LOCATIONAL ANALYSIS OF PUBLIC FACILITIES
J.-F. Thisse and H.G. Zoller (editors)

SPATIAL MODELS OF COLLECTIVE CHOICE

Gabrielle Demange*

Ecole Polytechnique, Paris
Paris, France

The majority-rule voting is a common mean of deciding a collective action but unfortunately a stable issue-called a Condorcet winner- may not exist. The characterisation of the individual preferences which ensure the existence of such an issue has been a widely studied problem in social choice theory. We gather in this paper the obtained results when the space of alternatives admits a special structure (multidimensional space or graph). These very numerous results can in particular be applied to the usual models of location theory where they are nicely interpretable.

I. INTRODUCTION

The majority voting is a widely used procedure for taking collective decisions. This method may however be irrational in the sense that a) the majority relation is intransitive for some profiles of individual preferences, and b) a majority winner, namely an issue which is never defeated by another, does not necessarily exist. The well-known Condorcet paradox illustrates this observation : there are three individuals, called 1,2,3 and three alternatives a,b,c such that 1 prefers a to b to c, 2 prefers b to c to a and 3 prefers c to a to b; then no issue is stable since c defeats a, a defeats b and b defeats c.

A large body of the literature in social choice theory is concerned with formulating conditions on individual preferences which assure one of the two properties : the *transitivity of the majority rule* and the *existence of a majority winner*. Two different approaches have been followed. In the first one, restrictions on the family of individual preferences are determined in order to guarantee one of the properties; more precisely, a family of preferences is said to *guarantee* a property if, for any number of voters, this property is satisfied whenever the individual preferences are in this family. The most well-known result of this type is the characterisation of Sen (1966) - the so-called value restriction : a family of strict orders on a finite choice space X guarantees the transitivity of the strict majority relation if and only if, for every triple of alternatives in X, there is an alternative which is not best, or not worst, or not medium in the triple for all the strict orders considered.

The second approach is concerned with conditions on the *distribution* of the individuals over the family of preferences. One looks for the relations between the preferences of the individuals of a given society which are necessary or sufficient to ensure the required property. The Plott's conditions (1967) provide an example of this approach : if the choice set is the Euclidean space $\mathbb{R}^m$ and if the individual preferences are represented by strictly concave and differentiable utility functions on $\mathbb{R}^m$, then the directions of the gradients of the utility functions at a Plott equilibrium (a notion of majority winner) must be opposite two by two.

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The purpose of this paper is to survey and to discuss the results obtained in the particular context of *location theory*. This restriction seems to be interesting for two reasons. In the first place, within the general formulation, no "a priori" structure is given on the choice set nor on the preference family. The consequence is that the methods used are mostly combinatorial and the results not easily interpreted. (We except the single peakedness notion which probably explains the success of the Black's theorem). If, on the contrary, assumptions on the choice space and on the preferences are introduced, one may apply powerful mathematical tools as differential calculus, topology or convex analysis, and interpret the results in terms of the distribution of the individual locations (location is to be taken in a large sense : this is the most preferred point in the choice space). Traditionally, location theory considers two types of models : the discrete and the continuous models. In the discrete model, the choice space is a subset of a network; in the continuous model, it is a convex subset of a Euclidean space. (Incidently, note that the latter can also be used in the study of electoral competition : see for example the survey by Ordeshook (1974). Both formulations will be considered.

Secondly, the spatial approach allows to develop a new theory of public facility location (see Hansen and Thisse (forthcoming) and Rushton, McLafferty and Ghosh (forthcoming)). The choice set being included in $\mathbb{R}^2$, the results obtained enable us to characterise the location of a public service resulting from a voting process. This interpretation therefore raises the interest for the particular properties established in the plane.

The paper is organised as follows. Section 2 gives the basic definitions and presents the locational models. We provide the results concerning the transitivity of majority relations in section 3 and those on the existence of a majority winner in section 4. Section 5 briefly concludes with some remarks on possible extensions of the notion of majority winner.

In the following tables we sum up the main results. They are arranged according to the type of results obtained (i.e. transitivity and acyclicity of a majority relation or existence of a majority winner) and according to the approach used (restrictions on the individual preferences or distribution conditions).

	GUARANTEE CONDITIONS	DISTRIBUTION CONDITIONS
Transitivity and acyclicity of the majority relations (section 3)	<u>Continuous model</u> - Theorem 1 (Kramer) A necessary condition for the transitivity of the strict simple majority relation P to be satisfied by a set of quasi-concave and differentiable utility functions is that all the gradients of the utility functions at any interior point of the choice space belongs to a half-plane. - Theorem 3 (Grandmont) An intermediate preference orders family indexed by an interval guarantees the transitivity of P.	<u>Continuous model</u> - Theorem 5 (Grandmont) A profile composed with intermediate preferences indexed by an open convex subset of IR^P yields the transitivity of the simple majority relation when the distribution of the individual preferences is weakly symmetric around a point.
	<u>Discrete model</u> - Theorem 4 (Romero-Hansen-Thisse) The family of orders induced by a distance on a tree guarantees the transivity of P. Furthermore, the weak Condorcet points are the points which minimise the total distance to the individuals' locations.	

	GUARANTEE CONDITIONS	DISTRIBUTION CONDITIONS
Existence of a majority winner (section 4)	<u>Continuous model</u> - Theorem 9 (Demange) A family of pseudo-concave utility functions indexed by an interval and satisfying the local intermediate preferences assumption guarantees the extistence of a weak quasi-Condorcet winner. - Theorem 10 (Wendell-Thorson-Demange) A norm on $\mathbb{R}^2$ guarantees the eixstence of a quasi-Condorcet winner on $\mathbb{R}^2$ if and only if it is of the l^1-type.	<u>Continuous model</u> - Thoerem 12 (Plott) Let $(U_i)_{i \in S}$ be a family of concave and differentiable utility functions. If at most one utility function is maximised at $x \in \mathbb{R}^m$, then this point is weak Condorcet winner if and only if there exists a pairing T on S such that the gradients at x of U_i and $U_{T(i)}$ have opposite directions for any $i \in S$. - Theorem 13 (McKelvey-Wendell) Let $\Vert\ \Vert$ be a strictly convex norm on $\mathbb{R}^m$ admitting a differentiable representation. If at most one individual is located at $x \in \mathbb{R}^m$, then this point is a weak Condorcet winner if and only if x is a weak symmetry center. - Theorem 14 (Demange) A profile composed with local intermediate preferences represented by pseudo-concave utility functions ensures the existence of a weak quasi-Condorcet winner if the distribution of the individual utilities satisfies the condition of symmetry (C2).
	<u>Discrete model</u> - Theorem 11 (Demange) A set of singlepeaked orders on a tree guarantees the existence of a weak Condorcet winner.	<u>Discrete model</u> - Theorem 15 (McKelvey-Wendell) Let N be a network endowed with a distance inducing the preference orders of the individuals located at vertices. A point x in N is a weak Condorcet winner if there exists a pairing T on S such that x belongs to a shortest route between the locations of any pair $\{i, T(i)\}$ of individuals.

2. DEFINITIONS AND MODELS

2.1 Definitions

(a) Binary Relations

A binary relation $\mathcal{B}$ on X is said to be

- reflexive : if $x \mathcal{B} x$ for every $x \in X$;
- irreflexive : if not $x \mathcal{B} x$ for every $x \in X$;
- transitive : if $x \mathcal{B} y$ and $y \mathcal{B} z$ imply $x \mathcal{B} z$ for every $x, y, z \in X$;
- acyclical : if there is no finite sequence $x_1, \dots x_k \in X$ such that

  $x_1 \mathcal{B} x_2$, $x_2 \mathcal{B} x_3, \dots,$ and $x_{k-1} \mathcal{B} x_k$ and $x_k \mathcal{B} x_1$.
- complete : if $x \mathcal{B} y$ or $y \mathcal{B} x$ for every $x, y \in X$, $x \neq y$;

From a binary relation $\mathcal{B}$ we may define two new relations I and S called respectively *indifference relation* and *strict relation* : for every $x, y \in X$,

- $x \, I \, y \Leftrightarrow x \mathcal{B} y$ and $y \mathcal{B} x$.
- $x \, S \, y \Leftrightarrow x \mathcal{B} y$ and not $y \mathcal{B} x$.

If $\mathcal{B}$ is a complete binary relation, then we have :

$\mathcal{B}$ transitive $\Rightarrow S$ transitive $\Rightarrow S$ acyclical.

We say that $x \in X$ is maximal for $\mathcal{B}$ if not $y \, S \, x$ for every $y \, \mathcal{B} \, X$.

A (preference) *order* on X, denoted by $\geq$, is a reflexive, transitive and complete relation on X. Let $\sim$ be its indifference component and $>$ its strict component which is a *strict order* (i.e. an irreflexive, transitive and complete relation).

(b) Majority Winners

Let $S = \{1, \dots, n\}$ be a society, that is a set of n individuals, and X a set of alternatives (or choice space). Each individual i has a preference order, denoted $\geq_i$, on X. The n-tuple $(\geq_i)_{i \in S}$ is called the *profile* of the society.

Several notions of majority winner will be useful. An alternative x in X is called a *strong* (resp. *weak*) *Condorcet winner* for the profile $(\geq_i)_{i \in S}$ if : for every $y \in X$ distinct of x,

$$\# \; \{i \in S \; ; \; x >_i y\} > \# \; \{i \in S \; ; \; y >_i x\},$$

(resp. $\# \; \{i \in S \; ; \; x >_i y\} \geq \# \; \{i \in S \; ; \; y >_i x\}$)

where $\#$ denotes the cardinality.

An alternative x in X is said to be a *strong* (resp. *weak*) *quasi-Condorcet winner* for the profile $(\geq_i)_{i \in S}$ if : for every $y \in X$,

$$\# \; \{i \in S \; ; \; y >_i x\} < \frac{n}{2}$$

(resp. $\# \; \{i \in S \; ; \; y >_i x\} \leq \frac{n}{2}$).

Obviously, a strong (resp. weak) Condorcet winner is a strong (resp. weak) quasi-Condorcet winner.

(c) Majority Relations

We associate with every profile $(\geq_i)_{i \in S}$ on X three relations on X. The first one is the *simple majority relation* R : for every x, y $\in$ X,

$$x \, R \, y \Leftrightarrow \# \{i \in S \,;\, x >_i y\} \geq \#\{i \in S \,;\, y >_i x\}.$$

The second one is the strict component P of R, called the *strict simple majority relation.*

The third one is denoted by Q and defined as follows : for every x, y $\in$ X,

$$x \, Q \, y \Leftrightarrow \# \{i \in S \,;\, x >_i y\} > \frac{n}{2}.$$

Clearly, we have : x Qy $\Rightarrow$ x Py $\Rightarrow$ x Ry ; R is complete ; and

$$R \text{ transitive} \Rightarrow P \text{ transitive} \Rightarrow P \text{ acyclical} \Rightarrow Q \text{ acyclical}.$$

A weak Condorcet (resp. quasi-Condorcet) winner is a maximal element for P (resp. Q) in X ; thus, if X is a finite set the acyclicity of P (resp. of Q) implies the existence of a weak Condorcet (resp. quasi-Condorcet) winner.

(d) The Notion of Guarantee

A set E of preference orders (or of utility functions) on X *guarantees* a property if, for every integer n and every profile in E^n, the property is satisfied.

In this paper, we shall consider the following properties : the transitivity of R or of P, the acyclicity of Q, the existence of a strong or weak Condorcet winner, the existence of a strong or weak quasi-Condorcet winner.

2.2 Spatial Models

In the context of spatial analysis, the choice space is to be interpreted as the set of feasible "locations". Two types of models are considered according to the properties of X.

(a) The Continuous Model

The choice space is a *convex subset of* $\mathbb{R}^m$ with a non empty interior X. The distance on X may be the same for all individuals in the society, as in model I, or distinct according to the individuals, as in model II.

Consider, first, *model I.* We assume a *norm,* denoted $\| \ \|$, to be given on the Euclidean space $\mathbb{R}^m$. The preference of an individual i is characterised by his most preferred point in $\mathbb{R}^m$, say a_i, which is called his location : for every x, y $\in$ X,

$$x \geq_i y \Leftrightarrow \| x-a_i \| \leq \| y-a_i \| .$$

The profile of a society S is therefore described by the norm $\| \ \|$ on $\mathbb{R}^m$ and by the n-tuple $(a_i)_{i \in S}$ of the locations of the individuals in S.

The model is easy to understand in the two-dimensional frame of location theory but it can also be interpreted in terms of welfare economics. In location theory, the choice space X represents the set of feasible locations in $\mathbb{R}^2$ for a public good. The society of the n individuals located respectively at points $a_1, \dots a_n$ has to choose a place in X for this good. Each individual wishes to have the good as close as possible to his location and $\| x-a \|$ is interpreted as the transportation cost from a to x. The majority voting is then a decentralised, non manipulable choice procedure for selecting the place where to set up the public facility (see Hansen and Thisse (forthcoming)).

In welfare economics, the choice space is often assumed to be multidimensional : a point represents the quantities consumed of public and private goods by the individuals, and or the levels of taxes. Usually, the norm used in this context is the Euclidean one ($\| x \|_2 = (\sum_{i=1}^{m} x_i^2)^{1/2}$) ; $\| x-a \|$ then expresses the loss of utility of the individual whose most preferred point is a (see Kramer (1977) and Tullock (1967)).

Let us now turn to *model II*. The individual preferences on X are assumed to be represented by quasi-concave(1), (or pseudo-concave(2), or concave, etc.) and differentiable utility functions on an open convex set Ω which includes X. If the function U_i represents the order $\geq_i$ of the individual $i \in S$, we have : for every $x, y \in X$,

$$x \geq_i y \Leftrightarrow U_i(x) \geq U_i(y).$$

(In the sequel, $\nabla U_i(x)$ denotes the gradient of U_i at x, with $x \in \Omega$).

If each function U_i achieves its maximum in X in a unique point (called a peak or an ideal point or a location) model II can be interpreted exactly as model I. The difference is only with the possibility of distinct distance evaluations for distinct individuals. Conversely, when a norm $\| \ \|$ on $\mathbb{R}^m$ admits a differentiable representation Φ (i.e. Φ is a strictly increasing function from $\mathbb{R}^+$ to $\mathbb{R}$ and $\Phi(\| x \|)$ is differentiable on $\mathbb{R}^m$) model I appears as a special case of model II. This holds for many norms, for example the l^p norms,

$$(\| x \|_p = (\sum_{i=1}^{m} |x_i|^p)^{1/p}) \text{ when } p > 1.$$

Note that, when the utility functions U_i are strictly quasi-concave(3) every strong (resp.weak) quasi-Condorcet winner is a strong (resp.weak) Condorcet winner (see McKelvey and Wendell (1976)).

(b) The Discrete Model

The choice space is a *subset of a network*. A *network* is the union of a finite number of arcs homeomorphic to [0,1]. Formally, a *network* N is a subset of $\mathbb{R}^m$ which satisfies the following conditions :

(i) $N = \bigcup_{i=1}^{n} h_i([0,1])$ where $n \geq 1$ and h_i a continuous injection from [0,1] in $\mathbb{R}^m$, $i = 1 \dots n$;

(ii) $h_i(\Theta) \neq h_{i'}(\Theta')$ for any $i \neq i'$, with $i, i' \in \{1 \dots n\}$, and any $\Theta \neq \Theta'$, with $\Theta, \Theta' \in [0,1]$;

(iii) is connected.

The set of *vertices* associated with N is given by $V = \{v \in N\ ;\ \exists\ i \in \{1...n\}/ v = h_i(0)$ or $v = h_i(1)\}$. A subset $h_i([0,1])$ of is called an *arc* and denoted $A(v,v')$; the set of arcs is L. A connected subset of $h_i([0,1])$ is called *sub-arc*. Finally, a *route* linking $x \in N$ and $y \in N$ is defined as a smallest connected subset of N containing x and y. Stated differently, a route is formed by the union of a finite number of arcs and sub-arcs which is the image of $[0,1]$ by a continuous injection.

A network N is a *tree* if, for every x and y in N, there is a single route linking x and y.

A network may be endowed with a metric structure thanks to a length mapping l :

$$L \to \mathbb{R}^+ - \{0\}$$

$$A(v,v') \to l(v,v').$$

where $l(v,v')$ represents the length of the arc $A(v,v')$.

The length mapping can be extended to the sub-arcs and to the routes : the length of a sub-arc $A(v,x)$, where x is such that $A(v,x) = h_i([0,t])$, is equal to $t.l(v,v')$ and the length of a route $\bigcup_{i\in\{0,...k\}} A(v_i,v_{i+1})$ to $\sum_{i\in\{0,...k\}} l(v_i,v_{i+1})$.

We then define a *distance* d on N : $d(x,y)$, for $x,y \in N$, $x \neq y$, is the length of the shortest route connecting x and y ; by convention, $d(x,x)$ is put equal to 0. It is then easily seen that d is a metric on N.

A discrete model(4) is thus characterised by the choice space X, which is a subset of a network N, and by the profile of the society on X. When N is endowed with a distance d, a natural order on N for the individual i located at a_i is as follows : for every $x, y \in N$

$$x \geq_i y \Leftrightarrow d(x,a_i) \leq d(y,a_i).$$

This last model is called *model I* by comparison with what we did for continuous models. Similarly, we can consider *model II* in which the utility U_i defined on N admits a unique maximiser a_i and is such that :

$$U_i(x) \geq U_i(y) \Leftrightarrow x \text{ belongs to a shortest route between } a_i \text{ and } y.$$

The difference with model I is that $d(a_i,x) = d(a_i,y)$ does not imply $U_i(x) = U_i(y)$.

3. TRANSITIVITY AND ACYCLICITY OF THE MAJORITY RELATIONS

3.1 The Guarantee Conditions

(a) The Necessary Conditions of Kramer

Inada (1969) and Sen and Pattanaik (1969) have characterised the sets of preference relations which guarantee the transitivity of the strict majority relation P (see Fishburn (1973) for a detailed discussion). From this characterisation, Kramer (1973) has deduced necessary conditions on the gradients of the utility functions for the transitivity of P to hold. As the preference orders are assumed to be represented by quasi-concave and differentiable utility functions on a convex subset of the Euclidean space, the Kramer's approach is of the type "continuous model II".

THEOREM 1. [Kramer (1973)]

The choice space is a convex subset X *of* $\mathbb{R}^m$, $m \geq 2$. *If three quasi-concave and differentiable utility functions* U_1, U_2, U_3 *satisfy the following condition at a point* x *in* $\mathring{X}$ *: no gradient belongs to the closed convex cone generated by the two other gradients, that is there exist no* $\lambda_i \geq 0$, $\lambda_j \geq 0$ *such that*

$$\nabla U_k(x) = \lambda_i \nabla U_i(x) + \lambda_j \nabla U_j(x) \text{ and } \{i,j,k\} = \{1,2,3\} \quad ,$$

then, in any neighbourhood of x *in* $\mathring{X}$, *there are three points* {a,b,c} *forming a Condorcet cycle, i.e.*

$$U_1(a) > U_2(b) > U_3(c), \; U_2(b) > U_2(c) > U_2(a)$$

and

$$U_3(c) > U_3(a) > U_3(b).$$

Thus, if a set of quasi-concave and differentiable utility functions on X guarantees the transitivity of the strict majority, all the gradients $\nabla U(x)$, for every x in X, must belong to a two dimensional half-space of $\mathbb{R}^m$. This result shows that guarantee conditions in continuous model II are very restrictive. For instance, if ϕ is a differentiable representation of a norm $\| \; \|$ of $\mathbb{R}^m$ the family $(-\phi(\| x-a \|))_{a \in A}$ does not satisfy the Kramer's condition when A is not an interval. In section 4, we shall see some significative examples for which the condition of Theorem 1 holds. In those examples, however, the existence of a majority winner, and not the transitivity of relation P, is guaranteed. Recall, indeed, that the condition is necessary but not sufficient. For that, an additional assumption on the convexity of the preferences is required.

(b) The Intermediate Preferences

The notion of intermediate preference provides us with non trivial families of preference relations for which the transitivity of P is guaranteed. Note that the orders are not assumed to be represented by utility functions; however, the Kramer's condition is satisfied when utility functions exist. We first introduce the concept of intermediate preferences.

DEFINITION 1.

a) Consider three orders $\geq_1$, $\geq_2$, $\geq_3$ defined on X. We say that $\geq_3$ is *between* $\geq_1$ and $\geq_2$ if, for every x and y in X,

$$x \geq_1 y \text{ and } x \geq_2 y \text{ imply } x \geq_3 y,$$

and

$$(x \geq_1 y \text{ and } x >_2 y) \text{ or } (x >_1 y \text{ and } x \geq_2 y) \text{ imply } x >_3 y.$$

b) A family of preference orders $(\geq_a)_{a \in A}$ on X (or of utility functions $(U_a)_{a \in A}$ if this one exists) indexed by a convex open set A in $\mathbb{R}^p$ satisfies the *intermediate preferences assumption* if the following two conditions hold:

- *for every* $x \in X$ *and* $y \in X$, *the set* $\{a \in A;\ x \geq_a y\}$ *is closed in* A,
- *for every* $a' \in A$ *and* $a'' \in A$, *the order* $\geq_a$ *is between* $\geq_{a'}$ *and* $\geq_{a''}$ *whenever* $a \in]a', a''[$.

Some examples will clarify this definition.

Example 1 : The norm induced by a scalar product in $\mathbb{R}^m$.

Let X and A be two convex subset of $\mathbb{R}^m$ and B a symmetric matrix of order m. The mapping $x \to \| x \| = Bx.Bx$ is a norm on $\mathbb{R}^m$. Then, the family of utility functions $(-\| x-a \|)_{a \in A}$ satisfies the intermediate preferences assumption.

Indeed, the sets $\{a \in A\ ;\ x \geq_a y\}$ are convex and closed in A since

$$\{a \in A\ ;\ x \geq_a y\} = \{a \in A\ ;\ Ba.\ B(x-y) \geq \frac{Bx.Bx - By.By}{2}\}$$

Example 2 : The linear preference orders

Let a be a non null vector in $\mathbb{R}^m$. We assign to a the preference order $\geq_a$ in $\mathbb{R}^m$ - called linear order - as follows : for every x, $y \in \mathbb{R}^m$

$$x \geq_a y \Leftrightarrow x.a \geq y.a$$

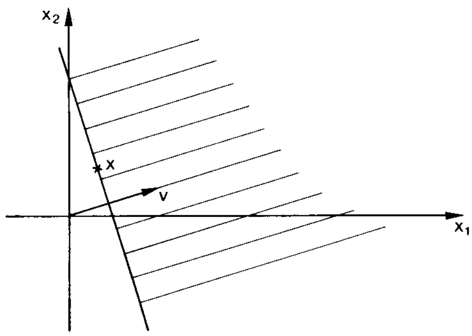

Figure 1

In Fig. 1, the shaded set indicates the set of the outcomes preferred to x for $\geq_a$, i.e. $\{y \in \mathbb{R}^2\ ;\ y \geq_a x\}$.

Note that $\geq_a = \geq_{\lambda a}$ for every positive scalar λ. It is easily seen that the family $(\geq_a)_{a \in A}$ for every convex open set A in $\mathbb{R}^m - \{0\}$, is an intermediate

preferences family.

Nitzan (1976) has directly shown that the family of linear orders guarantees the transitivity of P when v runs over the set $\{v;\ v > 0\}$. This result will appear in the sequel as a special case of Theorem 3. Before that, the following characterisation of the assumption (H1) will be useful.

LEMMA 2. [Grandmont (1978)]

Let A be an open convex subset of $\mathbb{R}^p$. The family $(\geq_a)_{a \in A}$ of orders on the choice set X satisfies the assumption (H1) if, for every x and y in X, one of the following conditions holds :

(1) Either $x >_a y$, for all a, or $x \sim_a y$, for all a, or $x <_a y$, for all a.

(2) There exists q in $\mathbb{R}^p - \{0\}$ and a real number c such that

$\{a \in A\ ;\ q.\ a > c\} = \{a \in A\ ;\ x >_a y\}$, $\{a \in A\ ;\ q.a = c\} = \{a \in A\ ;\ x \sim_a y\}$

and $\{a \in A\ ;\ q.\ a < c\} = \{a \in A\ ;\ y >_a x\}$,

that is to say the sets $\{a \in A\ ;\ x >_a y\}$ and $\{a \in A\ ;\ x <_a y\}$ are the intersections of A with the open half-spaces delimited by the same hyperplane of $\mathbb{R}^p$. The next theorem can then be deduced from Lemma 2.

THEOREM 3. [Grandmont (1978)]

Every family $(\geq_a)_{a \in A}$ *of intermediate preference orders on a choice space* X *guarantees the transitivity of relation* P *when* A *is an open interval of* $\mathbb{R}$.

It remains to show how the Nitzan's result can be deduced from Theorem 3 ; that is, how the family of linear orders $(\geq_a)_{a \in A}$ can be indexed by an open interval. It is known that A is an open convex subset of $(R^+ - \{0\})^n$ and that $\geq_a = \geq_{\lambda a}$ for every $\lambda > 0$. If we therefore choose a vector q in $\mathbb{R}^n$ such that $q.\ a > 0$ for every a in A (such a vector always exists by a separation theorem in $\mathbb{R}^n$), then the family $(\geq_a)_{a \in A}$ is equal to the family $(\geq_a)_{a \in I}$ where I is the set

$\{ \frac{a}{q.a}\ ;\ a \in A\}$ (see Fig. 2). Obviously, I is an open interval on the line $\Delta = \{x \in \mathbb{R}^2;\ q.\ x = 1\}$.

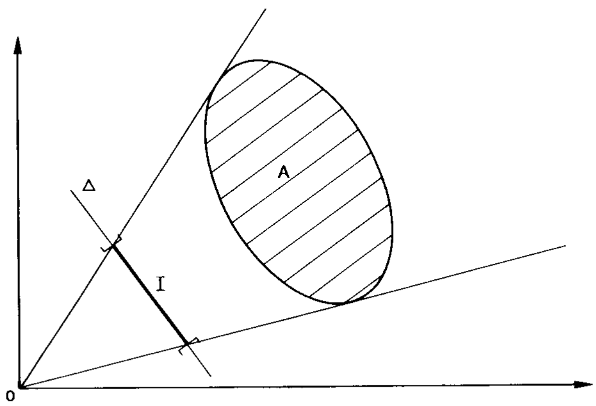

Figure 2

(c) The Pseudo-Singlepeakedness Condition

The next result, obtained in discrete model I, extends the property of single-peakedness on a line. This notion having several meanings in social choice theory, we introduce another label, that of "pseudo-singlepeakedness" due to Romero (1978).

DEFINITION 2.

A family of preference orders $(\geq_a)_{a \in A}$ on X is said to be *pseudo-singlepeaked* if, for every triple of distinct alternatives x, y, z in X, one of them, say x, is not worst in the triple for all the orders, i.e. for every $a \in A$

$$x >_a y \text{ or } x >_a z.$$

A pseudo-singlepeaked family of preference orders guarantees the transitivity of P, since it satisfies the value restriction condition. The proof is very simple. Let x, y, z be three distincts alternatives in X, where x is never worst. Then, for every profile $(\geq_i)_{i \in S}$ and $a_i \in A$

$$y\ R\ x \Rightarrow y\ R\ z, \quad y\ P\ x \Rightarrow y\ P\ z, \quad z\ R\ x \Rightarrow z\ R\ y \quad \text{and } z\ P\ x \Rightarrow z\ Py\ .$$

Indeed, $y >_i x$ implies $x >_i z$ and $y >_i z$ by transitivity of $>_i$, so that

$\{i\ ;\ y >_i x\} \subset \{i\ ;\ y >_i z\}$.

Similarly, $z >_i y$ implies $x >_i y$ since $y \geq_i x$ would imply $z \geq_i x$;

hence, $\{i\ ;\ z >_j y\} \subset \{i\ ;\ x >_i y\}$.

Then the implications easily follow.

Let now a, b, c be three alternatives such that a P b and b P c. If a is not the worst in the triple {a,b,c}, a P c is true since c R a would imply c R b. If b is not the worst, a P b implies a P c.

Finally, if c is not the worst, b P c would imply b P a which is impossible. Thus, in all cases, a P c is true and P is transitive. This completes the proof. The sets of singlepeaked orders on a line are pseudo-singlepeaked.

DEFINITION 3.

A set of preference orders on X is said to be *singlepeaked* on a line if X can be ordered in such a way that, when we go from the left to the right on the line, every preference strictly increases up to a peak and then strictly decreases.

For example, if a distance is given on a line and if the alternatives are finite and ranged as $x_1 < \ldots < x_m$, the set of orders $(\geq_i)$ is singlepeaked where $\geq_i$ is defined by :

$$x_j \geq_i x_k \Leftrightarrow d(x_i, x_j) \leq d(x_i, x_k).$$

This example can be generalised by replacing the line by a tree. We then get the next theorem which is valid in discrete model I.

THEOREM 4. [Romero (1978)] (5)

Let N be a tree, the set of vertices of which is v. *The family of orders*

$(\geq_v)_{v \in V}$ *induced by the distance on* N *is pseudo-singlepeaked and, therefore, guarantees the transitivity of* P *on* N .

The existence of a weak Condorcet winner in N for a profile $(\geq_i)_{i \in S}$ with $a_i \in V$, can be easily deduced from this theorem. Indeed, as N is a tree, a point x not in V and belonging to an arc A(v,v') is a weak Condorcet winner in N if and only if v and v' are also weak Condorcet winners. Moreover, v and v' are weak Condorcet winners if N if and only if they have the same property in V. Thus the problem can be restricted to the finite choice space V and the existence follows from the transitivity of P. This last proposition has been independently proved by Hansen and Thisse (forthcoming). Their result is very interesting from another point of view; they show that the weak Condorcet winners coincide with the points which minimise the total distance from the locations of the individuals, i.e. $\sum_{i \in S} d(a_i,x)$. Thus when the network is a tree and when the orders are derived from the distance on N , the majority rule provides not only a stable issue but also an optimal one.

These positive results are recent. Recall that the first attempts to generalise the singlepeakedness notion have taken place in $\mathbb{R}^m$ for $m \geq 2$ and have led to extremely limited results (see Kramer (1976) and Wagstaff (1976).

3.2 The Distribution Conditions

A distribution condition is stated in terms of relations between the preferences inside a profile which are necessary or sufficient for a property to be satisfied by that profile. Here, we shall consider the following two properties : the transitivity of R and the acyclicity of Q. Of course the guranteee conditions may appear as sufficient distribution conditions. However, in contrast to the former approach, the latter does not restrict the set of admissible preferences orders *a priori*.

Only the result of Grandmont (1978) about the intermediate preference orders is concerned with such distribution conditions. These ones can be deduced from the characterisation of the intermediate preferences assumption given by Lemma 2. For that, we need some notation and definitions.

Let A be an open convex set of $\mathbb{R}^p$. Given a finite sequence $(a_i)_{i \in S}$ in A, we denote by μ the probability distribution $\frac{1}{n} \sum_{i \in S} \delta_{a_i}$ where δ_{a_i} is the discrete probability on $\mathbb{R}^m$ supported by a_i. Let H be an hyperplane of $\mathbb{R}^p$; we denote by A' and A" the intersections of A with the two closed half-spaces delimited by H.

Given $(a_i)_{i \in S}$ we say that a point a* in A satisfies the condition (C1) when :

(C1) *For every hyperplane* H *in* $\mathbb{R}^p$, $\mu(A') = \mu(A'')$ *if and only if* a* *belongs to* H.

We can now state :

THEOREM 5. [Grandmont (1978)]

Let $(\geq_a)_{a \in A}$ *be an intermediate preference orders family on the choice set* X, *where* A *is open and convex in* $\mathbb{R}^p$. *If the profile* $(\geq_a)_{i \in S}$, *where* $a_i \in A$, *is such that there exists a point* a* *in* A *which satisfies condition* (C1), *then the majority relation associated with* $(\geq_{a_i})_{i \in S}$ *is the order* $\geq_{a^*}$ *and is therefore transitive.*

The following definition will help us to understand condition (C1).

DEFINITION 5.

A bijection T from $S = \{1,\dots n\}$ onto S such that $T[T(i)] = i$, for every $i \in S$, is called a *pairing* on S. A point $c \in \mathbb{R}^p$ is said to be a *weak symmetry center* of the sequence $(a_i)_{i \in S}$, with $a_i \in \mathbb{R}^p$, if there exists a pairing on S such that c belongs to $[a_i, a_{T(i)}]$ for every $i \in S$.

It can then be shown that :

* if a^* satisfies (C1) for the sequence $(a_i)_{i \in S}$, there exists $i \in S$ such that $a_i = a^*$;

* if a point $a^* \in A$ is identical to one point a_i, a^* satisfies (C1) if and only if a^* is a weak symmetry center.

This last property suggests that condition (C1) is very restrictive. Note that a similar condition on the individual locations is obtained when the continuous model I is assumed (see McKelvey and Wendell (1981)). However, when the utility functions family $(-\| x-a \|)_{a \in A}$ does not satisfy the intermediate preferences assumption (H1), the fact that condition (C1) is met for a point implies in general only the existence of a majority winner. This shows that the intermediate preferences allow us to pass from the majority winner existence - which is a local notion when the utility functions are concave - to the transitivity of R - which is a global notion.

By slightly weakening condition (C1), we may obtain a result on the acyclicity of Q :

Given a sequence $(a_i)_{i \in S}$, with $a_i \in A$, a point a^* in A satisfies the condition (C2) if :

(C2) *For every intersection* B *of* A *with an open half-space of* $\mathbb{R}^p$, $\mu(B) > \frac{1}{2}$ *implies that* a^* *belongs to* B.

THEOREM 6.

Let $(\geq_a)_{a \in A}$ *be a family of intermediate preference orders on* X *where* A *is open and convex in* $\mathbb{R}^p$. *If the profile* $(\geq_{a_i})_{i \in S}$, *with* $a_i \in A$, *is such that there exists* a *in* A *which satisfies* (C2), *then the relation* Q *associated with* $(\geq_{a_i})_{i \in S}$ *is acyclical.*

Proof : Suppose that Q is not acyclical. Then there exists a k-typle $(x_1 \dots, x_j \dots, x_k)$, $x_j \in X$, such that $x_1 \, Q \, x_2, \dots, x_{k-1} \, Q \, x_k \, Q \, x_1$. By Lemma 2, the sets $B(x_j, x_{j+1}) = \{a_i \in A \; ; \; x_j >_{a_i} x_{j+1}\}$, for $j \in \{1,\dots, k-1\}$, and $B(x_k, x_1) = \{a_i \in A \; ; \; x_k >_{a_i} x_1\}$ are either equal to A or to the intersection of A with an open half-space of $\mathbb{R}^p$ or to the empty set. This last case cannot occur since $x_j \, Q \, x_{j+1} \Leftrightarrow \mu(B(x_j, x_{j+1}))$ is greater than $\frac{1}{2}$. Therefore, only the first two cases may arise. Consequently, by condition (C2), a^* may be found in

$$\bigcap_{j \in \{1,\dots k-1\}} B(x_j, x_{j+1}) \cap B(x_k, x_1) \; .$$

But we arrive at a contradiction since $>_{a*}$ must verify $x_1 >_{a*} x_2, \ldots x_{k-1} >_{a*} x_k$ and $x_k >_{a*} x_1$ which contradicts the transitivity of this relation. Q.E.D.

The following two properties characterise the relationships between the weak symmetry center and condition (C2) :

* a weak symmetry center of $(a_i)_{i \in S}$ satisfies (C2) ;
* conversely, when a* is different form any point a_i, a* satisfies (C2) only if a* is a weak symmetry center of $(a_i)_{i \in S}$ (which implies that n is necessarily even).

Let us now illustrate those two theorems with some examples which are pertinent both under the intermediate preferences assumption and for the continuous model I. The choice set is $\mathbb{R}^2$ and the preferences are induced by the Euclidear norm $\| \|_2$. In Fig. 3, we consider a model ($\mathbb{R}^2$, $\| \|_2$, $(a_1, \ldots a_5)$) ; point a_5 satisfies (C1), so that R is given by $\geq_{a_5}$ and a_5 is a strong Condorcet winner. Fig. 4 is associated with the model ($\mathbb{R}^2$, $\| \|_2$, $(a_1, \ldots a_5)$) ; the intersection of $[a_1, a_3]$ and $[a_2, a_4]$, say a, verifies (C2). Hence, Q is acyclical and a is a weak Condorcet winner and a weak symmetry center. Finally, in the model ($\mathbb{R}^2$, $\| \|_2$, $(a_1, \ldots a_4)$) corresponding to Fig. 5, (C2) is satisfied by point a_4. This implies that Q is acyclical and that a_4 is a weak Condorcet winner, but here a_4 is not a weak symmetry center.

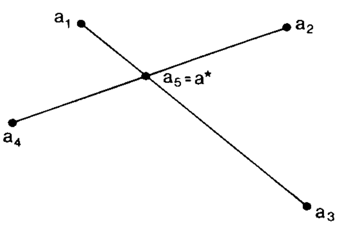

Figure 3

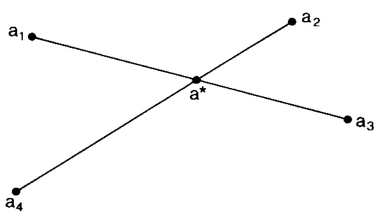

Figure 4

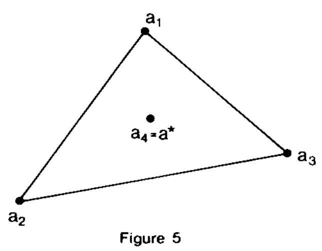

Figure 5

4. EXISTENCE OF A MAJORITY WINNER

To start with, let us notice that the existence of a weak Condorcet winner (resp. a weak quasi-Condorcet winner) is in general implied by the acyclicity of P (resp. the acyclicity of Q) : when X is finite these implications are always satisfied, and they remain true when X is finite under general topological assumptions. For example, we have :

THEOREM 7. [Bergstrom (1975)]

If X is a compact space and $\mathcal{B}$ *an acyclical relation on X and if for every x in X the set* $\{y \in X \; ; \; x \mathcal{B} y\}$ *is open in X, then there exists* $\bar{x}$ *in X such that* $\bar{x}$ *is a maximal element for* $\mathcal{B}$ *in X, i.e. for every y in X, no* $y \mathcal{B} \bar{x}$.

In the continuous models, the sets $\{y \in X \; ; \; x P y\}$ and $\{y \in X \; ; \; x Q y\}$ are open in X since the preferences are continuous. Moreover, even if X is not a compact set, the set of Pareto optima is often compact so that Theorem 7 may be applied.

4.1 The Guarantee Conditions

Until now, no general characterisation of the guarantee conditions has been found. Nevertheless, we can give some interesting examples for each type of model.

(a) Continuous Model II

Again, we use an intermediate preferences assumption, but slightly different from (H1).

DEFINITION 4.

Let A be an open convex subset of $\mathbb{R}^p$ and Ω an open convex subset of $\mathbb{R}^m$. A family $(U_a)_{a \in A}$ of differentiable functions defined on Ω satisfies the *local intermediate preferences assumption* if the following two conditions are met : for every $x \in X \subset \Omega$ and every $v \in \mathbb{R}^m - \{0\}$,

(H2) - *the set* $\{a \in A \; ; \; \nabla U_a(x).v > 0\}$ *is an open convex subset of* A,

- *the set* $\{a \in A \; ; \; \nabla U_a(x).v \geq 0\}$ *is a closed convex subset of* A.

While assumption (H1) implies that the sets $\{a \in A \; ; \; x >_a y\}$ and $\{a \in A \; ; \; x \geq_a y\}$ are convex for every x and y in X (see Lemma 2), assumption (H2) states a similar condition only for x and y "close" to each other. This explains why the hypothesis is said to be "local". Furthermore, when the family $(\geq_a)_{a \in A}$ is represented by a family of pseudo-concave utility functions $(U_a)_{a \in A}$, (H1) implies (H2). Indeed, $\{a \in A \; ; \; \nabla U_a(x).v > 0\} = \bigcup_{\lambda > 0} \{a \in A \; ; \; U_a(x+\lambda v) > U_a(x)\}$ i.e. the union of an increasing family of open convex subsets of A. Similarly, $\{a \in A \; ; \; \nabla U_a(x).v \geq 0\} = \bigcap_{\lambda > 0} \{a \in A \; ; \; U_a(x+\lambda v) \geq U_a(x)\}$ is the intersection of a decreasing family of closed convex subsets of A. On the other hand, (H2) does not imply (H1). For instance, the family $(-\|x-a\|_p^p)$, for $p > 1$ and $p \neq 2$, satisfies (H2) but not (H1) in $\mathbb{R}^2$.

Note, however, that (H2) is associated with a more restrictive context than (H1) : it is indeed supposed that the preference orders are represented by differentiable utility functions. Moreover, these functions are assumed pseudo-concave and the choice space convex to show that (H1) implies (H2).

LEMMA 8(6)

Let Ω be an open subset of $\mathbb{R}^m$ and A an open convex set in $\mathbb{R}^p$. If a family $(U_a)_{a \in A}$ of differentiable functions on Ω satisfies the assumption (H2), then, for every $a_1, a_2 \in A$, $a \in]a_1, a_2[$ and $x \in \Omega$, the gradient $\nabla U_a(x)$ is positively dependent of the two gradients $\nabla U_{a_1}(x)$, $\nabla U_{a_2}(x)$, that is :

$$\exists(\lambda_1,\lambda_2) \neq (0,0),\ \lambda_1 \geq 0,\ \lambda_2 \geq 0, \text{ such that } \nabla U_a(x) = \lambda_1 \nabla U_{a_1}(x) + \lambda_2 \nabla U_{a_2}(x).$$

Proof. Let $a_1 \in A$, $a_2 \in A$, $x \in \Omega$ and the two cones

$$K = \{\lambda_1 \nabla U_{a_1}(x) + \lambda_2 \nabla U_{a_2}(x) \ ;\ \lambda_1 \geq 0,\ \lambda_2 \geq 0\}$$

and $K^* = \{\lambda_1 \nabla U_{a_1}(x) + \lambda_2 \nabla U_{a_2}(x) \ ;\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ (\lambda_1,\lambda_2) \neq (0,0)\}$.

For the proof, we have to show that $\nabla U_a(x)$ belongs to K^* for any $a \in]a_1,a_2[$.
α) Suppose that 0 is not in K^*. We then have to show that $\nabla U_a(x)$ is not equal to zero and that $\nabla U_a(x)$ belongs to K.

First $\nabla U_a(x) \neq 0$. The set K is a closed convex pointed cone, that is $K \cap -K = \{0\}$, since K^* does not contain 0. Hence, there exists a vector $\bar{v}$ in $\mathbb{R}^m - \{0\}$ such that $k.\bar{v} > 0$ for every k in $K - \{0\}$. Therefore, we have $\nabla U_{a_1}(x).\bar{v} > 0$ and $\nabla U_{a_2}(x).\bar{v} > 0$. From the convexity of $\{a \in A \ ;\ \nabla U_a(x).\bar{v} > 0\}$, we deduce that $\nabla U_a(x).\bar{v} > 0$, so that $\nabla U_a(x)$ is not null.

Second $\nabla U_a(x) \in K$. Assume, on the contrary, that $\nabla U_a(x) \notin K$. Then, by the separation theorem, there exists w in $\mathbb{R}^m$ such that $\inf_{k\in K} w.k > \nabla U_a(x).k$. Since K is a cone, we have $\inf_{k\in K} w.k = 0$.
Consequently, $\nabla U_{a_1}(x).w \geq 0$, $\nabla U_{a_2}(x).w \geq 0$ and $\nabla U_a(x).w < 0$; but this contradicts the convexity of $\{a \in A \ ;\ \nabla U_a(x).w \geq 0\}$.

β) Suppose now that 0 belongs to K^*. Then, K^* is equal to K. Assuming that $\nabla U_a(x) \notin K$, we can apply a separation argument to the closed convex cone K^* and to $\nabla U_a(x)$, and the result follows by contradiction. Q.E.D.
This result allows us to construct some non trivial examples of utility functions which satisfy the Kramer's condition. Indeed, if A is an open interval of $\mathbb{R}$ and if $(U_a)_{a \in A}$ is a family of differentiable functions which satisfies (H2), the gradients $\nabla U_a(x)$, for every x in Ω, belong to the same two dimensional half-space when a varies in A.

THEOREM 9.

Let A be an interval of $\mathbb{R}$ and Ω an open convex subset of $\mathbb{R}^m$. A family $(U_a)a \in A$ of pseudo-concave utility functions on Ω which satisfies (H2) guarantees the existence of a weak quasi-Condorcet winner on every compact convex X in Ω.

Proof. Let $(\geq_{a_i})_{i \in S}$ be the profile of the society, where $\geq_{a_i}$ is the order represented by U_{a_i}, $a_i \in A$. Since A is an interval, we can order the points a_i in increasing order : $a_1 \leq a_2 \ldots \leq a_n$.

Let a_k be a median of the sequence $(a_1, \ldots, a_n)$; for example $k = p + 1$ if $n = 2p$ or $n = 2p + 1$.

By Lemma 8, the gradients $\nabla U_{a_i}(x)$, $i \in s$, are in the convex cone generated by $\nabla U_{a_1}(x)$ and $\nabla U_{a_n}(x)$. Moreover, for i, j, 1 in S with $i < j < 1$, $\nabla U_{a_j}(x)$ belongs to the convex cone generated by $\nabla U_{a_i}(x)$ and $\nabla U_{a_1}(x)$.

Three cases may then arise (they are illustrated in Fig. 6). In the first one, $\nabla U_{a_1}(x)$ and $\nabla U_{a_n}(x)$ are linearly independent ; in the second one they have the same direction ; in the third one, they have opposite direction or at least one of them is null.

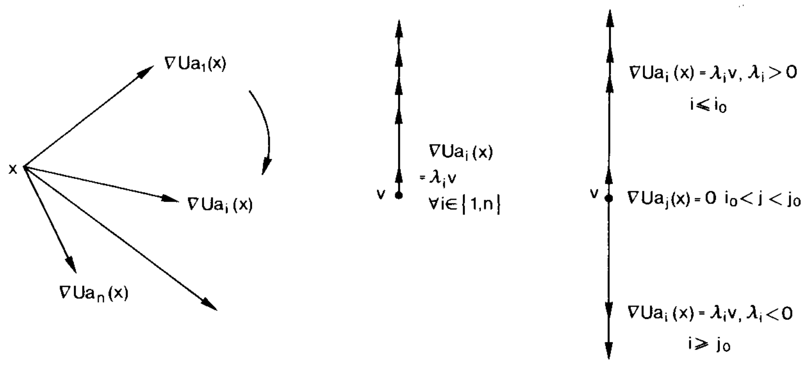

Figure 6

In the three cases, each open half-space of $\mathbb{R}^m$ which contains strictly more than $\frac{n}{2}$ gradients $\nabla U_{a_i}(x)$ must also contain $\nabla U_{a_k}(x)$. As U_{a_k} is continuous on the compact X, it admits a maximizer x* in X. Moreover, by the differentiability of U_{a_k} on Ω , we have :

$$\text{for every } x \in X, \quad U_{a_k}(x).(x-x^*) \leq 0. \qquad (*)$$

Let us now show that x* is a weak quasi-Condorcet winner in X for $(\geq_{a_i})_{i \in S}$. Since every function U_{a_i} is pseudo-concave, the set $\{i \in S \; ; \; U_{a_i}(x) > U_{a_i}(x^*)\}$ is included in $\{i \in S \; ; \nabla U_{a_i}(x^*).(x-x^*) > 0\}$ for every $x \in X$.

By (*), we know that the half-space $\{z \in \mathbb{R}^m \; ; \; z.(x-x^*) > 0\}$ does not contain

$\nabla U_{a_k}(x^*)$. It therefore contains less than $\frac{n}{2}$ gradients $\nabla U_{a_i}(x^*)$. Consequently, for every $x \in X$, $\#\{i \in S ; U_{a_i}(x) > U_{a_i}(x^*)\} \leq \frac{n}{2}$ and x^* is a weak quasi-Condorcet winner. Q.E.D.

When the preference orders are represented by a family $(U_a)_{a \in]\alpha, \beta[}$ of pseudo-concave utility functions, Theorems 3 and 9 can be grouped in the following table :

Intermediate preference assumption	$\Rightarrow$	Local intermediate preferences assumption
$\Downarrow$		$\Downarrow$
The transitivity of P is guaranteed	$\Rightarrow$	The existence of a weak quasi-Condorcet winner on every compact convex X in Ω is guaranteed.

(b) Continuous Model I

Wendell and Thorson (1974) have shown that the l^1-norm on $\mathbb{R}^2$ $(\| x \|_1 = |x_1|+|x_2|)$ guarantees a weak quasi-Condorcet winner. The following result is a sort of converse (the proof is given in Appendix).

THEOREM 10.

A norm $\| \ \|$ *on* $\mathbb{R}^2$ *guarantees a weak quasi-Condorcet winner if and only if its unit ball is a parallelogramm, i.e. there exists a basis* (v_1, v_2) *of* $\mathbb{R}^2$ *such that* $\| x \| = |x_1| + |x_2|$ where $x = x_1 v_1 + x_2 v_2$.

For instance, the l^∞-norm ($\| x \|_\infty = \sup (|x_1|, |x_2|)$) satisfies this condition but not the weighted one-infinity norm ($\| x \| = \alpha_1 \| x \|_1 + \alpha_2 \sqrt{2} \| x \|_\infty$; see Ward and Wendell (1980). Note also that in $\mathbb{R}^m$, for $m \geq 3$, even the I-norm does not guarantee the existence of a weak quasi-Condorcet winner (see Wendell and Thorson (1974)).

(c) Discrete Model II

When the network is a tree, the notion of singlepeakedness is naturally generalised as follows :

DEFINITION 5.

Let N be a tree and V the set of vertices. A family of preference orders on N is said to be *singlepeaked* if for every $x \in V$ and every order $\geq$ having x for top alternative, $y \in N$ is strictly preferred to $z \in N$, $z \neq y$, whenever y belongs to *the* route between x and z, i.e.

if x is a top alternative for $\geq$, $y \neq z$ and $d(x,z) = d(x,y) + d(y,z) \Rightarrow y > z$.
(Note that this definition does not depend on the distance d).

When N is a line, we fall back on Definition 3. For a tree N which is not a line, the set of singlepeaked orders (in the sense of Definition 5) is not pseudo-singlepeaked (in the sense of Definition 2). Moreover, the value restriction conditions may fail to hold. Indeed, if the tree is not a line, there exist four distinct vertices x, y, z, t such that (x,y) and (x,t) are arcs on N (see e.g. Fig. 7). Thus, we can find three singlepeaked orders on N say $>_1, >_2, >_3$, with the following restriction on {x,y,z,t}:

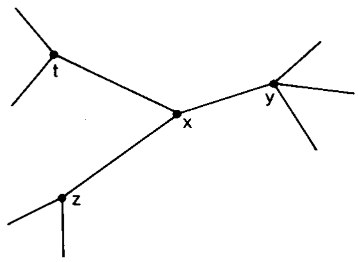

Figure 7

$$x >_1 y >_1 z >_1 t , \quad x >_2 z >_2 t >_2 y, \quad x >_3 t >_3 y >_3 z$$

Clearly, the three points y, z and t form a Condorcet cycle for the profile $(>_1, >_2, >_3)$.

THEOREM 11. [Demange (1980)]

A family of singlepeaked preference orders on a tree guarantees the existence of a weak quasi-Condorcet winner. Furthermore, any set of orders on N which strictly includes the set of singlepeaked orders does not guarantee the existence of such a point.

4.2 The Distribution Conditions

(a) The Continuous Model

The basis result is the Plott's theorem (1967).

THEOREM 12. [Plott (1967), McKelvey and Wendell (1976)]

Let X be an open convex subset of $\mathbb{R}^m$ and $(U_i)_{i \in S}$ a family of utility functions where U_i is differentiable, concave and has a unique maximiser in X . If no individual (resp. exactly one) has his top alternative at x, then x is a weak (resp. strong) Condorcet winner if an only if there exists a pairing T on S such that $\nabla U_i(x)$ and $\nabla U_{T(i)}(x)$ have opposite directions for any $i \in S$.

When applied to the continuous model I, these conditions can be expressed on the individual locations. We then find similar conditions to (C1) and (C2) (see Theorems 5 and 6).

THEOREM 13. [McKelvey and Wendell (1976)]

Let $\| \ \|$ *be a strictly convex norm on* $\mathbb{R}^m$ *admitting a differentiable representation. If* $x \in \mathbb{R}^m$ *is the location of no (resp. only one) individual,* x *is a weak (resp. strong) Condorcet winner for the profile* $(-\| x-a_i \|)_{i \in S}$ *if and only if* x *is a weak symmetry center.*

In the appendix, we show that the hypothesis on the differentiable representation may be dropped in $\mathbb{R}^2$. The proof of Theorem 9 can be adapted to yield the alternative result :

THEOREM 14.

Let A *be an open convex subset of* $\mathbb{R}^p$, Ω *an open convex subset of* $\mathbb{R}^n$ *and* $(U_a)_{a \in A}$ *a family of pseudo-concave utility functions on* Ω *which satisfies* (H2). *If, for a given n-tuple* $(a_i)_{i \in S}$ *in* A *, there exists* $a^* \in A$ *which satisfies* (C2), *then the profile* $(U_{a_i})_{i \in S}$ *admits a weak quasi-Condorcet winner on every compact convex subset* X *of* Ω *.*

Proof. Let x^* be a maximizer of $U_{a^*}(x)$ on X. We then have

$$\nabla U_{a^*}(x^*).\ (y-x^*) \quad 0 \text{ for every } y \in X. \qquad (*)$$

If x^* is not a weak quasi-Condorcet winner in X for $(U_{a_i})_{i \in S}$, there exists $y \in X$ such that :

$$\mu(\{a \in A \ ; \ U_a(x^*) < U_a(x)\}) > \frac{1}{2} ,$$

μ being defined as in 3.2, and, therefore, by the pseudo-concavity of the functions U_a:

$$\mu(\{a \in A \ ; \ \nabla U_a(x^*).(x-x^*) > 0\}) > \frac{1}{2}.$$

Furthermore, $\{a \in A \ ; \ \nabla U_a(x^*).(x-x^*) > 0\})$ is the intersection of A with an open half-space of $\mathbb{R}^m$ (see Demange (forthcoming) or the whole set A. Its μ-measure being strictly greater than $\frac{1}{2}$, condition (C2) implies that $\nabla U_{a^*}(x^*).(x-x^*) > 0$ which contradicts (*). Q.E.D.

Note that Theorem 14 implies Theorem 10 since there exists always a point a^* which satisfies (C2) when the points a_i belong to an interval : a^* is a median of $(a_i)_{i \in S}$.

Interestingly, Theorems 6 and 14 can be compared in the following way. In short, Theorem 6 says that :

Intermediate preference orders (H1) + distribution condition (C2)	⇒	Acyclicity of Q.

while Theorem 14 states :

Local intermediate utility functions (H2) + distribution condition (C2)	⇒	Existence of a weak quasi-Condorcet on every compact convex.

(b) The Discrete Model

Let N be a network on which a metric d is defined. Consider the case when the choice space is N and when the individual $i \in S$ is located at vertex a_i and endowed with the preference order

$$x, y \in N, x \geq_i y \Leftrightarrow d(a_i,x) \leq d(a_i,y).$$

We have :

THEOREM 15. [Wendell and McKelvey (1981)]

If there exists a pairing T *on* S *and a point* x *in* N *such that* x *belongs to a shortest route between* a_i *and* $a_{T(i)}$ *for every* $i \in S$, *then* x *is a weak Condorcet winner.*

The existence of a point which satisfies the condition of Theorem 15 is not necessary for the existence of a Condorcet winner : this is a first difference with continuous model I. Moreover, the existence of T is more likely than in the continuous model. Some problems in which such a mapping exists are considered in Hansen, Thisse and Wendell (1981).

5. CONCLUSIONS

Despite some interesting properties, most of the available results appear to be negative. This suggests that *the majority winner is a too restrictive solution concept.* Two questions then arise : What does occur when a majority winner fails to exist? Which concepts do generalise(7) that of majority winner? We shall here mention only some of the attempts made to deal with these two problems.

The first question is solved by the very netative results of McKelvey (1979), Cohen (1979) and Schofield (1978) : for example, if a model ($\mathbb{R}^m$, $\| \|_2$, $(a_i)_{i \in S}$) does not admit a weak Condorcet winner, then for every pair x, y in $\mathbb{R}^m$ there exists a finite sequence $x_1, \ldots x_k$ in $\mathbb{R}^m$ such that

$$x \; P \; x_1, \; x_1 \; P \; x_2, \ldots \; x_k \; P \; y.$$

In other words, by suitably choosing the agenda of the binary votes, a clever organiser can obtain as a final result his most preferred alternative, this one being Pareto optimal or not.

This result is directly related to our second question. Indeed, one of the most studied generalisations of the majority winner is the "top cycle", that is : the set of the alternatives x which are such that for every other y there exists a finite sequence $x_1, \ldots, x_k$ in X with $x \; P \; x_1, \; x_1 \; P \; x_2, \ldots \; x_k \; P \; y$. But then, the McKelvey-Cohen result implies that, in most models, the top cycle is the entire choice space when a majority winner does not exist. Needless to say, the top cycle does not therefore seem to be useful for our purpose.

Among the many other generalisations (see for example Young (1977)), let us mention the Copeland winners and the minmax set. The former are defined for finite choice set X in the following way : to every alternative x in X we assign its "score" s(x) as the number of alternatives defeated by x : $s(x) = \#\{y \in X, \; x \; Q y\}$. A Copeland winner is then defined as an alternative which maximises s on X. This concept has not been extended to the case of an infinite choice set. On the contrary the minmax set, introduced by Kramer (1977), applies to infinite spaces : for every ordered pair (x,y) in X^2, n(x,y) is defined as the number of individuals who strictly prefer y to x and $\Theta(x)$ as the maximum of n(x,y) when y runs over X ($\Theta(x)$) is always well defined since $n(x,y) \leq n, \; \forall \; x,y \in X$). The minmax set is then the set of alternatives minimising Θ on X. It generalises the Condorcet winners ; it is never empty ; it is included in the set of Pareto alternatives if this one is nonempty and it enjoys good axiomatic features (see Blair (1979)). Furthermore, it appears as a natural concept in a political dynamical model (see Kramer (1977)) and it tends to a unique point when the size of the society increases, the preferences being spread smoothly enough under an assumption of local intermediate preferences (see Demange (forthcoming)).

Another interesting concept, that of local Condorcet winner, has been introduced by Wendell and McKelvey (1981), in the case of discrete model I : a point $x \in N$ is a local Condorcet winner if there exists a neighbourhood of x whose points do not beat x. As shown by the above-mentioned authors, there always exists a local Condorcet winner whatever the network is. Moreover, Hansen, Thisse and Wendell (1981) have established that the set of local Condorcet winners is identical to the set of local minimisers of the function

$$\sum_{i \in S} d(a_i, x).$$

To sum-up : The spatial structure has proved to be very useful for social choice theory. Nevertheless, the exploitation of this structure is far from being achieved; in particular, the analysis of the extensions of the Condorcet winner concept is still to be performed.

APPENDIX

Our purpose is to prove the following two results :

(1) Theorem 13 is *valid for each strictly convex norm on* $\mathbb{R}^2$.

(2) *A norm in* $\mathbb{R}^2$ *guarantees the existence of a weak quasi-Condorcet winner if and only if its unit ball is a parallelogram.*

For that, we need Lemmas A.1 ; A.2 and A.3 which characterises the weak quasi-Condorcet winners in the model $(\mathbb{R}^2, \| \|, (a_i)_{i \in S})$. Before stating these lemmas, we introduce some notation.

Given $x \in \mathbb{R}^2$ and $\rho > 0$, $B(x,\rho)$ denotes the ball with center x and radius ρ :

$$B(x, \rho) = \{a \in \mathbb{R}^2 ; \| a-x \| \leq \rho \},$$

and $\partial B(x,\rho)$ its boundary :

$$\partial B(x,\rho) = \{a \in \mathbb{R}^2 ; \| a-x \| = \rho \}.$$

When x is the origin, we use $B(\rho)$ and $\partial B(\rho)$ instead of $B(0,\rho)$ and $\partial B(0, \rho)$ respectively.
If A and B are two subsets of $\mathbb{R}^2$, A-B is the set $\{a-b ; a \in A, b \in B\}$.

LEMMA A.1

A point x in $\mathbb{R}^2$ is a weak quasi-Condorcet winner in the model $(\mathbb{R}^2, \| \|, (a_i)_{i \in S})$ if and only if :

(*) for every $v \in \mathbb{R}^2 - \{0\}$, $\mu(\{a \in \mathbb{R}^2 ; v \in \mathring{T}(x,a) \}) \leq \frac{1}{2}$,

where $T(x,a)$ denotes the tangent cone at a to $B(x, \| a-x \|)$.

Proof. A point x in $\mathbb{R}^2$ is a weak quasi-Condorcet winner for the model $(\mathbb{R}^2, \| \|, (a_i)_{i \in S})$ if and only if for every $v \in \mathbb{R}^2 - \{0\}$,

$$\mu(\{a \in \mathbb{R}^2 ; \|a+v-x\| < \|a-x\|\}) \leq \frac{1}{2},$$

or

$$\mu(\{a \in \mathbb{R}^2 ; v \in \mathring{B}(x, \|a-x\|) - \{a\}\})) \leq \frac{1}{2}.$$

By a classical result of convex analysis, the tangent cone $T(x,a)$ to $B(x, \|a-x\|)$ is the closed cone generated by $\mathring{B}(x, \|a-x\|) - \{a\}$ and $\mathring{T}(x,a)$ is the cone generated by $\mathring{B}(x, \|a-x\|) - \{a\}$. Thus condition (*) is sufficient for x to be a weak quasi-Condorcet winner.

Conversely, suppose that $\bar{v} \in \mathbb{R}^2 - \{0\}$ exists such that

$$\mu(\{ a \in \mathbb{R}^2 ; \bar{v} \in T(x,a) \}) > \frac{1}{2}.$$

Since $T(x,a)$ is the union $\bigcup_{\lambda>0} \lambda(B(x, \|a-x\|) - \{a\})$, $\bar{v} \in T(x,a)$ implies that

there exists $\lambda_a > 0$ such that $\bar{v} \in \lambda(\mathring{B}(x, \|a-x\|) - \{a\})$ for every $\lambda \geq \lambda_a$.

The support of μ being finite, there exists $\lambda_o > 0$ for which

$$\mu(\{a \in \mathbb{R}^2 ; \bar{v} \in \mathring{T}(x,a)\}) = \mu(\{a \in \mathbb{R}^2 ; \bar{v} \in \lambda_o(B(x, \|a-x\|) - \{a\})\})$$
$$= \mu(\{a \in \mathbb{R}^2 ; \|a-x - \frac{\bar{v}}{\lambda_o}\| < \|a-x\| \})$$

and x is not a weak quasi-Condorcet winner. Q.E.D.

LEMMA A.2.

The set $\{a \in \mathbb{R}^2 ; v \in \mathring{T}(x,a)\}$ is the set $\{a \in \mathbb{R}^2 ; v \in \mathring{T}(o,a)\}$ translated by x. If $H(v)$ denotes the set $\{a \in \mathbb{R}^2 ; v \in \mathring{T}(o,a)\}$, $H(v)$ is an open cone with 0 for vertex, $H(v) \cap H(-v) = \emptyset$ and $H(-v) = -H(v)$.

Proof : We want to show that

$$\{a \in \mathbb{R}^2 ; v \in \mathring{T}(x,a)\} = \{a \in \mathbb{R}^2 ; v \in \mathring{T}(o,a)\} + \{x\} = H(v) + \{x\}.$$

Now, $B(x, \|a-x\|) - \{a\} = B(0, \|a-x\|) - \{a-x\}$

so that
$$\mathring{T}(x,a) = \bigcup_{\lambda>0} \lambda(B(x, \|a-x\|) - \{a\}) = \bigcup_{\lambda>0} \lambda(\mathring{B}(0, \|a-x\|) - \{a-x\})$$
$$= \mathring{T}(0,a-x).$$

Thus point a is such that $v \in T(x,a)$ if and only if $(a-x)$ is such that $v \quad T(o, a-x)$, that is :

$$\{a \in \mathbb{R}^2 ; v \in \mathring{T}(x,a)\} = \{a = b+x ; v \in \mathring{T}(o,b)\}$$
$$= H(v) + \{x\}. \quad \text{Q.E.D.}$$

Note that $H(v)$ is a cone and that $H(-v) = -H(v)$ since the cones tangent at a to B $(\|a\|)$ and at λa to $B(\|\lambda a\|)$ are identical if $\lambda > 0$ and symmetric around 0 if $\lambda < o$. Moreover, as 0 does not belong to $H(v)$, the intersection $H(v) \cap H(-v)$ is empty. Finally, $H(v)$ is open since we know that a belongs to $H(v)$ if and only if there is $\lambda > 0$ such that $\|a+\lambda v\| < \|a\|$. (Recall that every norm in $\mathbb{R}^2$ is continuous).

LEMMA A.3

The set $H(v)$ is a convex cone. If $\partial B(1)$ contains no segments parallel to v, then $H(v)$ is an open half-space.

If not, there are two symmetric segments $[a_o, b_o]$ and $[-a_o, -b_o]$, with $a_o \neq b_o$, parallel to v such that $H(v)$ and $H(-v)$ are the two convex cones $\{\lambda_1 b_o - \lambda_2 a_o ; \lambda_1 > 0 \text{ and } \lambda_2 > 0\}$ and $\{-\lambda_1 b_o + \lambda_2 a_o ; \lambda_1 > 0 \text{ and } \lambda_2 > 0\}$.

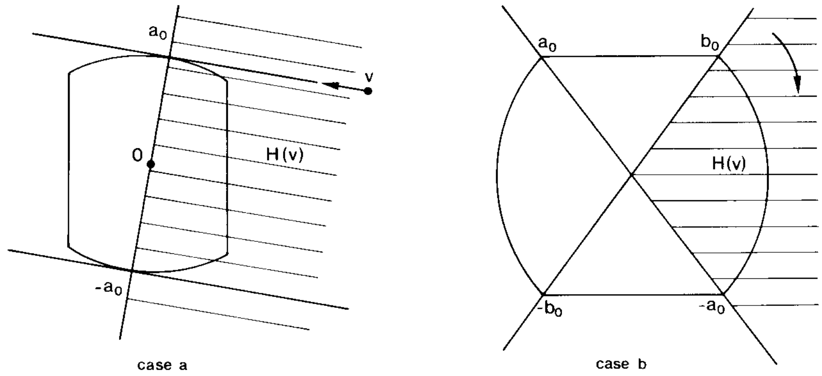

Figure A1

Proof : As $H(v)$ is a cone with vertex 0, for the proof, it is sufficient to study its intersection, denoted $K(v)$, with $\partial B(1)$.
From above, we have :

$$\begin{cases} a \in K(v) \cup K(-v) \\ a \in \partial B(1) \end{cases} \Leftrightarrow \quad \lambda \in \mathbb{R}, \ \| a+ v \| \geq \| a \| ,$$

which means that the line through a with direction v is a supporting line to $B(1)$ at a. This implies that $\partial B(1)$ is the disjoint union of $K(v)$, $K(-v)$ and of the set of points which admit a line parallel to v as a supporting line. This set is formed by

a) two symmetric points a_o and $-a_o$ if $\partial B(1)$ does not contain any segment parallel to v (see Fig. A.1.a) or by

b) the segments $[a_o, b_o]$ and $[-a_o, -b_o]$ parallel to v if these ones exist (see Fig. A.1.b).

In the plane, $\partial B(1)$ is in both cases the union of the following disjoint sets :

$[a_o, b_o]$, the arc $(b_o, -a_o)$, $[-a_o, -b_o]$ and the arc $(-b_o, a_o)$ (take $a_o = b_o$ for the first case). Thus, $K(v) \cap (b_o, -a_o)$ and $K(-v) \cap (b_o, -a_o)$ is a partition in two open sets of the connected set $(b_o, -a_o)$. Therefore, one of these sets, say $K(-v) \cap (b_o, -a_o)$, is empty and the other, say $K(v) \cap (b_o, -a_o)$ and

$K(v) = (-b_o, a_o)$.

Consequently, $H(v)$ and $H(-v)$ are defined by the two open half-spaces delimited by the line joining a_o to $-a_o$ in case a and by the two convex cones $\{\lambda_1 b_o - \lambda_2 a_o ; \lambda_1 > 0, \lambda_2 > 0\}$ and $\{-\lambda_1 b_o + \lambda_2 a_o ; \lambda_1 > 0, \lambda_2 > 0\}$ in case b. Q.E.D.

Proof of (1)

If the norm is strictly convex, B(1) contains no linear segment. It then follows from Lemma A.3 that $(H(v))_{v \in \mathbb{R}^2 - \{o\}}$ is the family of all the open half-spaces whose boundary contains 0. Hence, by Lemmas A.1 and A.2, we know that a point x is a weak quasi-Condorcet if and only if

$$\mu(H(v) + \{x\}) \leq \frac{1}{2}, \text{ for every } v \in \mathbb{R}^2 - \{0\},$$

that is

$$\mu(\{a \in \mathbb{R}^2 ; (a-x). v > 0\}) \leq \frac{1}{2} \text{ for every } v \in \mathbb{R}^2 - \{0\}.$$

As this last characterisation is independent of the norm chosen, the statement follows. Q.E.D.

Proof of (2)

We first note that, if the family $(H(v))_{v\in \mathbb{R}^2-\{0\}}$ contains three half-spaces $H(v_1)$, $H(v_2)$, $H(v_3)$ whose boundaries are parallel to distinct u_1, u_2 and u_3, the norm does not guarantee a weak quasi-Condorcet winner.

To see it, consider three lines D_1, D_2, D_3 with an empty intersection and five individuals located as in Fig. A.2.

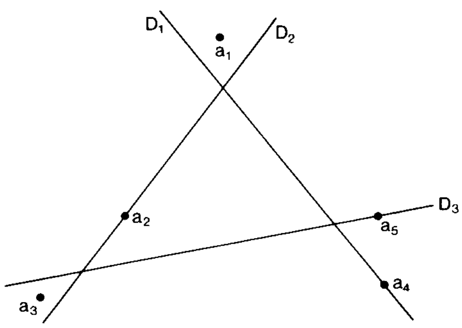

Figure A 2

Every point x which satisfies $\mu(H(v_i) + \{x\}) \leq \frac{1}{2}$ and $\mu(H(-v_i) + \{x\}) \leq \frac{1}{2}$ must be on D_i. A weak quasi-Condorcet winner should therefore belong to the intersection $D_1 \cap D_2 \cap D_3$, which is impossible. We now prove that if the family $(H(v))_{v \in \mathbb{R}^2 - \{0\}}$ contains only two half-spaces having distinct boundaries $\partial B(1)$ is a parallelogramm.

First, $\partial B(1)$ is necessarily a polygon since otherwise the family $(H(v)_{v \in \mathbb{R}^2-\{0\}}$ would contain an infinity of half-space by Lemma A.3.

Second, if a is a vertex of this polygon, any half-space orthogonal to a is a set of the form H(v). Indeed, it suffices to choose v outside of the closed convex cone of vertex a generated by the polygon (see Fig. A.3).

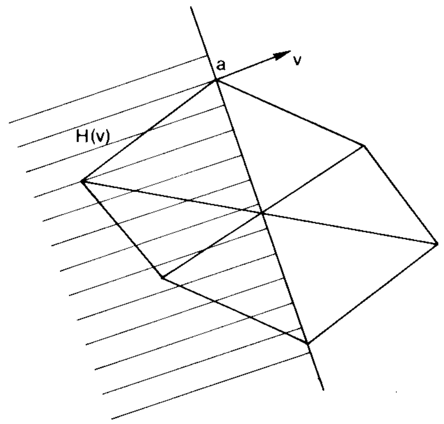

Figure A3

Consequently, $H(v))_{v \in \mathbb{R}^2-\{0\}}$ contains no more than two-half spaces with distinct boundaries if and only if $\partial B(1)$ has four vertices, i.e. B(1) is a parallelogram.

Furthermore, in that case, the norm guarantees a weak quasi-Condorcet winner since every cone H(v), $v \in \mathbb{R}^2 - \{0\}$, is included in a set H_i or $- H_i$, $i \in \{(1,2\}$ where $H_i = \{a \in \mathbb{R}^2 ; a.b_i > 0\}$, where a_1, a_2, $-a_1$, $-a_2$ are the vertices of B(1) and where b_i is orthogonal to a_i.

Thus, from Lemma A.1, the set of weak quasi-Condorcet winners is given by $\{x \in \mathbb{R}^2 ; \mu(H_i + \{x\}) \leq \frac{1}{2}, \quad \mu(-H_i + \{x\}) \leq \frac{1}{2}, i \in \{1,2\}$ which is a parallelogram whose sides are parallel to a_1 and a_2.

In the basis (a_1, a_2) of $\mathbb{R}^2$, the norm which has the parallelogram of vertices a_1, a_2, $-a_1$, $-a_2$ as unit ball is $\| x \|_1 = | x_1 | + |x_2|$. Q.E.D.

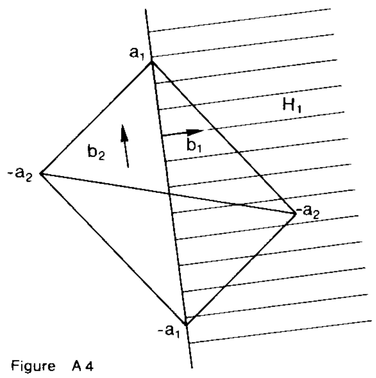

Figure A4

FOOTNOTES

[1] A function U on a convex set X is *quasi-concave* if, for every $\alpha \in \mathbb{R}$ the set $\{x \in X \; ; \; U(x) \geq \alpha\}$ is convex.

[2] A function U on an open convex set Ω is *pseudo-concave* if U is differentiable and if, for every x and y in Ω, $U(x) > U(y)$ implies $\nabla U(y) . (x-y) > 0$ where $\nabla U(y)$ is the gradient of U at y.

[3] A function U on a convex set X is *strictly quasi-concave* if for every x, y in X $U(x) > U(y)$ implies $U(z) > U(y)$ for each z in $]x,y[$. Note that : pseudo-concave $\Rightarrow$ strictly quasi-concave $\Rightarrow$ quasi-concave.

[4] This is because the set of points can often be reduced to the set of vertices in many optimisation problems (see, e.g. Wendell and Hurter (1973). This reduction is also possible in model I when the number of individuals is odd (see Hansen and Thisse (forthcoming)).

[5] Romero has in fact proved a slightly weaker result in which the strict orders are defined on V and not on N .

[6] The lemma is proved in Demange (forthcoming) under more restrictive assumptions.

[7] A solution concept generalises that of majority winner if this solution coincides with the majority winners when these ones exist.

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