechanism design. Our analysis demonstrates that per-consumer information disclosure, especially in scenarios where hazard rates are non-monotonic, can significantly enhance the manufacturer's revenue even when there is no intertemporal dependency, challenging previous findings. This highlights the dynamics of information asymmetry and strategic disclosure in driving consumer behavior and market outcomes.

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6 Appendix

6.1 Manufacturer with Limited Commitment

In this section, unlike the main body, we assume that the manufacturer lacks commitment power over two periods and can only commit to the selling scheme within each period. The consumers' problem remains the same as illustrated in section 2.1, but the manufacturer's problem differs. We discuss it in a backward way.

Period 2

Given the consumer's reports in both periods and the allocation of P1 in period 1, the manufacturer's revenue in period 2 is as follows:

$$R_2(v_1, v_2) = (1 - x_1(v_1))t_2^0(v_1, v_2) + x_1(v_1)t_2^1(v_1, v_2)$$

The manufacturer's objective is to maximize its revenue

$$\Pi_2(y_2^0, y_2^1, x_2^0, x_2^1) = \int_0^{+\infty} R_2(v_1, v_2) dG(v_2|v_1)$$

with constraints

- $y_2^i(v_1, v_2),\ i \in \{0, 1\}$ is increasing in v_2 for any v_1.

We denote $\Pi_2^*(v_1)$ as the maximized revenue in period 2.

Period 1

Given the consumer's report in period 1, the manufacturer's revenue in period 1 along with the expected revenue in period 2 is as follows:

$$R_1(v_1) = t_1(v_1) + \Pi_2^*(v_1)$$

The manufacturer's objective is to maximize its revenue

$$\Pi_1(x_1) = \int_0^{+\infty} R_1(v_1) dF(v_1)$$

with constraints

- $\forall v_1', v_1'',$

$$\int_{v_1'}^{v_1''} \left(\frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} - \frac{\partial u_1(v_1 = t, \tilde{v}_1 = v_1')}{\partial v_1} \right) dt \geq 0$$

Then, we present the optimal allocation rule.

Proposition 8. *Under the following assumption,*

1. $G(\cdot|v_1)$ *is regular for any* v_1*,*

2. *Either* $I_2(v_1, v_2) \leq \delta$*,* $\forall v_1, v_2$ *and* $I_2(v_1, v_2)$ *is increasing in both* v_1 *and* v_2*, or* $I_2(v_1, v_2) \geq \delta$*,* $\forall v_1, v_2$ *and* $I_2(v_1, v_2)$ *is decreasing in both* v_1 *and* v_2*,*

the allocation rule of the optimal mechanism is as follows:

1. $$x_1(v_1) = \begin{cases} 0 & v_1 \leq v_1^* \\ 1 & \textit{otherwise} \end{cases}$$

2. $$y_2^i(v_1, v_2) = \begin{cases} 0 & v_2 \leq \hat{v}_2(v_1) \\ 1 & \textit{otherwise} \end{cases},$$

   where $\hat{v}_2(v_1)$ *is implicitly defined by the equation*

   $$\hat{v}_2 - \frac{1 - G(\hat{v}_2|v_1)}{g(\hat{v}_2|v_1)} - \delta v_1 = 0,$$

   and

   $$x_2^i(v_1, v_2) = 1 - y_2^i(v_1, v_2).$$

Due to the manufacturer's limited commitment, the allocation rule resembles selling two products across two periods. In period 1, the consumer receives P1 if the virtual value is positive. In period 2, the consumer receives P2 if its virtual value exceeds the outside option, which is the depreciated value of P1. Specifically, $\hat{v}_2 - \delta v_1 = \frac{1-G(\hat{v}_2|v_1)}{g(\hat{v}_2|v_1)} > 0$ by the definition.

If $v_1 < v_1^*$, in period 2, the transfer is

$$t_2^0(v_1, v_2) = \begin{cases} \delta v_1 & v_2 \leq \hat{v}_2(v_1) \\ \hat{v}_2(v_1) & \textit{otherwise} \end{cases}$$

In period 1, the consumer does not get P1 and his monetary transfer is

$$\begin{aligned} t_1(v_1) &= \mathbb{E}_{v_2|v_1}[\hat{u}_2^0(v_1, v_1, v_2)] - \int_0^{v_1} \mathbb{E}_{v_2|t}[\delta x_2^0(t, v_2) + y_2^0(t, v_2) I_2(t, v_2)]dt \\ &= \int_{\hat{v}_2(v_1)}^{+\infty} (v_2 - \hat{v}_2(v_1))dG(v_2|v_1) - \int_0^{v_1} \mathbb{E}_{v_2|t}[\delta x_2^0(t, v_2) + y_2^0(t, v_2) I_2(t, v_2)]dt \end{aligned}$$

The value of t_1 is equivalent to the price of a call option in which the manufacturer guarantees the consumer to get P1 in period 2 along with an option to get P2 with a higher price.

If $v_1 > v_1^*$, in period 2, the transfer is

$$t_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \leq \hat{v}_2(v_1) \\ \hat{v}_2(v_1) - \delta v_1 & otherwise \end{cases}$$

If the consumer owns P1 in period 2, he can upgrade to P2 with a discount price, which is similar to a trade-in policy.

In period 1, the consumer does not get P1 and his monetary transfer is

$$\begin{aligned} t_1(v_1) = v_1 + \mathbb{E}_{v_2|v_1}[\hat{u}_2^1(v_1, v_1, v_2)] &- \int_0^{v_1^*} \mathbb{E}_{v_2|t}[\delta x_2^0(t, v_2) + y_2^0(t, v_2) I_2(t, v_2)]dt \\ &- \int_{v_1^*}^{v_1} \mathbb{E}_{v_2|t}[1 + \delta x_2^1(t, v_2) + y_2^1(t, v_2) I_2(t, v_2)]dt \\ = (1+\delta)v_1^* + &\int_{\hat{v}_2(v_1)}^{+\infty} (v_2 - \hat{v}_2(v_1)) dG(v_2|v_1) \\ &- \int_0^{v_1^*} \mathbb{E}_{v_2|t}[\delta x_2^0(t, v_2) + y_2^0(t, v_2) I_2(t, v_2)]dt \\ &+ \int_{v_1^*}^{v_1} \mathbb{E}_{v_2|t}[(\delta - I_2(t, v_2)) y_2^1(t, v_2)]dt \end{aligned}$$

6.2 Proofs

Proof of lemma 1:

"$\Rightarrow$": For any v_2' and v_2'',

$$\begin{aligned} & u_2^i(v_1, \tilde{v}_1 = v_1, v_2', v_2') \geq u_2^i(v_1, \tilde{v}_1 = v_1, v_2', v_2'') \\ \Leftrightarrow\ & \delta v_1 x_2^i(v_1, v_2') + v_2' y_2^i(v_1, v_2') - t_2^i(v_1, v_2') \geq \delta v_1 x_2^i(v_1, v_2'') + v_2' y_2^i(v_1, v_2'') - t_2^i(v_1, v_2'') \end{aligned}$$

and

$$\begin{aligned} & u_2^i(v_1, \tilde{v}_1 = v_1, v_2'', v_2`') \geq u_2^i(v_1, \tilde{v}_1 = v_1, v_2'', v_2') \\ \Leftrightarrow\ & \delta v_1 x_2^i(v_1, v_2'') + v_2'' y_2^i(v_1, v_2'') - t_2^i(v_1, v_2'') \geq \delta v_1 x_2^i(v_1, v_2') + v_2'' y_2^i(v_1, v_2') - t_2^i(v_1, v_2') \end{aligned}$$

With the above two inequities,

$$\begin{aligned}
&v_2'[y_2^i(v_1,v_2') - y_2^i(v_1,v_2'')] \\
&\geq \delta v_1 x_2^i(v_1,v_2'') - t_2^i(v_1,v_2'') - \delta v_1 x_2^i(v_1,v_2') + t_2^i(v_1,v_2') \\
&\geq v_2''[y_2^i(v_1,v_2') - y_2^i(v_1,v_2'')] \\
&\Leftrightarrow (v_2' - v_2'')[y_2^i(v_1,v_2') - y_2^i(v_1,v_2'')] \geq 0
\end{aligned}$$

Given the arbitrary values of v_2' and v_2'', $y_2^i(v_1,v_2)$ is increasing in v_2 for any v_1.

By applying the envelop theorem,

$$\frac{d\hat{u}_2^i(v_1,\tilde{v}_1 = v_1, v_2)}{dv_2} = \frac{\partial \hat{u}_2^i(v_1,\tilde{v}_1 = v_1, v_2)}{\partial v_2} = y_2^i(v_1,v_2)$$

Thus,

$$\delta v_1 x_2^i(v_1,v_2) + v_2 y_2^i(v_1,v_2) - t_2^i(v_1,v_2) = \hat{u}_2^i(v_1,v_1,v_2) = \hat{u}_2^i(v_1,v_1,0) + \int_0^{v_2} y_2^i(v_1,t)dt$$

Therefore,

$$t_2^i(v_1,v_2) = \underline{t}_2^i(v_1) + \delta v_1 \cdot x_2^i(v_1,v_2) + v_2 \cdot y_2^i(v_1,v_2) - \int_0^{v_2} y_2^i(v_1,t)dt$$

where $\underline{t}_2^i(v_1) = -\hat{u}_2^i(v_1,v_1,0)$.

"$\Leftarrow$": For v_2' and v_2'',

$$\begin{aligned}
&u_2^i(v_1,\tilde{v}_1 = v_1, v_2', v_2') - u_2^i(v_1,\tilde{v}_1 = v_1, v_2', v_2'') \\
&= \int_{v_2''}^{v_2'} y_2^i(v_1,t)dt + (v_2' - v_2'')y_2^i(v_1,v_2'') \geq (v_2' - v_2'')y_2^i(v_1,v_2'') - (v_2' - v_2'')y_2^i(v_1,v_2'') \\
&= 0
\end{aligned}$$

The inequity comes from $y_2^i(v_1,v_2)$ increasing in v_2. ■

Proof of lemma 2:

The expression of t_1 comes from the envelop theorem. Then, with t_1,

$$
\begin{aligned}
IC_1 \Leftrightarrow & u_1(v_1'', v_1'') \geq u_1(v_1'', v_1'), \forall v_1', v_1'' \\
\Leftrightarrow & \int_{v_1'}^{v_1''} \frac{\partial u_1(v_1 = t, \tilde{v}_1 = t)}{\partial v_1} dt \geq (v_1'' - v_1') x_1(v_1') \\
& \quad + (1 - x_1(v_1'))(\mathbb{E}_{v_2|v_1''}[\hat{u}_2^0(v_1'', v_1', v_2)] - \mathbb{E}_{v_2|v_1'}[\hat{u}_2^0(v_1', v_1', v_2)]) \\
& \quad + x_1(v_1')(\mathbb{E}_{v_2|v_1''}[\hat{u}_2^1(v_1'', v_1', v_2)] - \mathbb{E}_{v_2|v_1'}[\hat{u}_2^1(v_1', v_1', v_2)])
\end{aligned}
$$

Note that

$$
\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} = x_1(\tilde{v}_1) + (1 - x_1(\tilde{v}_1)) \frac{\partial \mathbb{E}_{v_2|v_1}[\hat{u}_2^0(v_1, \tilde{v}_1, v_2)]}{\partial v_1} + x_1(\tilde{v}_1) \frac{\partial \mathbb{E}_{v_2|v_1}[\hat{u}_2^1(v_1, \tilde{v}_1, v_2)]}{\partial v_1}
$$

Therefore, the right-hand side of inequality is $\int_{v_1'}^{v_1''} \frac{\partial u_1(v_1 = t, \tilde{v}_1 = v_1')}{\partial v_1} dt$. ■

Proof of proposition 1:

By applying the envelop theorem,

$$
\begin{aligned}
\left. \frac{du_1}{dv_1} \right|_{\tilde{v}_1 = v_1} &= \left. \frac{\partial u_1}{\partial v_1} \right|_{\tilde{v}_1 = v_1} \\
&= (1 - x_1(\tilde{v}_1)) \frac{\partial \mathbb{E}_{v_2}[\hat{u}_2^0(v_1, \tilde{v}_1, v_2)]}{\partial v_1} + x_1(\tilde{v}_1) \left(1 + \frac{\partial \mathbb{E}_{v_2}[\hat{u}_2^1(v_1, \tilde{v}_1, v_2)]}{\partial v_1} \right) \Bigg|_{\tilde{v}_1 = v_1} \\
&= (1 - x_1(v_1)) \delta \mathbb{E}_{v_2}[x_2^0(v_1, v_2)] + x_1(v_1) \left(1 + \delta \mathbb{E}_{v_2}[x_2^1(v_1, v_2)] \right)
\end{aligned}
$$

The Lagrangian function of the manufacturer's optimization problem is

$$
\begin{aligned}
\mathcal{L} =& \Pi(x_1, y_2^0, y_2^1, x_2^0, x_2^1) - \int_0^{+\infty} \int_0^{+\infty} x_1(v_1) \lambda(v_1, v_2)(y_2^1(v_1, v_2) + x_2^1(v_1, v_2) - 1) dG(v_2) dF(v_1) \\
& - \int_0^{+\infty} \int_0^{+\infty} (1 - x_1(v_1)) \gamma(v_1, v_2)(y_2^0(v_1, v_2) + x_2^0(v_1, v_2) - 1) dG(v_2) dF(v_1)
\end{aligned}
$$

First, we solve $y_2^1(v_1, v_2)$ and $x_2^1(v_1, v_2)$.

$$
\frac{\partial \mathcal{L}}{\partial y_2^1(v_1, v_2)} = [v_2 - \lambda(v_1, v_2)] x_1(v_1) f(v_1) g(v_2)
$$

$$
\frac{\partial \mathcal{L}}{\partial x_2^1(v_1, v_2)} = [\delta v_1 f(v_1) - \delta(1 - F(v_1)) - \lambda(v_1, v_2) f(v_1)] x_1(v_1) g(v_2)
$$

With the constraint $y_2^1(v_1, v_2) + x_2^1(v_1, v_2) \leq 1$, the solution is

$$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \leq \delta\phi(v_1) := \delta\left(v_1 - \frac{1-F(v_1)}{f(v_1)}\right) \\ 1 & otherwise \end{cases}$$

and

$$x_2^1(v_1, v_2) = 1 - y_2^1(v_1, v_2)$$

One can repeat these steps to solve $y_2^0(v_1, v_2)$ and $x_2^0(v_1, v_2)$, and will find that they have the same expression as $y_2^1(v_1, v_2)$ and $x_2^1(v_1, v_2)$.

Then, we solve $x_1(v_1)$.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1(v_1)} &= v_1 f(v_1) + (1 - F(v_1))\delta\mathbb{E}_{v_2}[x_2^0(v_1, v_2)] - (1 - F(v_1))\left(1 + \delta\mathbb{E}_{v_2}[x_2^1(v_1, v_2)]\right) \\ &\quad - f(v_1)\mathbb{E}_{v_2}[\delta v_1 x_2^0(v_1, v_2) + v_2 y_2^0(v_1, v_2)] \\ &\quad + f(v_1)\mathbb{E}_{v_2}[\delta v_1 x_2^1(v_1, v_2) + v_2 y_2^1(v_1, v_2)] \\ &= \phi(v_1) f(v_1)\end{aligned}$$

The last equality comes from all things canceled given the same expression of y_2^0 and y_2^1 (x_2^0 and x_2^1). Consider regular distribution F, and

$$\phi(v_1^*) = 0$$

Then,

$$x_1(v_1) = \begin{cases} 0 & v_1 \leq v_1^* \\ 1 & otherwise \end{cases}$$

Under this result, $y_2^0 = 1$ for any $v_1 \leq v_1^*$. In other words, the consumer not owning P1 in period 1 always gets P2 in period 2.

The constraint 1 holds. Then, to check constraint 2, it is sufficient to verify that

$$\begin{aligned}\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} &= x_1(\tilde{v}_1)\left(1 + \delta\mathbb{E}_{v_2}[x_2^1(\tilde{v}_1, \hat{v}_2^1)]\right) + (1 - x_1(\tilde{v}_1))\delta\mathbb{E}_{v_2}[x_2^0(\tilde{v}_1, \hat{v}_2^0)] \\ &= x_1(\tilde{v}_1) + \delta\mathbb{E}_{v_2}[x_2^0(\tilde{v}_1, \hat{v}_2^0)]\end{aligned}$$

is increasing in $\tilde{v}_1$ for any v_1, or the single crossing property holds. The second equation comes from the same expression of x_2^1 and x_2^0.

Since $x_2^0(\tilde{v}_1, \tilde{v}_2) = 1 - y_2^0(\tilde{v}_1, \tilde{v}_2)$, according to the FOC with respect to $\hat{v}_2^0$, we have $\hat{v}_2^0 = v_2 - \delta(v_1 - \tilde{v}_1)$.

$$\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} = x_1(\tilde{v}_1) + \delta G(\delta\phi(\tilde{v}_1) + \delta(v_1 - \tilde{v}_1)) = x_1(\tilde{v}_1) + \delta G\left(\delta v_1 - \delta \frac{1 - F(\tilde{v}_1)}{f(\tilde{v}_1)}\right)$$

which is increasing in $\tilde{v}_1$. ■

Proof of proposition 2:

$$\begin{aligned}
R(v_1) = {} & \underline{t}_1 + v_1 x_1(v_1) \\
& - \int_0^{v_1} x_1(t)\left(1 + \delta \mathbb{E}_{v_2|t}\left[x_2^1(t, v_2)\right]\right) dt - \int_0^{v_1} (1 - x_1(t))\delta \mathbb{E}_{v_2|t}\left[x_2^0(t, v_2)\right] dt \\
& + (1 - x_1(v_1))\mathbb{E}_{v_2|v_1}\left[\delta v_1 \cdot x_2^0(v_1, v_2) + v_2 \cdot y_2^0(v_1, v_2)\right] \\
& + x_1(v_1)\mathbb{E}_{v_2|v_1}\left[\delta v_1 \cdot x_2^1(v_1, v_2) + v_2 \cdot y_2^1(v_1, v_2)\right] \\
& - \int_0^{v_1} (1 - x_1(t))\mathbb{E}_{v_2|t}\left[y_2^0(t, v_2) I_2(t, v_2)\right] dt \\
& - \int_0^{v_1} x_1(t)\mathbb{E}_{v_2|t}\left[y_2^1(t, v_2) I_2(t, v_2)\right] dt
\end{aligned}$$

Again, we apply the Lagrangian approach and get

$$\frac{\partial \mathcal{L}}{\partial y_2^1(v_1, v_2)} = \left[v_2 - \frac{1 - F(v_1)}{f(v_1)} I_2(v_1, v_2) - \lambda(v_1, v_2)\right] x_1(v_1) f(v_1) g(v_2|v_1)$$

$$\frac{\partial \mathcal{L}}{\partial x_2^1(v_1, v_2)} = \left[\delta v_1 - \delta \frac{1 - F(v_1)}{f(v_1)} - \lambda(v_1, v_2)\right] x_1(v_1) f(v_1) g(v_2|v_1)$$

$\left[v_2 - \frac{1-F(v_1)}{f(v_1)} I_2(v_1, v_2)\right]$ is increasing in v_2 and $\phi(v_1) < 0$ if and only if $v_1 < v_1^*$. Thus, if $v_1 < v_1^*$,

$$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \le \bar{v}_2^0(v_1) \\ 1 & otherwise \end{cases}$$

where $\bar{v}_2^0 - \frac{1-F(v_1)}{f(v_1)} I_2(v_1, \bar{v}_2^0) = 0$, and $x_2^1(v_1, v_2) = 0$.

If $v_1 \ge v_1^*$,

$$y_2^1(v_1, v_2) = \begin{cases} 0 & v_2 \le \bar{v}_2^1(v_1) \\ 1 & otherwise \end{cases}$$

where $\bar{v}_2^1 - \frac{1-F(v_1)}{f(v_1)} I_2(v_1, \bar{v}_2^1) = \delta\phi(v_1)$, and $x_2^1(v_1, v_2) = 1 - y_2^1(v_1, v_2)$.

One can repeat these steps to solve $y_2^0(v_1, v_2)$ and $x_2^0(v_1, v_2)$, and will find that they have the same expression as $y_2^1(v_1, v_2)$ and $x_2^1(v_1, v_2)$.

Then, we solve $x_1(v_1)$.

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1(v_1)} &= v_1 f(v_1) - (1-F(v_1))\left(1+\delta\mathbb{E}_{v_2|v_1}[x_2^1(v_1,v_2)]\right) + (1-F(v_1))\delta\mathbb{E}_{v_2|v_1}[x_2^0(v_1,v_2)] \\ &\quad - f(v_1)\mathbb{E}_{v_2}[\delta v_1 x_2^0(v_1,v_2) + v_2 y_2^0(v_1,v_2)] + f(v_1)\mathbb{E}_{v_2}[\delta v_1 x_2^1(v_1,v_2) + v_2 y_2^1(v_1,v_2)] \\ &\quad - (1-F(v_1))\mathbb{E}_{v_2|v_1}\left[y_2^0(v_1,v_2)I_2(v_1,v_2)\right] + (1-F(v_1))\mathbb{E}_{v_2|v_1}\left[y_2^1(v_1,v_2)I_2(v_1,v_2)\right] \\ &= f(v_1)\phi(v_1)\end{aligned}$$

The last equality comes from all things canceled given the same expression of y_2^0 and y_2^1 (x_2^0 and x_2^1). Thus,

$$x_1(v_1) = \begin{cases} 0 & v_1 \leq v_1^* \\ 1 & otherwise \end{cases}$$

Then, to check constraint 2, we need to verify that

$$\begin{aligned}\frac{\partial u_1(v_1,\tilde{v}_1)}{\partial v_1} &= x_1(\tilde{v}_1)\left(1+\delta\mathbb{E}_{v_2|v_1}[x_2^1(\tilde{v}_1,\hat{v}_2^1)]\right) + (1-x_1(\tilde{v}_1))\delta\mathbb{E}_{v_2|v_1}[x_2^0(\tilde{v}_1,\hat{v}_2^0)] \\ &\quad + x_1(\tilde{v}_1)\mathbb{E}_{v_2|v_1}[y_2^1(\tilde{v}_1,\hat{v}_2^1)I_2(v_1,v_2)] + (1-x_1(\tilde{v}_1))\mathbb{E}_{v_2|v_1}[y_2^0(\tilde{v}_1,\hat{v}_2^0)I_2(v_1,v_2)] \\ &= x_1(\tilde{v}_1)\left(1+\delta+\mathbb{E}_{v_2|v_1}[y_2^1(\tilde{v}_1,\hat{v}_2^1)(I_2(v_1,v_2)-\delta)]\right) \\ &\quad + (1-x_1(\tilde{v}_1))\mathbb{E}_{v_2|v_1}[y_2^0(\tilde{v}_1,\hat{v}_2^0)I_2(v_1,v_2)]\end{aligned}$$

is increasing in $\tilde{v}_1$ for any v_1, or the single crossing property. Since $x_2^1(\tilde{v}_1,\tilde{v}_2) = 1-y_2^1(\tilde{v}_1,\tilde{v}_2)$ if $\tilde{v}_1 \geq v_1^*$, according to the FOC with respect to $\hat{v}_2^1$, we have $\hat{v}_2^1 = v_2 - \delta(v_1-\tilde{v}_1)$. Since $x_2^0(\tilde{v}_1,\tilde{v}_2) = 0$ if $\tilde{v}_1 < v_1^*$, according to the FOC with respect to $\hat{v}_2^0$, we have $\hat{v}_2^0 = v_2$.

If $I_2(v_1,v_2) \leq \delta,\ \forall v_1, v_2$ and $I_2(v_1,v_2)$ is increasing in both v_1 and v_2, the condition $y_2^1(\tilde{v}_1,\hat{v}_2^1) = 1$ is equivalent to

$$\hat{v}_2^1 > \delta\phi(\tilde{v}_1) + I_2(\tilde{v}_1,\hat{v}_2^1)\frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)} \Leftrightarrow v_2 > \delta v_1 + (I_2(\tilde{v}_1,\hat{v}_2^1)-\delta)\frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)}$$

The right-hand side of the inequality is increasing in $\tilde{v}_1$. Under this case, $\mathbb{E}_{v_2|v_1}[y_2^1(\tilde{v}_1,\hat{v}_2^1)(I_2(v_1,v_2)-\delta)]$ is increasing to $\tilde{v}_1$.

If $I_2(v_1,v_2) \geq \delta,\ \forall v_1, v_2$ and $I_2(v_1,v_2)$ is decreasing in both v_1 and v_2, the right-hand side of the inequality is decreasing in $\tilde{v}_1$. Under this case, $\mathbb{E}_{v_2|v_1}[y_2^1(\tilde{v}_1,\hat{v}_2^1)(I_2(v_1,v_2)-\delta)]$ is increasing to $\tilde{v}_1$.

If $\tilde{v}_1 < v_1^*$, $1 - x_1(\tilde{v}_1) = 1$ and

$$\mathbb{E}_{v_2|v_1}[y_2^0(\tilde{v}_1, \hat{v}_2^0) I_2(v_1, v_2)] = \int_{\bar{v}_2(\tilde{v}_1)}^{+\infty} I_2(v_1, v_2) dG(v_2|v_1)$$

To have $\mathbb{E}_{v_2|v_1}[y_2^0(\tilde{v}_1, \hat{v}_2^0) I_2(v_1, v_2)]$ increasing in $\tilde{v}_1$, $\bar{v}_2(\tilde{v}_1)$ should decrease in $\tilde{v}_1$. With $\bar{v}_2^0(\tilde{v}_1) - \frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)} I_2(\tilde{v}_1, \bar{v}_2^0(\tilde{v}_1)) = 0$,

$$\frac{\partial \bar{v}_2^0}{\partial \tilde{v}_1} = -\frac{\frac{\partial\left(-\frac{1-F(v_1)}{f(v_1)} I_2(v_1, v_2)\right)}{\partial v_1}}{\frac{\partial\left(v_2 - \frac{1-F(v_1)}{f(v_1)} I_2(v_1, v_2)\right)}{\partial v_2}} < 0$$

The inequity comes from the first and the second assumption in the proposition. Therefore, constraint 2 is satisfied. ■

Proof of proposition 3:

$$\begin{aligned}
&\Pi(x_1, y_2^0, x_2^0, y_2^1, x_2^1) = \int_0^{+\infty} R(v_1) dF(v_1) \\
&= \int_0^{+\infty} \phi(v_1) x_1(v_1) dF(v_1) \\
&\quad + \int_0^{+\infty} x_1(v_1) \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2^1(v_1, v_2) + v_2 y_2^1(v_1, v_2)\right] dF(v_1) \\
&\quad + \int_0^{+\infty} (1 - x_1(v_1)) \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2^0(v_1, v_2) + v_2 y_2^0(v_1, v_2)\right] dF(v_1) \\
&= \int_0^{+\infty} \phi(v_1) x_1(v_1) dF(v_1) + \int_0^{+\infty} \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2(v_1, v_2) + v_2 y_2(v_1, v_2)\right] dF(v_1)
\end{aligned}$$

The last equation comes from the same expression of the expectation, so we do not distinguish between x_2^0 and x_2^1 (y_2^0 and y_2^1). Then, the solution $x_1 = 1$ if $\phi(v_1) \geq 0$ or $v_1 \geq v_1^*$. Otherwise, $x_1^* = 0$. Moreover, if $\phi(v_1) < 0$ or $v_1 < v_1^*$, $x_2^0(v_1, v_2)$ should be 0 and $y_2^0(v_1, v_2)$ should be 1 obviously, and thus the monotonicity constraint holds. We can just focus on the case $v_1 \geq v_1^*$.

If $v_1 \geq v_1^*$, $\phi(v_1)$ is non-negative. Thus, the feasibility condition must be binding, $y_2^1 + x_2^1 = 1$. Then, Π is linear in y_2^1. According to the extreme point theorem (Bauer's Maximum Principle), for any $v_1 \geq v_1^*$, each of the solution $y_2^1(v_1, \cdot)$ must be extreme points in the space of increasing functions $[0, +\infty) \to [0, 1]$, which are step functions. The cutoff of $y_2^1(v_1, \cdot)$ is denoted as $\bar{v}_2(v_1) := \delta v_1 + \kappa(v_1)$.

To guarantee the incentive compatibility condition, one needs to have $\mathbb{E}_{v_2}[x_2^1(\tilde{v}_1, \hat{v}_2)] = \mathbb{E}_{v_2}[1 - y_2^1(\tilde{v}_1, \hat{v}_2)] = G(\delta v_1 + \kappa(\tilde{v}_1))$ being increasing in $\tilde{v}_1$, which means $\kappa(v_1)$ should be increasing in v_1, or $\dot{\kappa}(v_1) \geq 0$.

Then, we check the $\kappa(v_1)$ characterized is the solution of the manufacturer's optimization problem. We focus on the Hamiltonian (only with regard to y_2) of the problem,

$$\begin{aligned}\mathcal{H}(\dot{\kappa}, \kappa, \nu, v_1) &= f(v_1) \int_{\bar{v}_2(v_1)}^{+\infty} (v_2 - \delta\phi(v_1)) dG(v_2) + \nu(v_1)\dot{\kappa}(v_1) \\ &= f(v_1) \int_{\kappa(v_1)}^{+\infty} \left(v_2 + \delta \frac{1 - F(v_1)}{f(v_1)} \right) g(v_2 + \delta v_1) dv_2 + \nu(v_1)\dot{\kappa}(v_1)\end{aligned}$$

where $\dot{\kappa}(v_1)$ is the control variable and $\kappa(v_1)$ is the state variable. The necessary conditions of the optimization are

1. $\dot{\kappa}(v_1)$ maximizes $\mathcal{H}(\dot{\kappa}, \kappa, \nu, v_1)$;

2. $\kappa(v_1)$ and $\nu(v_1)$ solve the following system,

$$\dot{\nu}(v_1) = -\frac{\partial \mathcal{H}}{\partial \kappa} = f(v_1) \left(\kappa(v_1) + \delta \frac{1 - F(v_1)}{f(v_1)} \right) g(\kappa(v_1) + \delta v_1).$$

We next prove that $\kappa(v_1)$ characterized in the proposition satisfies the necessary conditions.

For an interval of v_1 such that $\dot{\kappa}(v_1) > 0$, or the monotone condition is not binding, $\nu(v_1)$ should be 0 and $\dot{\nu}(v_1) = 0$ which leads to $\kappa(v_1) = -\delta \frac{1-F(v_1)}{f(v_1)}$. Moreover, apparently, $\dot{\kappa}(v_1)$ maximizes $\mathcal{H}$ under this case.

For an interval $(\underline{v}_1, \bar{v}_1)$ such that $\dot{\kappa}(v_1) = 0$, or $\kappa(v_1)$ is a constant, we should have

$$0 = \nu(\bar{v}_1) - \nu(\underline{v}_1) = \int_{\underline{v}_1}^{\bar{v}_1} \dot{\nu}(v_1) dv_1 = \int_{\underline{v}_1}^{\bar{v}_1} \left(\kappa(v_1) + \delta \frac{1 - F(v_1)}{f(v_1)} \right) g(\kappa(v_1) + \delta v_1) dF(v_1)$$

Moreover, we claim that $\nu(v_1) \leq 0, v_1 \in (\underline{v}_1, \bar{v}_1)$ because otherwise, $\dot{\kappa}(v_1) = 0$ does not maximize $\mathcal{H}$. It is equivalent to for any $v_1 \in (\underline{v}_1, \bar{v}_1)$,

$$\int_{\underline{v}_1}^{v_1} \left(\kappa(v_1) + \delta \frac{1 - F(v_1)}{f(v_1)} \right) g(\kappa(v_1) + \delta v_1) dF(v_1) \leq 0.$$

Next, we use the Arrow sufficient theorem to show that $\kappa(v_1)$ also satisfies the sufficient condition by checking that $\hat{\mathcal{H}}(\kappa, \nu, v_1) = \max_{\dot{\kappa}} \mathcal{H}(\dot{\kappa}, \kappa, \nu, v_1)$ is concave in

κ. This can be verified by that $\frac{\partial \hat{\mathcal{H}}}{\partial \kappa} = 0$ almost everywhere. Thus, $\hat{\mathcal{H}}$ is concave in κ. ■

Proof of proposition 4:

The proof is similar to that of proposition 3. We point out the subtle difference. First, the revenue function is

$$\begin{aligned}
R(v_1) = v_1 x_1(v_1) &- \int_0^{v_1} (1 - x_1(t))\mathbb{E}_{v_2|t}\left[\delta x_2^0(t, v_2) + I_2(t, v_2) y_2^0(t, v_2)\right] dt \\
&- \int_0^{v_1} x_1(t)\left(1 + \mathbb{E}_{v_2|t}\left[\delta x_2^1(t, v_2) + I_2(t, v_2) y_2^1(t, v_2)\right]\right) dt \\
&+ (1 - x_1(v_1))\mathbb{E}_{v_2|v_1}\left[\delta v_1 \cdot x_2^0(v_1, v_2) + v_2 \cdot y_2^0(v_1, v_2)\right] \\
&+ x_1(v_1)\mathbb{E}_{v_2|v_1}\left[\delta v_1 \cdot x_2^1(v_1, v_2) + v_2 \cdot y_2^1(v_1, v_2)\right]
\end{aligned}$$

and the objective function is

$$\begin{aligned}
&\Pi(x_1, y_2^0, x_2^0, y_2^1, x_2^1) = \int_0^{+\infty} R(v_1) dF(v_1) \\
&= \int_0^{+\infty} \phi(v_1) x_1(v_1) dF(v_1) \\
&\quad + \int_0^{+\infty} x_1(v_1)\mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2^1(v_1, v_2) + \left(v_2 - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right) y_2^1(v_1, v_2)\right] dF(v_1) \\
&\quad + \int_0^{+\infty} (1 - x_1(v_1))\mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2^0(v_1, v_2) + \left(v_2 - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right) y_2^0(v_1, v_2)\right] dF(v_1)
\end{aligned}$$

If $v_1 \geq v_1^*$, the feasibility condition must be binding, $y_2^1 + x_2^1 = 1$. According to the extreme point theorem, for any $v_1 \geq v_1^*$, each of the solution $y_2^1(v_1, \cdot)$ must be extreme points. The cutoff of $y_2^1(v_1, \cdot)$ is denoted as $\bar{v}_2^1(v_1) := \delta v_1 + \kappa^1(v_1)$. To guarantee the incentive compatibility condition, one needs to have

$$\mathbb{E}_{v_2|v_1}[(I_2(v_1, v_2) - \delta) y_2^1(\tilde{v}_1, \hat{v}_2^1)] = \int_{\kappa^1(\tilde{v}_1) + \delta v_1}^{+\infty} (I_2(v_1, v_2) - \delta) g\left(v_2|v_1\right) dv_2$$

being increasing in $\tilde{v}_1$. Take the derivative over $\tilde{v}_1$. One gets $\kappa^1(v_1)$ should be increasing in v_1, or $\dot{\kappa}^1(v_1) \geq 0$.

The Hamiltonian (only with regard to y_2^1) of the problem,

$$\begin{aligned}
&\mathcal{H}(\dot{\kappa}^1, \kappa^1, \nu, v_1) = f(v_1) \int_{\bar{v}_2^1(v_1)}^{+\infty} \left(v_2 - \delta\phi(v_1) - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right) dG(v_2|v_1) + \nu(v_1)\dot{\kappa}^1(v_1) \\
&= f(v_1) \int_{\kappa^1(v_1)}^{+\infty} \left(v_2 + (\delta - I_2(v_1, v_2 + \delta v_1))\frac{1 - F(v_1)}{f(v_1)}\right) g(v_2 + \delta v_1 | v_1) dv_2 + \nu(v_1)\dot{\kappa}^1(v_1)
\end{aligned}$$

If $v_1 < v_1^*$, $\phi(v_1) < 0$, so $x_2^0(v_1, v_2)$ should be 0. According to the extreme point theorem, for any $v_1 < v_1^*$, each of the solution $y_2^0(v_1, \cdot)$ must be extreme points. The cutoff of $y_2^0(v_1, \cdot)$ is denoted as $\bar{v}_2^0(v_1) := \delta v_1 + \kappa^0(v_1)$. To guarantee the incentive compatibility condition, one needs to have

$$\mathbb{E}_{v_2|v_1}[I_2(v_1, v_2)y_2^0(\tilde{v}_1, \hat{v}_2^0)] = \int_{\kappa^0(\tilde{v}_1)}^{+\infty} I_2(v_1, v_2)g\left(v_2|v_1\right)dv_2$$

being increasing in $\tilde{v}_1$. Take the derivative over $\tilde{v}_1$. One gets $\kappa^0(v_1)$ should be decreasing in v_1, or $\dot{\kappa}^0(v_1) \leq 0$.

The Hamiltonian (only with regard to y_2^0) of the problem,

$$\mathcal{H}(\dot{\kappa}^0, \kappa^0, \nu, v_1) = f(v_1)\int_{\bar{v}_2^0(v_1)}^{+\infty}\left(v_2 - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right)dG(v_2|v_1) - \gamma(v_1)\dot{\kappa}^0(v_1)$$
$$= f(v_1)\int_{\kappa^0(v_1)}^{+\infty}\left(v_2 - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right)g(v_2|v_1)dv_2 - \gamma(v_1)\dot{\kappa}^0(v_1) \blacksquare$$

Proof of proposition 5:

$$\begin{aligned}
&\Pi'(x_1, y_2^0, x_2^0, y_2^1, x_2^1, \alpha) = \int_0^{+\infty} R(v_1)dF(v_1) \\
&= \int_{v_1^*}^{+\infty} \phi(v_1)dF(v_1) \\
&\quad + \int_{v_1^*}^{+\infty} \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1)x_2^1(v_1, v_2) + \left(v_2 - \alpha(v_1)\frac{1 - F(v_1)}{f(v_1)}\right)y_2^1(v_1, v_2)\right]dF(v_1) \\
&\quad + \int_0^{v_1^*} \mathbb{E}_{v_2|v_1}\left[\left(v_2 - \alpha(v_1)\frac{1 - F(v_1)}{f(v_1)}\right)y_2^0(v_1, v_2)\right]dF(v_1) \\
&:= \int_{v_1^*}^{+\infty} \phi(v_1)dF(v_1) + \int_0^{+\infty} T(v_1, \alpha(v_1))dF(v_1)
\end{aligned}$$

If $v_1 \leq v_1^*$,

$$
\begin{aligned}
T(v_1, \alpha(v_1)) &= \mathbb{E}_{v_2|v_1}\left[\left(v_2 - \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}\right) y_2^0(v_1, v_2)\right] \\
&= \int_{\alpha(v_1)\frac{1-F(v_1)}{f(v_1)}}^{+\infty} \left(v_2 - \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}\right) dG(v_2|v_1) \\
&= \int_{\alpha(v_1)\frac{1-F(v_1)}{f(v_1)}}^{+\infty} \left(v_2 - \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}\right) dH\left(\frac{v_2 - \alpha(v_1)v_1}{1-\alpha(v_1)}\right) \\
&= (1-\alpha(v_1)) \int_{-\frac{\alpha(v_1)}{1-\alpha(v_1)}\phi(v_1)}^{+\infty} \left(t + \frac{\alpha(v_1)}{1-\alpha(v_1)}\phi(v_1)\right) dH(t) \\
&= (1-\alpha(v_1)) \int_{-\frac{\alpha(v_1)}{1-\alpha(v_1)}\phi(v_1)}^{+\infty} (1-H(t))dt
\end{aligned}
$$

Since $\phi(v_1) < 0$, $-\frac{\alpha(v_1)}{1-\alpha(v_1)}\phi(v_1)$ increases in α. Thus, T decreases in α. The optimal $\alpha^*(v_1) = \underline{\alpha}$.

If $v_1 > v_1^*$,

$$
\begin{aligned}
T(v_1, \alpha(v_1)) &= \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1)x_2^1(v_1, v_2) + \left(v_2 - \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}\right) y_2^1(v_1, v_2)\right] \\
&= \delta\phi(v_1) + \mathbb{E}_{v_2|v_1}\left[\left(v_2 - \delta\phi(v_1) - \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}\right) y_2^1(v_1, v_2)\right] \\
&= \delta\phi(v_1) + \int_{\bar{v}_2^1(v_1)}^{+\infty} \left(v_2 - \bar{v}_2^1(v_1)\right) dG(v_2|v_1)
\end{aligned}
$$

where $\bar{v}_2^1(v_1) = \delta\phi(v_1) + \alpha(v_1)\frac{1-F(v_1)}{f(v_1)}$.

If $\alpha(v_1) > \delta$,

$$
v_2 - \bar{v}_2^1(v_1) = (\alpha(v_1) - \delta)\phi(v_1) + (1-\alpha(v_1))\omega > 0
$$

Thus,

$$
\begin{aligned}
T(v_1, \alpha(v_1)) &= \delta\phi(v_1) + \int_{\bar{v}_2^1(v_1)}^{+\infty} \left(v_2 - \bar{v}_2^1(v_1)\right) dG(v_2|v_1) \\
&= \delta\phi(v_1) + \int_{\alpha(v_1)v_1}^{+\infty} \left(v_2 - \bar{v}_2^1(v_1)\right) dG(v_2|v_1) \\
&= \delta\phi(v_1) + \alpha(v_1)v_1 - \bar{v}_2^1(v_1) + \int_{\alpha(v_1)v_1}^{+\infty} (1 - G(v_2|v_1))dv_2 \\
&= \alpha(v_1)\phi(v_1) + (1-\alpha(v_1)) \int_0^{+\infty} (1-H(t))dt
\end{aligned}
$$

T is linear in $\alpha(v_1)$ when $\alpha(v_1) > \delta$.

If $\alpha(v_1) < \delta$,

$$
\begin{aligned}
T(v_1, \alpha(v_1)) &= \delta\phi(v_1) + \int_{\bar{v}_2^1(v_1)}^{+\infty} \left(v_2 - \bar{v}_2^1(v_1)\right) dG(v_2|v_1) \\
&= \delta\phi(v_1) + \int_{\bar{v}_2^1(v_1)}^{+\infty} (1 - G(v_2|v_1))dv_2 \\
&= \delta\phi(v_1) + (1 - \alpha(v_1)) \int_{\gamma(v_1,\alpha(v_1))}^{+\infty} (1 - H(t))dt
\end{aligned}
$$

where $\gamma(v_1, \alpha(v_1)) = \frac{\delta-\alpha(v_1)}{1-\alpha(v_1)}\phi(v_1)$.

$$
\begin{aligned}
\frac{\partial T}{\partial \alpha} &= -\int_{\gamma(v_1,\alpha)}^{+\infty} (1 - H(t))dt - (1 - \alpha)[1 - H(\gamma(v_1, \alpha))]\frac{\delta - 1}{(1 - \alpha)^2}\phi(v_1) \\
&= \gamma(v_1, \alpha)(1 - H(\gamma(v_1, \alpha)) - \int_{\gamma(v_1,\alpha)}^{+\infty} th(t)dt + [1 - H(\gamma(v_1, \alpha))]\frac{1 - \delta}{1 - \alpha}\phi(v_1) \\
&= -\int_{\gamma(v_1,\alpha)}^{+\infty} th(t)dt + [1 - H(\gamma(v_1, \alpha))]\phi(v_1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 T}{\partial \alpha^2} &= \gamma(v_1, \alpha)h(\gamma(v_1, \alpha))\frac{\partial \gamma}{\partial \alpha} - h(\gamma(v_1, \alpha))\phi(v_1)\frac{\partial \gamma}{\partial \alpha} \\
&= \frac{1}{1 - \alpha}\left(\frac{\delta - 1}{1 - \alpha}\phi(v_1)\right)^2 h(\gamma(v_1, \alpha)) > 0
\end{aligned}
$$

Thus, T is convex in $\alpha(v_1)$ when $\alpha(v_1) < \delta$ and $v_1 > v_1^*$.

Then, we can focus on $\alpha(v_1)$ being three values: $\underline{\alpha}$, δ and 1. Correspondingly,

$$
T(v_1, \underline{\alpha}) = \delta\phi(v_1) + (1 - \underline{\alpha}) \int_{\gamma(v_1,\underline{\alpha})}^{+\infty} (1 - H(t))dt,
$$

$$
T(v_1, \delta) = \delta\phi(v_1) + (1 - \delta) \int_0^{+\infty} (1 - H(t))dt,
$$

and

$$
T(v_1, 1) = \phi(v_1).
$$

Deduce $\delta\phi(v_1)$ from all of them and divide them by $1 - \delta$,

$$
\tilde{T}(v_1, \underline{\alpha}) = \frac{1 - \underline{\alpha}}{1 - \delta} \int_{\gamma(v_1,\underline{\alpha})}^{+\infty} (1 - H(t))dt,
$$

$$
\tilde{T}(v_1, \delta) = \mathbb{E}[\omega],
$$

and

$$\tilde{T}(v_1, 1) = \phi(v_1).$$

$\tilde{T}(v_1, \underline{\alpha})$ decreases in v_1 and $\tilde{T}(v_1, 1)$ increases in v_1. Suppose $\tilde{T}(v_1, \delta)$ and $\tilde{T}(v_1, 1)$ cross at v_1^b, $\mathbb{E}[\omega] = \phi(v_1^b)$.

$$(1-\delta)(\tilde{T}(v_1^b, \underline{\alpha}) - \tilde{T}(v_1^b, \delta)) = (1-\underline{\alpha}) \int_{\frac{\delta-\underline{\alpha}}{1-\underline{\alpha}}\mathbb{E}[\omega]}^{+\infty} (1-H(t))dt - (1-\delta)\mathbb{E}[\omega]$$

The derivative with regard to δ is

$$-\left(1 - H\left(\frac{\delta-\underline{\alpha}}{1-\underline{\alpha}}\mathbb{E}[\omega]\right)\right)\mathbb{E}[\omega] + \mathbb{E}[\omega] > 0$$

which indicates that $(1-\delta)(\tilde{T}(v_1^b, \underline{\alpha}) - \tilde{T}(v_1^b, \delta))$ is always positive, and further indicates that $\tilde{T}(v_1, \delta)$ is always lower than the maximum of $\tilde{T}(v_1, \underline{\alpha})$ and $\tilde{T}(v_1, 1)$ for any v_1. Thus, when $v_1 < v_1^a$, $\alpha^*(v_1) = \underline{\alpha}$. Otherwise, $\alpha^*(v_1) = 1$.

Finally, we need to verify consumers' incentive compatibility holds.

$$\begin{aligned}\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} &= x_1(\tilde{v}_1)\left(1 + \mathbb{E}_{v_2|v_1}[\delta x_2^1(\tilde{v}_1, \hat{v}_2^1) + y_2^1(\tilde{v}_1, \hat{v}_2^1)I_2(v_1, v_2; \tilde{v}_1)]\right) \\ &\quad + (1 - x_1(\tilde{v}_1))\left(\mathbb{E}_{v_2|v_1}[\delta x_2^0(\tilde{v}_1, \hat{v}_2^0) + y_2^0(\tilde{v}_1, \hat{v}_2^0)I_2(v_1, v_2; \tilde{v}_1)]\right)\end{aligned}$$

If $\tilde{v}_1 < v_1^*$,

$$\begin{aligned}\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} &= \mathbb{E}_{v_2|v_1}[\delta x_2^0(\tilde{v}_1, \hat{v}_2^0) + y_2^0(\tilde{v}_1, \hat{v}_2^0)I_2(v_1, v_2; \tilde{v}_1)] \\ &= \mathbb{E}_{v_2|v_1}[\underline{\alpha} y_2^0(\tilde{v}_1, v_2)] \\ &= \underline{\alpha}\left(1 - G\left(\underline{\alpha}\frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)}\Big| v_1\right)\right)\end{aligned}$$

Since the hazard rate is monotone, the right-hand side increases in $\tilde{v}_1$.

If $\tilde{v}_1 \in (v_1^*, v_1^a)$,

$$\begin{aligned}\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} &= 1 + \mathbb{E}_{v_2|v_1}[\delta x_2^1(\tilde{v}_1, \hat{v}_2^1) + y_2^1(\tilde{v}_1, \hat{v}_2^1)I_2(v_1, v_2; \tilde{v}_1)] \\ &= 1 + \delta + \mathbb{E}_{v_2|v_1}[(\underline{\alpha} - \delta)y_2^1(\tilde{v}_1, \hat{v}_2^1)] \\ &= 1 + \delta + (\underline{\alpha} - \delta)\left(1 - G\left(\delta v_1 + (\underline{\alpha} - \delta)\frac{1-F(\tilde{v}_1)}{f(\tilde{v}_1)}\Big| v_1\right)\right)\end{aligned}$$

The right-hand side increases in $\tilde{v}_1$.

If $\tilde{v}_1 > v_1^a$,

$$\begin{aligned}\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} &= 1 + \mathbb{E}_{v_2|v_1}[\delta x_2^1(\tilde{v}_1, \hat{v}_2^1) + y_2^1(\tilde{v}_1, \hat{v}_2^1) I_2(v_1, v_2; \tilde{v}_1)] \\ &= 2\end{aligned}$$

The last equation comes from the fact that $v_2 > \delta v_1$ and thus y_2^1 always equals 1.

In summary, for any report $\tilde{v}_1$, $\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1}$ increases in $\tilde{v}_1$ for any v_1. So, consumers' incentive compatibility holds. ■

Proof of proposition 6:

With the proof in proposition 5, the value of δ matters only when $v_1 > v_1^*$ and $\alpha(v_1) < \delta$, in which case

$$T(v_1, \alpha(v_1)) = \delta\phi(v_1) + (1 - \alpha(v_1)) \int_{\gamma(v_1, \alpha(v_1))}^{+\infty} (1 - H(t))dt$$

where $\gamma(v_1, \alpha(v_1)) = \frac{\delta - \alpha(v_1)}{1 - \alpha(v_1)}\phi(v_1)$.

$$\frac{\partial T}{\partial \delta} = \phi(v_1) - (1 - H(\gamma(v_1, \alpha))\phi(v_1) > 0$$

Thus, $\int_0^{+\infty} R(v_1)dF(v_1)$ is increasing in δ and $\delta = 1$ is optimal.

To verify the manufacture has no incentive to deviate in period 2, one can check the monetary transfer in period 2 increases in δ. When $v_1 < v_1^*$, the monetary transfer in period 2 is irrelevant to δ. We can focus on the situation $v_1 > v_1^*$.

$$\begin{aligned}R_2(v_1) = \mathbb{E}_{v_2|v_1}[t_2^1(v_1, v_2)] &= \mathbb{E}_{v_2|v_1}\left[(\alpha(v_1) - 1)\frac{1 - F(v_1)}{f(v_1)}\mathbb{1}_{\{v_2 > \bar{v}_2'(v_1)\}}\right] \\ &= (\alpha(v_1) - 1)\frac{1 - F(v_1)}{f(v_1)} Pr[v_2 > \bar{v}_2'(v_1)|v_1] \\ &= (\alpha(v_1) - 1)\frac{1 - F(v_1)}{f(v_1)}\left(1 - H\left(\frac{\delta' - \alpha(v_1)}{1 - \alpha(v_1)}v_1 - \phi(v_1)\right)\right)\end{aligned}$$

where $\bar{v}_2'(v_1) = \delta' v_1 + (\alpha(v_1) - 1)\frac{1 - F(v_1)}{f(v_1)}$, δ' is the depreciation level chosen in period 2 and $v_2 = \alpha(v_1)v_1 + (1 - \alpha(v_1))\omega$. It is easy to see that $R_2(v_1)$ increases in δ.

In summary, even without commitment in period 1, the manufacture will not deviate from choosing $\delta = 1$ in period 2. ■

Solution of Example 1:

The Incentive Compatibility condition for consumer with $v_1 = a$, IC_1^a, is $u_1(a,a) \geq u_1(a,b)$:

$$IC_1^a: \ -t_a + t_b + a(r_a - r_b) + \int_{a+r_a}^{1} (v_2 - r_a) dv_2 - \int_{a+r_b}^{1} (v_2 - r_b) dv_2 \geq 0$$

For consumer with $v_1 = b$,

$$IC_1^b: \ -t_b + t_a + b(r_b - r_a) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 - \int_{b+r_a}^{1} (v_2 - r_a) dv_2 \geq 0$$

The Individual Rationality condition for consumer with $v_1 = a$, IR_1^a, is $u_1(a,a) \geq 0$:

$$IR_1^a: \ -t_a + a(1 + a + r_a) + \int_{a+r_a}^{1} (v_2 - r_a) dv_2 \geq 0$$

For consumer with $v_1 = b$,

$$IR_1^b: \ -t_b + b(1 + b + r_b) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 \geq 0$$

The sum of the both sides of IC_1^b and IR_1^a indicates

$$\begin{aligned}
0 \leq & - t_b + b(r_b - r_a) + a(1 + a + r_a) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 + \int_{a+r_a}^{b+r_a} (v_2 - r_a) dv_2 \\
= & - t_b + b(r_b - r_a) + a(1 + a + r_a) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 + \frac{1}{2}(b^2 - a^2) \\
= & - t_b + b(r_b - r_a) + a(1 + r_a) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 + \frac{1}{2}(b^2 + a^2) \\
\leq & - t_b + b(r_b - r_a) + b(1 + r_a) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2 + b^2 \\
= & - t_b + b(1 + b + r_b) + \int_{b+r_b}^{1} (v_2 - r_b) dv_2
\end{aligned}$$

which is IR_1^b. Thus, IR_1^b is redundant and IR_1^a should be binding.

The sum of the left-hand sides of IC_1^a and IC_1^b indicates

$$(a - b)(r_a - r_b)$$

which is irrelevant to t_b. To maximize the manufacturer's revenue, t_b should reach to maximum. Thus, IC_1^b should be binding and IC_1^a is redundant.

With IC_1^b and IR_1^a, we can rewrite the expression of t_a and t_b in r_a and r_b.

Then, take the derivative of the manufacturer's revenue with regard to r_a and r_b and get the solution. It is easy to verify that $v_1 + r_{\tilde{v}_1}$ is in the range $[0, 1]$.

Before strategic disclosure,

$$\begin{aligned} u_1(b, a) &= -t_a + b[1 + H(b + r_a)] + \int_{b+r_a}^{1} (v_2 - r_a) dH(v_2) \\ &= -t_a + b(1 + a) + \frac{1}{2}\left[(1 - a + b)^2 - b^2\right] \end{aligned}$$

After strategic disclosure,

$$\begin{aligned} u_1'(b, a) &= -t_a + b[1 + \hat{H}(b + r_a|a)] + \int_{b+r_a}^{1} (v_2 - r_a) d\hat{H}(v_2|a) \\ &= -t_a + b(1 + 2a - b) + \frac{1}{2}(1 - 2a + b)(1 + b) \end{aligned}$$

And,

$$u_1(b, a) - u_1'(b, a) = \frac{1}{2}(a - b)^2 > 0 \blacksquare$$

Proof of proposition 7:

We focus on the allocation rule and monetary transfer discussed in proposition 1.

First, the IC_2, IR_2 and IR_1 still hold under the binary-partition disclosure $\hat{H}$. Then, we check IC_1.

$$u_1(v_1, \tilde{v_1}) = v_1 - t_1(\tilde{v}_1) + \mathbb{E}_{\hat{H}(v_2|\tilde{v}_1)}[\hat{u}_2(v_1, \tilde{v}_1, v_2)]$$

where

$$t_1(\tilde{v}_1) = v_1^* + \int_{\delta\phi(\tilde{v}_1)}^{+\infty} (v_2 - \delta\phi(\tilde{v}_1)) dH(v_2) + \int_0^{\tilde{v}_1} \delta[1 - H(\delta\phi(t))] dt$$

and given the assumptions in the proposition, the consumer should act according to the signal obediently, upgrading if and only if knowing that v_2 is above $\delta\phi(\tilde{v}_1)$. Thus,

$$\mathbb{E}_{\hat{H}(v_2|\tilde{v}_1)}[\hat{u}_2(v_1, \tilde{v}_1, v_2)] = \delta v_1 H(\delta\phi(\tilde{v}_1)) + \int_{\delta\phi(\tilde{v}_1)}^{+\infty} \left(v_2 + \delta\frac{1 - F(\tilde{v}_1)}{f(\tilde{v}_1)}\right) dH(v_2)$$

Then, $\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1} = 1 + \delta H(\delta\phi(\tilde{v}_1))$. Since F is regular, $\frac{\partial u_1(v_1, \tilde{v}_1)}{\partial v_1}$ increases in $\tilde{v}_1$ for any v_1. Therefore, IC_1 holds.

If the hazard rate is not monotone, the revenues coming from selling P2 under these two disclosures are

$$\begin{aligned}\int_{\delta\phi(v_1)}^{+\infty} (v_2 - \delta\phi(v_1))\, d\hat{H}(v_2|v_1) &= \int_{\delta\phi(v_1)}^{+\infty} (v_2 - \delta\phi(v_1))\, dH(v_2) \\ &> \int_{\delta v_1+\kappa(v_1)}^{+\infty} (v_2 - \delta\phi(v_1))\, dH(v_2),\ \forall v_1\end{aligned}$$

Thus, the binary-partition disclosure generates higher revenue than full disclosure. ■

Proof of proposition 8:

$$\begin{aligned}&\Pi_2(y_2^0, x_2^0, y_2^1, x_2^1) = \int_0^{+\infty} R_2(v_1, v_2) dG(v_2|v_1) \\ =&(1 - x_1(v_1)) \int_0^{+\infty} \left[\delta v_1 \cdot x_2^0(v_1, v_2) + v_2 \cdot y_2^0(v_1, v_2) - \int_0^{v_2} y_2^0(v_1, t)dt\right] dG(v_2|v_1) \\ &+ x_1(v_1) \int_0^{+\infty} \left[-\delta v_1 + \delta v_1 \cdot x_2^1(v_1, v_2) + v_2 \cdot y_2^1(v_1, v_2) - \int_0^{v_2} y_2^1(v_1, t)dt\right] dG(v_2|v_1) \\ =&(1 - x_1(v_1)) \int_0^{+\infty} \left[\delta v_1 \cdot x_2^0(v_1, v_2) + \left(v_2 - \frac{1 - G(v_2|v_1)}{g(v_2|v_1)}\right) \cdot y_2^0(v_1, v_2)\right] dG(v_2|v_1) \\ &+ x_1(v_1) \int_0^{+\infty} \left[-\delta v_1 + \delta v_1 \cdot x_2^1(v_1, v_2) + \left(v_2 - \frac{1 - G(v_2|v_1)}{g(v_2|v_1)}\right) \cdot y_2^1(v_1, v_2)\right] dG(v_2|v_1)\end{aligned}$$

Under $x_2^i(v_1, v_2) + y_2^i(v_1, v_2) \le 1$, if $v_2 - \frac{1-G(v_2|v_1)}{g(v_2|v_1)} > \delta v_1$, $y_2^i(v_1, v_2) = 1$. Otherwise, $x_2^i(v_1, v_2) = 1$. Since G is regular, one can find a threshold $\hat{v}_2(v_1)$ defined by

$$\hat{v}_2 - \frac{1 - G(\hat{v}_2|v_1)}{g(\hat{v}_2|v_1)} - \delta v_1 = 0$$

$$\begin{aligned}&\Pi_1(x_1) = \int_0^{+\infty} R_1(v_1) dF(v_1) \\ =& \int_0^{+\infty} \left(\phi(v_1) x_1(v_1) + \mathbb{E}_{v_2|v_1}\left[\delta\phi(v_1) x_2(v_1, v_2) + \left(v_2 - I_2(v_1, v_2)\frac{1 - F(v_1)}{f(v_1)}\right) y_2(v_1, v_2)\right]\right) dF(v_1)\end{aligned}$$

We ignore the superscript of x_2^i and y_2^i because the expressions are the same for $i = 0, 1$. Then, $x_1(v_1)$ should be 1 if $\phi(v_1) > 0$. Otherwise, it should be 0.

The verification of monotonicity condition follows the same procedure in the proof of proposition 2. ■