Macroeconomic Dynamics, **7**, 2003, 63–88. Printed in the United States of America.
DOI: 10.1017.S1365100502010301

DYNAMIC EQUIVALENCE PRINCIPLE IN LINEAR RATIONAL EXPECTATIONS MODELS

Stéphane Gauthier
ERMES, Université Paris 2
and
CREST-Laboratoire de Macroéconomie

Linear models with infinite horizon generally admit infinitely many rational expectations solutions. Consequently, some additional selection devices are needed to narrow the set of relevant solutions. The viewpoint of this paper is that a solution will be more likely to arise if it is locally determinate (i.e., locally isolated), locally immune to sunspots, and locally stable under learning. These three criteria are applied to solutions of linear univariate models along which the level of the state variable evolves through time. In such models the equilibrium behavior of the level of the state variable is described by a linear recursive equation characterized by the set of its coefficients. The main innovation of this paper is to define new perfect-foresight dynamics whose fixed points are these sets of coefficients, thus allowing us to study the property of determinacy of these sets, or, equivalently, of the associated solutions. It is shown that only one solution is locally determinate in the new dynamics. It is also locally immune to sunspots and locally stable under myopic learning. This solution corresponds to the saddle path in the saddle-point case.

Keywords: Rational Expectations, Determinacy, Sunspots, Learning

1. INTRODUCTION

Linear models with infinite horizon generally admit infinitely many perfect-foresight solutions. In particular, this implies that there are infinitely many ways the economy may react to unanticipated shocks. Some claim that this makes such models useless for economic predictions or policy evaluations. Some even claim that nonuniqueness undermines the very concept of perfect foresight itself since there is then no clear reason why economic agents should manage to choose one particular solution [Kehoe and Levine (1985)]. This paper describes three alternative conditions that should be met for public opinion to focus so sharply. Following Guesnerie (1993), we argue that a solution should be locally determinate, locally

This work has been undertaken at DELTA (Joint Research Unit CNRS-EHESS-ENS) and GREQAM (Université Aix-Marseille 2). I wish to thank Russell Davidson, Gabriel Desgranges, Jean-Michel Grandmont, Roger Guesnerie, and two anonymous referees of this journal for many helpful comments. Of course the usual disclaimers apply.
Address correspondence to: Stéphane Gauthier, CREST-LMA, Bâtiment de Malakoff 2—timbre J360, 15 bd Gabriel Péri, 92245 Malakoff CEDEX, France; e-mail: gauthier@ensae.fr.

immune to sunspots, and locally stable under learning in order to be a possible outcome of a decentralized process through which agents try to coordinate their behavior on perfect foresight. The first two criteria are based on local multiplicity properties, as they respectively rule out any solution arbitrarily close to which there exists some other perfect-foresight solution, or some stationary stochastic sunspot solution along which the system fluctuates in response to random events unrelated to economic fundamentals. In contrast, the last criterion recommends eliminating any solution that economic agents would fail to discover through a simple adaptive learning process. These three criteria have clearly very different status. It will be shown, however, that in the field of models covered in this paper they select the same solution, the one commonly referred to as the fundamental solution in the literature [McCallum (1999)].

In nonstochastic models these three criteria have been applied primarily to steady states, that is, particular trajectories along which the level of the state variable remains constant over time or periodically cycles [see, e.g., Azariadis (1981), Azariadis and Guesnerie (1982), Woodford (1984), Grandmont and Laroque (1986), Marcet and Sargent (1989), Grandmont and Laroque (1991), Guesnerie and Woodford (1991), Chiappori et al. (1992), Farmer and Woodford (1997), or Grandmont (1998)].[1] As emphasized by Guesnerie (1993), if economic agents forecast one period ahead and if there are no predetermined variables in the model, then a steady state is locally determinate if and only if it is locally immune to sunspots and locally stable for some reasonable learning rules. This suggests that these three criteria may consequently be equivalent, at least when attention is focused on a certain class of solutions to rational expectations models. This paper shows that this equivalence, suitably reinterpreted, may be viewed as a part of a *dynamic equivalence principle*, a property that provides, when it holds, a rather compelling selection device on the set of rational expectations equilibria of general linear models where there are arbitrarily many leads in expectations or lags in memory.

The relevant class of solutions to which this principle applies links the current state of the system with the same number of past states as the number of initial conditions to the economic system (which is, in turn, equal to the number of predetermined variables). An appealing characteristic of these *minimal state variable solutions*[2] is that they can be fully defined by the set of the coefficients of the linear difference equation that governs the intertemporal behavior of the system in equilibrium. The main innovation of this paper is to apply the three criteria described above to these sets of coefficients in order to choose among minimal state variable solutions. Namely, we study whether these sets of coefficients can be locally determinate, in new dynamics with perfect foresight induced by the usual dynamics on the level of the state variable, locally free of finite Markovian stationary sunspot equilibria, and locally stable under a specified myopic learning process that fits the iterative version of the expectational stability device, extensively used by, for example DeCanio (1979) or Evans (1985). Surprisingly, it turns out that these three devices select the same minimal state variable solution, the so-called fundamental one. In particular, this solution coincides with the saddle-path trajectory in the saddle-point configuration, in accordance with the recommendations of the main selection devices used in the literature, such as the stability-based device by Blanchard and Kahn (1981), the minimal state variable criterion by McCallum (1983), or the minimal variance criterion by Taylor (1977). Unlike these devices, however, this principle does not depend on particular features of the model, for example, some stability properties, and there are quite natural reasons why the economy should apply it.

The paper is organized in the following way: Section 2 examines the relevance of the dynamic equivalence principle to the simple class of models with only one lead in expectations and one lag in memory. Section 3 reviews some of the results obtained by Desgranges and Gauthier (in press) and Gauthier (2002) in the more general class of one-step-forward-looking models where the number of lags in memory is arbitrary. Section 4, which contains the new results of the paper, is a preliminary exploration of the case in which agents forecast beyond the next period, but where there is only one lag memory. Finally, Section 5 concludes.

2. A SIMPLE FRAMEWORK

Let us first focus attention on the class of linear models where the current state of the economic system is a real number (e.g., a price) that depends on both the previous state and the forecast of the next state (we assume that this forecast is the same for all agents). Such a class encompasses some overlapping generations models with production, and some infinite-horizon models with cash-in-advance constraints. Unfortunately, even in so simple a framework, there are multiple minimal state variable (MSV) solutions, and our purpose is to study whether some of them can be locally determinate, immune to sunspots, and stable under some adaptive learning rules. In general, in much of the literature, these devices have been primarily applied to steady states. Here, however, MSV solutions involve deterministic changes in the level of the state variable over time. Each of these solutions actually expresses the current state of the economic system as a linear function of the previous state. Therefore, each of them can be fully characterized by a single coefficient, giving the constant ratio of two consecutive realizations of the level of the state variable, that is, the growth rate of the level of the state variable. In this section, we apply the dynamic equivalence principle to each of these growth rates, or, equivalently, to the corresponding MSV solutions.[3]

Let the current state of the system be represented by a real number x_t at time $t \geq 0$ that depends on the past state x_{t-1} and on the common forecast $E(x_{t+1} \mid I_t)$ conditional to the information set I_t held by economic agents at time t, through the temporary equilibrium relation

$$\gamma_1 E(x_{t+1} \mid I_t) + x_t + \delta_1 x_{t-1} = 0, \tag{1}$$

where γ_1 and δ_1 are real parameters. Perfect-foresight solutions to (1) are sequences (x_t) associated with the initial condition x_{-1} and satisfying the difference equation

$$\gamma_1 x_{t+1} + x_t + \delta_1 x_{t-1} = 0, \tag{2}$$

which results from replacing the forecast $E(x_{t+1} \mid I_t)$ with the actual realization x_{t+1} in (1). We restrict our attention to the particular class of MSV solutions to (2), which relate, by definition, the current state of the system to as many lagged variables as the number of initial conditions to (2). Therefore, along any of these solutions, the current state x_t will only depend on the previous state x_{t-1} through a law of motion of the form

$$x_t = \beta x_{t-1}. \tag{3}$$

Following Grandmont and Laroque (1991), the growth rate β can be obtained by interpreting (3) as the outcome of a process where agents a priori believe that the economic system will evolve according to (3), whatever x_{t-1} and t are, and where this belief is self-fulfilling. According to this interpretation, traders form their forecast by leading (3) forward and set $E(x_{t+1} \mid I_t)$ equal to βx_t in (1). Thus, the actual law of motion corresponding to this forecast, directly obtained from (1), yields

$$x_t = -(\delta_1/(1 + \gamma_1 \beta))x_{t-1}, \tag{4}$$

provided that $\beta \neq -1/\gamma_1$. It is clear that the actual law (4) coincides with the a priori guess (3), for any x_{t-1} and any t, if and only if the actual growth rate in (4) is equal to the a priori guess on the growth rate in (3), or, equivalently,

$$-(\delta_1/(1 + \gamma_1 \beta)) = \beta \Leftrightarrow \gamma_1 \beta^2 + \beta + \delta_1 = 0. \tag{5}$$

Let λ_1 and λ_2 be the roots of (5). Assume that they are *real* and *distinct*; that is, $1 - 4\gamma_1\delta_1 > 0$. Also, let λ_1 be the root of least modulus; that is, $|\lambda_1| < |\lambda_2|$. The process described in this section highlights the point that, given the previous state of the system, the occurrence of an MSV solution depends on whether traders will be able to choose the root $\lambda_i \, (i = 1, 2)$ corresponding to it. As in Guesnerie (1993), we argue that economic agents may not succeed in coordinating their behavior on a root that fails to be locally determinate, immune to sunspots, or stable under some learning rule.

2.1. Determinacy of Growth Rates

According to the determinacy criterion, coordination on a root of (5) will be necessarily disturbed if there is at least another perfect-foresight solution (x_t) to (2) that displays a sequence of growth rates $(\beta_t \equiv x_t/x_{t-1})$ arbitrarily close to this root. To assert whether some of these roots may be locally determinate, it must be noticed that the perfect-foresight dynamics of the level of the state variable, described by (2), induces new dynamics on the growth rate whose fixed points are the two roots λ_1 and λ_2, which solve (5). Those new dynamics are obtained by assuming that the level of the state variable evolves according to (2) and is bound to satisfy in addition the relation

$$x_t = \beta_t x_{t-1} \tag{6}$$

 https://doi.org/10.1017/S1365100502010301

whatever x_{t-1} and t are. It follows from (6) that (2) can be rewritten as

$$\gamma_1 \beta_{t+1} \beta_t x_{t-1} + \beta_t x_{t-1} + \delta_1 x_{t-1} = 0,$$

since $x_{t+1} = \beta_{t+1} x_t$ in virtue of (6). Thus, if $x_{t-1} \neq 0$, that is, if the economic system is not at the steady-state level of the state variable, one can define the growth-rate perfect-foresight dynamics induced by (2) as a nonlinear sequence of growth rates (β_t) such that

$$\gamma_1 \beta_{t+1} \beta_t + \beta_t + \delta_1 = 0 \quad \mathbf{(7)}$$

in each period $t \geq 0$. These dynamics are well defined if $\beta_t \neq 0$ and $\beta_t \neq -1/\gamma_1$ whatever t is. Its fixed points are the roots λ_1 and λ_2. Hence (7) is locally well defined if and only if $\lambda_i \neq 0$ $(i = 1, 2)$, which holds true if both $\gamma_1 \neq 0$ and $\delta_1 \neq 0$ (since $\lambda_1 + \lambda_2 = -1/\gamma_1$ and $\lambda_1 \lambda_2 = \delta_1/\gamma_1$). Observe now that the current growth rate β_t defined in (6), that is, the ratio x_t/x_{t-1}, is not predetermined in (7) at time t because x_t is not predetermined in (2) at this date. Therefore, (7) has a classical one-step-forward-looking structure without predetermined variables, and the root λ_i is locally determinate in (7) if and only if it is locally unstable in the forward perfect-foresight dynamics (7). The following result is due to Gauthier (2002).

PROPOSITION 1. *Let* $\gamma_1 \neq 0$ *and* $\delta_1 \neq 0$ *in* (1). *Then, the root of lowest modulus* (λ_1) *is locally determinate in the growth rate perfect foresight dynamics* (7), *while the root of highest modulus* (λ_2) *is locally indeterminate in this dynamics.*

Proof. The dynamics (7) arbitrarily close to λ_i $(i = 1, 2)$ are governed by the first-order approximation of (7) at point $\beta_t = \beta_{t+1} = \lambda_i$ $(i = 1, 2)$,

$$\gamma_1 \lambda_i (\beta_{t+1} - \lambda_i) + (\gamma_1 \lambda_i + 1)(\beta_t - \lambda_i) = 0. \quad \mathbf{(8)}$$

The root λ_i is locally unstable in (8) if and only if $|(\gamma_1 \lambda_i + 1)/(\gamma_1 \lambda_i)| > 1$. Since $(\lambda_1 + \lambda_2) = -1/\gamma_1$, this inequality is rewritten as

$$|(\gamma_1 \lambda_i + 1)/(\gamma_1 \lambda_i)| = |(\lambda_i + 1/\gamma_1)/\lambda_i| = |\lambda_j/\lambda_i| > 1,$$

for $j \neq i$, $j = 1, 2$. Thus, the root λ_1 is locally unstable $(|\lambda_2/\lambda_1| > 1)$ and the root λ_2 is locally stable $(|\lambda_1/\lambda_2| < 1)$. ■

Proposition 1 recommends picking out the MSV solution corresponding to the root of least modulus, independently of the stability properties of the dynamics (2), that is, independently of $|\lambda_1|$ and $|\lambda_2|$. However, if (2) is to be interpreted as dynamics restricted to an arbitrarily small neighborhood of the steady state $(x_t = 0)$, then the stability condition $|\lambda_1| < 1$ should hold for the corresponding MSV solution to be locally feasible. This condition covers both the saddle-point configuration $(|\lambda_1| < 1 < |\lambda_2|)$, where the locally determinate MSV solution coincides with the stable saddle path $(x_t = \lambda_1 x_{t-1})$, and the sink configuration $(|\lambda_2| < 1)$ where there are infinitely many stable solutions to (2).

2.2. Sunspots on Growth Rates

Suppose now that traders form their expectations about the rate of change of the level of the state variable conditionally to a k-state Markovian sunspot process associated with a Markov matrix Π. Note that traders are not directly concerned here with the level of the state variable itself, as is usually the case in the literature. Instead, if the current sunspot event s_t is s $(s = 1, \ldots, k)$, they a priori believe that $\beta_t = \beta_s$ and expect $\beta_{t+1} = \beta_{s'}$ to occur with probability $\pi_{ss'}$ $(s' = 1, \ldots, k)$. Their forecast, conditional to this sunspot event, then is written

$$E(x_{t+1} \mid \{x_t, s_t\}) = \left(\sum_{s'=1}^{k} \pi_{ss'} \beta_{s'} \right) x_t \equiv \bar{\beta}_s x_t, \tag{9}$$

where $\bar{\beta}_s$ is a one-period average growth rate. The actual dynamics, obtained by reintroducing the forecast (9) into the temporary equilibrium map (1), that is,

$$x_t = -[\delta_1/(1 + \gamma_1 \bar{\beta}_s)]\, x_{t-1}, \tag{10}$$

are consistent with the a priori belief of traders whenever the actual growth rate in (10) is equal to the a priori guess about the growth rate in sunspot event s, or, equivalently,

$$-[\delta_1/(1 + \gamma_1 \bar{\beta}_s)] = \beta_s \Leftrightarrow (1 + \gamma_1 \bar{\beta}_s)\beta_s + \delta_1 = 0. \tag{11}$$

Following Desgranges and Gauthier (in press), we define a stationary sunspot equilibrium on the growth rate as a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$ associated with Π such that (a) there are s and $s' \neq s$ such that $\beta_s \neq \beta_{s'}$ $(s, s' = 1, \ldots, k)$, and (b) the pair $(\boldsymbol{\beta}, \Pi)$ satisfies (11) for any s $(s = 1, \ldots, k)$. Condition (a) ensures that the growth rate truly changes according to sunspot events, and condition (b) makes traders' beliefs consistent with rational expectations. Observe that the vector $\boldsymbol{\beta} = (\lambda_i, \ldots, \lambda_i)$ satisfies condition (b) whatever Π is, but violates (a). Thus, there can exist stationary sunspot equilibria on the growth rate $(\boldsymbol{\beta}, \Pi)$ such that each component β_s of $\boldsymbol{\beta}$ stands arbitrarily close to λ_i only if condition (b) does not implicitly define $\boldsymbol{\beta}$ as a smooth function of Π at point $(\lambda_i, \ldots, \lambda_i)$, that is, only if the implicit-function theorem does not apply at this point.

PROPOSITION 2. *Let a root λ_i $(i = 1, 2)$ be immune to sunspots if there do not exist stationary sunspot equilibria on the growth rate $(\boldsymbol{\beta}, \Pi)$ such that each component β_s $(s = 1, \ldots, k)$ of $\boldsymbol{\beta}$ stands arbitrarily close to λ_i. Then, if $\gamma_1 \neq 0$ and $\delta_1 \neq 0$ in* (1), *the root of lowest modulus* (λ_1) *is the only one to be immune to sunspots.*

Proof. This follows from standard results in the class of one-step-forward-looking models without predetermined variables. See, e.g., Chiappori et al. (1992) or Desgranges and Gauthier (in press) for precise statements. ■

2.3. Learning Growth Rates

Let us now focus attention on the case in which traders try to discover some MSV solution through an adaptive learning process. Their a priori belief is assumed to be still consistent with (3) but the parameter β in this equation is no longer necessarily equal to some root λ_1 or λ_2 of (5). It is instead estimated by β_t at time t. Conditional to this belief, the forecast $E(x_{t+1} \mid I_t)$ is equal to $\beta_t x_t$ in (1) , and, as (4) shows, the actual law of motion of the state variable corresponding to this forecast is

$$x_t = -[\delta_1/(1+\gamma_1\beta_t)]x_{t-1} \quad \textbf{(12)}$$

as long as $\beta_t \neq -1/\gamma_1$. It is clear that the actual growth rate $-(\delta_1/(1+\gamma_1\beta_t))$ in (12) will differ from the estimated growth rate β_t as long as β_t is not equal to either λ_1 or λ_2. This spread between the actual and the estimated growth rates should urge agents to revise their estimate in the next period. Here, we use one of the simplest learning rules, the myopic learning rule, which recommends setting the new estimate of the growth rate equal to the actual growth rate at time t; that is, $\beta_{t+1} = -[\delta_1/(1+\gamma_1\beta_t)]$, or

$$\gamma_1\beta_t\beta_{t+1} + \beta_{t+1} + \delta_1 = 0. \quad \textbf{(13)}$$

Since the dynamics with learning (13) are merely the time mirror of the perfect-foresight dynamics of growth rates (7), the local stability properties of its fixed points λ_1 and λ_2 are the reverse of those given in Proposition 1.[4]

PROPOSITION 3. *Let $\gamma_1 \neq 0$ and $\delta_1 \neq 0$ in* (1). *Then the root* (λ_1) *is locally stable in the dynamics with myopic learning* (13) *while the root of highest modulus* (λ_2) *is locally unstable in these dynamics.*

Proof. Obvious from Proposition 1. ■

The dynamic equivalence principle directly follows from Propositions 1, 2, and 3, provided that both $\gamma_1 \neq 0$ and $\delta_1 \neq 0$ (i.e., expectations and past matter, respectively). Both conditions are actually necessary for there to exist multiple MSV solutions. In particular, in the case in which $\delta_1 = 0$ in (1), the steady state ($x_t = 0$) is the unique MSV solution to the model, under the additional regularity condition that $\gamma_1 \neq -1$. As emphasized by Guesnerie (1993), it is then locally determinate in the usual dynamics with perfect foresight on the level of the state variable, that is, (2) with $\delta_1 = 0$, if and only if it is locally immune to Markovian stationary sunspot equilibria on the level of the state variable, and locally stable under a myopic learning rule.

3. ONE-STEP-FORWARD-LOOKING MODELS

We now study whether the dynamic equivalence principle still holds true if $L \geq 1$ predetermined variables are introduced into the temporary equilibrium map (1).

The evolution of the level of the state variable along any MSV solution is then described by an L-dimensional linear difference equation. Hence, given the L initial conditions to the economic system, a solution of this type will be no longer characterized by a single coefficient, a constant growth rate, but instead by the set of the L coefficients of this linear difference equation, to be called a *stationary extended growth rate*. This section describes how to apply the three tests used in Section 2 to such sets of coefficients, or, equivalently, to the corresponding MSV solutions. As shown by Gauthier (2002), only one MSV solution is locally determinate in the suitable dynamics with perfect foresight, whose fixed points are now these stationary extended growth rates. Moreover, as shown by Desgranges and Gauthier (in press), this solution is the only one to be locally immune to sunspots. Finally, as the local stability properties of this solution in the dynamics with myopic learning obtain again as a simple corollary of its property of local determinacy, the form taken here by the dynamic equivalence principle is still satisfied in the more general framework under consideration, and, furthermore, still allows us to pick out a unique MSV solution, the fundamental one.

Let the temporary equilibrium relation (1) be transformed as

$$\gamma_1 E(x_{t+1} \mid I_t) + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0. \tag{14}$$

Perfect-foresight solutions are now sequences (x_t) associated with the initial condition $(x_{-1}, \dots, x_{-L})$ and satisfying the $(L+1)$-dimensional difference equation

$$\gamma_1 x_{t+1} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0, \tag{15}$$

obtained by replacing the forecast $E(x_{t+1} \mid I_t)$ with the actual realization x_{t+1} in (14). These dynamics are governed by the $(L+1)$ roots $(\lambda_1, \dots, \lambda_{L+1})$ of the characteristic polynomial associated with (15). Let these roots be real and distinct, with $|\lambda_1| < \cdots < |\lambda_{L+1}|$.[5] Unlike Section 2, however, we are not primarily concerned with these roots since the presence of $L \geq 1$ predetermined variables in (14) now makes the current state x_t linearly related to the $L \geq 1$ past realizations $(x_{t-1}, \dots, x_{t-L})$ along any MSV solution. These solutions actually satisfy not only (15) but also the difference equation

$$x_t = \sum_{l=1}^{L} \beta_l^* x_{t-l}, \tag{16}$$

where the vector of coefficients $\boldsymbol{\beta}^* \equiv (\beta_1^*, \dots, \beta_L^*)$ is called a *stationary extended growth rate*. As in Section 2, one can characterize such a vector by interpreting (16) as a situation where (16) is a self-fulfilling belief about the evolution of the economic system. Assume accordingly that traders believe that the law of motion of the level of the state variable is given by (16) for some set of coefficients $(\beta_1, \dots, \beta_L)$. In this case, they form their forecast by iterating (16) once,

$$E(x_{t+1} \mid \{x_t, \ldots, x_{t-l+1}\}) = \sum_{l=1}^{L} \beta_l x_{t-l+1}, \quad \textbf{(17)}$$

so that the actual evolution of the level of the state variable, which results from reintroducing (17) into (14), determines the actual current state x_t as a function of past history, namely,

$$x_t = -\sum_{l=1}^{L} [(\gamma\beta_{l+1} + \delta_l)/(1 + \gamma\beta_1)]x_{t-l}, \quad \textbf{(18)}$$

with the convention that $\beta_{L+1} \equiv 0$. The belief (16) is then self-fulfilling if and only if (16) and (18) coincide whatever t and x_{t-l} are, or, equivalently, $\beta_l = \beta_l^*$ for $l = 1, \ldots, L$, with

$$\beta_l^* = -\sum_{l=1}^{L} [(\gamma\beta_{l+1}^* + \delta_l)/(1 + \gamma\beta_1^*)], \quad \textbf{(19)}$$

whatever l is ($l = 1, \ldots, L$), and $\beta_{L+1}^* \equiv 0$. How the solutions $(\beta_1^*, \ldots, \beta_L^*)$ to (19) depend on economic fundamentals, summarized here by the parameters γ_1 and δ_l in (14), is fully described in Gauthier (2002). For our purpose, it is sufficient to notice that (19) admits in fact $(L + 1)$ different solutions; that is, there are $(L + 1)$ stationary extended growth rates to the model (16). Indeed, since (15) and (16) must be consistent (otherwise the sequence of realizations of the level of the state variable induced by (16) would not form a perfect-foresight equilibrium), it must be the case that (16) restricts the level of the state variable to evolve in one L-dimensional eigensubspace of (15), as x_t actually relies on L lagged variables in (16). Let W_i be the L-dimensional eigensubspace of (15) spanned by the L eigenvectors associated with the L roots in the set $\mathcal{L}_i$ of all the roots but λ_i ($i = 1, \ldots, L + 1$). Since the total number of different L-dimensional eigensubspaces of (15) is equal to $(L + 1)$, there are $(L + 1)$ different MSV solutions [each of these solutions governs the behavior of the level of the state variable in a given the L-dimensional eigensubspaces of (15)]. Now, it follows from (16) that each MSV solution is in turn uniquely defined by a given set of the coefficients $(\beta_1^*, \ldots, \beta_L^*)$ solution to (19) and by the L initial conditions to the economic system. As a result, there are also $(L + 1)$ different stationary extended growth rates to the model (16). In the sequel, these vectors of coefficients will be denoted $\boldsymbol{\beta}^*(\mathcal{L}_i)$, where $\boldsymbol{\beta}^*(\mathcal{L}_i)$ actually corresponds through (16) to the MSV solution that governs the behavior of the level of the state variable in W_i.

3.1. Determinacy of Extended Growth Rates

Our aim is now to examine whether some of these stationary extended growth rates pass the tests used in Section 3. Here we start applying the determinacy criterion by noticing that the usual dynamics on the level of the state variable, that is, (15), trigger new dynamics on the vectors $\boldsymbol{\beta}(t) = (\beta_1(t), \ldots, \beta_L(t))$ whose fixed points are the $(L+1)$ stationary extended growth rates. In these new dynamics, a stationary extended growth rate $\boldsymbol{\beta}^*(\mathcal{L}_i)$ is locally determinate whenever there is no other vector $\boldsymbol{\beta}(t)$ remaining arbitrarily close to it in each period. Otherwise it is locally indeterminate. As shown in Gauthier (2002), the stationary extended growth rate $\boldsymbol{\beta}^*(\mathcal{L}_{L+1})$ corresponding to the set $\mathcal{L}_{L+1}$ of the L roots of lowest modulus is in fact the only one to be locally determinate.

The *extended growth-rate perfect-foresight dynamics* come by assuming that the level of the state variable is bound to satisfy (15) and, further, to evolve according to

$$x_t = \sum_{l=1}^{L} \beta_l(t) x_{t-l}, \tag{20}$$

whatever $t \geq 0$ and x_{t-l} $(l = 1, \ldots, L)$ are. Perfect foresight requires that the belief (20) be self-fulfilling, that is, consistent with the actual law of motion of the system. This actual law is derived by replacing $E(x_{t+1} \mid I_t)$ in (15) with

$$x_{t+1} = \sum_{l=1}^{L} \beta_l(t+1) x_{t+1-l},$$

which is obtained by iterating (20) once. It is then straightforward to verify that the actual law corresponding to (20) is

$$x_t = -\sum_{l=1}^{L} \{[(\gamma_1 \beta_{l+1}(t+1) + \delta_l)]/[(1 + \gamma_1 \beta_1(t+1))]\} x_{t-l}, \tag{21}$$

with the convention that $\beta_{L+1}(t+1) \equiv 0$. We are now in a position to define the extended growth-rate perfect-foresight dynamics. It is a nonlinear sequence of extended growth rates $(\boldsymbol{\beta}(t))$ such that (20) coincides with (21) whatever t and x_{t-l} are, or, equivalently,

$$\beta_l(t) = -[\gamma_1 \beta_{l+1}(t+1) + \delta_l]/[1 + \gamma_1 \beta_1(t+1)], \tag{22}$$

for $l = 1, \ldots, L$. Its fixed points $\boldsymbol{\beta}(t) = \boldsymbol{\beta}(t+1)$ are the $(L+1)$ stationary extended growth rates [which solve (19)]. These dynamics are well defined arbitrarily close to any stationary extended growth rate if and only if all the roots λ_i $(i = 1, \ldots, L+1)$ differ from 0, which is true if and only if $\delta_L \neq 0$ [see Gauthier (2002)]. Since (22) has a one-step-forward-looking structure without predetermined variables, a stationary extended growth rate will be locally determinate in (22) if and only if all the L eigenvalues of the Jacobian matrix that governs the forward dynamics (22) in its immediate vicinity have moduli greater than 1.[6]

PROPOSITION 4. *The stationary extended growth rate* $\boldsymbol{\beta}^*(\mathcal{L}_{L+1})$ *which governs the state variable perfect-foresight dynamics* (15) *restricted to the L-dimensional subspace corresponding to the L roots of lowest modulus*

$(\lambda_1, \ldots, \lambda_L)$ *is the only one to be locally determinate in the extended growth-rate perfect-foresight dynamics* (22).

Proof. See Gauthier (2002). ■

Although Proposition 4 is independent of the stability properties of the dynamics with perfect foresight on the level of the state variable, the MSV solution corresponding to $\boldsymbol{\beta}^*(\mathcal{L}_{L+1})$ will be locally feasible if and only if the condition $|\lambda_L| < 1$ is satisfied, which encompasses both the saddle-point configuration ($|\lambda_L| < 1 < |\lambda_{L+1}|$) for (15), where $\boldsymbol{\beta}^*(\mathcal{L}_{L+1})$ supports the saddle-path trajectory, and the sink configuration ($|\lambda_{L+1}| < 1$) for these dynamics, where there are infinitely many stable solutions.

3.2. Sunspot Equilibria on Extended Growth Rates

We now come to study whether $\boldsymbol{\beta}^*(\mathcal{L}_{L+1})$ is also the only stationary extended growth rate to be locally immune to sunspots. Suppose, consequently, that agents observe a k-state Markovian process associated with a Markov matrix Π and believe that, in the sunspot event $s_t = s$ $(s = 1, \ldots, k)$, the current state should be linked to the L previous states according to the law

$$x_t = \sum_{l=1}^{L} \beta_l^s x_{t-l}, \qquad \textbf{(23)}$$

where the current set of coefficients $\boldsymbol{\beta}^s \equiv (\beta_1^s, \ldots, \beta_L^s)$ is allowed to depend on the current sunspot event. The belief (23) is intended to hold whatever t and the past history $(x_{t-1}, \ldots, x_{t-L})$ are. Therefore, agents also deduce from the occurrence of the event $s_t = s$ that the next extended growth rate should be equal to $\boldsymbol{\beta}^{s'}$ with probability $\pi_{ss'}$ $(s' = 1, \ldots, k)$, and, consequently, their forecast can be written as

$$E(x_{t+1} \mid \{s_t = s, x_t, \ldots, x_{t-L+1}\}) = \sum_{l=1}^{L} \left(\sum_{s'=1}^{k} \pi_{ss'} \beta_l^{s'} \right) x_{t+1-l}. \qquad \textbf{(24)}$$

In a rational expectations equilibrium, the a priori belief (23) must be self-fulfilling whatever s_t is. If $s_t = s$, the actual dynamics corresponding to (23) are obtained by reintroducing the forecast (24) into (14), which leads to

$$x_t = -\sum_{l=1}^{L} \left\{ \left[\gamma \left(\sum_{s'=1}^{k} \pi_{ss'} \beta_{l+1}^{s'} \right) + \delta_l \right] \Big/ \left[\gamma \left(\sum_{s'=1}^{k} \pi_{ss'} \beta_1^{s'} \right) + 1 \right] \right\} x_{t-l}, \quad \textbf{(25)}$$

with the convention that $\beta_{L+1}^{s'} \equiv 0$ whatever s' is. Thus, in view of (23) and (25), one can define a stationary sunspot equilibrium on extended growth rates as a kL-dimensional vector $\boldsymbol{\beta} \equiv (\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^k)$ associated with Π such that (a) there exist l $(l = 1, \ldots, L)$ and $s \neq s'$ $(s, s' = 1, \ldots, k)$ such that $\beta_l^s \neq \beta_l^{s'}$, and (b) β_l^s $(s = 1, \ldots, k)$ satisfies

$$\beta_l^s = -\left(\gamma \sum_{s'=1}^{k} \pi_{ss'} \beta_{l+1}^{s'} + \delta_l\right) \bigg/ \left(\gamma \sum_{s'=1}^{k} \pi_{ss'} \beta_1^{s'} + 1\right)$$

for $l = 1, \ldots, L$. Condition (b) makes the a priori belief (23) self-fulfilling since it ensures that (23) and (25) coincide whatever s_t is. For the extended growth rate to fluctuate according to sunspots, condition (a) must also hold. Otherwise, if (b) holds true but (a) fails, then $\boldsymbol{\beta}^s = \boldsymbol{\beta}^{s'} = \boldsymbol{\beta}^*(\boldsymbol{\pounds}_i)$ for any $s, s' = 1, \ldots, k$, and i $(i = 1, \ldots, L+1)$.

The next result, which bears on Theorem 3 in Chiappori et al. (1992), studies whether there exist stationary sunspot equilibria on extended growth rates arbitrarily close to a stationary extended growth rate $\boldsymbol{\beta}^*(\boldsymbol{\pounds}_i)$, i.e., such that the lth component β_l^s $(l = 1, \ldots, L)$ of $\boldsymbol{\beta}^s$ is arbitrarily close to the lth component $\beta_l^*(\boldsymbol{\pounds}_i)$ of $\boldsymbol{\beta}^*(\boldsymbol{\pounds}_i)$ whatever s and l are. A stationary extended growth rate $\boldsymbol{\beta}^*(\boldsymbol{\pounds}_i)$ is said to be locally immune to sunspots when this is not the case.

PROPOSITION 5. *The stationary extended growth rate* $\boldsymbol{\beta}^*(\boldsymbol{\pounds}_{L+1})$ *which governs the state variable perfect-foresight dynamics* (15) *restricted to the* L*-dimensional subspace corresponding to the* L *roots of lowest modulus* $\lambda_1, \ldots, \lambda_L$ *is the only one to be locally immune to sunspots.*

Proof. See Chiappori et al. (1992) for a general study and Desgranges and Gauthier (in press) for an application to stationary sunspot equilibria on extended growth rates. ■

3.3. Learning Extended Growth Rates

It follows from Propositions 4 and 5 that a stationary extended growth rate is locally determinate if and only if it is locally immune to sunspots. We show in this section that the dynamic equivalence principle is satisfied since a stationary extended growth rate is locally stable under myopic learning if and only if it is locally determinate. Assume, therefore, that the a priori belief on the law of motion still fits (16) whatever $t \geq 0$ and x_{t-l} $(l = 1, \ldots, L)$ are, but that agents are no longer aware of the entire set of stationary extended growth rates, and try to learn them. Let $\beta_l(t)$ be the estimate of the lth component of some stationary extended growth rate at date t. Given this vector of estimates, the traders' forecast $E(x_{t+1} \mid I_t)$ is written as

$$E(x_{t+1} \mid \{x_t, \ldots, x_{t-L}\}) = \sum_{l=1}^{L} \beta_l(t) x_{t-l+1}. \tag{26}$$

The actual law of motion of the level of the state variable, namely,

$$x_t = -\sum_{l=1}^{L} (\gamma \beta_{l+1}(t) + \delta_l)/(1 + \gamma \beta_1(t)) x_{t-l}, \tag{27}$$

is obtained by reintroducing the forecast (26) into (14). In the myopic learning dynamics, traders compare their initial estimate $\beta_l(t)$ to the actual lth coefficient $(l = 1, \ldots, L)$ given in (27), and revise this estimate according to the rule

$$\beta_l(t+1) = -\sum_{l=1}^{L}[\gamma\beta_{l+1}(t) + \delta_l]/[1 + \gamma\beta_1(t)] \qquad \mathbf{(28)}$$

for any l, and with the convention that $\beta_{L+1}(t) \equiv 0$. The algorithm (28) can be thought of as a learning rule updated in real time in which agents keep fixed their forecast rule until the implied actual law-of-motion coefficients could be learned from the data, that is, typically after L periods.[7] As discussed by Evans (1985), the learning rule (28) is then identical to the iterative version of the expectational stability; it belongs nevertheless to a rather special class of learning rules. As (19) shows, the fixed points of these dynamics are the $(L+1)$ stationary extended growth rates. A stationary extended growth rate is locally stable under learning if and only all the eigenvalues of the Jacobian matrix that governs (28) in its immediate vicinity have modulus less than 1. However, (28) is simply the time mirror of the extended growth-rate perfect-foresight dynamics (22). Thus, the property of local determinacy of a stationary extended growth rate in (22) is equivalent to its property of local stability in (28), which establishes that the dynamic equivalence principle holds for the class of models (14).

PROPOSITION 6. *The stationary extended growth rate* $\boldsymbol{\beta}^*(\boldsymbol{\mathcal{L}}_{L+1})$, *which governs the state variable perfect-foresight dynamics* (15) *restricted to the L-dimensional subspace corresponding to the L roots of lowest modulus* $\lambda_1, \ldots, \lambda_L$, *is the only one to be locally stable in the dynamics with myopic learning* (28).

Proof. Obvious from Proposition 4. ■

4. ONE-LAG IN MEMORY MODELS

So far only models where agents forecast one period ahead have been considered. In this section, we explore the remaining polar configuration where the number of leads in expectations is arbitrary. However, our analysis is restricted to the simple case in which only one predetermined variable matters. As shown in Section 2, models of this class admit multiple MSV solutions. Each one makes the current state of the economic system linearly related to the previous state only, and, therefore, each one can be characterized by a constant growth rate of the level of the state variable. The question is whether the dynamic equivalence principle, when applied to such growth rates, is saved in this new framework. It will be shown that only one MSV solution is still both locally determinate and locally immune to sunspots, but this solution is no longer necessarily locally stable under myopic learning.

Let us consider the temporary equilibrium relation

$$\sum_{h=1}^{H}\gamma_h E(x_{t+h} \mid I_t) + x_t + \delta_1 x_{t-1} = 0, \qquad \mathbf{(29)}$$

where $E(x_{t+h} \mid I_t)$ represents the forecast formed at date t about the level of the state variable in period $t+h$ $(h=1,\ldots,H)$, and where x_{t-1} is predetermined at date t. A perfect-foresight solution is a sequence (x_t) associated with the initial condition x_{-1} and satisfying the $(H+1)$-dimensional recursive equation

$$\sum_{h=1}^{H} \gamma_h x_{t+h} + x_t + \delta_1 x_{t-1} = 0, \tag{30}$$

obtained by assuming that $E(x_{t+h} \mid I_t)$ is equal to x_{t+h} in (29) whatever $h \geq 1$ and $t \geq 0$ are. Let λ_i $(i=1,\ldots,H+1)$ be the $(H+1)$ roots of the characteristic equation associated with (30). As in the previous sections, we assume that these roots are real, distinct, and labeled in the order of increasing modulus; that is, $|\lambda_i| < |\lambda_j|$ whenever $i < j$ $(i, j = 1, \ldots, H+1)$. Since there is only one predetermined variable in (29), the law of motion of the level of the state variable along any MSV solution to (30) is governed by a first-order difference equation that still fits (3) and, now, the perfect-foresight dynamics (30). The constant growth rate β in (3) ensures perfect foresight when (3) is a self-fulfilling belief, that is, when it coincides with actual observations generated by (29). In view of (3), the forecast of traders is given by

$$E(x_{t+h} \mid I_t) = \beta E(x_{t+h-1} \mid I_t) = \cdots = \beta^h x_t, \quad \text{for } h = 1, \ldots, H, \tag{31}$$

so that the actual dynamics, resulting from reinserting (31) into (29), are written

$$x_t = -\left[\delta_1 \Big/ \left(1 + \sum_{h=1}^{H} \gamma_h \beta^h\right)\right] x_{t-1}. \tag{32}$$

Thus, the belief (3) on the law of motion coincides with the actual law (32), whatever x_{t-1} and t are, if and only if

$$\beta = -\left[\delta_1 \Big/ \left(1 + \sum_{h=1}^{H} \gamma_h \beta^h\right)\right] \Leftrightarrow \sum_{h=1}^{H} \gamma_h \beta^{h+1} + \beta + \delta_1 = 0, \tag{33}$$

or, equivalently, β is a root λ_i $(i=1,\ldots,H+1)$ of the characteristic polynomial associated with (30).

4.1. Determinacy of Growth Rates

The sequence of growth rates induced by the perfect-foresight dynamics on the level of the state variable is obtained whenever the relation $x_t = \beta_t x_{t-1}$ holds in (30) whatever t and x_{t-1} are, so that $x_{t+h} = \beta_{t+h} x_{t+h-1} = (\beta_{t+h} \cdots \beta_t) x_{t-1}$ for $h = 0, \ldots, H$. Under these requirements, the dynamics (30) can be rewritten as

$$\sum_{h=1}^{H} \gamma_h \prod_{j=0}^{h} \beta_{t+j} x_{t-1} + \beta_t x_{t-1} + \delta_1 x_{t-1} = 0, \tag{34}$$

or, provided that $x_{t-1} \neq 0$,

$$\sum_{h=0}^{H} \gamma_h \prod_{j=0}^{h} \beta_{t+j} + \delta_1 = 0, \tag{35}$$

with the convention that $\gamma_0 \equiv 1$. The dynamics of (35) are well-defined if $\beta_t \neq 0$ whatever t is. Its fixed points are the $(H+1)$ roots λ_i $(i = 1, \ldots, H+1)$ of (33). Hence it is well defined arbitrarily close to λ_i, if and only if $\lambda_i \neq 0$. This is the case whatever i is if and only if $\delta_1 \neq 0$. The dynamics of (35) arbitrarily close to λ_i are governed by a linear recursive equation of order H, obtained by linearizing (35) at point $\beta(t+s) = \lambda_i$ whatever s is $(s = 0, \ldots, H)$, namely,

$$\sum_{h=0}^{H} \sum_{j=h}^{H} \gamma_j \lambda_i^j (\beta_{t+h} - \lambda_i) = 0. \tag{36}$$

Since (35) determines the current growth rate β_t as a function of future growth rates β_{t+h} $(h = 1, \ldots, H)$, a fixed point λ_i of (35) is locally determinate if and only if all the H roots of the characteristic polynomial associated with (36) have moduli greater than 1.

PROPOSITION 7. *Assume that $\gamma_H \neq 0$ for $H \geq 1$ and $\delta_1 \neq 0$. Then the root of lowest modulus* (λ_1) *is the only fixed point of the growth-rate perfect-foresight dynamics* (35) *to be locally determinate in this dynamics.*

Proof. See Appendix A. ■

4.2. Sunspots on Growth Rates

Assume now that all the agents believe that the growth rate of the level of the state variable is perfectly correlated to a k-state Markovian sunspot process whose probability law is described by a $k \times k$ Markov matrix Π. Hence agents expect the current growth rate β_t $(t \geq 0)$ to be equal to β_s $(s = 1, \ldots, k)$ whenever they observe at date t some sunspot signal s. In a stationary sunspot equilibrium on the growth rate, this belief must be consistent with the actual law of motion of the system. This section is devoted to study whether stationary sunspot equilibria of this type exist when all the growth rates β_s are bound to lie arbitrarily close to a given root λ_i $(i = 1, \ldots, H+1)$. The root of lowest modulus (λ_1) will be shown to be the only one to be immune to sunspot fluctuations, thus making local determinacy equivalent to the lack of local stationary sunspot equilibria on the growth rate.

To set out the actual temporary equilibrium dynamics that make sunspot beliefs self-fulfilling, one must specify how traders forecast future at any date, given their a priori belief on the law of motion of the economic system. When the sunspot signal s_t at date t is s, traders expect (i) the period t growth rate to be β_s and (ii) the growth rates in the subsequent periods to be distributed conditionally to the event $s_t = s$;

that is, the probability that $\beta_{t+h} = \beta_{s'}$ ($s' = 1, \ldots, k$) is given by the ss'th element of Π^h. Since (i) holds true at any date, the expectation $E(x_{t+h} \mid \{s_t = s\})$ is equal to $E[(\beta_{t+h} \cdots \beta_{t+1})x_t \mid \{s_t = s\}]$. What is important, therefore, is the probability law of the product $(\beta_{t+h} \cdots \beta_{t+1})$. The following lemma describes this probability law.

LEMMA 1. *Let* B *be the* $k \times k$ *diagonal matrix whose ssth element is* β_s ($s = 1, \ldots, k$), *that is, the growth rate that traders expect to occur in sunspot state s. Let* $x_t(s)$ *be the temporary equilibrium state in sunspot state s at date t* ($t \geq 0$). *Let* X_t *be the* $k \times k$ *diagonal matrix whose ssth element is* $x_t(s)$. *Let* $E(\mathbf{x}_{t+h} \mid \{s_t\})$ *be the* $k \times 1$ *vector whose sth component is the forecast* $E(x_{t+h} \mid \{s_t = s\})$. *Finally, let* $\mathbf{1}_k$ *be the* $k \times 1$ *unitary vector. Then we have*

$$E(\mathbf{x}_{t+h} \mid \{s_t\}) = \mathrm{X}_t(\Pi \mathrm{B})^h \mathbf{1}_k.$$

Proof. See Appendix B. ∎

The actual dynamics are then obtained by reintroducing the forecast $E(x_{t+h} \mid \{s_t = s\})$, given in Lemma 1, into (29); that is, for any s ($s = 1, \ldots, k$),

$$\sum_{h=1}^{H} \gamma_h \mathrm{X}_t(\Pi \mathrm{B})^h \mathbf{1}_k + \mathrm{X}_t \mathbf{1}_k + \delta_1 \mathbf{1}_k x_{t-1} = \mathbf{0}_k, \tag{37}$$

where $\mathbf{0}_k$ stands for the $k \times 1$ null vector. The sth equation ($s = 1, \ldots, k$) of the system (37) determines implicitly the current state $x_t(s)$ in the event $s_t = s$ as a function of Π, B, and the relevant past history, here summarized by the previous state x_{t-1} of the system. By definition, in equilibrium, $x_t(s)$ must be equal to $\beta_s x_{t-1}$ in (37) whatever s is ($s = 1, \ldots, k$); that is, $\mathrm{X}_t = \mathrm{B}x_{t-1}$. Under this requirement, (37) is rewritten as

$$\sum_{h=1}^{H} \gamma_h \mathrm{B}x_{t-1}(\Pi \mathrm{B})^h \mathbf{1}_k + \mathrm{B}x_{t-1}\mathbf{1}_k + \delta_1 \mathbf{1}_k x_{t-1} = \mathbf{0}_k. \tag{38}$$

Let us now define a sunspot equilibrium on the growth rate as a vector $\boldsymbol{\beta} \equiv (\beta_1, \ldots, \beta_k)$ and a Markov matrix Π such that (a) there exist two components β_s and $\beta_{s'}$ of the vector $\boldsymbol{\beta}$ such that $\beta_s \neq \beta_{s'}$ and such that (b) the $k \times 1$ vector $\boldsymbol{\beta} \equiv \mathrm{B}\mathbf{1}_k$ satisfies

$$\sum_{h=0}^{H} \gamma_h \mathrm{B}(\Pi \mathrm{B})^h \mathbf{1}_k + \delta_1 \mathbf{1}_k = \mathbf{0}_k \tag{39}$$

with the convention that $\gamma_0 \equiv 1$. Equation (39) is obtained from (38) when $x_{t-1} \neq 0$. Condition (a) ensures that the growth rate is changing with sunspot events, while (b) makes the actual growth rate equal to the initial guess β_s in the sunspot event s. If (b) holds true, but (a) is violated, then the solutions to (39) are the $k \times 1$ vectors $\boldsymbol{\beta}$ whose each component β_s is equal to λ_i ($i = 1, \ldots, H + 1$).

The mere existence of sunspot equilibria associated with a $k \times 1$ vector $\boldsymbol{\beta}$ whose components β_s $(s = 1, \ldots, k)$ stand arbitrarily close to some root λ_i would weaken the likelihood that traders succeed in coordinating their forecasts on this root. In view of the fact that any $k \times k$ diagonal matrix $\mathrm{B} = \mathrm{L}_i$ whose ssth entry is λ_i is a solution to (39) whatever Π is, a $k \times k$ diagonal matrix B whose ssth element is arbitrarily close to λ_i will form a sunspot equilibrium on the growth rate associated with Π only if the implicit function theorem breaks at $\mathrm{B} = \mathrm{L}_i$ for the system (39), namely,

$$\det\ \mathrm{D}\left[\sum_{h=0}^{H} \gamma_h \mathrm{L}_i (\Pi \mathrm{L}_i)^h \mathbf{1}_k + \delta_1 \mathbf{1}_k\right] = 0,$$

where $\mathrm{D}(\bullet)$ is the Jacobian matrix of $(\bullet)$ and $\det \mathrm{D}(\bullet)$ stands for the determinant of this Jacobian matrix. The following result shows that the root of lowest modulus (λ_1) is in fact the only one to be locally immune to sunspots.

PROPOSITION 8. *Let $x_{t-1} \neq 0$ whatever $t \geq 0$ is. Then there do not exist stationary sunspot equilibria on the growth rate $(\boldsymbol{\beta}, \Pi)$ associated with a $k \times 1$ vector $\boldsymbol{\beta}$ each of whose component β_s $(s = 1, \ldots, k)$ stands arbitrarily close to the root of lowest modulus (λ_1). On the contrary, there do exist Markov matrices Π such that $(\boldsymbol{\beta}, \Pi)$ is a stationary sunspot equilibrium whenever each component β_s $(s = 1, \ldots, k)$ of $\boldsymbol{\beta}$ stands arbitrarily close to any remaining root λ_i $(i = 2, \ldots, H + 1)$.*

Proof. See Appendix C. ■

4.3. Learning Growth Rates

Assume now that traders try to estimate the law of motion of the level of the state variable restricted to some MSV solution. Their a priori belief on the law of motion still fits (3) but β is now estimated by β_t at date t, where β_t is possibly different from some root λ_i of the characteristic polynomial of (30). Therefore, their time t forecast $E(x_{t+h} \mid I_t)$, $h \geq 1$, obtained by leading (3) h times forward, is equal to $\beta_t^h x_t$. The actual dynamics then result from reinserting this forecast into the temporary equilibrium map (29):

$$x_t = -\left[\delta_1 \Bigg/ \left(1 + \sum_{h=1}^{H} \gamma_h \beta_t^h\right)\right] x_{t-1}. \qquad \mathbf{(40)}$$

According to the myopic learning rule, traders must revise at time $(t + 1)$ their time t estimate β_t by choosing as a new estimate β_{t+1} the actual growth rate at period t, such as given by (40)[8]:

$$\beta_{t+1} = -\left[\delta_1 \Bigg/ \left(1 + \sum_{h=1}^{H} \gamma_h \beta_t^h\right)\right]. \qquad \mathbf{(41)}$$

The $(H+1)$ roots λ_i are the fixed points of (41). The dynamics with learning (41) arbitrarily close to λ_i are governed by the H roots of the characteristic polynomial associated with the H-dimensional difference equation obtained by linearizing (41) at point $\beta_t = \lambda_i$. As usual, a fixed point λ_i is locally stable under learning if and only if all these H roots have moduli less than 1. The following result provides a condition under which λ_i is locally stable in the dynamics (41).

PROPOSITION 9. *A root λ_i $(i = 1, \ldots, H+1)$ is locally stable in the dynamics with learning* (41) *if its modulus $|\lambda_i|$, which measures the speed of convergence of the level of the state variable toward its steady state value along the corresponding MSV solution, is low enough.*

Proof. See Appendix D. ■

In general, unlike the previous sections, the local stability under learning of a solution is not equivalent to its local determinacy. Nevertheless, if the level of the state variable converges rapidly enough toward its steady state along the MSV solution corresponding to the root λ_1, which is satisfied if, for example, the influence of past onto the economic system is low enough (δ_1 is close enough to 0), then agents will locally learn this solution, thus making local stability under learning consistent with local determinacy and local immunity to sunspots. However, of course, it is still possible that several MSV solutions to (29) are locally stable in the learning dynamics (41),[9] but, in this case, the set of stable solutions necessarily includes the MSV solution corresponding λ_1.

5. CONCLUDING COMMENTS

Rational expectations models typically feature a multiplicity of equilibrium paths. The aim of this paper was to describe how to apply three selection devices, respectively based on local determinacy, local immunity to sunspots, and local stability under learning, to MSV solutions of general linear models. The results are clear: All these devices are generally equivalent, and, furthermore, recommend selecting a unique MSV solution. This solution is actually the one that is usually believed to be of practical relevance in macroeconomics (in particular, it is the saddle stable path in the saddle-point configuration, i.e., the only stable solution in this configuration). There are some open questions that may deserve to be the subject of further work. First, formal extensions include the case of the general linear model with an arbitrary number of leads and lags, where the state variable is multidimensional. In view of the results presented in this paper, one can expect most of the dynamic equivalence principle to be saved in such frameworks. Another direction consists in applying the techniques of the paper to bubble solutions, along which the current state of the system depends on more lagged variables than the number of initial conditions to the economic system. In this case the actual equilibrium trajectory is characterized by not only a vector of stationary extended growth rate, but also by some arbitrary forecasts about the level of the state variable in the initial periods. Intuitively, coordination on this type of solutions is much more demanding than coordination on MSV solutions, but there are examples in the literature where bubble solutions are locally stable under learning [see, e.g., Evans and Honkapohja (1994)]. Therefore, it would be interesting to know whether these solutions can be locally determinate or immune to sunspots. If this is not the case, as one may conjecture, the relevance of the MSV solution come whose role has been recurrently emphasized throughout this paper, would still come up in practical analysis.

NOTES

1. In linear models subject to extrinsic shocks, there is a substantial literature on the stability under least-squares learning of various dynamic rational expectations solutions. See, e.g., Evans and Honkapohja (1999) for a survey on this topic.

2. See McCallum (1983) for closely related terminology. McCallum (1983) actually defines minimal state variable solutions using both a primary principle and a second principle that isolates a particular minimal state variable solution as bubble-free. Here, as in Evans and Honkapohja (1999), we define minimal state variable solutions using only the primary principle. It is worth noticing that, in fact, the dynamic equivalence principle recommends choosing the solution identified as bubble-free, or fundamental, by McCallum (1983).

3. In fact, local stability under least-squares learning of MSV solutions has been extensively considered in the literature. See, e.g., Evans and Honkapohja (1999) for a survey. Whether these solutions can be locally determinate or locally free of sunspots, however, has not been studied so far.

4. The algorithm (13) obviously belongs to a special class of learning rules. In fact, it is identical to the iterative version of the expectational stability device used by DeCanio (1979) and Evans (1985). As shown by Evans and Guesnerie (1993), stability of a rational expectations solution in (13) is necessary, but not sufficient, for this solution to be strongly rational. Note also that if a solution is locally stable in the learning dynamics (13), then it is also locally expectationally stable in the sense of Evans (1989) and, accordingly, locally stable for the least-squares learning rule, as first shown by Marcet and Sargent (1989).

5. Theorem 6 in Gantmacher (1966, Ch. 15, Sect. 9) gives the number of real and distinct roots of a polynomial as a function of its real coefficients. A simple necessary condition for all these roots to be real is that $\gamma_1\delta_1 \le 1$, $\delta_2 \le \delta_1^2$, $\delta_1\delta_3 \le \delta_2^2, \ldots, \delta_{L-2}\delta_L \le \delta_{L-1}^2$ [see Du Gua-Huat-Euler theorem in Mignotte (1989)].

6. Equivalently, a stationary extended growth rate is locally indeterminate in (22) if at least one eigenvalue of the Jacobian matrix that governs the *backward* perfect-foresight dynamics corresponding to (22) arbitrarily close to this stationary extended growth rate falls outside the unit disk. See, e.g., Chiappori et al. (1992, p. 1098).

7. The updating (28) is feasible every L periods provided that the $L \times L$ matrix generated is nonsingular [see Evans (1985)].

8. Here again, note that algorithm (41) is identical to the iterative version of the expectational stability criterion.

9. Evans and Honkapohja (1994) have actually shown that the class of models (29) with $H = 2$ and $L = 1$ may have two distinct MSV solutions that are locally stable under least-squares learning.

10. Consider the polynomial with real coefficients

$$P(z) = z^n + a_{n-1}z^{n-1} \cdots + a_0$$

whose roots are z_i^* $(i = 1, \ldots, n)$. By definition, we have

$$P(z) = 0 \Leftrightarrow \prod_{i=1}^{n}(\lambda - z_i^*) = 0 \Leftrightarrow z^n - \left(\sum_{i=1}^{n} z_i^*\right) z^{n-1} + \cdots + (-1)^n \prod_{i=1}^{n}(z_i^*) = 0$$

Identifying the preceding equations term by term leads to the relations

$$-a_n\left(\sum_{i=1}^{n} z_i^*\right) = a_{n-1}, \ldots, a_n(-1)^n \prod_{i=1}^{n}(z_i^*) = a_0.$$

If one denotes $\sigma_l(z_1^*, \ldots, z_n^*)$ the sum over all the different products of l distinct elements ($l = 1, \ldots, n$) of the set $\{z_1^*, \ldots, z_n^*\}$; i.e., $\sigma_1(z_1^*, \ldots, z_n^*)$ is, for instance, the sum of the n roots and $\sigma_n(z_1^*, \ldots, z_n^*)$ is the product of these n roots. The formula (A.3) directly comes.

REFERENCES

Azariadis, C. (1981) Self-fulfilling prophecies. *Journal of Economic Theory* 25, 380–396.

Azariadis, C. & R. Guesnerie (1982) Prophéties créatrices et persistence des théories. *Revue Economique* 33, 787–806.

Blanchard, O.J. & C. Kahn (1981) The solution of linear difference models under rational expectations. *Econometrica* 48, 1305–1311.

Chiappori, P.-A., P.-Y. Geoffard, & R. Guesnerie (1992) Sunspot fluctuations around a steady state: The case of multidimensional one-step forward looking models. *Econometrica* 60, 1097–1126.

DeCanio, S. (1979) Rational expectations and learning from experience. *Quarterly Journal of Economics* 93, 47–58.

Desgranges, G. & S. Gauthier (in press) Uniqueness of bubble-free solution in linear rational expectations models. *Macroeconomic Dynamics*.

Evans, G. (1985) Expectational stability and the multiple equilibria problem in linear rational expectations models. *Quarterly Journal of Economics* 100, 147–157.

Evans, G. (1989) The fragility of sunspots and bubbles. *Journal of Monetary Economics* 23, 297–317.

Evans, G. & R. Guesnerie (1993) Rationalizability, strong rationality, and expectational stability. *Games and Economic Behavior* 5, 632–646.

Evans, G. & S. Honkapohja (1994) Learning, convergence, and stability with multiple rational expectations equilibria. *European Economic Review* 38, 1071–1098.

Evans, G. & S. Honkapohja (1999) Learning dynamics. In J.B. Taylor & M. Woodford (eds.), *Handbook of Macroeconomics*, vol. 1, pp. 449–542. Amsterdam: Elsevier Science.

Farmer, R. & M. Woodford (1997) Self-fulfilling prophecies and the business cycle. *Macroeconomic Dynamics* 1, 740–769.

Gantmacher, F.R. (1966) *Théorie des Matrices*, Paris: Dunod.

Gauthier, S. (2002) Determinacy and stability under learning of rational expectations equilibria. *Journal of Economic Theory* 102, 354–374.

Grandmont, J.-M. (1998) Expectations formation and stability of large socioeconomic systems. *Econometrica* 66, 741–783.

Grandmont, J.-M. & G. Laroque (1986) Stability of cycles and expectations. *Journal of Economic Theory* 40, 138–151.

Grandmont, J.-M. & G. Laroque (1991) Economic dynamics with learning: Some instability examples. In W. Barnett (ed.), *Equilibrium Theory and Applications: Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*. Cambridge, UK: Cambridge University Press.

Guesnerie, R. (1993) Theoretical tests of the rational expectations hypothesis in economic dynamic models. *Journal of Economic Dynamics and Control* 17, 847–864.

Guesnerie, R. & M. Woodford (1991) Stability of cycles under adaptive learning rules. In W. Barnett (ed.), *Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*. Cambridge, UK: Cambridge University Press.

Kehoe, T.J. & D. Levine (1985) Comparative statics and perfect foresight in infinite horizon. *Econometrica* 53, 433–454.

Magnus, J.R. & H. Neudecker (1988) *Matrix Differential Calculus with Applications to Statistics and Econometrics*. New York: Wiley.

Marcet, A. & T. Sargent (1989) Convergence of least-squares learning mechanisms in self-referential stochastic models. *Journal of Economic Theory* 48, 337–368.
McCallum, B.T. (1983) On non-uniqueness in rational expectations models: An attempt at perspective. *Journal of Monetary Economics* 11, 139–168.
McCallum, B. (1999) Role of the Minimal State Variable Criterion in Rational Expectations Models. NBER Working Paper 7087.
Mignotte, M. (1989) *Mathématiques pour le Calcul Formel*. Paris: Presses Universitaires de France.
Ramis, E., C. Deschamps, & J. Odoux (1974) *Cours de Mathématiques Spéciales*, vol. 1. Paris: Masson.
Taylor, J.B. (1977) Conditions for unique solutions in stochastic macroeconomic models with rational expectations. *Econometrica* 45, 1377–1385.
Woodford, M. (1984) Indeterminacy of Equilibrium in the Overlapping Generations Model: A Survey. Mimeo, Columbia University.

APPENDIX A. PROOF OF PROPOSITION 7

The characteristic polynomial Q of the Hth-order linear recursive equation (36) is

$$Q(\mu) = \sum_{h=0}^{H} \sum_{j=h}^{H} \gamma_j \lambda_i^j \mu^h. \tag{A.1}$$

The root λ_i is locally determinate in the dynamics (35) if and only if all the roots of Q have moduli greater than 1. To find these roots, observe first that

$$Q(\mu) = 0 \Leftrightarrow \frac{Q(\mu)}{\gamma_H (\lambda_i)^H} = \sum_{h=0}^{H} \sum_{j=h}^{H} \frac{\gamma_j}{\gamma_H} \lambda_i^{j-H} \mu^h = 0, \tag{A.2}$$

which is allowed when $\gamma_H \neq 0$ and $\delta \neq 0$ since then $\lambda_i \neq 0$ $(i = 1, \ldots, H+1)$. We now use the relations between the coefficients of the characteristic polynomial corresponding to (30) and the roots λ_i $(i = 1, \ldots, H+1)$, namely [see, e.g., Ramis et al. (1974)],[10]

$$\frac{\gamma_{H-j}}{\gamma_H} = (-1)^j \sigma_j(\Lambda) \quad \text{for any } j \ (j = 1, \ldots, H), \tag{A.3}$$

where Λ denotes the set of the roots $\{\lambda_1, \ldots, \lambda_{H+1}\}$ and $\sigma_j(\Lambda)$ is the jth symmetric polynomial in the set Λ; that is,

$$\sigma_h(\Lambda) = \sum_{1 \leq j_1 < \cdots < j_h} \lambda_{j_1} \ldots \lambda_{j_h}.$$

One can show that

$$\sum_{j=h}^{H} \frac{\gamma_j}{\gamma_H} (\lambda_i)^{j-H} = (-1)^{H-h} \sigma_{H-h}(\Lambda(l/i)),$$

where $\Lambda(l/i)$ is the set of all possible ratio λ_l/λ_i for any $l \neq i$ $(l = 1, \ldots, H+1)$ with i given $(i = 1, \ldots, H+1)$. It follows that

$$Q(\mu) = 0 \Leftrightarrow \sum_{h=0}^{H} (-1)^{H-h} \sigma_{H-h}(\Lambda(l/i)) \mu^h = 0.$$

The same argument as in the derivation of (A.3) shows that the H roots of Q are the H ratios λ_l/λ_i for any $l \neq i$. All these roots have moduli larger than 1 if and only if $i = 1$. ■

APPENDIX B. PROOF OF LEMMA 1

Let $\pi_{ss'}$ denote the probability that $s_{t+1} = s'$ $(s' = 1, \ldots, k)$ if $s_t = s$ $(s = 1, \ldots, k)$. This probability is the ss'th element of the $k \times k$ matrix Π.

Step 1. Here we prove that Lemma 1 is true for $h = 1$. For instance, let $s_t = s$ $(s = 1, \ldots, k)$. In this case, $\pi_{ss'}$ is the probability that $\beta_{t+1} = \beta_{s'}$ $(s' = 1, \ldots, k)$. Hence, the expected growth rate $\bar{\beta}_s$ between t and $t+1$ is given by

$$\bar{\beta}_s = \sum_{s'=1}^{k} \pi_{ss'} \beta_{s'}.$$

Since $E(x_{t+1} \mid \{s_t = s\}) = E(\beta_{t+1} \mid \{s_t = s\}) x_t(s)$, we also have

$$E(x_{t+1} \mid \{s_t = s\}) = \bar{\beta}_s x_t(s).$$

Observe now that $\bar{\beta}_s$ is the sth component of the $k \times 1$ vector $\Pi \mathrm{B} \mathbf{1}_k$ and $x_t(s)\bar{\beta}_s$ is consequently the sth component of the $k \times 1$ vector $\mathrm{X}_t \Pi \mathrm{B} \mathbf{1}_k$, which shows Lemma 1 for $h = 1$.

Step 2. Recall first that $E(x_{t+h} \mid \{s_t = s\}) = E(\beta_{t+1} \cdots \beta_{t+h} \mid \{s_t = s\}) x_t(s_t)$. Let our recursion hypothesis be that Lemma 1 is true for some given $h \geq 1$; that is, $E(\beta_{t+1} \cdots \beta_{t+h} \mid \{s_t = s\})$ is the sth component of $(\Pi \mathrm{B})^h \mathbf{1}_k$. The result follows by exploiting a simple vector recursion over growth rates expected values. By definition, it is indeed true that

$$E(\beta_{t+1} \cdots \beta_{t+h+1} \mid \{s_t = s\}) = \sum_{s'=1}^{k} \pi_{ss'} \beta_{s'} E(\beta_{t+2} \cdots \beta_{t+h+1} \mid \{s_{t+1} = s'\}). \qquad \textbf{(B.1)}$$

However, $E(\beta_{t+2} \cdots \beta_{t+h+1} \mid \{s_{t+1} = s'\}) = E(\beta_{t+1} \cdots \beta_{t+h} \mid \{s_t = s'\})$. Thus (B.1) can be rewritten as

$$\begin{pmatrix} E(\beta_{t+1} \cdots \beta_{t+h+1} \mid \{s_t = 1\}) \\ \vdots \\ E(\beta_{t+1} \cdots \beta_{t+h+1} \mid \{s_t = k\}) \end{pmatrix} = \Pi \mathrm{B} \begin{pmatrix} E(\beta_{t+1} \cdots \beta_{t+h} \mid \{s_t = 1\}) \\ \vdots \\ E(\beta_{t+1} \cdots \beta_{t+h} \mid \{s_t = k\}) \end{pmatrix}$$

The result comes directly from both our recursion hypothesis and the fact that $E(x_{t+h+1} \mid \{s_t = s\}) = E(\beta_{t+1} \cdots \beta_{t+h+1} \mid \{s_t = s\}) x_t(s)$; that is, $E(\mathbf{x}_{t+h+1} \mid \{s_t\}) = \mathrm{X}_t(\Pi \mathrm{B})^{h+1}\mathbf{1}_k$. This proves Lemma 1 for any $h \geq 1$. ■

APPENDIX C. PROOF OF PROPOSITION 8

Equation (37) describes a system of k equations whose k unknowns are the k growth rate components β_s ($s = 1, \ldots, k$) of the $k \times 1$ vector $\boldsymbol{\beta}$. Therefore, the Jacobian matrix of this system is of dimension $k \times k$. This matrix is derived by using basic matrix differential calculus. We first define

$$F(\mathrm{B}) = \sum_{h=0}^{H} \gamma_h \mathrm{B}(\Pi \mathrm{B})^h \mathbf{1} + \delta \mathbf{1} \quad \text{and} \quad \mathrm{B}(\boldsymbol{\beta}) = \begin{pmatrix} \beta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_k \end{pmatrix}.$$

A sunspot equilibrium is consequently a $k \times 1$ vector $\boldsymbol{\beta}$ associated with a $k \times k$ matrix $B(\boldsymbol{\beta})$ such that $\boldsymbol{\beta}$ belongs to the kernel of $F(B(\bullet))$. The Jacobian matrix $\mathrm{D}F(B(\beta))$ at some point $\boldsymbol{\beta}$ of IR^k is equal to $\mathrm{D}F(B)\mathrm{D}B(\boldsymbol{\beta})$, where $\mathrm{D}F(B)$ and $\mathrm{D}B(\boldsymbol{\beta})$ are as follows [see Magnus and Neudecker (1988, Sect. 9.4.1)]:

$$\mathrm{D}F(B) = \frac{\partial \mathrm{vec} F(B)}{\partial \mathrm{vec} B}, \quad \text{and} \quad \mathrm{D}B(\beta) = \frac{\partial \mathrm{vec} B(\beta)}{\partial \mathrm{vec} \beta},$$

where the symbol ∂ stands for the differential and where the vec operator transforms a matrix into a vector by stacking its columns one underneath the other. Of course, $\mathrm{vec}F(B)$ is the $k \times 1$ vector $F(B)$. Given that B is a $k \times k$ matrix, vecB is a $k^2 \times 1$ vector. Thus, $\mathrm{D}F(B)$ is a $k \times k^2$ matrix. The remaining Jacobian matrix $\mathrm{D}B(\boldsymbol{\beta})$ is of dimension $k^2 \times k$ since $\mathrm{vec}\boldsymbol{\beta} = \boldsymbol{\beta}$ is a $k \times 1$ vector (and recall that vecB is a $k^2 \times 1$ vector). Then, by the chain rule, it follows that $\mathrm{D}F(B(\boldsymbol{\beta}))$ is a $k \times k$ matrix.

We prove Proposition 8 in two steps. First, we provide an expression for $\mathrm{D}F(B(\boldsymbol{\beta}))$. Second, we show that $\det \mathrm{D}F(B(\boldsymbol{\beta}))$ is always different from zero if β_s ($s = 1, \ldots, k$) is equal to λ_1 while there are some stochastic matrices Π that make $\mathrm{D}F(B(\boldsymbol{\beta}))$ singular as soon as β_s ($s = 1, \ldots, k$) is equal to λ_i ($i = 2, \ldots, H+1$). Proposition 8 is then a straightforward consequence of the implicit function theorem. These two steps are the subject of two different lemmas.

LEMMA C.1. *The Jacobian matrix* $\mathrm{D}F(B(\boldsymbol{\beta}))$ *is given by*

$$\mathrm{D}F(B(\beta)) = \sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} \Pi^j, \quad \textit{at point } \beta = (\lambda_i, \ldots, \lambda_i)'.$$

Proof. The map F has the following differential

$$\partial F(B) = \sum_{h=0}^{H} \gamma_h \partial[B(\Pi B)^h]1, \tag{C.1}$$

where $(\Pi B)^h$ stands for the product $(\Pi B)\cdots(\Pi B)$ h times. A mild adaptation of results in Magnus and Neudecker (1988, Ch. 9, Table 7) allows us to get

$$\partial[B\Pi B\cdots\Pi B] = \sum_{j=0}^{h}(B\Pi)^j \partial B(\Pi B)^{h-j}.$$

As a result, (C.1) can be rewritten as

$$\partial F(B) = \sum_{h=0}^{H}\gamma_h \sum_{j=0}^{h}(B\Pi)^j \partial B(\Pi B)^{h-j}1$$

$$\Rightarrow \text{vec}\partial F(B) = \partial\text{vec}F(B) = \sum_{h=0}^{H}\gamma_h \sum_{j=0}^{h}\text{vec}[(B\Pi)^j\partial B(\Pi B)^{h-j}1].$$

Now it follows from Theorem 2.2 in Magnus and Neudecker (1988) that

$$\partial\text{vec}F(B) = \sum_{h=0}^{H}\gamma_h\sum_{j=0}^{h}[((\Pi B)^{h-j}1)' \otimes (B\Pi)^j]\partial\text{vec}B,$$

where the prime represents the matrix transpose and $\otimes$ stands for the Kronecker product. Let us take now $B = \Lambda_i = \lambda_i \mathrm{I}_k$, where λ_i is some root of the characteristic equation associated with (30). Then, we obtain

$$\partial\text{vec}F(\Lambda_i) = \sum_{h=0}^{H}\gamma_h\lambda_i^h\sum_{j=0}^{h}[(\mathbf{1}'(\Pi')^{h-j})\otimes\Pi^j]\partial\text{vec}\Lambda_i.$$

Observe that the inner product between $\mathbf{1}'$ and the sth column of $(\Pi')^{h-j}$ is merely equal to the sum over all the components of the sth row of Π^{h-j}, which is equal to 1 by definition of a Markov matrix, i.e., $\mathbf{1}'(\Pi')^{h-j} = \mathbf{1}'$. Hence,

$$\partial\text{vec}F(\Lambda_i) = \sum_{h=0}^{H}\gamma_h\lambda_i^h\sum_{j=0}^{h}\overbrace{(\Pi^j\cdots\Pi^j)}^{k}\partial\text{vec}\Lambda_i, \tag{C.2}$$

where $(\Pi^j\cdots\Pi^j)$ is the $k\times k^2$ matrix whose ss'th element $(s,s'=1,\ldots,k)$; that is, the ss'th element of the $k\times k$ matrix Π^j, is the same as its $s(s'+k \bmod n)$th element $(n=1,\ldots,k)$. Given the way the Jacobian matrix of F at point Λ_i is defined, it directly follows from (C.2) that

$$\mathrm{D}F(\Lambda) = \sum_{h=0}^{H} \gamma_h \lambda^h \sum_{j=0}^{h} \overbrace{(\Pi^j \cdots \Pi^j)}^{k}.$$

We now calculate the remaining Jacobian matrix $\mathrm{D}B(\beta)$. Observe that

$$\partial \mathrm{vec} B(\beta) = \begin{pmatrix} \partial\beta_1 \\ 0 \\ \vdots \\ 0 \\ \partial\beta_2 \\ 0 \\ \vdots \\ \partial\beta_k \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & \vdots & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial\beta_1 \\ \partial\beta_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \partial\beta_k \end{pmatrix} \equiv \Phi \partial \mathrm{vec}\beta.$$

By definition, it follows that $\mathrm{D}B(\beta) = \Phi$, where Φ is a $k^2 \times k$ matrix. Recall now that the chain rule applies and notice that the ss'th element of the $k \times k$ matrix $(\Pi^j \cdots \Pi^j)\Phi$ is the ss'th element of the $k \times k$ matrix Π^j. Therefore, for $\beta = (\lambda_i, \ldots, \lambda_i)'$, we have

$$\mathrm{D}F(B(\beta)) = \sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} (\Pi^j \Pi^j)\Phi = \sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} \Pi^j.$$

and Lemma C.1 follows. ■

LEMMA C.2. *Let π_s $(s = 1, \ldots, k)$ be an eigenvalue of the Markov matrix Π. Let $\mathbf{l}_i$ be the $k \times 1$ vector all of whose components are equal to a given root λ_i $(i = 1, \ldots, H+1)$. Assume that $\lambda_i \neq 0$. Then, the $k \times k$ Jacobian matrix $\mathrm{D}F(\Lambda(\bullet))$ is singular at point $\mathbf{l}_i$ if and only if there exists an eigenvalue π_s of Π such that*

$$\prod_{\substack{j=1 \\ j \neq i}}^{H+1} \left(\pi_s - \frac{\lambda_j}{\lambda_i} \right) = 0.$$

Proof. Let ϕ_s $(s = 1, \ldots, k)$ be an eigenvalue of $\mathrm{D}F(\Lambda(\bullet))$ at point $\mathbf{l}_i$ and recall that the matrix $\mathrm{D}F(\Lambda(\bullet))$ is singular at point $\mathbf{l}_i$ if and only if its determinant $\det \mathrm{D}F(\Lambda(\bullet))$ equals 0 at this point. Now, let $\boldsymbol{\pi}$ be an eigenvector of Π associated with the eigenvalue π. Observe that

$$\mathrm{D}F(\Lambda(\mathbf{l}_i))\boldsymbol{\pi} = \sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} \Pi^j \boldsymbol{\pi} = \left(\sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} \pi^j \right) \boldsymbol{\pi},$$

which shows that the eigenvalues of $\mathrm{D}F(\Lambda(\bullet))$ at point $\mathbf{l}_i$ are of the following form:

$$\phi = \sum_{h=0}^{H} \gamma_h \lambda^h \sum_{j=0}^{h} \pi^j.$$

Of course, $\det \mathrm{D}F(\Lambda(\bullet))$ equals 0 at point $\mathbf{l}_i$ if and only if one of the eigenvalues ϕ of $\mathrm{D}F(\Lambda(\mathbf{l}_i))$ is equal to 0, namely,

$$\sum_{h=0}^{H} \gamma_h \lambda_i^h \sum_{j=0}^{h} \pi^j = 0 \Leftrightarrow \sum_{h=0}^{H} \pi^h \sum_{j=h}^{H} \gamma_h \lambda_i^h = 0. \tag{C.3}$$

Now, let us divide the right member of (C.3) by $\gamma_H \lambda_i^H$ (which is assumed to differ from 0). We get

$$\sum_{h=0}^{H} \pi^h \sum_{j=h}^{H} \frac{\gamma_h \lambda_i^h}{\gamma_H \lambda_i^H} = 0 \Leftrightarrow \prod_{\substack{j=1 \\ j \neq i}}^{H+1} \left(\pi - \frac{\lambda_j}{\lambda_i}\right) = 0,$$

where the last equality comes from Proposition 7 [see equation (A.2) in particular], which proves Lemma C.2. ■

Given that the real part of any eigenvalue of a Markov matrix has modulus less than 1, i.e., $|\pi_s| < 1$, and given that the roots λ_i are labeled in the order of increasing modulus, the ratio $|\lambda_j/\lambda_1|$ is strictly greater than 1 for $j \neq 1$ $(j = 1, \ldots, H+1)$. Proposition 8 follows from the implicit function theorem since $\lambda_1 \neq 0$ under the assumption that $\delta_1 \neq 0$, and standard local bifurcation theory [see Chiappori et al. (1992) for a general argument]. ■

APPENDIX D. PROOF OF PROPOSITION 9

The dynamics with learning (41) arbitrarily close to the root λ_i $(i = 1, \ldots, H+1)$ are obtained by linearizing (41) at point $\beta_{t+1} = \beta_t = \lambda_i$. These dynamics are governed by the ratio

$$\frac{d\beta_{t+1}}{d\beta_t} = \delta_1 \frac{\sum_{h=1}^{H} h \gamma_h \lambda_i^{h-1}}{\left(1 + \sum_{h=1}^{H} \gamma_h \lambda_i^h\right)^2}.$$

The root λ_i is locally stable under learning if and only if this ratio is less than 1 in modulus. This is the case when λ_i is close enough to 0 because then, $|d\beta_{t+1}/d\beta_t|$ also is close to 0. ■