Université d'Orléans

Master International Economics

Competition policy and game theory

Exercise Set 2 : game theory 's basic elements

Fall 2019

You should know the definition of a mixed strategy and of Nash equilibrium.

A mixed strategy of a player is a distribution about its strategies. Instead of choosing a pure strategy, a player can commit to choose a mixed strategy. That extend the set of strategy, and in certain situations, allow to find an equilibrium when there were no equilibrium in pure strategy. Remark that a pure strategy is a (degenerated) mixed strategy	A set of strategies (that could be mixed strategies) is a Nash equilibrium whenever there is no unilateral deviation of the players.

1 Rock-Paper-Scissor

This is about showing that the famous Rock-Paper-Scissor game has only one equilibrium in mixed strategy, in which each player plays at random.

	r	p	s
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Rock-Paper-Scissor

1) Prove that if one player chooses the mix strategy putting the same weight on R, P, S, then the payoff of each of the two player is zero.

When each firm put the same weight on R, P, S, then, the frequency of all the cells of the table is the same equal to 1/9, and then, player 1 's payoff is

$$\pi_1 = \frac{1}{9}(0 - 1 + 1 + 1 + 0 - 1 - 1 + 1 + 0) = 0,$$

then, also player 2 's payoff is

$$\pi_1 = \frac{1}{9}(0 + 1 - 1 - 1 + 0 + 1 + 1 - 1 + 0) = 0$$

2) Prove that we are at an equilibrium when both players chooses the mix strategy putting the same weight on the three actions.

Is there a déviation for player 1, denote it (p_R, p_P, p_S) with $p_R + p_P + p_S = 1$, when player 2 puts the same weight on the three actions ?

Following this deviation, player 1 's profit is

$$\pi_1^d = \frac{p_R}{3}(0 - 1 + 1) + \frac{p_P}{3}(1 + 0 - 1) + \frac{p_S}{3}(-1 + 1 + 0) = 0,$$

which does not improve player 1 's profit.

We then conclude that when both players chooses the mix strategy putting the same weight on the three actions, we are at the equilibrium

3) Prove that there is only one equilibrium of the game.

ROAD MAP Proving that if there is a strategy in which player 2 put more weight, then, there are strategies that player 1 will not choose ; Then to prove that such a strategy choice can never be an equilibrium

Let suppose that ROCK is the most weighted action by player 2 ($p_r \geq \max(p_p, p_s)$)and that $p_r > \frac{1}{3}$. Then we prove that if Player 1 anticipates such a player 2 's choice, he would at least choose not to weight Scissor ($p_S = 0$)

Let denote by (α, β, γ) the mixed strategy of player 1

When player 2 chooses (p_r, p_p, p_s) and when player 1 chooses (α, β, γ), then, player 1 's payoff is :

$$\pi_1 = \alpha(-p_p + p_s) + \beta(p_r - p_s) + \gamma(-p_r + p_p)$$

Notice that in that expression

- **the sign of the coefficient of α, $-p_p + p_s$, is unknown a priori**
- **the sign of the coefficient of β, $p_r - p_s$, is non negative**
- **the sign of the coefficient of γ, $-p_r + p_p$, is non positive**
- **At least one of the three coefficient is positive (as we do not consider the case $p_r = p_p = p_s = 1/3$.**

Then, when choosing efficiently her action, with the preceding anticipations on the form of her payoff π_1, player 1 will put weight only on thee strategy that increase strictly her profit (with a positive coefficient). Then, as $\gamma \leq 0$, she will *always* choose $\gamma = 0$, that is not to put any weight on scissor.

The intuition is that if Player 1 anticipates that player 2 put more weight on Rock, she has no incentive to put any weight on Scissor, that is strategy that drive to loss if the other player plays rock. We can prove in a same flavour that if Player 1 anticipates that player 2 put more weight on Paper, she has no incentive to put any weight on Rock, and that if Player 1 anticipates that player 2 put more weight on Scissors, she has no incentive to put any weight on Paper.

Then let suppose that there would be an equilibrium such that ROCK is the most weighted action by player 2 ($p_r \geq \max(p_p, p_s)$)and that $p_r > \frac{1}{3}$. Then, in such an equilibrium, necessarily, $p_S = 0$. But then, if player 2 anticipates that $p_S = 0$, player 2 has no incentive to put any weight on Rock, a contradiction. The same contradiction happen if we were to consider an equilibrium such that Paper is the most weighted action by player 2 and also an equilibrium such that Scissor is the most weighted action by player 2. That achieve to prove that a necessary condition on players'2 choice at the equilibrium is that $p_r = p_p = p_s$.

A symmetric reasoning can be done on player 1 's choice. The only possible equilibrium candidate is such that every player puts the same weight on Rock Paper Scissor

Think about it, you probably made such an analysis of the game when, young, you were playing at that game.

2 Four finite Games

Compute Nash Equilibrium for each following game When the strategy space for the players are $S_1 = S_2 = [0, 1]$ and with the pay-off functions :

a) ▷
$$g_1(x, y) = 5xy - x^2 - y^2 + 2$$
$$g_2(x, y) = 5xy - 3x^2 - 3y^2 + 5$$

b) ▷
$$g_1(x, y) = -2x^2 + 7y^2 + 4xy$$
$$g_2(x, y) = (x + y - 1)^2$$

c) ▷
$$g_1(x, y) = 5xy - x - y + 2$$
$$g_2(x, y) = 5xy - 3x - 3y + 5$$

d) ▷
$$g_1(x, y) = -2x^2 + 7y^2 + 4xy$$
$$g_2(x, y) = (x - y)^2$$

acFor the four games, the analysis is similar, we compute a best response for each player. More precisely, for player 1, we compute $\partial g_1/\partial x$ and for player 2, we compute $\partial g_2/\partial y$, and the conditions for which those two derivatives are nul. There is at the end a last check, to ensure that we are at the maximum

Game a) $g_1(x,y) = 5xy - x^2 - y^2 + 2$

$g_2(x,y) = 5xy - 3x^2 - 3y^2 + 5$

We compute two relevant derivatives :

$$\frac{\partial g_1}{\partial x} = 5y - 2x \qquad \frac{\partial g_2}{\partial y} = 5x - 6y$$

Those two derivatives are nul when

$$\begin{cases} 5y - 2x &= 0 \\ 5x - 6y &= 0 \end{cases} \iff x = y = 0$$

We check that g_1 is locally concave in x as $\dfrac{\partial^2 g_1}{\partial x^2} = -2 < 0$, and that g_2 is locally concave in y as $\dfrac{\partial^2 g_1}{\partial y^2} = -6 < 0$. That achieve to prove that the corresponding x and y chosen are the best choice for the two players.

In conclusion, then, $x = 0$, $y = 0$ are the unique equilibrium of the game.

Game b) $g_1(x,y) = -2x^2 + 7y^2 + 4xy$

$g_2(x,y) = (x + y - 1)^2$

We compute two relevant derivatives :

$$\frac{\partial g_1}{\partial x} = -4x + 4y \qquad \frac{\partial g_2}{\partial y} = 2(x + y - 1)$$

Those two derivatives are nul when

$$\begin{cases} x &= y \\ 5x - 6y &= 0 \end{cases} \iff x = y = 0$$

We check that g_1 is locally concave in x as $\dfrac{\partial^2 g_1}{\partial x^2} = -2 < 0$, and that g_2 is locally concave in y as $\dfrac{\partial^2 g_2}{\partial y^2} = -6 < 0$. That achieve to prove that the corresponding x and y chosen are the best choice for the two players.

In conclusion, then, $x = 0$, $y = 0$ are the unique equilibrium of the game.

3 Stop at the pedestrian crossing

Should a pedestrian cross at the zebra crossing in a country when motorists arrive on the zebra crossing. We show that there exists an economic answer to that question by modelizing a basic problem in game theory.

Consider the folllowing game with two players, a pedestrian and a car : The payoffs (pedestrian, car) are as follows : if the pedestrian crosses and the car passes (-1, *alpha*) ; if the pedestrian crosses and the car does not pass (1.0) ; if the pedestrian does not cross and the car passes (0,1) ; if the pedestrian does not cross and the car does not pass (0, 0).

1) Represent that game in normal form and find all the Nash equilibria, depending on the value of the parameter α. Interpret what you obtain.