Chapter 1
Diffusion of Defaults Among Financial Institutions

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Abstract The paper proposes a simple unified model for the diffusion of defaults across financial institutions and presents some measures for evaluating the risk imposed by a bank on the system. So far the standard contagion processes might not incorporate some important features of financial contagion.

1.1 Introduction

Financial institutions use the interbank market to develop relationships that protect them against liquidity risk. These interbank claims are an important concern for regulators as some argue that they have played a large role in the dissemination of the financial crisis starting in 2007. However this is still a controversial issue in part because of the dual role played by these claims as risk sharing and risk spreading instruments.

Results are rather inconclusive at both theoretical and empirical levels. On one hand, the theoretical studies about the role and rationale of these relationships, though pointing to important phenomena such as the insurance against liquidity shocks and the limitation of bank runs, are conducted in rather simple frameworks (see e.g. Allen and Gale [2], Freixas et al. [8]). Given the complexity of the existing architecture, the robustness of the analysis can be questioned. On the other hand, empirical studies based on simulation methods have so far been unable to reproduce the extent of the crisis, even though they do not introduce the frequent bail-out interventions observed in practice (for a detailed recent survey, see Upper [15]). Simulations however shed some light on the role of the network in systemic risk and on how the assessment of the 'systemic' importance of a bank varies with the chosen measure.

Systemic risk can be understood as a diffusion process of defaults. This paper, building partly on previous studies on diffusion processes, presents a simple model of diffusion of defaults to discuss some questions about measuring systemic risk.

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Indeed, diffusion processes have been introduced in various areas, ranging from sociology to study the diffusion of innovation or ideas through social networks, in epidemiology, viral marketing and so on.

The paper is organized as follows. Section 1.2 presents the model, gives examples, and defines the diffusion processes. Section 1.3 discusses some measures of systemic risk and analyze in more details one measure.

1.2 The Framework

I describe a simple model in which financial institutions draw some risky revenues from their activities, are endowed with capital and portfolios, and are linked through claims on each other.

Consider n financial institutions, called banks for simplicity and denote $N = \{1, \ldots, n\}$. A bank i is endowed with some capital e_i. Let $\tilde{z}_i$ represent[1] the (risky) revenue that i expects from its activities excluding the interbank relationships. In the sequel z_i is called the *net worth*. Examples are described below. From a balance sheet perspective, $\tilde{z}_i$ is equal to the asset values (stocks + loans to consumers) minus the consumers' deposits. The interbank liabilities are described by (ω_{ij}) where ω_{ij} represents the magnitude of i's nominal debt obligation towards j.

When dealing with a large number of banks, the pattern of their relationships is quite stable and specific, with some banks having regular and large relationship while others having none. In such a situation, the interpretation of financial interlinkages as a network, where banks are nodes and bilateral exposures are the links, is very compelling. It may be useful to think of the graph G formed with the set of links (i, j) where i has an obligation toward j, $\omega_{ij} > 0$.

Let us describe the timing. The contagion process takes place ex post once the net worth values $\mathbf{z} = (z_i)$ are realized. In the process described below, the creditors to a defaulting bank receive nothing but the repayment from their not-in-default debtors. The loss incurred by the defaulting bank can engender defaults among its creditors. Defaults can then spread sequentially through the system, and affect, perhaps, a significant number of banks. The resulting set of defaulting banks will be denoted by $D(\mathbf{z})$ and called defaulting set. Thus, given $\mathbf{z}$, the process is well-defined and deterministic. Ex ante however, the defaulting set is random given by $D(\tilde{\mathbf{z}})$, hence with a distribution driven by the distribution of $\mathbf{z}$.

The risk of contagion is affected by various factors such as the magnitude of the inter-bank linkages, the risk distribution of the banks' net worth values, the sensitivity of the asset prices to distress sales. Let us clarify this by considering three examples with increasing complexity.

(i) The *pure network* model.
The elements of $\tilde{\mathbf{z}}$ are independent. Defaults can spread only through the interbank liabilities.

[1] In the following, a $\tilde{}$ on variable a means that the variable is random.

(ii) *Aggregate shocks.*
The presence of aggregate shocks induces correlation in the $\tilde{z}_i$ hence the possibility of simultaneous defaults. This is described by

$$\tilde{z}_i = \tilde{y}_i + \beta_i \tilde{\eta},$$

where the $(\tilde{y}_i)_{i \in N}$ are independent across banks and independent of $\tilde{\eta}$. $\tilde{\eta}$ is interpreted as a common macroeconomic factor and β_i the sensitivity of bank i's assets to that factor. If the value of the macroeconomic shock is known at the beginning of the process, and is not affected subsequently by the contagion process, the analysis of the contagion process itself is unaffected since it takes place for each realization of the $\mathbf{z}$.

(iii) *Amplification effects.*
It has been argued that the impact of 'distress' sales from defaulting banks amplifies crises in case of illiquidity. 268 positions and the sensitivity of prices to sales determine the strength of this effect. A simple description captures this effect. Letting x_i be i's asset holding in the market portfolio, i's net worth is

$$\tilde{z}_i = \tilde{y}_i + \beta_i \tilde{\eta} + x_i p_0$$

in which p_0 is the asset's price when no default occurs. As previously, the $(\tilde{y}_i)_{i \in N}$ are independent across banks and independent of $\tilde{\eta}$. For example, taking $\beta_i = x_i$, the macroeconomic shock $\tilde{\eta}$ is interpreted as the unexpected variation in the asset's price. The liquidation of assets by the defaulting banks triggers a decrease in asset's price, hence in the asset value of the balance sheets of *all* banks. Thus, defaults generate a correlated change in the banks' net worth levels through their assets' positions along the contagion process.[2]

1.2.1 Contagion Process

We assume that there is no possibility of partial default.[3] Let us describe the process by which the default of some banks propagates to other banks, first in the pure network model.

The Pure Network Model Once $\mathbf{z} = (z_i)$ are realized, each bank faces a solvency constraint. Given the bank's endowment, the net worth and the amount of its incoming and outgoing liabilities, the constraint for bank i assuming no default from its debtors is:

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[2] In a model with partial default as in [6], Cifuentes et al. [4] introduce a different mechanism in which the non-defaulting banks have to sell in order to satisfy some solvability ratio constraints.

[3] Eisenberg and Noe [6] introduce a model in which default can be partial, represented by a default level on liabilities. For a measure of the threat index of a bank in that model, see Demange [5].

$$z_i + e_i + \sum_{j \in N} \omega_{ji} - \sum_{j \in N} \omega_{ij} \geq 0. \tag{1.1}$$

The left hand side is referred to as i's *net equity*. The initial assumption on the absence of default is indeed satisfied if the net equity of each bank is non-negative, namely if the solvency condition (1.1) holds for each i. To simplify notation, let us write it as $z_i \geq v_i$ where v_i is defined as the initial net value of interbank liabilities minus the capital

$$v_i = \sum_{j \in N} \omega_{ij} - \sum_{j \in N} \omega_{ji} - e_i. \tag{1.2}$$

If some defaults occur, non-defaulting banks suffer a loss and the solvency condition is modified. Let D denotes the set of defaulting banks at some point in time. We assume no recovery, so that a bank that has not failed incurs a loss that amounts to the total of its loans to the banks in D, $\sum_{j \in D} \omega_{ji}$. Thus, given that banks in D have failed, the solvency condition for bank i not in D is:

$$z_i \geq v_i + \sum_{j \in D} \omega_{ji}. \tag{1.3}$$

The process of contagion follows from the initial defaults. Here we simply assume that all the banks that are insolvent at some step are declared defaulting (this assumption is relaxed afterwards). Given the realized values for the $\mathbf{z}$, the initial set of defaulting banks is

$$D_0 = \{i, \text{ for which } z_i < v_i\}.$$

At the beginning of step t, $t = 1, 2, \ldots,$ for which the process has not stopped, there is a set D_{t-1} of banks that have failed. The solvency conditions (1.3) for the other banks, given the total loss they have incurred due to the failures in D_{t-1}, determine the set of defaulting banks at t:

$$D_t = \left\{ i, \text{ for which } z_i < v_i + \sum_{j \in D_{t-1}} \omega_{ji} \right\}.$$

The sequence $\{D_t\}$ is increasing, $D_{t-1} \subset D_t$, until no additional failure occurs, that is $D_{t-1} = D_t$ (there are at most $n - 1$ steps).

The solvency condition (1.3) on a bank only depends on its defaulting debtors, that is, the j with a positive ω_{ji}. Hence, starting with a single default, the defaulting banks along the process are connected.[4]

Let us now allow for correlations in the net asset values and amplification effects.

[4]The process fits in the class of linear 'threshold' models as introduced by Granovetter [9]. More precisely, the linear model assumes a uniform distribution on the threshold, here the $\tilde{\mathbf{z}}$, and an influence of j on i, here ω_{ji}. A node not yet 'active' at step t becomes active in step t if the influence of its neighbors in step $t-1$ is larger than its threshold, here if the sum of their liabilities, $\sum_{j \in D} \omega_{ji}$ to i is larger than the threshold z_i.

Macroeconomic Risk In the presence of a macroeconomic risk, the diffusion process is defined for each realized value of $\mathbf{z}$ where $z_i = y_i + \beta_i \eta$ as in the pure network model. There is a change in the ex ante evaluation of the defaulting set because the correlation in the distribution of the $\tilde{z}_i$ affects the distribution of the initial defaulting set.

Amplification Effects The presence of an illiquid asset needs some explanation. Given the realizations of the y_i and η, assuming no default, let p_0 denote the asset's price in that case (recall that the variation in the prices is incorporated in the factor η, so p_0 can be interpreted as the expected value under normal conditions). Denote by z_i the known value $y_i + \beta_i \eta + x_i p_0$. The solvency condition for i writes

$$(y_i + \beta_i \eta + x_i p_0) + e_i + \sum_{j \in N} \omega_{ji} - \sum_{j \in N} \omega_{ij} \geq 0 \tag{1.4}$$

or

$$z_i \geq v_i \quad \text{where } v_i = \sum_{j \in N} \omega_{ij} - \sum_{j \in N} \omega_{ji} - e_i. \tag{1.5}$$

If the solvency condition (1.5) holds for each i, the initial assumption of no default is justified. If instead it is not met for a bank, this bank will be unable to fulfill its obligations, even by selling its asset at price p_0 and a fortiori at any lower price (we use here that x_i is non-negative). The bank defaults and liquidates its asset.

Thus, the default of banks in D results in an aggregate amount of asset's sales equal to $x_D = \sum_{i \in D} x_i$. The sale triggers a decrease in the asset's price equal to $p_0 - P(x_D)$. Accounting for this decrease, the solvency condition (1.3) for non-defaulting banks is replaced by

$$z_i \geq v_i + \underbrace{\sum_{j \in D, (j,i) \in G} \omega_{ji}}_{\text{direct loss}} + \underbrace{x_i \big(p_0 - P(x_D)\big)}_{\text{indirect loss}}. \tag{1.6}$$

Bank i's incremental loss incurred by the default of D is composed of the direct loss due to the non-reimbursement of its defaulting neighbors and the indirect loss due to the variation in prices. To simplify notation, the solvency condition (1.6) on bank i under defaulting set D writes

$$z_i \geq V_i(D) \quad \text{where } V_i(D) = v_i + \sum_{j \in D, (j,i) \in G} \omega_{ji} + x_i \big(p_0 - P(x_D)\big). \tag{1.7}$$

Thus, with amplification effects, the default of a bank is influenced not only by its defaulting neighbors as in a pure network but also by a term reflecting the whole set of defaulting banks, neighbors or not: the value $V_i(D)$ is strictly increasing in D for each i when each bank holds a positive quantity of the asset and the asset price is sensitive to sales. As a result, each bank is affected by the default of any other banks, even they are not its debtors. The network is made 'complete' by the presence of the price effect.

A general formulation that encompasses previous examples is described as follows.

Activation Model *For each i there is an activation function V_i defined over subsets D of N, nondecreasing in D, so that the solvency condition on bank i under defaulting set D writes*

$$z_i \geq V_i(D). \tag{1.8}$$

Denote $v_i = V_i(\emptyset)$.

Given $\mathbf{z}$, a diffusion process is defined as follows. At $t = 0$, define

$$D_0 = D_0(\mathbf{z}) = \{i, \text{ for which } z_i < v_i\}.$$

At the beginning of step t, $t = 1, 2, \ldots$, for which the process has not stopped, there is a set D_{t-1} of banks that have failed. Similarly at time t, the set of defaulting banks D_t is updated from D_{t-1} by checking the solvency conditions (1.8) for banks not in D_{t-1}:

$$D_t = \big\{i, \text{ for which } z_i < V_i(D_{t-1})\big\}.$$

If $D_{t-1} = D_t$, each bank not in D_t is solvent and the process stops. The reached set is called the defaulting set.

Alternative processes can be contemplated in which all the banks that are insolvent at some step are not necessarily eliminated but at least one of them must be. Specifically a process is said *without stop* if at a step t for which a 'new' bank is not solvable, that is, a bank i not in D_{t-1} for which $z_i < V_i(D_{t-1})$, then at least one such a bank is declared defaulting, i.e. D_t strictly includes D_{t-1}. The process surely stops in at most $n - 1$ steps, though it may involves more steps than in the one described previously.

1.2.2 Characterization of the Defaulting Set

The defaulting set is characterized by a property, which is independent of the process (provided the process is without stop). Hence, whatever the speed and the order of eliminations, the same defaulting set is reached.

The characterization is based on the notion of closed set.

Definition 1.1 A set C is said to be *closed* at $\mathbf{z}$ if no i outside C defaults in case of a failure of each bank in C. Specifically C is closed at $\mathbf{z}$ if

$$z_i \geq V_i(C) \quad \text{for each } i \text{ not in } C. \tag{1.9}$$

The condition requires banks outside to be solvent when they incur losses corresponding to the failure of C. Observe that no condition bears on elements in C and some banks in C can be solvent at $\mathbf{z}$. By construction, a process without stop can only settle at a set that is closed and contains $D_0(\mathbf{z})$. Furthermore,

Claim 1 *Let the solvency conditions be described by* (1.8). *Given* $\mathbf{z}$, *the defaulting set is the smallest set that is closed and contains* $D_0(\mathbf{z})$, *whatever the process without stop. Denote it by* $D(\mathbf{z})$.

Proof Let us first show that there is a smallest set among those that are closed and contain $D_0(\mathbf{z})$. Since the value $V_i(C)$ does not increase with C, the non-empty intersection of two closed sets is closed as well.[5] This readily implies that there is a smallest set among those that are closed and contain $D_0(\mathbf{z})$.

Given $\mathbf{z}$ denote by D_∞ the defaulting set reached by a process without stop. D_∞ is closed at $\mathbf{z}$ and contains $D_0(\mathbf{z})$. It thus suffices to show that D_∞ is a subset of any C that is closed at $\mathbf{z}$ and contains $D_0(\mathbf{z})$. Since D_∞ coincides with D_t for t large enough, it suffices to show that D_t is included in C for any t. We prove this by induction on t.

The induction assumption is true at $t = 0$ by the assumption on C: $D_0(\mathbf{z}) \subset C$.

Let us assume $D_t \subset C$ at t. By monotony of the V_j this implies that $V_j(D_t)$ is less than $V_j(C)$. By definition, a bank in D_{t+1} defaults at the level given $V_j(D_t)$ hence *a fortiori* each one of them defaults at level $V_j(C)$:

$$\text{for each } j \in D_{t+1}, \quad z_j < V_j(D_t) \leq V_j(C).$$

Since C is closed, $z_j < V_j(C)$ implies that j belongs to C: D_{t+1} is a subset of C, which proves the induction assumption. □

1.3 Measuring Losses and Externalities

Now that a diffusion process has been defined, it remains to evaluate the impact of a bank on the risk of the system, in particular, how it influences the reached defaulting set. Alternative measures of the impact of a bank on the risk of the system have been proposed (see Elsinger et al. [7], Tarashev et al. [14], Adrian and Brunnermeier [1], the survey of Upper [15] among a few). They differ in the following three dimensions at least.

(i) The time—and the available information—at which the expected loss imposed by a bank is evaluated: this loss can be evaluated ex ante, thereby accounting for the probability of initial insolvency of the bank or ex post, conditional on its insolvency. Also the loss can be computed conditional on some macroeco-

[5] For closed sets C and C' and i not in $C \cap C'$, surely $z_i \geq V_i(C)$ or $z_i \geq V_i(C')$. Since both $V_i(C)$ and $V_i(C')$ are larger or equal to $V_i(C \cap C')$, $z_i \geq V_i(C \cap C')$: $C \cap C'$ is closed.

nomic events. For example, in the model with macroeconomic shocks, one can use the distribution conditional on the macroeconomic variable η being smaller than some value. This is in spirit with proposals to compute a VaR measure (or an expected shortfall) conditional on 'systemic events' (to be made precise).

(ii) The risk that is measured, as an initiator of the default, or as a propagator of defaults, or both as in the 'contribution' approach. Some use the term top-down versus bottom-up.

(iii) The cost evaluation associated to the reached defaulting set. One may take the point of view of stockholders and evaluate the equity of the defaulting banks or rather be concerned with the loss to the non-financial creditors.

I describe very roughly some measures in the context of this paper and then study in more detailed one ex-post measure.

1.3.1 Some Measures

The measures are built on the cost associated to the set of defaulting banks. Let $C(D)$ be the cost associated to the defaulting set D.[6] For example, taking the point of view of stockholders, one has $C(D) = \sum_{j \in D} e_j$ to evaluate the loss in capital. If one is concerned with the loss to the non-financial creditors, the loss possibly includes some measurement of the impact on economic activity. In such a case, it is reasonable to assume that the cost C has some form of increasing returns to default (super-modularity as defined in Sect. 1.3.2).

In the sequel I focus on the loss in capital to fix the idea. In some cases it simplifies the presentation owing to the linearity of C in that case.

Loss and Externality Conditional on a Bank's Default (Bottom Up) The expected loss conditional on a bank i's default is assessed by assuming that i is the only bank to initiate default. Thus, it is computed by considering the (random) defaulting set following the initial insolvency of i only. Specifically, the loss conditional on the failure of i is

$$L(\{i\}) = E\big[C\big(D(\mathbf{z})\big) | D_0(\mathbf{z}) = \{i\}\big] \tag{1.10}$$

in which the expectation is taken over the distribution of the defaulting set $D(\mathbf{z})$ following i's default. The external loss is obtained by considering the loss imposed on other banks, that is

$$L^*\big(\{i\}\big) = L\big(\{i\}\big) - e_i.$$

Let us illustrate the computation in a simple hierarchical graph, as depicted in Fig. 1.1 in which an arrow represents a liability, that is, a bank is indebted to a

[6]This assumes that the cost only depends on the set D and not on the precise values of $\mathbf{z}$.

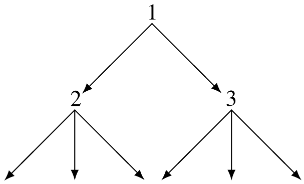

Fig. 1.1 Hierarchical structure

successor. L is given by a simple recursive expression:[7]

$$L\big(\{i\}\big) = e_i + \sum_{(i,j)\in G} q_j L\big(\{j\}\big)$$

where q_j is the probability that j fails if its unique debtor i fails. It is thus given by $q_j = \frac{F_j(v_j+\omega_{ij})-F_j(v_j)}{1-F_j(v_j)}$ where ω_{ij} is the loan made by j to i. In the particular case where each bank has the same capital value e, the same liability to each of its creditor ω, and the same distribution F one obtains: $L(\{i\}) = e(1 + q\cdot$ number of i's creditors $+q^2\cdot$ number of creditors of i's creditors $+\cdots)$.

The smaller the probability of inducing default, the closer the measure is to the number of creditors. When q is larger, indirect default starts to be determinant. In the case depicted in Fig. 1.1, $L(\{2\})$ is larger than $L(\{1\})$ for small enough q and the reverse for q large enough.

For more complex networks with cycles, there is not such a simple recursive expression. One may also consider simultaneous defaults. The (equity) loss to the initial default of a subset A is defined by

$$L(A) = E\Bigg[\sum_{j\in D(\mathbf{z})} e_j | D_0(\mathbf{z}) = A\Bigg] \tag{1.11}$$

and the externality (external loss) induced by A as

$$L^*(A) = L(A) - \sum_{i\in A} e_i. \tag{1.12}$$

These computations are performed according to some distribution of the payoffs of the non-initially defaulting banks. As said previously one may want to account for some information say on macroeconomic events.

Top Down Ex Ante Evaluation I present here a simple form of the contribution approach.[8] The basic idea here is first to evaluate the total risk in the system and second to define a contribution of a particular bank or of a subset of banks to that risk.

[7] Here the expression is similar to some measures of 'power' or 'prestige' developed in sociology such as the Katz prestige index [11].

[8] [14] also consider the loss imposed by a bank to each subsystem and derives the contribution to risk of that bank by taking an average (the Shapley value).

Fig. 1.2 A simple example

$$1 \longrightarrow 2 \longrightarrow 3$$

The total risk is evaluated as the expected loss accounting for all possibilities of failure. Let $\Pi(A)$ denote the probability that A is the initial insolvent set. In the pure network model for example $\Pi(A) = \prod_{k\in A} F_k(v_k) \prod_{k\notin A}(1 - F_k(v_k))$. The total risk is

$$T(N) = \sum_{A\subset N} \Pi(A)L(A). \tag{1.13}$$

With macroeconomic shocks, as said above, the cost can be evaluated conditional on a low enough value for the macroeconomic variable.

The cost that i imposes on the system, also called the 'contribution' of i to systemic risk, is defined as the ex ante benefit of making i totally safe. It is given by the difference

$$T(N) - T\big(N - \{i\}\big)$$

where $T(N - \{i\})$ is the risk in the system with i totally safe. Let us examine the cost in more detail. Specifically let $L_{-i}(A)$ denote the loss due to A defaulting in the system where i is totally safe; we have $T(N - \{i\}) = \sum_{A\subset N-\{i\}} \Pi(A)L_{-i}(A)$. We may thus write

$$T(N) - T\big(N - \{i\}\big) = \underbrace{\sum_{A\subset N, i\in A} \Pi(A)L(A)}_{\text{direct cost}} + \underbrace{\sum_{A\subset N-\{i\}} \Pi(A)\big[L(A) - L_{-i}(A)\big]}_{\text{indirect cost}}.$$

The cost imposed by i is composed of two terms: a direct one associated to all the events in which i defaults, and an indirect one that reflects the role of i as a vector in propagating defaults. Indeed the indirect cost only accounts for events in which i defaults but is not as an initiator; making it totally safe prevents i to spread default in those events. To illustrate, consider the simple example in Fig. 1.2. Take the same capital levels e, an identical probability for each of becoming insolvent alone, and an identical probability q of triggering default on a creditor. To simplify the presentation, let us assume that the events with several banks simultaneously initiating defaults are negligible. The direct cost is of course decreasing going down in the hierarchy ($L(\{1\}) = (1 + q + q^2)e$, $L(\{2\}) = (1 + q)e$, $L(\{3\}) = e$) but the indirect cost is larger for 2 than for 1 because 1 never spreads default. Taking the sum of the direct and indirect impact, simple computation shows that the system with 2 safe is safer than with 1 safe.

1.3.2 Some Properties of the Conditional Loss

I concentrate here on some properties of the measure L, still assuming the cost to be given by equity loss. The diffusion processes defined in the previous section fit in known classes in some special cases, such as the linear 'threshold' models (see footnote 4). The aim of this section is to draw insights from previous works on diffusion processes, and to see whether their main assumptions and results are adapted to our context. The focus is on properties referred to as sub-modularity or super-modularity. They describe concavity or convexity properties for a function defined over subsets.

Definition 1.2 Let Φ be a function defined over the subsets of N. Φ is *sub-modular* if for each i in N,

$$\Phi\big(A+\{i\}\big)-\Phi(A)\geq\Phi\big(B+\{i\}\big)-\Phi(B)\quad\text{for any sets } A\subset B\subset N. \tag{1.14}$$

Φ is *super-modular* if the reverse inequality in (1.14) holds.

Sub- or super-modularity on a function bears on its *incremental variation* and not on its level, and indicates decreasing or increasing incremental variations. The sub-modularity of Φ is interesting from a computational point of view in problems of maximization[9] of Φ.

Applied to the loss function $\Phi = L$, subsets are the initial sets of defaulting banks. Sub-modularity (resp. super-modularity) indicates decreasing (resp. increasing) returns to default: the incremental effect of a bank initially defaulting on others' defaults is decreasing with the set of banks that have already failed. Consider the problem of finding a subset A that maximizes $L(A)$ under some constraints, say the cardinality of A less than a number m. Since L is the loss conditional on the initial default of A, the solution to the problem may be thought as the set of most 'dangerous' m banks in terms of the level of the losses (using L^* instead of L, the problem is to find the subset of m banks that inflict the largest external loss on others). This type of problem has been investigated in the context of diffusion to find the nodes in a network that are the most important in terms of spreading some property. In viral marketing for example, these nodes are useful to spread the adoption of a new product.

In our context, however, the following situation looks more relevant. Let a regulation agency have some information on the value of $\mathbf{z}$ and in particular knows that the banks in D_0 will fail without intervention. Contemplating the possibility of rescuing a subset A, it evaluates the benefit according to $\Phi(A) = L(D_0) - L(D_0 - A)$. Φ inherits the modularity properties of L. Hence starting rescuing j, the value of rescuing i as well is decreasing or increasing depending on whether L being super-modular or sub-modular. Thus, here also, the modularity property of L matters.[10]

[9]Consider for example the problem of finding a subset A that maximizes $\Phi(A)$ under some constraints, say the cardinality of A less than a number m. Under sub-modularity, a fast algorithm provides an approximation for the problem. The algorithm is called 'greedy': it first looks for i with the largest value for Φ among the singletons, say i_1 that maximizes $\Phi(\{i\})$, then for j with the largest incremental change over i_1, say i_2 that maximizes $\Phi(\{i, i_1\}) - \Phi(\{i_1\})$, and so on m times. The exact problem is known to be NP-hard in the size of n.

[10]Intuitively, there should be a link between the sub-modularity or super-modularity of L and the properties of the diffusion process itself, in particular the distribution of $D(\tilde{z})$ given A. Sub-

The next proposition directly follows from Kempe, Kleinberg, and Tardos [12]. They consider the expected size of the contaminated set (or any positive linear function of it) and prove its sub-modularity in a linear threshold model. The result directly applies to the pure network model when each net worth z_i is uniformly distributed on some interval and the loss incurred by the default of a single debtor can trigger default.

Proposition 1.1 *In the pure network model, assuming the $\tilde{z}_i$ uniformly distributed on $[a_i, b_i]$, L and L^* are sub-modular if each bank i defaults with a positive probability when one of its debtor fails*:

$$z_i < v_i + \omega_{ji} \quad \textit{for each } j, \textit{ with } \omega_{ji} > 0. \tag{1.15}$$

Observe that sub-modularity is a fortiori true if the cost associated with a default set is itself sub-modular, and not additive as in the case of equity. However, sub-modularity in the cost function is surely not an appropriate assumption in our context. For example, if the cost represents the loss to creditors outside the financial system, it is more plausible that the converse assumption holds: the larger the loss, the more costly it is to absorb an additional loss.

Also, it is important to note that Proposition 1.1 may not apply in the presence of information effects, for example due to correlation in the net worth values. To be more specific, when D has defaulted and a bank has not yet defaulted, the only relevant information is that it has survived the loss induced by D, i.e., that z_i is larger than $V_i(D)$. No information is drawn from the fact that D has defaulted, as would be the case if the net worths were correlated.[11]

Condition (1.15) says that each node can contaminate each one of its neighbor.[12] This assumption is taken somewhat implicitly in the standard linear threshold model (which assumes the threshold to be uniformly distributed on [0, 1] and the influences factors of neighbors to be positive) and not emphasized. However it cannot be dispensed with as shown in the example below.

Example of Non-sub-modularity A simple example illustrates why sub-modularity may fail. There are three nodes, with 2 linked with 1 and 3. The influence weights of 1 and 3 on 2 are equal to 1 and the threshold value for 2 is bounded below by 1.5. Hence, if only 1 or 3 fails, 2 remains safe, whereas if both fail, 2 fails with some positive probability. We thus have $L^*(\{1\} + \{3\}) > 0$ and $L^*(\{1\}) = L^*(\{3\}) = 0$.

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modularity of L suggests a non-explosive dynamics of contagion. However I do not know of any result of this kind.

[11] What matters is the correlation that is unknown at the time of the evaluation.

[12] The condition is satisfied if there is a chance for the bank to default alone. This occurs for example if its initial values of the capital and liabilities are set so that a Value at Risk requirement is just binding. Specifically, given a level α, say 99 %, the level of capital, e_i, and the interbank assets and liabilities are set so that the probability of default is equal to the level $1 - \alpha$: $F_i(v_i) = 1 - \alpha$. But VaR does not make much sense in a context with a uniform distribution.

This contradicts sub-modularity: adding 3 to 1 increases more the expected size of the defaulting set than adding 3 to the empty set.

The example is not pathological and extends to any situation in which the default of a single neighbor is not enough to threaten bank i because the loss of its default inflict to i is smaller than i's minimum payoff. This explains the condition (1.15).

The assumption of a uniform distribution is of course not adequate for net worth. I examine here what are the conditions on a distribution that favor or deter sub-modularity. Basically, sub-modularity may fail if an additional loss to a bank may have a sudden large impact on the incremental probabilities of failure conditional on the fact that it has not yet defaulted as suggested in the example. This is avoided by assuming the concavity of the distribution functions, as stated is the following proposition (see our comment later on). Proposition 1.2 considers the more general formulation for a diffusion process through activation functions, as described by (1.8). It is derived by using an extension of the sub-modularity property of the expected contaminated size when the activation functions are not linear but sub-modular, still under a uniform distribution (Mossel and Roch [13]). The proof uses a transformation of the variable $\tilde{z}_i$.

Let us say that j *directly influences* i if $V_i(\{j\}) > v_i$. In other words, i incurs a loss under the single default of j. In a pure network model i is directly influenced by its debtors and in a model with price effects by all other banks (assuming all positions x_j positive).

Proposition 1.2 *Let the solvency conditions be described by* (1.8). *Assume that, for each* i,

- *the functions* V_i *are sub-modular.*
- *the cumulative distribution function of* $\tilde{z}_i$ *is strictly increasing on its support, concave and*

$$F_i\big(V_i(\{j\})\big) > 0 \quad \text{for each } j, \text{ that directly influences } i. \tag{1.16}$$

Then L *and* L^* *are sub-modular.*

The sub-modularity of V_i extends the linearity of the V_i in the pure network model. In the model with amplification effects, intuitively, sub-modularity should be satisfied if the decrease in the price due to some sales, $p_0 - P(x_D)$, diminishes with the amount of sales. This is indeed the case, as stated in the next claim

Claim 2 *Consider the model with amplification effects in which the activation functions are given by* (1.7): $V_i(D) = v_i + \sum_{j\in D,(j,i)\in G} \omega_{ji} + x_i(p_0 - P(x_D))$. *If the drop in the price* $p_0 - P(x)$ *is concave in the sales* x, *the functions* V_i *are sub-modular.*

When there are panic effects however, the drop in prices might be convex.

Consider now the assumptions on the distributions. Condition (1.16) has the same interpretation as Condition (1.15). It says that i fails with positive probability under the single default of a bank that has a direct influence on i. The concavity of a distribution however is a strong assumption. The larger the amount of losses that have already been incurred, the less likely it is that an additional amount of loss generates default. Most standard distributions are not concave.[13] For example, a normal distribution is not concave but convex for values below the mean. These values are precisely the ones that matter when considering the diffusion of the defaults.

The reverse assumptions, a convex distribution and a super-modular activation functions, do not guarantee super-modularity of the functions L. The reason is the following, that I call the ground effect. Starting with a larger initial default set, there are less banks susceptible to default. This effect is true not only at the beginning of the diffusion but also all along the process. This effect is a force towards sub-modularity, independently of the distribution. A formulation of this idea is the following proposition, stated in the pure network model for simplification.

Definition 1.3 Consider the pure network model. D is said to be a threat if for each i not in D

$$Proba\big(z_i < V_i(D)\big) > 0 \quad \text{each } i, \tag{1.17}$$

where $V_i(D) = v_i + \sum_{j \in D} \omega_{ji}$.

Proposition 1.3 *Let D be a threat. Assume in addition the distribution F_i to be concave for $z_i > V_i(D)$ for each i. Then L is sub-modular on larger sets, that is* (1.14) *holds for any sets A and B containing D.*

Condition (1.17) says that each bank not in D has some chances to fail if all banks in D default. If whatever situation the bank never fails, no set D is a threat. Otherwise, each bank is threatened when all its neighbors fail, so (1.17) is satisfied for 'large' enough sets. Observe also that a superset of a threat is also a threat. The second condition, which requires the concavity of the upper tail of the distributions, is satisfied by most distributions.

The proof is easy: It suffices to apply Proposition 1.2 to the diffusion process assuming that D has already defaulted. The distribution for the net worth of the banks that have not yet defaulted is adjusted only conditional on their absence of default, that is conditional on $z_i > V_i(D)$.

Some Concluding Remarks In conclusion, the properties of the diffusion processes studied so far in the (non-financial) literature might not be relevant to financial crises, and there is a need for further development. They focus on some properties on the distribution functions that are not appropriate. The analysis here

[13] Most standard distributions have a log-concave density. Then the ratio $\frac{F(x+\delta)-F(x)}{1-F(x)}$, related to the hazard rate (see for example Bergstrom and Bagnoli [3]) is increasing. Though this is compatible with the concavity of F, this suggests that concavity is far from being guaranteed.

distinguishes two influences of the distributions of the net worth on the diffusion of risks. On one hand, the lower tails of the distributions of the net worth matter for determining the initial defaults. On the other hand, the reaction of the diffusion to additional defaults, as embodied by the incremental values of the loss L, is instead affected by the shape of the distributions for larger values of the net worth, and this holds especially if interbank liabilities are large. Under standard assumptions on the distributions, this second effect might play an important role in shaping the loss. Finally, diffusion processes do not account for information effects.

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