Université d'Orléans

Master International Economics

Competition policy and game theory

Exercise Set 5 : Cournot and Bertrand, advanced

Fall 2019

You should know all about the basic Cournot and Bertrand oligopoly models. The first exercise is long and tedious, but you should come back every day to do a question more, and check for the first question. Like that, it is a very, very good training for that class.

Bertrand competition usually designs an oligopoly where firms compete in price. There is a demand addressed to each firm, that depends on the price posted by each firm ; let denote it $q_1^d(p_1, p_2)$ and $q_2^d(p_1, p_2)$. The basic model have been developed by Bertrand, in the case of an homogeneous good, and similar firms without capacity constraints, and marginal constant cost equal to c. The unique Nash equilibrium is such that $p_1^* = p_2^* = c$. Usually, people think that Bertrand competition is the toughest possible degree of product market competition.	Cournot competition usually designs an oligopoly where each firms posts simultaneouly quantity. It is supposed that there is implicitly an auction market clearing price, and that the selling price is the one that equilibrates demand and the produced goods, q_1^* and q_2^* ; we denote it p^*. In the case of an homogeneous good, and similar firms without capacity constraints, and marginal constant cost equal to c, the unique equilibrium is symmetric, such that the corresponding price $p^* > c$, and the price cost margin is decreasing with the number of competing firms. When $n \to +\infty$, the price tends to be c.

1 Bertrand Competition with capacity constraint

A capacity constraint for firm i, denoted k_i, is the maximum amount of good that she can produce. We study in this exercise TWO firms, i and j, which marginal cost is constant, equal to 1 and that differ in their capacity constraint. In the first part of the exercise, we suppose that capacity constraints are exogeneous, that the aggregate (inverse) demand is $p = a - q$, with $a > 1$. We consider the Bertrand competition game between those two firms.

1) Make a synthetic table in which you put the price and quantity for that economy in the pure competition case and in the monopoly (or cartel) case. After a general presentation, make a second table in the case $a = 2$ and $c = 1$.

Pure competition $p = c$ and $q = a - c$

Monopoly given that for the demand $q = a - p$, the price elasticity is $\varepsilon = \frac{\partial q}{\partial p}\frac{p}{q} = -1 * \frac{p}{a-p}$, and the optimal choice follows from

$$\frac{p-c}{p} = \frac{a-p}{p} \iff p - c = a - p \iff p = \frac{a+c}{2} \iff q = \frac{a-c}{2}$$

A synthetic presentation is

Price	Quantity
$p^* = c$	$q^* = a - c$
$p^M = \frac{a+c}{2}$	$q^M = \frac{a-c}{2}$

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Price	Quantity
$p^* = 1$	$q^* = 1$
$p^M = \frac{3}{2}$	$q^M = \frac{1}{2}$

For the case being, we suppose that both firms have a capacity constraint, respectively k_1 and k_2 and that $k_1 + k_2 = \frac{2}{3}(a - c)$. In other word, the firms could not serve the whole market in the pure competition setting, but they could act as a cartel.

2) Compute the price such that both firms work on the basis of their entire capacity (and sell all their stock). Show that this price (that we denote p^k) is between p^* and p^M.

Firm 1 works on the basis of its entire capacity when it produces k_1, Firm 2 works on the basis of its entire capacity when it produces k_2

The markets buys $k_1 + k_2$ when the price is (smaller or) equal to

$$p^k = a - k_1 - k_2 = a - \frac{2}{3}(a-c) = c + \frac{1}{3}(a-c)$$

This price is greater than c, as $a > c$ by assumption

This price is smaller than p^M. Indeed :

$$p^M - p^k = \frac{a+c}{2} - c\frac{1}{3}(a-c) = \frac{1}{6}(a-c) > 0$$

3) Show that $k_1 < a - c$. Interpret that condition and then show that $(p_i^*, p_i^*) = (c, c)$ is not a Bertrand equilibrium.

First, $k_1 \leq k_1 + k_2 = \frac{2}{3}(a-c) < a - c$

Interpretation The condition $k_1 < a - c$ means that firm 1 cannot respond alone to the demand of the market when $p = c = 1$. Indeed, when $p = c$, the demand is $a - 1$ and the maximum amount that firm 1 can produce is k_1 which is by assumption less than $a - 1$.

Proof Let consider that $p_1 = c$, what is Firm 2 's best response. If firm 2 posts $p_2 = c$, then she makes zero profit. If she posts $p_1 = c + \varepsilon$, with ε small enough such that $k_1 < a - 1 - \varepsilon$. Then, for this price, the demand will exceed the capacity of firm 1, and there will be some marginal demand for firm 2, $\delta q_2 > 0$, which makes for that price the profit $\pi_2 = \varepsilon * \delta q_2 > 0$. This achieves to prove that $p_2 = c$is not the best response of firm 2 when $p_1 = c$, and that $(p_i^*, p_i^*) = (c, c)$ is not a Bertrand-Nash equilibrium.

> *We want to study thereafter under which condition (p^k, p^k) is an equilibrium of the usual price competition between firm 1 and 2.*

4) Show that when firm 2 anticipates that firm 1 post $p_1 = p^k$, then firm 2 has no interest to undercut firm 1 (that is to lower the prices).

Remark that when firm 2 posts also $p_2 = p^k$, then the whole demand of the market is $k_1 + k_2$, and firm 2 works on the basis of its capacity. If firm 2 was to reduce the price, posting $p_2 = p^k - \varepsilon$, then, immediately the whole market would address its demand to firm 2, a demand greater than $k_1 + k_2$ and firm 2 could not respond to it, serving only k_2 : that would decrease the profit of firm 2, as the same quantity is produced and sold, at a lower selling price. This is not a profitable deviation.

> *To study the price competition, we have to make some assumption on the demand, when there is some rationing. In particular we suppose that in situations in which there are two prices $p_i < p_j$, then consumers first buys at the lowest price, and then, the rationed consumers could accept to pay p_j if their reservation price is above p_j : they buy if their reservation price is above p_j and if they were rationed at the price p_i.*

5) Analyze when firm 2 anticipates that firm 1 post $p_1 = p^k$, how firm 2 could have interest to propose to increase its prices and propose $p_2 > p_1$. Analyze with details that situation. Show in particular that the residual demand to firm 2 is $k_2 - \varepsilon$, write the deviation profit of firm 2 and think about it : under which condition a deviation could be profitable ?

If firm 2 proposes $p_2 = p^k + \varepsilon$, then, there are two things to remark. First, that firm 1 will serve the k_1 first consumers. Then, there remain a residual demand $k_2 - \varepsilon$. Indeed, if there was only one price, $p^k + \varepsilon$, the whole demand, at that price would be $k_1 + k_2 - \varepsilon$. But, as in our situation k_1 consumers goes to firm 1, the residual demand to firm 2 is $k_1 + k_2 - \varepsilon - k_1 = k_2 - \varepsilon$

The deviation profit of firm 2 is then

$$\pi_2^d = (p^k + \varepsilon - c)(k_2 - \varepsilon),$$

which we rewrite, given the value of p^k :

$$\pi_2^d = (\frac{1}{3}(a-c)+\varepsilon)(k_2-\varepsilon),$$

a quadratic function which maximum is obtained when

$$\varepsilon = \frac{1}{2}(k_2 - \frac{1}{3}(a-c)),$$

a value that is positive only if $k_2 > \frac{1}{3}(a-c)$

6) Analyze when firm 1 anticipates that firm 1 post $p_1 = p^k$, how firm 1 could have interest to propose to increase its prices and propose $p_1 > p_2$. Analyze with details that situation. Show in particular that the residual demand to firm 1 is $k_1 - \varepsilon$, write the deviation profit of firm 1 and think about it : under which condition a deviation could be profitable ?

The analysis is similar. When firm 1 proposes $p_1 = p^k + \varepsilon$, firm 1 will serve k_2 consumers. Then, there remain a residual demand $k_1 - \varepsilon$ inducing the deviation profit of firm 1

$$\pi_1^d = (\frac{1}{3}(a-c)+\varepsilon)(k_1-\varepsilon),$$

a quadratic function which maximum is obtained when

$$\varepsilon = \frac{1}{2}(k_1 - \frac{1}{3}(a-c)),$$

a value that is positive only if $k_1 > \frac{1}{3}(a-c)$

7) From the two preceding questions, deduce that whenever $k_1 = k_2$, then, (p^k, p^k) is an equilibrium of the price competition game

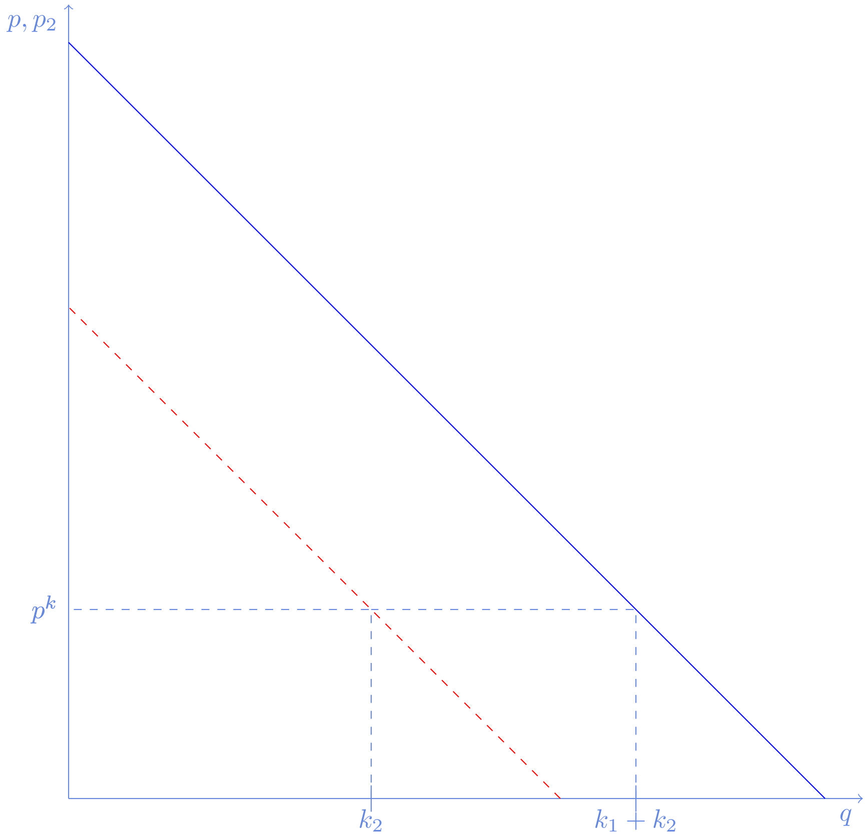
Notice that when $k_1 = k_2$, then, $k_1 = k_2 = \frac{1}{3}(a-c)$, then, the maximum of the deviating profit, for both firms is whenever $\varepsilon = 0$, which achieves to prove that (p^k, p^k) is an equilibrium.

8) Suppose that $k_1 < \frac{1}{3}(a-c) < k_2$ and prove that under that condition, (p^k, p^k) is not an equilibrium.

Whenever $k_2 > \frac{1}{3}(a-c)$ we know from the preceding questions that Firm 2 could increase its profit when increasing its price, independently of the price p^k offered by firm 1. That profitable deviation proves that (p^k, p^k) is not an equilibrium.

the remaining part is to show that this is not possible to sustain, under the demand behavior that we stated above, an asymmetric equilibrium. We stick in particular to the case $p_1 < p_2$.

9) In a graphic q, p draw the whole demand addressed to the market representing in particular the point $(k_1 + k_2, p^k)$ and as a dashed line, the residual demand to firm 2 , representing in particular the point $(k_2, p_2 = p^k)$



10) Show that if there is an asymmetric equilibrium $p_1 < p_2$, then a necessary condition for equilibrium is that consumers are rationed at p_1

Indeed, if consumers are not rationed at p_1, then, the residual demand to firm 2 is 0, and firm 2 has interest to deviate at least proposing the same price of firm 1, in order to share the profits with firm 1, a profitable deviation

11) Show then that if there is an asymmetric equilibrium $p_1 < p_2$, the best proposal by firm 2 would be $p_2^{**} = p^M - \frac{1}{2}k_1$

Indeed, suppose that consumers are rationed at p_1, then, the residual demand to firm 2 is $q_2 = a - p_2 - k_1$, which implies that

$$\pi_2 = (a - p_2 - k_1)(p_2 - c)$$

a function that is maximum when

$$p_2^{**} = \frac{a + c - k_1}{2} = p^M - \frac{1}{2}k_1$$

Remark that the preceding computation is true whenever $p_1 \leq p_2^{**}$. We suppose in what follows that this condition could is verified, and it is under this assumption that we will now investigate firm 1 's behavior.

12) check that the value of the quantity sold by firm 2 when she sticks to the preceding condition $p_2 = p_2^{**}$, at the conditions that $q_1 = k_1$ is equal to $q_2^{**} = \dfrac{a - k_1 - c}{2}$, and that thee quantity sold by the market is $a - p_2^{**}$. Check in particular what could happen if firm 1 increase slightly its price when firm 2 does not modify $p_2 = p_2^{**}$.

We already computed this quantity, $q_2 = a - p_2^{**} - k_1 = \dfrac{a - k_1 - c}{2}$. It follows that the quantity sold to the whole market is $a - p_2^{**} - k_1 + k_1$: it depends on the higher price. Then, the global amount of quantity sold to the market will not be modified when Firm 1 increase slightly its price, and also, all the consumers able to buy at p_2^{**} will prefer to buy at the lower price of Firm 1 : so she keeps her whole capacity.

13) Do you think that if $p_2 = p_2^{**}$, firm 1 will accept to price $p_1 = p^k$? Said differently, is (p^k, p_2^{**}) a Nash equilibrium ?

So, firm 1 is in a position in which she could sell her entire capacity, by posting a price lower that p_2^{**}. We see that a natural deviation for firm 1 would be to increase its price from $p_1 = p^k$ to $p_1 = p^k + \varepsilon$, which achieve to prove that we cannot have an asymmetric equilibrium.

2 Cournot with prior investment

Let consider two firms which technology is $y = \sqrt{L}\sqrt{K}$, competing in a Cournot setting, in which prior, the decision of capital is a long term one, while the decision on the level of work is a short term 1. Their decision of L is done in a Cournot framework. The price of Capital and of work are normalized to 1. We suppose that the inverse demand is $p = a - q$.

1) Show (it is an accountant equation) that the cost of producing x_i when the level of capital is K_i is $C(x_i, K_i) = K_i + \dfrac{x_i^2}{K_i}$.

When the level of capital is K_i, in order to produce the quantity x_i, the firm will need at least L_i such that $\sqrt{L_i}\sqrt{K_i} \geq x_i$ that is such that $L_i K_i \geq (x_i)^2$ or $L_i \geq \dfrac{x_i^2}{k_i}$. Then, the (minimal) cost to produce x_i gienve that the price of the factor is equal to 1 is :

$$C = K_i + L_i = K_i + \frac{x_i^2}{K_i}$$

2) Let suppose that firm i has chosen K_i and that firm j has chosen K_j as their long term investment.

We have to write the profit of each firm, the price being at the Cournot equilibrium $a - x_i - x_j$:

$$\pi_i = (a - x_i - x_j)x_i - K_i + \frac{x_i^2}{K_i}$$

this is a function of the variables x_i and x_j and of the «parameter» K_i.

3) Compute the Cournot equilibrium in the short term, and then write for each firm write their optimal production; Check that the equilibrium price is $p^* = \frac{a}{2} + a\dfrac{2 - \frac{1}{2}K_1K_2}{4 + 4K_1 + 4K_2 + 3K_1K_2}$.

As Firm i at this stage chooses x_i, we compute the derivative of the profit function relative to x_i.

$$\frac{\partial \pi_i}{x_i} = a - x_i - x_j - x_i + 2\frac{x_i}{K_i} \tag{1}$$

Then $\dfrac{\partial \pi_i}{x_i} = 0$ is equivalent to

$$x_i = \frac{1}{2}\frac{(a - x_j)}{1 + \frac{1}{K_i}}$$

The two first order condition turns out to be

$$2x_1 + \frac{K_1}{1 + K_1}x_2 = \frac{aK_1}{1 + K_1} \tag{2}$$

$$\frac{K_2}{1 + K_2}x_1 + 2x_2 = \frac{aK_2}{1 + K_2} \tag{3}$$

The determinant is :

$$\Delta = 4 - \frac{K_1}{1 + K_1}\frac{K_2}{1 + K_2} = \frac{4(1 + K_1)(1 + K_2) - K_1K_2}{(1 + K_1)(1 + K_2)}$$

which is always positive and the solutions are

$$x_1\Delta = 2\frac{aK_1}{1 + K_1} - \frac{aK_2}{1 + K_2}\frac{K_1}{1 + K_1} = \frac{aK_1}{1 + K_1}\frac{2 - K_2}{1 + K_2} \tag{4}$$

$$x_2\Delta = 2\frac{aK_2}{1 + K_2} - \frac{aK_1}{1 + K_1}\frac{K_2}{1 + K_2} = \frac{aK_2}{1 + K_2}\frac{2 - K_1}{1 + K_1} \tag{5}$$

or

$$x_1 = a\frac{K1(2 - K_2)}{4(1 + K_1)(1 + K_2) - K_1K_2} \tag{6}$$

$$x_2 = a\frac{K2(2 - K_1)}{4(1 + K_1)(1 + K_2) - K_1K_2} \tag{7}$$

From x_1 and x_2, we compute the equilibrium price :

$$p^* = a - x_1 - x_2 = a\frac{4(1+K_1)(1+K_2) - K_1K_2 - K1(2-K_2) - K2(2-K_1)}{4(1+K_1)(1+K_2) - K_1K_2} \quad (8)$$

$$= a\frac{4 + 2K_1 + 2K_2 + K1K_2}{4(1+K_1)(1+K_2) - K_1K_2} \quad (9)$$

$$= a\frac{4 + 2K_1 + 2K_2 + K1K_2}{4 + 4K_1 + 4K_2 + 3K_1K_2} \quad (10)$$

$$= \frac{a}{2} + a\frac{2 - \frac{1}{2}K_1K_2}{4 + 4K_1 + 4K_2 + 3K_1K_2} \quad (11)$$

4) From the previous computations show that $k_i \leq k_j$ is equivalent to $x_i \leq x_j$. Comment.

$x_1 \leq x_2$ is equivalent to $K1(2-K_2) \geq K2(2-K_1)$, which is equivalent to $K1 \geq K_2$: the firm with more capital will produce more. That is not surprising as the level of capital affects the productivity.

5) Compare the x_1^* and x_2^* that you obtained with the situation that would have been in pure competition : In particular, what should you conclude, given that the firm are constant return to scale ?

CRS firms should produce infty. Here, the cournot model reduce the production of each firm

6) From the previous computation, show that the equilibrium price is increasing with the level of capital invested [No need to compute the precise derivative] Comment

From the expression of p^* we see that if either K_1 or K_2 increase, then, the numerator decrease, the denominator increase, and then, the equilibrium price unambigously decrease. When one or the other of the two firms increase her capital, the price decrease, and then the global production unambigously increases.

We should consider now, in the first period, the game of the two firms, choosing in the long term their investment in capital. We model that situation as a game in which in the first stage, each firm anticipates what will be done in the next (Cournot) stage (analyzed precedingly). So after writting the profit as a function of K_1 and K_2 we would compute the derivative $\frac{\partial \pi_i}{\partial K_i}$ and then solve for the optimal level of production and then solve for the equilibrium. That exercise is tedious, and we will not do it entirely here

7) Is there an indirect path to Verify that the optimum corresponds to overinvestment ? [Write $\pi_1 = (a - x_1 - x_2)x_1 - c$]

We show a way to that conclusion with not so many computations

Write $\pi_1 = (a - x_1 - x_2)x_1 - c$, x_1, x_2 and c being functions of K_1. Then the derivative is :

$$\frac{\partial \pi_1}{\partial K_1} = \frac{\partial x_1}{\partial K_1}(a - 2x_1 - x_2) - \frac{\partial x_2}{\partial K_1}x_1 - \frac{\partial c}{\partial K_1}$$

Notice that the term in red is equal to ZERO, the choice of x_1 being maximized in the second stage. Then, from the equation $\frac{\partial \pi_1}{\partial K_1} = 0$ we deduce

$$\frac{\partial c}{\partial K_1} = -\frac{\partial x_2}{\partial K_1}x_1 > 0,$$

a term that would be optimaly equal to zero : Under Cournot firm overinvests.

************************* End of the Exercise set 5*************************