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Determinacy in linear rational expectations models

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Abstract

The purpose of this paper is to assess the relevance of rational expectations solutions to the class of linear univariate models where both the number of leads in expectations and the number of lags in predetermined variables are arbitrary. It recommends to rule out all the solutions that would fail to be locally unique, or equivalently, locally determinate. So far, this determinacy criterion has been applied to particular solutions, in general some steady state or periodic cycle. However solutions to linear models with rational expectations typically do not conform to such simple dynamic patterns but express instead the current state of the economic system as a linear difference equation of lagged states. The innovation of this paper is to apply the determinacy criterion to the sets of coefficients of these linear difference equations. Its main result shows that only one set of such coefficients, or the corresponding solution, is locally determinate. This solution is commonly referred to as the fundamental one in the literature. In particular, in the saddle point configuration, it coincides with the saddle stable (pure forward) equilibrium trajectory.

JEL classification: C32; E32

Keywords: Rational expectations; Selection; Determinacy; Saddle point property

1. Introduction

The rational expectations hypothesis is commonly justified by the fact that individual forecasts are based on the relevant theory of the economic system. According to this viewpoint, the actual evolution of the economy coincides with the expected one provided that agents refer precisely to this actual law when they form their forecasts. Such an argument is appealing as long as there exists a well defined reference; namely, a unique rational expectations outcome. Indeed, in this case, if one a priori accepts the rational expectations

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hypothesis, then one may argue that this unique outcome is the only possible focal point for the process through which agents try to coordinate their beliefs. On the contrary, in the remaining case where there are several competing rational expectations solutions, it is more likely that agents do not succeed to refer to the same theory of the functioning of the economy, at least in the absence of any other selection device.

Unfortunately, it is by now well known that intertemporal models with rational expectations typically admit infinitely many equilibrium trajectories (see, e.g., Blanchard, 1979 for an early reference). This should accordingly prevent agents to make determinate predictions, and as underlined by Kehoe and Levine (1985) for instance, this should even call into question the very concept of rational expectations. However, following Guesnerie (1993), one may wonder whether some solutions to these models can be still locally unique, or locally determinate in the terminology advocated by Woodford (1984). Such solutions would then provide a locally undisputable theory to economic agents. Indeed, a locally unique solution is an obvious anchor for any expectations coordination process that a priori accepts, as before, the rational expectations hypothesis, and under the additional requirement that agents a priori restrict their attention to some arbitrary immediate neighborhood of this solution.

So far, the determinacy property has been successfully applied to very special equilibrium trajectories, such as steady states or periodic cycles (for a recent survey on this topic, see, e.g., Benhabib and Farmer, 1999). Nevertheless, equilibrium laws of motion do not conform in general to so simple dynamic patterns. In linear models, for instance, these laws fit instead autoregressive processes which express the current state of the economic system as a linear difference equation in past states. The purpose of this paper is to describe how one can apply the determinacy criterion to such trajectories.

Our general methodology bears on characterizing any of these solutions by the vector of the coefficients of the difference equation associated with it, and not, as is usually the case in the main strand of the literature, by the infinite sequence of successive states that is generated by it. In other words, any possible relevant economic theory will be defined by these vectors of coefficients, to be called *steady extended growth rates*, thus implying that if agents succeed to refer to one of these steady extended growth rates when they form their forecasts, then the corresponding solution will govern the actual evolution the economic system. With this interpretation, it seems rather natural to study whether some these vectors of coefficients can be locally determinate. It may be important to emphasize here that the determinacy criterion will be, therefore, no longer applied to the levels of the state variable itself, in sharp contrast with what is usually done in the literature. That is, we shall say that a given steady extended growth rate, or the corresponding solution, is locally determinate if and only if there is no other solution associated with a sequence of extended growth rates remaining arbitrarily close to it in each period. Otherwise this solution is locally indeterminate.

In the sequel, we shall be concerned with the general class of linear univariate models where both the number of leads in expectations and the number of predetermined variables are arbitrary. In these models, one can distinguish the set of bubble solutions, making the actual equilibrium trajectory driven in part by arbitrary forecasts of agents, from the set of minimal order solutions, along which forecasts are only determined from economic fundamentals (see, e.g., McCallum, 1999, Section 4 for this terminology). The main result of this paper is to show that, independently of the stability properties of equilibrium trajectories, only one solution of minimal order, or the corresponding vector of coefficients, is locally determinate, which extents previous results obtained by Gauthier (2002) in the special case of one-step forward looking models. This solution is identified as the *fundamental solution* in the literature (see, again, McCallum, 1999). In particular, in the saddle point configuration for the usual dynamics with perfect foresight on the level of the state variable, it coincides with the stable saddle path trajectory, the so-called pure forward solution that supports policy neutrality results highlighted in the early rational expectations literature, e.g., in Sargent and Wallace (1973) or Blanchard (1979).

The paper will be organized as follows. Section 2 presents the class of models under consideration and its rational expectations solutions. Section 3 describes how to apply determinacy to solutions of minimal order and shows that a unique equilibrium trajectory fits this requirement. Finally Section 4 concludes.

2. General framework

We shall consider the class of linear univariate models with $H \geq 1$ leads in expectations and $L \geq 1$ predetermined lagged variables in each period. By definition the current state of the economic system expresses as a polynomial of the L previous states along any minimal order solution (hereafter MO-solution) to such models. The purpose of this section is to characterize the vector of the L coefficients of these polynomials by appealing to the standard method of undetermined coefficients, which involves deriving the expression of a finite number of parameters in an interactive setting where there is a feedback from some a priori guess on the general form of the solution onto the actual law of the system. In a rational expectations solution, the a priori guess must coincide with the actual law of the system. Therefore, according to this method, an MO-solution can be thought of as a situation where agents would succeed to guess the vector of coefficients corresponding to it (see, e.g., Grandmont and Laroque, 1991 for such an interpretation). Whether they are likely to discover such coefficients is postponed to the next section.

Let the period t ($t \geq 0$) state of the economic system be a real number x_t determined through the following expectational recursive equation

$$\sum_{h=1}^{H} \gamma_h x^{e}_{t+h} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0, \tag{1}$$

where x^{e}_{t+h} ($h = 1, \ldots, H$) stands for the forecast about the period $(t + h)$ state, and x_{t-l} ($l = 1, \ldots, L$) is given at date t. Under the perfect foresight hypothesis, the forecast x^{e}_{t+h} is equal to the actual realization x_{t+h} whatever t and h are, so that (1) rewrites

$$\sum_{h=1}^{H} \gamma_h x_{t+h} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0, \tag{2}$$

for $t \geq 0$. A perfect foresight solution to (2) is a sequence (x_t) associated with a given initial condition $(x_{-1}, \ldots, x_{-L})$ and satisfying (2) in any period. Its intertemporal behavior is consequently governed by the $(H + L)$ perfect foresight roots λ_i $(i = 1, \dots, H + L)$ of the characteristic polynomial associated with (2). In the sequel, we shall assume that these roots have distinct moduli, except in the case where they are complex conjugate. We shall rank them in the order of increasing modulus, i.e., $|\lambda_i| \le |\lambda_j|$ whenever $i < j$ $(i, j = 1, \dots, H + L)$, with strict inequality if λ_i or λ_j is real valued.

Let finally $\gamma_H \neq 0$ in (1), so that the model (1) admits multiple perfect foresight solutions (Gouriéroux et al., 1982). In this paper, we shall focus attention on the class of MO-solutions to (2), along which the current state x_t is by definition related to the $L \times 1$ vector $\boldsymbol{x}_{t-1}$ of the L previous states $(x_{t-1}, \dots, x_{t-L})'$ through the relation

$$x_t = \bar{\boldsymbol{\beta}}'\boldsymbol{x}_{t-1}, \tag{3}$$

where the coefficients of the $L \times 1$ vector $\bar{\boldsymbol{\beta}} \equiv (\bar{\beta}_1, \dots, \bar{\beta}_L)'$ will be determined by using the method of the undetermined coefficients (Muth, 1961).[1] This method amounts to assume that agents expect the law of motion of the state variable to be consistent with equilibrium, i.e., with (3), and then to derive the conditions under which this belief is actually self-fulfilling. Let agents accordingly believe that $x_t = \boldsymbol{\beta}'\boldsymbol{x}_{t-1}$ for some guess $\boldsymbol{\beta} \equiv (\beta_1, \dots, \beta_L)'$ on the $L \times 1$ vector of coefficients, or, equivalently, that

$$x_t = \boldsymbol{e}_1'\mathbf{B}\boldsymbol{x}_{t-1}, \quad \text{with } \mathbf{B} = \begin{pmatrix} \boldsymbol{\beta}' & \\ \boldsymbol{I}_{L-1} & \boldsymbol{o}_{L-1} \end{pmatrix}, \tag{4}$$

whatever t and $\boldsymbol{x}_{t-1}$ are, where $\boldsymbol{e}_1$ is the first $L \times 1$ vector of the canonical basis, $\mathbf{B}$ is the $L \times L$ companion matrix associated with (3) for the guess $\boldsymbol{\beta}$, and $\boldsymbol{o}_{L-1}$ is the $L \times 1$ null vector. Given this belief, agents form their forecasts $x^{e}_{t+h} = \boldsymbol{e}_1'\mathbf{B}^{h+1}\boldsymbol{x}_{t-1}$ by leading (4) forward. Reintroducing these forecasts into (1) generates an actual law of motion for the state variable, which corresponds to the belief (4),

$$x_t = -\Big(\sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1'\mathbf{B}^{h+1} + \boldsymbol{\delta}'\Big)\boldsymbol{x}_{t-1}, \tag{5}$$

where $\boldsymbol{\delta}$ represents the $L \times 1$ vector $(\delta_1, \dots, \delta_L)'$. The belief (4) is then self-fulfilling whenever it coincides with (5) whatever t and $\boldsymbol{x}_{t-1}$ are, i.e.,

$$\boldsymbol{e}_1'\mathbf{B} = -\sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1'\mathbf{B}^{h+1} - \boldsymbol{\delta}'. \tag{6}$$

Each matrix $\bar{\mathbf{B}}$ solution to (6) is characterized by a vector $\bar{\boldsymbol{\beta}}$ through (4). All the components of $\bar{\boldsymbol{\beta}}$ must be real for the MO-solution (3) to exist. In this case, $\bar{\boldsymbol{\beta}}$ will be called *steady extended growth rate of order L* (hereafter steady EGR(L)). It is clear that, given the vector

[1] The remaining perfect foresight solutions to (2) make the current state linked with $M > L$ (and $M \le H$) lagged state variables, i.e., $\bar{\boldsymbol{\beta}}$ is a M-dimensional vector in (3). Since the economic system has only L initial conditions (given by the L values of the predetermined variables in the initial period), the initial state of the system x_0 is determined by these L initial conditions and also by some *arbitrary* initial forecasts of agents $(x_1, \dots, x_{M-L})$. As a result, in general, there is no reason to focus attention on one among these bubble solutions (see for instance McCallum, 1999 for further developments).

$\boldsymbol{x}_{-1}$ of the L initial conditions of the system, the knowledge of a steady EGR(L) is sufficient to characterize an MO-solution through (3). The purpose of this part is to relate such a vector of steady EGR(L) to economic fundamentals, summarized here by the $(H+L)$ perfect foresight roots λ_i $(i = 1, \dots, H+L)$. An intuition for this connection proceeds as follows. Observe that the law (3) restricts the dynamics with perfect foresight (2) to one of its L-dimensional eigensubspaces. Each of these subspaces is spanned by L eigenvectors associated with L different perfect foresight roots among $(H + L)$. Therefore, in a given L-dimensional eigensubspace of (2), or along the corresponding MO-solution (3), the evolution of the state variable only depends on the L perfect foresight roots that solve the characteristic equation associated with (3),

$$P(\lambda) \equiv \lambda^L - \sum_{m=1}^{L} \bar{\beta}_m \lambda^{L-m} = 0. \tag{7}$$

This observation enables us to link the mth $(m = 1, \dots, L)$ component $\bar{\beta}_m$ of $\bar{\boldsymbol{\beta}}$ to the L perfect foresight roots solving (7). Consider for instance the MO-solution associated with the L perfect foresight roots of lowest modulus $(\lambda_1, \dots, \lambda_L)$, that is, the MO-solution defined by (3) for a vector $\bar{\boldsymbol{\beta}}$ such that the roots of $P(\cdot)$ in (7) are $(\lambda_1, \dots, \lambda_L)$. In this case $P(\lambda) = 0$ in (7) is equivalent to

$$\prod_{i=1}^{L}(\lambda - \lambda_i) = 0 \Leftrightarrow \lambda^L - \left(\sum_{i=1}^{L} \lambda_i\right) \lambda^{L-1} + \dots + (-1)^L \left(\prod_{i=1}^{L} \lambda_i\right) = 0. \tag{8}$$

If one denotes $\sigma_m(\lambda_1, \dots, \lambda_L)$ the mth symmetric polynomial, i.e., the sum over all the different products of m distinct elements in the set $(\lambda_1, \dots, \lambda_L)$, then it follows from (7) and (8) that $\bar{\beta}_m$ is equal to $(-1)^{m+1}\sigma_m(\lambda_1, \dots, \lambda_L)$. Of course the same argument would apply as well to any other MO-solution. One can therefore state the following result.

Lemma 1. *Let both future forecasts and past history matter, i.e., $\gamma_H \neq 0$and $\delta_L \neq 0$ for $H, L \geq 1$ in (1). Let also $\sigma_m(£)$ represent the mth elementary symmetric polynomial of any given set £ of L different perfect foresight roots among $(H + L)$. Then, the law of motion of the state variable along an MO-solution to (2) is described by the L dimensional linear difference equation (3) if and only if the m th $(m = 1, \dots, L)$ component $\bar{\beta}_m$ of $\bar{\boldsymbol{\beta}}$ in (3) is equal to $(-1)^{m+1}\sigma_m(£)$, for any given subset £.*

Proof. See in Section 5.1. □

One must also impose the additional condition that $\delta_l \neq 0$ for some $l \geq 1$ in (1) in order to ensure existence of multiple MO-solutions. Indeed, if no predetermined variable enters the model ($\delta_l = 0$ for any $l \geq 1$), then by definition the level of the state variable must remain constant through time along an MO-solution. This determines, under a simple regularity assumption $(\gamma_H + \dots + \gamma_1 \neq -1)$, the steady state sequence $(x_t = \bar{x} \equiv 0)$ as the unique MO-solution to (2).

3. Determinacy of minimal order solutions

Even in the case where agents expect the state variable to evolve according to (4), their belief is self-fulfilling if only if they use in (4) one of the steady EGR(L) defined in Lemma 1. In general, unfortunately, there is no central mechanism imposing the use of a particular vector of such coefficients. One may consequently wonder whether some of these vectors could be more likely outcomes of decentralized processes through which agents would try to coordinate their beliefs on MO-solutions to (2). According to the local determinacy viewpoint, the fact that a steady EGR(L) fails to be locally unique, or locally determinate, is the main obstacle for agents to discover it. In order to assert the local determinacy properties of a steady EGR(L), it must noticed that the usual dynamics with perfect foresight on the levels of the state variable (2) triggers a new dynamics with perfect foresight on the vectors of coefficients $\boldsymbol{\beta}_t \equiv (\beta_1(t), \ldots, \beta_L(t))'$ whose fixed points are the steady EGR(L). In this new dynamics, a steady EGR(L) is locally determinate when there are no vector $\boldsymbol{\beta}_t$ remaining arbitrarily close to it in any period t, and is locally indeterminate otherwise.

This new dynamics with perfect foresight on $L \times 1$ vectors $\boldsymbol{\beta}_t$ is derived from (2) by imposing that the relation $x_t = \boldsymbol{\beta}_t' \boldsymbol{x}_{t-1}$, or equivalently,

$$x_t = \boldsymbol{e}_1' \mathbf{B}_t \boldsymbol{x}_{t-1}, \quad \text{with } \mathbf{B}_t = \begin{pmatrix} \boldsymbol{\beta}_t' & \\ \boldsymbol{I}_{L-1} & \boldsymbol{o}_{L-1} \end{pmatrix}, \tag{9}$$

be satisfied in (2) whatever t and $\boldsymbol{x}_{t-1}$ are. Iterating (9) forward makes x_{t+h} equal to $\boldsymbol{e}_1'(\mathbf{B}_{t+h} \cdots \mathbf{B}_t)\boldsymbol{x}_{t-1}$ in (2), so that the current state x_t in (2) actually writes

$$x_t = -\left(\sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' (\mathbf{B}_{t+h} \cdots \mathbf{B}_t) + \boldsymbol{\delta}' \right) \boldsymbol{x}_{t-1}. \tag{10}$$

For x_t to verify both (9) and (10) whatever t and $\boldsymbol{x}_{t-1}$ are, it must be the case that

$$\boldsymbol{e}_1' \mathbf{B}_t \boldsymbol{=} -\sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' (\mathbf{B}_{t+h} \cdots \mathbf{B}_t) - \boldsymbol{\delta}' \tag{11}$$

whatever $t \geq 0$ is. We shall define the *extended growth rate perfect foresight dynamics* as a sequence of $L \times 1$ vectors $(\boldsymbol{\beta}_t)$ associated, through (9), with a sequence of $L \times L$ matrices $(\mathbf{B}_t)$ such that (11) holds true in any period t ($t \geq 0$). It is clear that the fixed points of this dynamics are the vectors of steady EGR(L) defined in Lemma 1, or the corresponding matrices $\bar{\mathbf{B}}$ solutions to (6).

Our aim is to study the properties of (11) arbitrarily close to its fixed points, namely such that the mm'th ($m, m' = 1, \ldots, L$) entry of $\mathbf{B}_t$ in (11) stands arbitrarily close to the mm'th entry of $\bar{\mathbf{B}}$ in each period $t \geq 0$, or equivalently such that the mth ($m = 1, \ldots, L$) component $\beta_m(t)$ of $\boldsymbol{\beta}_t$ stands arbitrarily close to the mth component $\bar{\beta}_m$ of $\bar{\boldsymbol{\beta}}$ in each period. This dynamics is well defined around any matrix $\bar{\mathbf{B}}$ solution to (6) if and only if $\bar{\mathbf{B}}$ is regular (see in Section 5.2, which is satisfied whatever $\bar{\mathbf{B}}$ is if and only if all the perfect foresight roots λ_i ($i = 1, \ldots, H + L$) differ from 0, or equivalently $\delta_L \neq 0$ in (2). Under this requirement, (11) can be approximated around $\bar{\boldsymbol{\beta}}$ by a linear first order recursive equation linking the $LH \times 1$ vector $\boldsymbol{\beta}^{t+1} \equiv ((\boldsymbol{\beta}_{t+H} - \bar{\boldsymbol{\beta}})', \ldots, (\boldsymbol{\beta}_{t+1} - \bar{\boldsymbol{\beta}})')'$ to the $LH \times 1$ vector

$\boldsymbol{\beta}^t \equiv ((\boldsymbol{\beta}_{t+H-1}-\bar{\boldsymbol{\beta}})', \dots, (\boldsymbol{\beta}_t-\bar{\boldsymbol{\beta}})')'$ through a $LH\times LH$ Jacobian matrix **J**, i.e., $\boldsymbol{\beta}^{t+1} = \boldsymbol{J}\boldsymbol{\beta}^t$. By definition, a steady EGR(L) is said to be locally determinate in the dynamics (11) if and only if all the LH eigenvalues of the Jacobian matrix **J** have modulus greater than 1 (see, e.g., Chiappori et al., 1992). That is, if at least one eigenvalue of **J** lies inside the unit circle, then there are infinitely many solutions to (2) for which $\boldsymbol{\beta}_t$ remains arbitrarily close to $\bar{\boldsymbol{\beta}}$ in (11) whatever t is. The following result establishes that only one steady EGR(L) is locally determinate in (11).

Proposition 1. *Let both future forecasts and past history matter, i.e., $\gamma_H \neq 0$ and $\delta_L \neq 0$ for $H, L \geq 1$ in (1). Assume that the MO-solution corresponding to $(\lambda_1, \dots, \lambda_L)$ exists, i.e., $\sigma_m(\lambda_1, \dots, \lambda_L)$ is real valued whatever m is $(m = 1, \dots, L)$. Then, this MO-solution, which governs the dynamics with perfect foresight on the level of state variable (2) restricted to the L-dimensional eigensubspace corresponding to the L perfect foresight roots of lowest modulus $(\lambda_1, \dots, \lambda_L)$, is the only one to be locally determinate in the perfect foresight dynamics (11). If this MO-solution does not exist, i.e., $\sigma_m(\lambda_1, \dots, \lambda_L)$ is complex valued for some m is $(m = 1, \dots, L)$, then no MO-solution is locally determinate in the perfect foresight dynamics (11).*

Proof. See in Section 5.2. □

In the saddle point configuration for the dynamics (2), where $|\lambda_L| < 1 < |\lambda_{L+1}|$, the only solution to be locally determinate in (11) is also the only one along which the level of the state variable does not explode toward infinity, the so-called saddle path trajectory (Blanchard and Kahn, 1980). However, it worth emphasizing that Proposition 1 is independent of the stability properties induced by the $(H + L)$ perfect foresight roots, and thus it applies as well in the case where there are multiple stable equilibrium trajectories ($|\lambda_{L+1}| < 1$) to (2). In other words, the MO-solution corresponding to $(\lambda_1, \dots, \lambda_L)$ is still locally determinate in the dynamics (11) when the steady state sequence ($x_t = \bar{x} \equiv 0$) is locally indeterminate in the dynamics (2).

An intuitive explanation for this lack of links between the familiar concept of determinacy of the steady state and the novel concept of determinacy of MO-solutions rests on the observation that the level of the state variable is not relevant in (11), which obviously reduces the likelihood that stability properties of (2) and (11) be related to each other. It follows that, in order to reconcile both concepts, one should derive a dynamics with perfect foresight taking into account the determinacy of both EGR(L) and the level of the state variable. This can be done by restricting the level of the state variable to satisfy not only (2) in each period, but also the new relation

$$x_t = \boldsymbol{\beta}_t'\boldsymbol{x}_{t-1} + \alpha_t \tag{12}$$

which then replaces (9). In (12), the parameter α_t is a real number that stands for the level of the state variable at date t, and unlike (9), it may not equal $\bar{\alpha} \equiv 0$ in each period. The restriction (12) induces a new dynamics with perfect foresight on $(L + 1) \times 1$ vectors $(\boldsymbol{\beta}_t', \alpha_t)'$ that replaces (11), the fixed points of which are of the form $(\bar{\boldsymbol{\beta}}', \bar{\alpha})'$, where $\bar{\boldsymbol{\beta}}$ is a steady EGR(L) and $\bar{\alpha} = \bar{x} \equiv 0$. As Lemma 2 highlights, the stability properties of (2) play then a crucial role in this new dynamics.

Lemma 2. *Let both future forecasts and past history matter, i.e., $\gamma_H \neq 0$ and $\delta_L \neq 0$ for $H, L \geq 1$ in (1). Let the set of the $(H + L)$ perfect foresight roots be split into any given subset £ of L different perfect foresight roots and a complement subset $£_c$ of the H remaining roots. Assume additionally that all the perfect foresight roots differ from 1. Consider some* MO*-solution $(\bar{\boldsymbol{\beta}}, \bar{\alpha})$ associated with £; that is, the solution which governs the dynamics with perfect foresight on the level of state variable (2) restricted to the L-dimensional eigensubspace corresponding to the L perfect foresight roots in £. This solution is locally determinate if and only if $\bar{\boldsymbol{\beta}}$ is locally determinate in (11) and the H roots in the subset $£_c$ have modulus greater than 1.*

Proof. See in Section 5.3. □

An immediate corollary to Proposition 1 and Lemma 2 is that the MO-solution corresponding to $(\lambda_1, \ldots, \lambda_L)$ is locally determinate in the new dynamics induced by (12) if and only if $|\lambda_{L+1}| > 1$, or equivalently, if and only if the steady state of (2) is locally determinate; otherwise, there is no locally determinate MO-solution. A possible interpretation of these results goes as follows. If, on the one hand, all the agents a priori refer to the steady state $(x_t = \bar{x} \equiv 0)$ when they form their forecasts, and accordingly regard the level of the state variable in (2) as an actual deviation from its steady state value, then choosing a solution is equivalent to choosing the steady EGR(L) corresponding to it. In this case, the determinacy of this steady EGR(L) is the only relevant property, and Proposition 1 applies. If, on the other hand, there is no a priori agreement among agents to view the steady state as a benchmark, as in (12), then choosing a solution involves focusing on both the corresponding steady EGR(L) and the steady state level of the state variable. In this case, Lemma 2 should be applied.

4. Concluding comments

It has been shown that only one solution is locally unique in the set of minimal order solutions to the general class of linear univariate rational expectations models. One may argue that if agents have only to delineate a rational expectations solution, then they should focus on this particular solution, which coincides with the saddle path trajectory in the saddle point configuration. In addition to extensions to more general economic frameworks, e.g., nonlinear multidimensional stochastic models (see Evans and Guesnerie, 2000 for recent insights on this topic), one can suggest two different directions for future research.

(i) First, it would be interesting to study whether some bubble solutions can be locally determinate. As stressed in footnote 1, these solutions are not only characterized by a vector of steady extended growth rates, but also by initial agents' forecasts. This implies that determinacy in terms of extended growth rates give an account for the determinacy of a class of solutions, and not of a single solution, as is the case for minimal order solutions. Thus the method developed in this paper can not be directly applied to such a type of solutions.

(ii) Second, Proposition 1 and Lemma 2 may justify to focus on one solution provided that agents are already aware of the full set of possible solutions. This requirement could be relaxed for analyzing whether agents may eventually learn some solutions. It is known that there are connections between determinacy and stability under learning, but these concepts are not equivalent in general (see Chapters 8 and 9 in Evans and Honkapohja, 2001 for a recent synthesis); hence the MO-solution corresponding to $(\lambda_1, \dots, \lambda_L)$ should not necessarily be stable under learning. On the other hand, the close link between the lack of local stationary sunspot equilibria and the property of local determinacy (Chiappori et al., 1992) suggests that the MO-solution corresponding to $(\lambda_1, \dots, \lambda_L)$ may be also locally immune to sunspots (see Desgranges and Gauthier, 2003 or Gauthier, 2003 for preliminary analysis).

5. Proofs of the results

5.1. Proof of Lemma 1

Consider a $L \times L$ matrix $\bar{\mathbf{B}}$ solution to (6). Let the $L \times 1$ vector $\bar{\boldsymbol{\beta}}$ corresponding to $\bar{\mathbf{B}}$ through (4) be defined as in Lemma 1, i.e., the mth component $\bar{\beta}_m$ of $\bar{\boldsymbol{\beta}}$ is equal to $(-1)^{m+1}\sigma_m(£)$, so that the L eigenvalues of $\bar{\mathbf{B}}$ are the L perfect foresight roots of the set £. Let us first prove that if the mth component $\bar{\beta}_m$ of $\bar{\beta}$ is equal to $(-1)^{m+1}\sigma_m(£)$, then $\bar{\beta}$ solves (6). Consider the set £ of the L perfect foresight roots of lowest modulus $(\lambda_1, \dots, \lambda_L)$; the proof would apply as well to any other set of L distinct perfect foresight roots. Observe that, since the perfect foresight roots are assumed to be distinct, $\bar{\mathbf{B}}$ is diagonalizable as $\bar{\mathbf{B}} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$, where the $L \times L$ matrix $\boldsymbol{\Lambda}$ is diagonal (with $\text{diag}(\boldsymbol{\Lambda}) = (\lambda_1, \dots, \lambda_L)$), and where the $L \times L$ matrix $\mathbf{P}$ is the (non-singular) Vandermonde matrix of eigenvectors of $\bar{\mathbf{B}}$. With the convention that $\gamma_0 = 1$, one can now rewrite (6) as

$$\sum_{h=0}^{H} \gamma_h \boldsymbol{e}'_1 \mathbf{P}\boldsymbol{\Lambda}^{h+1} - \boldsymbol{\delta}'\mathbf{P} = \boldsymbol{o}'_L. \tag{13}$$

The ijth entry $(i, j = 1, \dots, L)$ of $\mathbf{P}$ is equal to $(1/\lambda_i)^{j-1}$ so that $\boldsymbol{e}'_1\mathbf{P}$ in (13) is the $1 \times L$ unit vector. Hence, the mth component of the $1 \times L$ vector $\boldsymbol{e}'_1\mathbf{P}\boldsymbol{\Lambda}^{h+1}$ is equal to $(\lambda_m)^{h+1}$. Since the mth component of the $1 \times L$ vector $\boldsymbol{\delta}'\mathbf{P}$ is $(\delta_1 + (1/\lambda_m)\delta_2 + \cdots + (1/\lambda_m)^{L-1}\delta_L)$, the left hand side of (13) is the $1 \times L$ vector whose mth component is

$$\sum_{h=0}^{H} \gamma_h (\lambda_m)^{h+1} + \sum_{l=0}^{L} \delta_l \left(\frac{1}{\lambda_m}\right)^{l-1}. \tag{14}$$

By multiplying (14) by $(\lambda_m)^{1-L}$ (which differs from zero when $\delta_L \neq 0$, since the product of all the perfect foresight roots is equal to $(-1)^{H+L}(\delta_L/\gamma_H)$), one gets the expression of the characteristic polynomial associated with (2) calculated at point λ_m. By definition, λ_m is a root of this polynomial. Hence, the expression in (14) is equal to zero whatever m is. This completes the first part of the proof.

In order to prove that components of $\bar{\boldsymbol{\beta}}$ are necessarily of the form given in Lemma 1, note that the general solution to (2) writes

$$x_t = \sum_{i=1}^{H+L} \alpha_i (\lambda_i)^t, \tag{15}$$

for some weights α_i ($i = 1, \ldots, H + L$). Since (3) is a solution to (2), the current state x_t in (3) satisfies (15). But (3) is a linear difference equation of order L only, so that its solutions are necessarily of the form

$$x_t = \sum_{\lambda_i \in £} \alpha_i (\lambda_i)^t, \tag{16}$$

where £ is any given subset of L perfect foresight roots among $(H + L)$. Hence, the roots of the characteristic polynomial associated with (3) are necessarily the L roots in £, which concludes the proof.

5.2. *Proof of Proposition 1*

We proceed in three steps. First we derive the dynamics with perfect foresight (11) close to its fixed points $\bar{\mathbf{B}}$. The resulting dynamics is given in Lemma 3. It depends on γ_h ($h = 1, \ldots, H$), δ_l ($l = 1, \ldots, L$), and on the $L \times L$ matrix $\bar{\mathbf{B}}$ around which (11) has been linearized. Lemma 4 relates this dynamics to the $(H + L)$ perfect foresight roots λ_i ($i = 1, \ldots, H + L$). Finally Lemma 5 expresses the HL eigenvalues that govern this dynamics in terms of the $(H + L)$ perfect foresight roots only. Determinacy properties of a steady EGR(L) is obtained whenever these HL eigenvalues have moduli greater than 1.

Lemma 3. *The dynamics with perfect foresight of extended growth rates (11) in the immediate vicinity of a steady* EGR(L), *i.e., when the $L \times L$ matrix $\mathbf{B}_t$ stands in the immediate vicinity of some $L \times L$ matrix $\bar{\mathbf{B}}$ defined in Lemma 1, expresses as*

$$\sum_{h=0}^{H} \sum_{j=h}^{H} \frac{\gamma_j}{\gamma_H} [\boldsymbol{e}_1' (\bar{\mathbf{B}}')^{j-h} \boldsymbol{e}_1] (\bar{\mathbf{B}}')^{h-H} (\partial \boldsymbol{\beta}_{t+h}) = \boldsymbol{o}_L,$$

where the mth component of the $L \times 1$ vector $\partial \boldsymbol{\beta}_{t+h}$ represents an arbitrarily small difference $(\beta_m(t) - \bar{\beta}_m)$.

Proof. Let the differential $\partial \mathbf{B}_{t+h}$ ($h = 1, \ldots, H$) represent an arbitrarily small difference $(\mathbf{B}_{t+h} - \bar{\mathbf{B}})$, i.e.,

$$\partial \mathbf{B}_{t+h} = \begin{pmatrix} \partial \boldsymbol{\beta}_{t+h}' & \\ \mathbf{0}_{L-1} & \boldsymbol{o}_{L-1} \end{pmatrix}, \tag{17}$$

where $\mathbf{0}_{L-1}$ is the $(L - 1) \times (L - 1)$ zero matrix, and $\boldsymbol{o}_{L-1}$ is the $(L - 1) \times 1$ zero vector. It follows from Magnus and Neudecker (1988), Section 9.13, that the differential of (11) with respect to $\mathbf{B}_{t+h}$ $(h = 1, \ldots, H)$ is

$$\sum_{h=0}^{H}\sum_{z=0}^{h}\gamma_{H-z}(\bar{\mathbf{B}}')^{H-h}(\partial\mathbf{B}'_{t+H-h})(\bar{\mathbf{B}}')^{h-z}\boldsymbol{e}_1 = \boldsymbol{o}_L. \tag{18}$$

The left hand side of (18) is a $L \times 1$ vector, so that it is identically equal to

$$\mathrm{vec}\left(\sum_{h=0}^{H}\sum_{z=0}^{h}\gamma_{H-z}(\bar{\mathbf{B}}')^{H-h}(\partial\mathbf{B}'_{t+H-h})(\bar{\mathbf{B}}')^{h-z}\boldsymbol{e}_1\right), \tag{19}$$

where the vec operator transforms a matrix into a vector by stacking the columns of the matrix one underneath the other. Using elementary properties of the vec operator (Magnus and Neudecker, 1988, Chapter 2), one can rewrite (18) as

$$\sum_{h=0}^{H}\sum_{z=0}^{h}\gamma_{H-z}(((\bar{\mathbf{B}}')^{h-z}\boldsymbol{e}_1)' \otimes ((\bar{\mathbf{B}}')^{H-h}))\mathrm{vec}(\partial\mathbf{B}'_{t+H-h}) = \boldsymbol{o}_L, \tag{20}$$

where the symbol $\otimes$ stands for the Kronecker product. Remark now that

$$\mathrm{vec}(\partial\mathbf{B}'_{t+H-h}) = \begin{pmatrix} \boldsymbol{I}_L \\ \mathbf{0}_L \\ \vdots \\ \mathbf{0}_L \end{pmatrix}(\partial\boldsymbol{\beta}_{t+H-h}), \tag{21}$$

so that (20) becomes

$$\sum_{h=0}^{H}\sum_{z=0}^{h}\gamma_{H-z}((\boldsymbol{e}'_1(\bar{\mathbf{B}}')^{h-z}\boldsymbol{e}_1)(\bar{\mathbf{B}}')^{H-h})(\partial\beta_{t+H-h}) = \boldsymbol{o}_L. \tag{22}$$

Lemma 3 comes by premultiplying (22) by $(1/\gamma_H)(\bar{\mathbf{B}}')^{-H}$ (which is allowed when $\delta_L \neq 0$, which makes $\bar{\mathbf{B}}$ non singular, and $\gamma_H \neq 0$) and by relabelling indices ($j = H-z, h' = H-h$ and then $h = h'$) in (22). □

The next result relates the dynamics described in Lemma 3 to the perfect foresight roots λ_i $(i = 1, \ldots, H + L)$. Recall that £ is the set of the L perfect foresight roots that are also the eigenvalues of the $L \times L$ matrix $\bar{\mathbf{B}}$. Let $£_c$ be the set of the H remaining perfect foresight roots.

Lemma 4. *Let $£_c$ be the complement set of £ relative to the set Λ of the $(H + L)$ perfect foresight roots. Let*

$$p(h) = \sum_{j=h}^{H}\frac{\gamma_j}{\gamma_H}\left[\boldsymbol{e}'_1(\bar{\mathbf{B}}')^{j-h}\boldsymbol{e}_1\right]$$

Then, $p(h) = (-1)^{H-h}\sigma_{H-h}(£_c)$, where $\sigma_{H-h}(£_c)$ is the (H-h)th $(h = 0, \ldots, H)$ elementary symmetric polynomial of the set $£_c$.

Proof. Using the fact that $\boldsymbol{e}_1'\bar{\mathbf{B}}' = (\bar{\beta}_1\boldsymbol{e}_1' + \boldsymbol{e}_2')$, $\boldsymbol{e}_2'\bar{\mathbf{B}}' = (\bar{\beta}_2\boldsymbol{e}_1' + \boldsymbol{e}_3')$, and so on, leads to the relation

$$p(h) = \frac{\gamma_h}{\gamma_H} + \sum_{i=1}^{M} \bar{\beta}_i p(h+i), \tag{23}$$

where $M = \min(L, H-h)$. For $h = H$ in (23), $p(H) = 1$. Lemma 4 holds true for $h = H$ with the convention that $\sigma_0(\cdot) = 1$. Let $h = H-1$, so that $M = 1$ (since $L \geq 1$). Then, $p(H-1) = (\gamma_{H-1}/\gamma_H) + \bar{\beta}_1 p(H)$. The relations between the coefficients of the characteristic polynomial associated with (2) and its roots λ_i ($i = 1, \ldots, H+L$) (as, e.g., in (7) and (8)) imply that $(\gamma_{H-1}/\gamma_H) = -\sigma(\Lambda)$. But it follows from Lemma 1 that $\bar{\beta}_1 = \sigma(£)$. Thus $p(H-1) = -\sigma(\Lambda) + \sigma(£)$, which is equal to $\sigma(£_c)$, thus proving Lemma 4 for $h = H-1$. Assume now that Lemma 4 holds true for some $0 < h+1 \leq H$. As stated in Queysanne (1964), in Chapter 11, Section 199.c, $\sigma_{h'}(\Lambda) = \lambda_i\sigma_{h'-1}(\Lambda-\lambda_i) + \sigma_{h'}(\Lambda-\lambda_i)$ for $h' = 1, \ldots, H$. Lemma 4 follows by proceeding inductively, i.e., by taking into account that $\sigma_{h'-1}(\Lambda-\lambda_i)$ is equal is turn to $\lambda_j\sigma_{h'-2}(\Lambda-\lambda_i-\lambda_j) + \sigma_{h'-1}(\Lambda-\lambda_i-\lambda_j)$ while $\sigma_{h'}(\Lambda-\lambda_i)$ is equal to $\lambda_j\sigma_{h'-1}(\Lambda-\lambda_i-\lambda_j) + \sigma_{h'}(\Lambda-\lambda_i-\lambda_j)$, until all the elements of the set £ are sent out of the set Λ. Indeed, this procedure leads to the relation

$$\sigma_{h'}(\Lambda) = \sigma_{h'}(£_c) + \sum_{i=1}^{M} \sigma_i(£)\sigma_{h'-i}(£_c), \tag{24}$$

for $M = \min(L, h')$, $h' = 1, \ldots, H$ (and $h' = H - h$), so that Lemma 4 holds true for $h \leq H$, which concludes the proof. □

Lemmas 3 and 4 imply that (11) in the immediate vicinity of a given steady EGR(L) can be rewritten

$$\sum_{h=0}^{H} (-1)^{H-h}\sigma_{H-h}(£_c)(\bar{\mathbf{B}}')^{h-H}(\partial\beta_{t+h}) = \boldsymbol{o}_L. \tag{25}$$

Let $\mathbf{A}_h$ stand the $L \times L$ matrix $(-1)^{H-h}\sigma_{H-h}(£_c)(\bar{\mathbf{B}}^T)^{h-H}$ and transform (25) into the first order vector recursive equation

$$\begin{pmatrix} \partial\boldsymbol{\beta}_{t+H} \\ \vdots \\ \vdots \\ \partial\boldsymbol{\beta}_{t+1} \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{H-1} & \cdots & \cdots & -\mathbf{A}_0 \\ \mathbf{I}_L & \mathbf{0}_L & \cdots & \mathbf{0}_L \\ \vdots & & \ddots & \vdots \\ \mathbf{0}_L & \cdots & \mathbf{I}_L & \mathbf{0}_L \end{pmatrix} \begin{pmatrix} \partial\boldsymbol{\beta}_{t+H-1} \\ \vdots \\ \vdots \\ \partial\boldsymbol{\beta}_t \end{pmatrix} \equiv \mathbf{J} \begin{pmatrix} \partial\boldsymbol{\beta}_{t+H-1} \\ \vdots \\ \vdots \\ \partial\boldsymbol{\beta}_t \end{pmatrix}.$$

The dynamics (11) in the immediate vicinity of a given steady EGR(L) is governed by the HL eigenvalues of the $HL \times HL$ matrix $\boldsymbol{J}$. The purpose of the next result is to relate these eigenvalues to the $(H+L)$ perfect foresight roots of the set Λ.

Lemma 5. *The HL eigenvalues of the $HL \times HL$ matrix* $\mathbf{J}$ *are of the form* (λ_j/λ_i) *where* λ_i *is any element of* £ *and* λ_j *is any element of* $£_c$ *(for* $i = 1, \ldots, L$ *and* $j = 1, \ldots, H$*), where* £ *is the set of the eigenvalues of* $\bar{\mathbf{B}}$, $£_c$ *is the complement set of* £ *in* Λ.

Proof. Let $\boldsymbol{a}$ be the $HL \times 1$ eigenvector associated with some eigenvalue α of the $HL \times HL$ matrix $\mathbf{J}$. Then, $\boldsymbol{a}$ is of the form

$$\boldsymbol{a}' = \left(\bar{\boldsymbol{a}}', \left(\frac{1}{\alpha}\right)\bar{\boldsymbol{a}}', \dots, \left(\frac{1}{\alpha}\right)^{H-1}\bar{\boldsymbol{a}}'\right), \tag{26}$$

where $\bar{\boldsymbol{a}}$ is a $L \times 1$ vector. Let now $\boldsymbol{b}_i$ ($i = 1, \dots, L$) be the $L \times 1$ eigenvector of $\bar{\mathbf{B}}'$ associated with a given perfect foresight root λ_i in the set £. The proof proceeds from the fact that $\bar{\boldsymbol{a}} = \boldsymbol{b}_i$. Observe indeed that $\mathbf{J}\boldsymbol{a} = \alpha\boldsymbol{a}$. Developing the L first rows of this system (the $H(L-1)$ remaining rows are identities), with $\bar{\boldsymbol{a}} = \boldsymbol{b}_i$, and using the expression of the $L \times L$ matrix $\mathbf{A}_h$ ($h = 0, \dots, H-1$) leads to

$$-(-1)\sigma_1(£_c)(\bar{\mathbf{B}}')^{-1}\boldsymbol{b}_i - \dots - \left(\frac{1}{\alpha}\right)^{H-1}(-1)^H\sigma_H(£_c)(\bar{\mathbf{B}}')^{-H}\boldsymbol{b}_i = \alpha\boldsymbol{b}_i. \tag{27}$$

By definition, $\bar{\mathbf{B}}'\boldsymbol{b}_i = \lambda_i\boldsymbol{b}_i$ so that $(\bar{\mathbf{B}}')^{-h}\boldsymbol{b}_i = (1/\lambda_i)^h\boldsymbol{b}_i$. Since $\sigma_h(£_c)(1/\lambda_i)^h$ is equal to $\sigma_h(£_c^i)$ where $£_c^i$ is the set of all the H perfect foresight roots in the set $£_c$ divided by a given, but arbitrary, perfect foresight root λ_i in the set £, (27) becomes

$$\alpha\boldsymbol{b}_i + (-1)\sigma_1(£_c^i)\boldsymbol{b}_i + \dots + \left(\frac{1}{\alpha}\right)^{H-1}(-1)^H\sigma_H(£_c^i)\boldsymbol{b}_i = \boldsymbol{o}_L, \tag{28}$$

and, for $\alpha \neq 0$ (which implies that $\alpha^{H-1} \neq 0$), (28) is equivalent to

$$[\alpha^H + (-1)\sigma_1(£_c^i)\alpha^{H-1} + \dots + (-1)^H\sigma_H(£_c^i)]\boldsymbol{b}_i = \boldsymbol{o}_L. \tag{29}$$

All the L components of $\boldsymbol{b}_i$ are different from 0 (see Lemma 1). Therefore, (28) is satisfied if and only if

$$\sum_{h=0}^{H}(-1)^{H-h}\sigma_{H-h}(£_c^i)\alpha^h = 0. \tag{30}$$

By analogy with (7) and (8), one can directly conclude that the H roots of (30) are the H elements of $£_c^i$, i.e., the H ratios (λ_j/λ_i) for any element λ_j of $£_c$ and a given, but arbitrary, element λ_i of £. It follows that the HL eigenvalues of $\mathbf{J}$ are the HL ratios (λ_j/λ_i) for any element λ_j of $£_c$ and any element λ_i of £. By hypothesis such roots are well defined and different from 0 (since the assumption that $\delta_L \neq 0$ implies that no perfect foresight root is equal to 0). The $HL \times 1$ eigenvector $\boldsymbol{a}$ of $\mathbf{J}$ that is associated with the eigenvalue $\alpha = (\lambda_j/\lambda_i)$ is defined by (26) with $\bar{\boldsymbol{a}} = \boldsymbol{b}_i$ and $\boldsymbol{b}_i$ is the eigenvector of $\bar{\mathbf{B}}'$ associated with the perfect foresight root λ_i in the set £. □

The dynamics (11) in the immediate vicinity of a steady EGR(L) corresponding to a set £ of perfect foresight roots is well defined if and only if all the HL eigenvalues of $\mathbf{J}$ are different from 0. This is the case for any steady EGR(L) if and only if no perfect foresight root is equal to 0, i.e., $\delta_L \neq 0$. Now, given the lack of predetermined variable (11), a steady EGR(L) is locally determinate in this dynamics if and only if all the HL eigenvalues of $\mathbf{J}$ have a modulus greater than 1. It follows from Lemma 5 that the modulus of the eigenvalues of $\mathbf{J}$ is greater than 1 if and only if the modulus of any perfect foresight root λ_j in the set $£_c$

is greater than the modulus of any perfect foresight root λ_i in the set £. This is the case if and only if £ is the set of the L perfect foresight roots of lowest modulus $(\lambda_1, \dots, \lambda_L)$. This completes the proof of Proposition 1.

5.3. Proof of Lemma 2

Observe that (12) can be alternatively rewritten $\boldsymbol{x}_t = \mathbf{B}_t \boldsymbol{x}_{t-1} + \alpha_t \boldsymbol{e}_1$ whatever $t \geq 0$ and $\boldsymbol{x}_{t-1}$ are. Leading this law forward implies that, for $h = 1, \dots, H$,

$$x_{t+h} \equiv \boldsymbol{e}_1' \boldsymbol{x}_{t+h} = \boldsymbol{e}_1' (\mathbf{B}_{t+h} \cdots \mathbf{B}_t) \boldsymbol{x}_{t-1} + \boldsymbol{e}_1' \left[\sum_{j=0}^{h-1} (\mathbf{B}_{t+h} \cdots \mathbf{B}_{t+j+1}) \alpha_{t+j} \boldsymbol{e}_1 + \alpha_{t+h} \boldsymbol{e}_1 \right]. \tag{31}$$

The expression of the actual current state x_t in (2) is then obtained by reintroducing (31) into (2), for $h = 1, \dots, H$, i.e.,

$$\begin{aligned} x_t = &- \left[\sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' (\mathbf{B}_{t+h} \cdots \mathbf{B}_t) + \boldsymbol{\delta}' \right] \boldsymbol{x}_{t-1} \\ &- \sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' \left[\sum_{j=0}^{h-1} (\mathbf{B}_{t+h} \cdots \mathbf{B}_{t+j+1}) \alpha_{t+j} \boldsymbol{e}_1 + \alpha_{t+h} \boldsymbol{e}_1 \right], \end{aligned} \tag{32}$$

For x_t to verify both (12) and (32) whatever t and $\boldsymbol{x}_{t-1}$ are, it must be the case that

$$\boldsymbol{e}_1' \mathbf{B}_t = - \sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' (\mathbf{B}_{t+h} \cdots \mathbf{B}_t) - \boldsymbol{\delta}', \tag{33}$$

$$\text{and } \alpha_t = - \sum_{h=1}^{H} \gamma_h \boldsymbol{e}_1' \left[\sum_{j=0}^{h-1} (\mathbf{B}_{t+h} \cdots \mathbf{B}_{t+j+1}) \boldsymbol{e}_1 \alpha_{t+j} + \alpha_{t+h} \boldsymbol{e}_1 \right]. \tag{34}$$

It is clear that (33) coincides with (11). Hence, the fixed points of the system formed by (33) and (34) are of the form $(\bar{\boldsymbol{\beta}}, \bar{\alpha})$, where $\bar{\boldsymbol{\beta}}' \equiv \boldsymbol{e}_1' \bar{\mathbf{B}}$ is a steady EGR(L) and $\bar{\alpha}$ is a scalar to be determined. To this aim, set $\mathbf{B}_t = \bar{\mathbf{B}}$ in (34), and let £ be the set of the L eigenvalues of $\bar{\mathbf{B}}$. Then, (34) becomes

$$\sum_{h=0}^{H} \sum_{j=h}^{H} \frac{\gamma_j}{\gamma_H} (\boldsymbol{e}_1' \bar{\mathbf{B}}^{j-h} \boldsymbol{e}_1) \alpha_{t+h} = 0, \tag{35}$$

with the convention that $\gamma_0 \equiv 1$ (and since $\gamma_H \neq 0$ by assumption). By using first the fact that $\boldsymbol{e}_1' \bar{\mathbf{B}}^{j-h} \boldsymbol{e}_1$ is identically equal to $\boldsymbol{e}_1' (\bar{\mathbf{B}}')^{j-h} \boldsymbol{e}_1$ and then Lemma 4, one can rewrite (35) as

$$\sum_{h=0}^{H} (-1)^{H-h} \sigma_{H-h}(\pounds_c) \alpha_{t+h} = 0, \tag{36}$$

where $\pounds_c = \Lambda - \pounds$ is the set of the H perfect foresight roots that are not in £. This defines a linear difference equation of order H whose only fixed point is $\alpha_t = \bar{\alpha} = 0$ if and only if the H roots of the characteristic polynomial associated with (36), i.e., the H perfect foresight roots of $\pounds_c$, do differ from 1; otherwise the steady states of (33) and (34)) would be not locally well-defined.

In the immediate neighborhood of some fixed point $(\bar{\boldsymbol{\beta}}, 0)$, the dynamics driven by (33) and (34) can be approximated by a linear first order recursive equation linking the $(HL + H) \times 1$ vector $(\boldsymbol{\beta}^{t+1}, \boldsymbol{\alpha}^{t+1}) \equiv ((\boldsymbol{\beta}_{t+H} - \bar{\boldsymbol{\beta}})', \dots, (\boldsymbol{\beta}_{t+1} - \bar{\boldsymbol{\beta}})', (\alpha_{t+H}, \dots, \alpha_{t+H-1}))'$ to the $(HL + H) \times 1$ vector $(\boldsymbol{\beta}^t, \boldsymbol{\alpha}^t)$ through a $(HL + H) \times (HL + H)$ Jacobian matrix $\tilde{\mathbf{J}}$, i.e., $(\boldsymbol{\beta}^{t+1}, \boldsymbol{\alpha}^{t+1})' = \tilde{\mathbf{J}}(\boldsymbol{\beta}^t, \boldsymbol{\alpha}^t)'$. By definition, $(\bar{\boldsymbol{\beta}}, 0)$ is locally determinate if and only if the $(HL + H)$ eigenvalues of $\tilde{\mathbf{J}}$ have moduli greater than 1. Since (33) does not depend on α_{t+h} $(h \geq 0)$, the $(HL + H)$ eigenvalues of $\tilde{\mathbf{J}}$ are in fact the LH eigenvalues of $\mathbf{J}$ and the H roots of (34), with $\mathbf{B}_t = \bar{\mathbf{B}}$, that is, the roots of the characteristic polynomial associated with (36). Hence, the MO-solution $(\bar{\boldsymbol{\beta}}, 0)$ corresponding to £ is locally determinate if and only $\bar{\boldsymbol{\beta}} \equiv \boldsymbol{e}_1' \bar{\mathbf{B}}$ is locally determinate in (11) and all the H elements of $\pounds_c$ have a modulus greater than 1. Lemma 2 follows.

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