Certainty Equivalence and Noisy Redistribution

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Abstract

This paper assesses the usefulness of stochastic contracts in the presence of informational asymmetries. It identifies circumstances where a stochastic redistribution policy is socially dominated by the deterministic policy where after-tax income lotteries are replaced with their certainty equivalent. It also provides a parametric example where every stochastic menu which has the optimal deterministic menu as certainty equivalent is dominated by the deterministic menu, while there exist feasible and incentive compatible lotteries improving locally upon the deterministic menu.

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1 Introduction

According to the `travelandleisure.com` website, 'discerning travelers know that the best tour operators in the world are like magicians. They might surprise you with a sundowner on a secluded beach, a helicopter ride to lunch, or a walking safari.' In the same vein customers using the Too Good To Go app also are not aware of the exact content of the goods they buy: they book at a given known price some unknown perishable surplus items that will remain unsold at the end of the day, and would otherwise be wasted. Sometimes firms instead release the good they sell but not its price; for instance buyers may only know that the good will be sold 'at the going price.' Buyers then end up in a position common in finance where short/long investors commit to buy/sell an asset at a given date at an unknown price.

These features accord well with the theoretical literature that provides us with examples where in the presence of asymmetric information a principal finds it profitable to design a random contracting scheme, rather than a deterministic one. This should not be viewed as a surprising property since the incentive compatibility constraints that enter optimization programs typically lead to failures of convexity. Randomizing over allocations located on the frontier of the set delimited by the constraints, if allowed, leads the principal to reach allocations on the convex hull of the constraints of the deterministic program. Non-convexities imply that the two sets, deterministic or random, differ and so open the possibility that some allocations in the convex hull perform better than the best deterministic alternative.

Early contributions exploiting these ideas can be found in Gjesdal (1982) and Stiglitz (1982). In industrial organization, the revenu maximizing rule of a monopolist selling multiple goods to a single buyer may involve lotteries when the amount that buyers are willing to pay is privately known; see, e.g., Thanassoulis (2004), Manelli and Vincent (2007) Manelli and Vincent (2010), Pycia and Unver (2015) Pavlov (2011), Hart and Reny (2015) and Rochet and Thanassoulis (2019). In public finance, Stiglitz (1982) and Brito, Hamilton, Slutsky, and Stiglitz (1995) show that deterministic redistribution sometimes is socially dominated. Suppose that the government would like to redistribute income from high to low skill in a population of risk-averse workers. Redistribution is potentially limited if individual skills remain private information to workers, as high skilled may mimic low skilled by reducing labor effort. Introducing randomness in the bundles designed for low skilled is detrimental to their welfare, but this also relaxes incentive constraints and thus expands the scope of redistribution.

As shown by Hellwig (2007), the expanded possible redistribution is not always enough to compensate low skilled agents for their welfare losses. In particular noisy redistribution may not be desirable if the socially-favored low skilled are more risk averse than high skilled types. In Gauthier and Laroque (2014) we find a necessary and sufficient condition for a deterministic optimum to be locally socially dominated by a random redistribution. When applied to a variant of the two-type Mirrlees (1971) model, noise is optimal when the socially-favored low skilled agents indeed display a low enough risk aversion, compared to the high skilled workers. The social acceptability of explicit randomness in taxes and transfers is likely to be disputable. Loose interpretations of randomness can be found in frequent tax and labor market reforms (Zangari, Caiumi, and Hemmelgarn (2017)), errors in administrating the tax (Slemrod (2007)) or when there is only a chance to evade taxes; also lotteries are commonly used as complementary means to levy funds by private charities and governments when taxation is difficult to implement or even not feasible (Morgan (2000)).

Gauthier and Laroque (2014) deals with discrete types. We made a first attempt to treat the continuum of types case in Gauthier and Laroque (2017). But our analysis was focused on qualification failures where the incentive compatibility constraints implied a uniform tax and transfer treatment. The current paper considers a standard version of Mirrlees (1971) with a continuum of types and qualified constraints.

We start from the observation that the social usefulness of random redistribution in this setup requires some bunching to occur in the deterministic optimal policy. The role of noise is therefore to be found in the possibility of a differential treatment of agents that cannot be achieved in a deterministic world. The early classic example of Lollivier and Rochet (1983) where there is bunching in the deterministic optimum is used to highlight the role of the certainty equivalent as a sufficient statistics for social usefulness of noise. In this example, any two different types facing the same income lottery have the same certainty equivalent incomes. This makes redistribution through lotteries socially dominated by menus where lotteries are replaced with their certainty equivalent. This property extends recent results in Chen, He, Li, and Sun (2019) beyond risk neutrality.

We build on the certainty equivalent to exhibit a sufficient condition for local randomization to be socially useless from a deterministic optimum. It is satisfied if the redistributive social tastes favor low risk aversion types. It is also satisfied in the absence of redistributive purpose: with a pure Bentham social welfare function, certainty equivalence always performs better than local lotteries.

Finally we assess the status of our sufficient condition for useless local randomization from random menus which have the optimal deterministic menu as certainty equivalent. We provide an explicit parametric example where these random menus are dominated by the deterministic optimum, while there exist other local random menus performing better than the deterministic optimum. We consider a Rawlsian government offering menus where a greater transfer comes with a greater noise. By introducing some noise at the bottom of the distribution, the government is able to deter high type agents from mimicking low types, which allows it to reduce the transfers to high types. When the types are distributed according to a variant of the Weibull distribution, we show that the budget saving from the reduction in these transfers is enough to overcome the change in the amount of collected tax on low types: eventually the available total tax resources increase. The government thus can raise the welfare of the least-utility type agents by using noisy redistribution.

The paper is organized as follows. The general framework is laid down in Section 2. Section 3 relates optimal randomization to deterministic bunching. Section 4 shows that the class of individual preferences used in Lollivier and Rochet (1983) yields a deterministic optimum. In Section 5, we deal with more general preferences and provide a sufficient condition for a deterministic optimum to dominate upon local randomizations. The main trade-offs at stake in this condition are spelled out in Section 6. Finally, in Section 7, we spell out the explicit parametric example of a random menu that locally improves upon the deterministic optimum.

2 General framework

There is a continuum of agents indexed by a real parameter θ in a unit size population. The type θ takes values in $\Theta = \left[\theta^{\inf}, \theta^{\sup}\right]$, with a cumulative distribution function $F : \Theta \to [0,1]$ and a well-behaved associated positive probability density function $f : \Theta \to \mathbb{R}_+^*$. The preferences of a type θ agent are represented by the quasilinear utility function

$$u(c, \theta) - y \tag{1}$$

when she earns a before-tax income y and pays the tax $y-c$, which yields an after-tax income c that is also her consumption. The function u is increasing and strictly concave in c. In addition, it satisfies the Spence-Mirrlees condition that the cross derivative $u''_{c\theta}(c,\theta)$ keeps a constant sign for all (c,θ).

The government offers a menu of contracts $(\tilde{c}(\theta),\tilde{y}(\theta))$ that consists of before and after-tax income lotteries. The expected utility of type θ endowed with lotteries $\tilde{c}(\theta)$ and $\tilde{y}(\theta)$ is

$$\mathbb{E}\left[u(\tilde{c}(\theta),\theta)-\tilde{y}(\theta)\right].$$

Feasibility requires

$$\int_{\Theta} \mathbb{E}\left[\tilde{c}(\theta)-\tilde{y}(\theta)\right] \mathrm{d}F(\theta) \leq 0. \tag{2}$$

If θ is private information to the agent, the government must ensure that every agent chooses the contract designed for her. This is met when the incentive constraints

$$\mathbb{E}\left[u(\tilde{c}(\theta),\theta)-\tilde{y}(\theta)\right] \geq \mathbb{E}[u(\tilde{c}(\tau),\theta)-\tilde{y}(\tau)] \tag{3}$$

are satisfied for all (θ,τ) in $\Theta\times\Theta$.

An optimal redistribution policy is a menu $(\tilde{c}(\theta),\tilde{y}(\theta))$ that maximizes the social welfare objective

$$\int_{\Theta} \mathbb{E}\left[u(\tilde{c}(\theta),\theta)-\tilde{y}(\theta)\right] \mathrm{d}G(\theta) \tag{4}$$

subject to the feasibility constraint (2) and the incentive constraints (3). In (4) the social weights in $G(\cdot)$ are normalized so that they sum up to 1.

3 Randomization and deterministic bunching

We are interested into conditions ensuring a deterministic optimal redistribution policy, where every contract $(\tilde{c}(\theta),\tilde{y}(\theta))$ yields the outcome $(c(\theta),y(\theta))$ with probability 1. From that perspective one can restrict attention to nonrandom before-tax income profiles $(y(\theta))$. Indeed, with quasilinear utility (1), replacing for every θ the lottery $\tilde{y}(\theta)$ with the sure outcome $\mathbb{E}[\tilde{y}(\theta)]$ affects neither the constraints nor the objective.

Using the standard quasilinear toolkit of contract theory developed by Myerson (1982), maximization of the social objective (4) subject to (2) and the local first-order conditions for (3) amounts to choose lotteries $(\tilde{c}(\theta))$ that maximize

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right] \,\mathrm{d}G(\theta) \tag{5}$$

where

$$W(c,\theta) = u(c,\theta) - c - \frac{G(\theta)-F(\theta)}{f(\theta)} u'_\theta(c,\theta)$$

is the deterministic virtual social surplus. If the optimal deterministic after-tax income profile involves no bunching, only the local first-order conditions for (3) are relevant to deal with incentives. Therefore every type θ must be offered some $c^*(\theta)$ that maximizes $W(c,\theta)$. It follows that $\mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right] \leq W(c^*(\theta),\theta)$ for all θ.

Proposition 1. *A random redistribution policy is useless if the optimal deterministic redistribution policy involves no bunching.*

Hence randomness can only be socially useful if incentive considerations force the government to use a deterministic policy treating uniformly different types of individuals.

4 The case of multiplicative utility

As is shown in Lollivier and Rochet (1983), the optimal deterministic redistribution policy may involve bunching if

$$u(c,\theta) = \theta v(c), \tag{6}$$

$\theta^{\inf} \geq 0$, and low types are highly favored by the government.

With this multiplicative utility formulation, however, the optimal policy is deterministic. The argument proceeds as follows. Consider a menu $(\tilde{c}(\theta), y(\theta))$ that satisfies the feasibility constraint (2) and the incentive constraints (3). Compare this menu with the menu where the lottery $\tilde{c}(\theta)$ is replaced with its certainty equivalent $\mathbb{C}(\tilde{c}(\theta))$ with probability 1 for every θ, while the before-tax income $y(\theta)$ is kept unchanged. The after-tax certainty equivalent for type θ when she chooses the lottery $\tilde{c}(\tau)$ designed for type τ is the sure after-tax income $\mathbb{C}(\tilde{c}(\tau))$ defined by $\mathbb{E}[\theta v(\tilde{c}(\tau))] = \theta v(\mathbb{C}(\tilde{c}(\tau)))$.

If individual preferences are represented by (6) the reference to the actual type θ of the agent drops out, and the certainty equivalent actually satisfies

$$\mathbb{E}[v(\tilde{c}(\tau))] = v(\mathbb{C}(\tilde{c}(\tau))).$$

Hence, when facing the deterministic contract $(\mathbb{C}(\tilde{c}(\theta)), y(\theta))$ offered to type θ, her utility

$$\theta\mathbb{E}\left[v(\tilde{c}(\theta))\right] - y(\theta) = \theta v(\mathbb{C}(\tilde{c}(\theta))) - y(\theta)$$

remains unchanged.

In addition, the deterministic menu satisfies the incentive constraints since (3) can be rewritten

$$\theta v(\mathbb{C}(\tilde{c}(\theta))) - y(\theta) \geq \theta v(\mathbb{C}(\tilde{c}(\tau))) - y(\tau)$$

for all θ and τ.

Finally the deterministic menu provides the government with positive net resources

$$\int_{\Theta} \left[y(\theta) - \mathbb{C}(\tilde{c}(\theta))\right] \mathrm{d}F(\theta) > \int_{\Theta} \left[y(\theta) - \mathbb{E}\left[\tilde{c}(\theta)\right]\right] \mathrm{d}F(\theta) \geq 0,$$

since the strict concavity of v implies the strict inequality $\mathbb{E}[\tilde{c}(\theta)] > \mathbb{C}(\tilde{c}(\theta))$ for all θ such that $\tilde{c}(\theta)$ is a non-degenerate lottery.

Proposition 2. *The optimal redistribution policy is deterministic if individual preferences for consumption are represented by the multiplicative utility function $u(c, \theta) = \theta v(c)$ for all $c \geq 0$ and θ* in Θ.

Proof. The deterministic menu yields a positive slack in the feasibility constraint (2). This surplus can be used to improve upon the social welfare (4) obtained in the random case without violating (3). For instance, a welfare improvement can be obtained by dividing equally the surplus across agents to get a uniform reduction in the before-tax income. ■

With a general utility $u(c, \theta)$, the actual type θ enters the definition of the after-tax income certainty equivalent. For type θ when she chooses the lottery $\tilde{c}(\tau)$, it is the sure consumption $\mathbb{C}(\tilde{c}(\tau), \theta)$ defined by: for all θ and τ,

$$\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] = u(\mathbb{C}(\tilde{c}(\tau), \theta), \theta). \tag{7}$$

For notational convenience, we continue to denote $\mathbb{C}$ the certainty equivalent, though it now depends on both the reported type τ through her choice of the lottery $\tilde{c}(\tau)$ designed for type τ, and the true taste θ of this agent. Lemma 1 characterizes the set of individual preferences preserving incentive compatibility when the government switches from a menu of random contracts $(\tilde{c}(\theta), y(\theta))$ satisfying the feasibility constraint (2) and the incentive constraints (3) to the deterministic menu $(\mathbb{C}(\tilde{c}(\theta), \theta), y(\theta))$.

Lemma 1. *Consider a menu of non-degenerate lotteries $(\tilde{c}(\theta), y(\theta))$ that satisfies the incentive constraints (3). The menu $(\mathbb{C}(\tilde{c}(\theta), \theta), y(\theta))$ also satisfies these constraints if and only if, for all τ and θ, $\mathbb{C}(\tilde{c}(\tau), \theta)$ does not depend on θ.*

Proof. Assume that $\mathbb{C}(\tilde{c}(\tau), \theta) = \mathbb{C}(\tilde{c}(\tau), \tau)$ for all (θ, τ). Since the menu $(\tilde{c}(\theta), y(\theta))$ satisfies (3), we have

$$\theta = \arg\max_{\tau} u\left(\mathbb{C}\left(\tilde{c}(\tau), \theta\right), \theta\right) - y(\tau)$$

for all θ. When switching to the deterministic menu, incentives are preserved if and only if

$$\theta = \arg\max_{\tau} u\left(\mathbb{C}\left(\tilde{c}(\tau), \tau\right), \theta\right) - y(\tau)$$

for all θ. This is satisfied if $\mathbb{C}(\tilde{c}(\tau), \theta)$ does not depend on θ.

To show necessity, we consider agents with CARA utility and Gaussian after-tax income lotteries with mean $m(\tau)$ and variance $v(\tau)$, $\tau \in \Theta$ with $\theta^{\inf} \geq 0$. The certainty equivalent is

$$\mathbb{C}\left(\tilde{c}(\tau), \theta\right) = m(\tau) - \frac{\theta}{2} v(\tau).$$

Consider a menu satisfying (3): for all θ and τ,

$$y(\tau) - y(\theta) \geq u(\mathbb{C}(\tilde{c}(\tau), \theta), \theta) - u(\mathbb{C}(\tilde{c}(\theta), \theta), \theta).$$

The incentive constraints associated with the deterministic menu are violated if in such a menu there is a couple (θ, τ) such that

$$y(\tau) - y(\theta) < u(\mathbb{C}(\tilde{c}(\tau), \tau), \theta) - u(\mathbb{C}(\tilde{c}(\theta), \theta), \theta).$$

This is satisfied whenever $\mathbb{C}(\tilde{c}(\tau), \tau) > \mathbb{C}(\tilde{c}(\tau), \theta)$ for some (θ, τ). Using the expression of the certainty equivalent, this is $(\theta - \tau)v(\tau) > 0$, a condition satisfied as soon as the menu $(\tilde{c}(\theta), y(\theta))$ allocates some random after-tax income. ■

If, for all τ, the certainty equivalent $\mathbb{C}(\tilde{c}(\tau), \theta)$ is independent of θ, one can reproduce the same argument as the one used to get Proposition 2, with the multiplicative utility formulation. Then random redistribution is useless.

5 Certainty equivalent domination

Considering the before-tax income profile $(y(\theta))$ as given is restrictive since adjustments in the before-tax income when we switch to deterministic contracts can help satisfying incentive compatibility requirements. In this section, we compare a random menu $(\tilde{c}(\theta), y(\theta))$ satisfying (2) and (3) with the deterministic menu $(\mathbb{C}(\tilde{c}(\theta), \theta), y(\theta) - \delta(\theta))$ for some well chosen differentiable profile $(\delta(\theta))$.

Since every agent gets its after-tax certainty equivalent, the change in social welfare reduces to

$$\int_{\theta^{\text{inf}}}^{\theta^{\text{sup}}} \delta(\theta)\ \mathrm{d}G(\theta). \tag{8}$$

That is, social welfare improves when offering the certainty equivalent consumption comes with a reduction in the aggregate before-tax income, presumably with a lower labor effort. This reduction must fit (2) and (3). When faced with the deterministic income tax schedule, the incentive constraints (3) rewrite

$$\theta = \arg\max_{\tau} u(\mathbb{C}(\tilde{c}(\tau), \tau), \theta) - y(\tau) + \delta(\tau)$$

for all θ and τ. Since, by assumption, the random menu $(\tilde{c}(\theta), y(\theta))$ satisfies (3), the preservation of incentives in the deterministic case requires

$$\delta'(\theta) = -\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta).$$

By summation the adjustment in before-tax income

$$\delta(\theta) = \delta(\theta^{\text{inf}}) - \int_{\theta^{\text{inf}}}^{\theta} \mathbb{C}'_\theta(\tilde{c}(z), z) u'_c(\mathbb{C}(\tilde{c}(z), z), z)\, \mathrm{d}z$$

for all θ meets incentive compatibility at the outcome of the reform. The change in social welfare (8) rewrites, after integrating by parts,

$$\delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - G(\theta)}{f(\theta)} \mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \, \mathrm{d}F(\theta). \tag{9}$$

The value of $\delta(\theta^{\inf})$ comes from the feasibility constraint (2) at equality. Replacing in (2) the sure after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ with the difference $\mathbb{E}[\tilde{c}(\theta)] - \pi(\tilde{c}(\theta), \theta)$, where $\pi(\tilde{c}, \theta)$ is the positive risk premium for type θ when facing the lottery $\tilde{c}$, the feasibility constraint takes the form

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta) - \mathbb{E}[\tilde{c}(\theta)] + \pi(\tilde{c}(\theta), \theta)] \, \mathrm{d}F(\theta) = 0.$$

Since the initial random menu $(\tilde{c}(\theta), y(\theta))$ is assumed to meet (2), this constraint reduces to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta) \, \mathrm{d}F(\theta) = \int_{\theta^{\inf}}^{\theta^{\sup}} \pi(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta)$$

Using (5) and the identity $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$ gives

$$\delta(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \pi(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \, \mathrm{d}F(\theta).$$

The change in total before-tax income then follows from (9).

Proposition 3. *Consider a menu $(\tilde{c}(\theta), y(\theta))$ that satisfies (2) and (3). Assume that $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-decreasing (resp. non-increasing) in θ if $u''_{c\theta}(c, \theta) > 0$ (resp., $u''_{c\theta}(c, \theta) < 0$) for all (c, θ). If*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\pi(\tilde{c}(\theta), \theta) - \frac{G(\theta) - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \right] \mathrm{d}F(\theta) > 0, \tag{10}$$

then there is a deterministic menu where every type θ gets the sure outcome $\mathbb{C}(\tilde{c}(\theta), \theta)$ that satisfies (2) and (3) and yields a higher social welfare than $(\tilde{c}(\theta), y(\theta))$.

Argument. Social welfare is higher when (8) is positive, which is satisfied if and only if (10) holds true. Under the Spence-Mirrlees condition, the monotonicity condition on $\mathbb{C}(\tilde{c}(\theta), \theta)$ is a sufficient monotonicity condition for incentive constraints to be satisfied globally. ■

Since $\pi(c, \theta)$ is positive, condition (10) is satisfied in the Utilitarian case $G(\theta) = F(\theta)$ for every θ. It is also satisfied if the risk premium is the same for every type, $\pi'_\theta(\tilde{c}, \theta) = 0$ for all $\tilde{c}$ and θ. More generally, as shown by Hellwig (2007), it is satisfied if both $G(\theta) \geq F(\theta)$ and $\pi'_\theta(\tilde{c}, \theta) \leq 0$ for all $\tilde{c}$ and θ, i.e., the socially favored agents display a higher risk aversion (captured by a higher risk premium).

CARA-Gaussian example. Type θ agents have CARA preferences

$$u(c, \theta) = -\frac{1}{\theta} \exp(-\theta c),$$

and they face Gaussian after-tax income lotteries $(\tilde{c}(\theta))$. Let the associated certainty equivalent $(\mathbb{C}(\tilde{c}(\theta), \theta))$ be the optimal menu of deterministic consumption. If $\theta^{\inf} \geq 1$, then the deterministic menu improves upon Gaussian lotteries, independently of the social tastes for redistribution embodied in $G(\theta)$. To show this property, recall that, with $\tilde{c}(\tau)$ normally distributed with mean $m(\tau)$ and variance $v(\tau)$, $v(\tau) > 0$,

$$\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] = -\frac{1}{\theta} \exp\left[-\theta\left(m(\tau) - \frac{\theta}{2} v(\tau)\right)\right].$$

This yields the certainty equivalent

$$\mathbb{C}(\tilde{c}(\tau), \theta) = m(\tau) - \frac{\theta}{2} v(\tau)$$

and the risk premium

$$\pi(\tau, \theta) = m(\tau) - \mathbb{C}(\tilde{c}(\tau), \theta) = \frac{\theta}{2} v(\tau).$$

Hence inequality (10) is equivalent to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} v(\theta)\left[\theta - \left[G(\theta) - F(\theta)\right] \exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right)\right] \mathrm{d}\theta > 0.$$

Since both $G(\theta) - F(\theta) \leq 1$ and $\exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right) \leq 1$ for all θ (the after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-negative), we have $\left[G(\theta) - F(\theta)\right] \exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right) \leq 1$ for all θ. For $\theta^{\inf} \geq 1$, every term in the sum is non-negative. Therefore (10) is satisfied. ■

6 A simple reform

The main trade-offs underlying (10) can be seen by considering a tax reform that removes after-tax income randomness for a small part of the population. Suppose that agents initially face a menu $(\tilde{c}(\theta), y(\theta))$ satisfying feasibility (2) and incentive compatibility requirements (3). Then replace the lottery $\tilde{c}(\theta)$ with the associated certainty equivalent after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ for every agent whose type is in some interval $[\underline{\theta}, \overline{\theta}]$, $\overline{\theta} = \underline{\theta} + \mathrm{d}\theta$ for positive small $\mathrm{d}\theta$. To keep incentive compatibility, we adjust the before-tax income $y(\theta)$ of every type θ in $[\theta^{\inf}, \theta^{\sup}]$ by some amount $\delta(\theta)$, so that the before-tax income becomes $y(\theta) - \delta(\theta)$. Thus the change in social welfare is given by (8).

Since the contracts offered to types θ outside the interval $[\underline{\theta}, \overline{\theta}]$ only change by the before-tax income adjustment $\delta(\theta)$, incentive compatibility requires that $\delta(\theta)$ be some uniform amount $\underline{\delta}$ for every $\theta \leq \underline{\theta}$, and $\overline{\delta}$ for every $\theta \geq \overline{\theta}$. The utility $u\left(\mathbb{C}(\tilde{c}(\theta), \theta), \theta\right) - y(\theta)$ of a type θ in $[\underline{\theta}, \overline{\theta}]$ must be greater than the utility $u\left(\mathbb{C}(\tilde{c}(\tau), \tau), \theta\right) - y(\tau) + \delta(\tau) - \delta(\theta)$ she would obtain by reporting τ, τ in $[\underline{\theta}, \overline{\theta}]$. We know that $u\left(\mathbb{C}(\tilde{c}(\theta), \theta), \theta\right) - y(\theta)$ is greater than $u\left(\mathbb{C}(\tilde{c}(\tau), \theta), \theta\right) - y(\tau)$ since incentive compatibility requirements (3) are initially satisfied. Therefore, restricting the reform so that

$$\mathbb{C}_\theta'(\tilde{c}(\tau), \tau) u_c'\left(\mathbb{C}(\tilde{c}(\tau), \tau), \theta\right) + \delta'(\tau) = 0 \tag{11}$$

evaluated at $\tau = \theta$, θ in $[\underline{\theta}, \overline{\theta}]$, ensures that incentive compatibility requirements are also satisfied at the outcome of the reform. From (11) we get

$$\overline{\delta} \simeq \underline{\delta} - \mathbb{C}_\theta'(\tilde{c}(\theta), \theta) u_c'\left(\mathbb{C}(\tilde{c}(\theta), \theta), \theta\right) \, \mathrm{d}\theta. \tag{12}$$

The quantity $\underline{\delta}$ follows from the feasibility constraint

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta)] \, \mathrm{d}F(\theta) = \int_{\theta^{\inf}}^{\underline{\theta}} \mathbb{E}\left[\tilde{c}(\theta)\right] \, \mathrm{d}F(\theta) + \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \mathbb{C}(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta) + \int_{\underline{\theta}+\mathrm{d}\theta}^{\theta^{\sup}} \mathbb{E}\left[\tilde{c}(\theta)\right] \, \mathrm{d}F(\theta).$$

Replacing $\mathbb{C}(\tilde{c}(\theta), \theta)$ with $\mathbb{E}\left[\tilde{c}(\theta)\right] - \pi(\tilde{c}(\theta), \theta)$ and using (2) this constraint reduces to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta) \, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \pi(\tilde{c}(\theta), \theta) \, \mathrm{d}F(\theta).$$

At the first-order for $\mathrm{d}\theta$ close to 0 we thus have

$$\underline{\delta} F(\underline{\theta}) + \overline{\delta} \left(1 - F(\underline{\theta})\right) \simeq \pi(\underline{\theta}) f(\underline{\theta}) \, \mathrm{d}\theta. \tag{13}$$

The system formed by (12) and (13) gives the two before-tax income adjustments $\underline{\delta}$ and $\bar{\delta}$ consistent with feasibility and incentive compatibility at the outcome of the reform. Substituting $\bar{\delta}$ from (12) into (13) yields

$$\underline{\delta} \simeq \left[\pi(\underline{\theta})f(\underline{\theta}) + (1 - F(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\theta), \theta)u'_c\, (\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\right] \, \mathrm{d}\theta,$$

and we get $\bar{\delta}$ from (12).

From (8), the reform improves social welfare if $\underline{\delta} G(\underline{\theta}) + \bar{\delta}\,(1 - G(\underline{\theta})) > 0$. Using (12), this is

$$\underline{\delta} - (1 - G(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\theta), \theta)u'_c\, (\mathbb{C}(\tilde{c}(\theta), \theta), \theta) \; \mathrm{d}\theta > 0, \tag{14}$$

or equivalently

$$\pi(\underline{\theta})f(\underline{\theta}) + (G(\underline{\theta}) - F(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\theta), \theta)u'_c\, (\mathbb{C}(\tilde{c}(\theta), \theta), \theta) > 0.$$

We recognize the expression that is summed up in (10), with $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$.

When $\pi'_\theta(\tilde{c}(\theta), \theta) < 0$, i.e., low type display greater risk aversion, (12) shows that incentive considerations require $\underline{\delta} < \bar{\delta}$. For $\mathrm{d}\theta$ small enough, (13) in fact gives $\underline{\delta} < 0 < \bar{\delta}$, so that low (resp. high) types work more (resp. less) at the outcome of the reform. The reason is simple. Type τ reporting θ, $\theta < \tau$, gets the deterministic after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$: type τ being less averse than θ thus becomes ready to mimic type θ. The government discourages type τ from mimicking type θ by reducing type τ labor effort. The induced before-tax income adjustment has to be higher when $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta)u'_c\, (\mathbb{C}(\tilde{c}(\theta), \theta), \theta) > 0$ is high, as the utility gain of type τ mimicking type θ then is high.

The expression (14) shows that the social welfare change eventually consists of the pain coming from the labor effort necessary to produce $\underline{\delta}$ net of the social gain from the lower effort asked from high type agents, $\theta \geq \bar{\theta}$.

7 An example of useful randomization

Proposition 3 assesses the robustness of deterministic contracts referring to the certainty equivalent. We now provide a parametric example where the optimal deterministic menu locally dominates random menus that have the deterministic menu as certainty equivalent, while it is still socially dominated by other random menus. The utility of type θ, $\theta \in \Theta$ with $0 \leq \theta^{\inf} < 1$, is

$$\ln(c + \theta) - y. \tag{15}$$

7.1 Deterministic optimum

Let $V(\theta) = \ln(c(\theta) + \theta) - y(\theta)$ be the indirect utility of type θ when she chooses the bundle $(c(\theta), y(\theta))$. The best deterministic menu of a Rawlsian government maximizes the virtual surplus

$$V(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c(\theta) + \theta) - c(\theta) - \frac{m(\theta)}{c(\theta) + \theta} \right] \mathrm{d}F(\theta) \tag{16}$$

subject to the monotonicity condition $c'(\theta) \leq 0$ for all θ, where

$$m(\theta) = \frac{1 - F(\theta)}{f(\theta)}$$

is the Mills ratio. If, for all θ,

$$m'(\theta) - [1 + 4m(\theta)]^{\frac{1}{2}} > 0, \tag{17}$$

the after-tax income $c^{**}(\theta)$ maximizing $V(\theta^{\inf})$ is increasing with θ and thus violates incentive compatibility requirements. Then the deterministic optimum involves bunching for every type (see Appendix A for a detailed derivation).

Generalized Weibull distribution example. Given two real positive shape parameters a and b, and a real positive scale parameter $s > 0$, θ is distributed according to a Generalized Weibull distribution if

$$1 - F(\theta) = \exp(1 - (1 + s\theta^b)^a)$$

for all $\theta > 0$. For $a < 1$ and $b \leq 1$ the associated hazard rate is monotone decreasing and therefore $m(\theta)$ is increasing for all $\theta > 0$. See Dimitrakopoulou, Adamidis, and Loukas (2007) for properties of this distribution. For $a = 0.9$, $b = 0.01$ and $s = 5$ the left-hand side of the inequality (17) is decreasing in θ, from positive to negative values. There is therefore a threshold $\bar{\theta}$ such that the inequality (17) is satisfied for all $\theta < \bar{\theta}$. With the values of the parameters a, b and s, the threshold is equal to 6.39. At the threshold $F(\bar{\theta}) = 98.4\%$, i.e., (17) is satisfied for all types of agents but a small negligible subset consisting of the highest types. ■

With utility (15) the best deterministic income tax schedule has a single bracket if the inequality (17) is met for all types. The constant after-tax income maximizing the social objective $V(\theta^{\inf})$ is

$$c^* = 1 - \theta^{\inf}.$$

Feasibility and incentive compatibility then require a constant before-tax income $y^* = c^*$. At the deterministic optimum, the Rawlsian social objective equals

$$V(\theta^{\inf}) = V^* = -(1 - \theta^{\inf}).$$

This outcome is disappointing for the government: neglecting the second-order monotonicity conditions, it would design a contract where types who display a high taste for consumption (θ is high) have to work more. But bunching prevails and prevents the government from extracting more resources from high type agents. Eventually, sticking to deterministic tools, every agent must be treated uniformly.

7.2 Introducing randomness on the after-tax income

We now examine whether lotteries $\tilde{c}(\theta) = c^* + \tilde{\varepsilon}(\theta)$, with every realization of $\tilde{\varepsilon}(\theta)$ close to 0, can improve upon the deterministic optimum. Let $\mathbb{E}\left[\tilde{\varepsilon}(\theta)\right] = \lambda\mu(\theta)$ and $\operatorname{var}\left[\tilde{\varepsilon}(\theta)\right] = \lambda v(\theta)$, with λ a positive real number close to 0, and $\mu(\theta)$ and $v(\theta) \geq 0$ arbitrary bounded real numbers. Referring to the coefficient of absolute risk aversion of type θ,

$$A(c, \theta) = -\frac{u''_{cc}(c, \theta)}{u'_c(c, \theta)} = \frac{1}{c + \theta} > 0,$$

the (second-order Taylor expansion of the) expected utility of type θ when she chooses $\tilde{c}(\theta)$ is

$$\mathbb{E}\left[u(c^* + \tilde{\varepsilon}(\theta), \theta)\right] \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta)\left(\mu(\theta) - \frac{A(c^*, \theta)}{2} v(\theta)\right). \qquad (18)$$

Using Proposition 3, any menu where the certainty equivalent consumption of the lotteries $\tilde{c}(\theta)$ is the optimal deterministic consumption c^* performs better than the random menu. Indeed $A'_\theta(c^*, \theta) < 0$, so that the socially favored agents with type $\theta^{\inf}$ display the highest risk aversion: in the case of Rawlsian social preferences ($G(\theta) = 1$ for all θ), every squared bracket term in the sum (10) of Proposition 3 is non-negative.

Still it remains possible that other lotteries, that do not have the optimal deterministic after-tax income c^* as certainty equivalent, improve upon the deterministic optimum. To address this issue we now let

$$\mu(\theta) = v(\theta) + \beta \tag{19}$$

for some real number β. Hence the larger the transfer, the larger the noise. From (18) the utility of type θ is

$$V(\theta) \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta)\beta + U(\theta), \tag{20}$$

where $U(\theta) = S(c^*, \theta)v(\theta) - y(\theta)$ and

$$S(c, \theta) = \lambda u'_c(c, \theta) \left(1 - \frac{A(c, \theta)}{2}\right).$$

From (20), the incentive constraints (3) simplify to

$$S(c^*, \theta)v(\theta) - y(\theta) \geq S(c^*, \theta)v(\tau) - y(\tau)$$

for all θ and τ. They thus have the familiar quasilinear textbook shape. They are satisfied locally if, for all θ,

$$U'(\theta) = S'_\theta(c^*, \theta)v(\theta), \tag{21}$$

$$S'_\theta(c^*, \theta)v'(\theta) \geq 0.$$

Furthermore, if

$$\int_\tau^\theta S'_\theta(c^*, z)v'(\tau)\, \mathrm{d}z$$

has the same sign as $\theta - \tau$, then the local conditions for incentive compatibility are sufficient for (3) to hold for all admissible τ and θ.

With utility (15), it is readily checked that

$$S(c^*, \theta) = \lambda u'_c(c^*, \theta) \left(1 - \frac{A(c^*, \theta)}{2}\right) = \frac{\lambda}{c^* + \theta} \left(1 - \frac{1}{2} \frac{1}{c^* + \theta}\right) \tag{22}$$

and

$$S'_\theta(c^*, \theta) = -\frac{\lambda}{(c^* + \theta)^2} \left(1 - \frac{1}{c^* + \theta}\right). \tag{23}$$

Hence $S'_\theta(c^*, \theta) \leq 0$ when evaluated at $c = c^* = 1 - \theta^{\inf}$. Therefore:

Lemma 2. *Consider a menu $(\mu(\theta), v(\theta))$ such that $\mu(\theta) = v(\theta) + \beta$ for some real number β. The incentive constraints (3) hold if the first-order conditions (21) are satisfied and $v'(\theta) \leq 0$ for all θ.*

A Rawlsian government continues to refer to type $\theta^{\inf}$ when choosing taxes since the derivative $V'(\theta)$ obtained by differentiating (20) and using (21),

$$V'(\theta) \simeq u'_\theta(c^*, \theta) + \beta\lambda u''_{c\theta}(c^*, \theta) + S'_\theta(c^*, \theta)v(\theta) \tag{24}$$

is positive if λ is close enough to 0 and $v(\theta)$ is bounded from above.

The conditions for incentive compatibility in Lemma 2 introduce a difficulty for the government: the socially favored agents have to bear the highest consumption volatility, measured by the variance of their after-tax income. Computing the least utility $V(\theta^{\inf})$ in the presence of a small after-tax income noise and comparing this level to the social welfare V^* at the deterministic optimum, one gets:

Proposition 4. *Consider a menu of lotteries $(\mu(\theta), v(\theta))$ such that $\mu(\theta) = v(\theta) + \beta$ for some real number β, with $v(\theta)$ non-increasing bounded from above. The random menu improves upon the deterministic optimum, where $c(\theta) = c^* = 1 - \theta^{\inf}$ for all types, if and only if*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \phi(c^*, \theta)v(\theta) \, \mathrm{d}F(\theta) > 0,$$

where

$$\phi(c, \theta) = \left(\frac{1}{c+\theta} - 1 + \frac{m(\theta)}{(c+\theta)^2}\right) - \frac{1}{(c+\theta)^2}\left(\frac{1}{2} + \frac{m(\theta)}{(c+\theta)}\right).$$

Proof. See Appendix B ■

An intuition for this condition obtains by considering the total collected tax

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - c^* - \lambda\mu(\theta)] \, \mathrm{d}F(\theta).$$

Using the definition of $U(\theta) = S(c^*, \theta)v(\theta) - y(\theta)$ and our modeling restriction that the average transfer $\mu(\theta)$ to type θ is $v(\theta) + \beta$, this tax can be rewritten as

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [S(c^*, \theta)v(\theta) - U(\theta) - c^* - \lambda v(\theta) - \lambda\beta] \, \mathrm{d}F(\theta).$$

Given $U(\theta)$ a marginal increase $\mathrm{d}v > 0$ in the variance of the after-tax income of types between θ and $\theta + \mathrm{d}\theta$, $\mathrm{d}\theta$ close to 0, yields a change in the total collected tax equal to $[S(c^*, \theta) - \lambda] \, f(\theta) \, \mathrm{d}v \, \mathrm{d}\theta$. Types located between θ and $\theta + \mathrm{d}\theta$ increase their labor supply, which yields an additional amount of tax resources $S(c^*, \theta) f(\theta) \, \mathrm{d}v \, \mathrm{d}\theta > 0$. However a greater noise comes with a greater mean transfer, which costs $\lambda f(\theta) \, \mathrm{d}v \, \mathrm{d}\theta > 0$ to the government budget.

Abstracting from incentives, the reform would yield more taxes if $S(c^*, \theta) > \lambda$. However introducing noise affects incentives. Indeed, by (21), a necessary condition for incentive compatibility is $U'(\theta) = S'_\theta(c^*, \theta)v(\theta)$ for all θ. The reform thus implies $\mathrm{d}U'(\theta) = S'_\theta(c^*, \theta) \, \mathrm{d}v$. It follows that the sub-utility $U(\theta)$ is also affected: we have $\mathrm{d}U = \mathrm{d}U'(\theta) \, \mathrm{d}\theta = S'_\theta(c^*, \theta) \, \mathrm{d}v \, \mathrm{d}\theta$ for all types above $\theta + \mathrm{d}\theta$. Neglecting the second-order utility change for types between θ and $\theta + \mathrm{d}\theta$, the impact on the total collected tax is $-(1 - F(\theta))S'_\theta(c^*, \theta) \, \mathrm{d}v \, \mathrm{d}\theta$. Since $S'_\theta \leq 0$ this is a positive change: agents with high types are discouraged to mimic types between θ and $\theta + \mathrm{d}\theta$, which allows the government to reduce the transfers to high types and thus raises the collected tax.

Finally the change in tax induced by the introduction of a small noise on the after-tax income is

$$[S(c^*, \theta) - \lambda] \, f(\theta) \, \mathrm{d}v \, \mathrm{d}\theta - (1 - F(\theta))S'_\theta(c^*, \theta) \, \mathrm{d}v \, \mathrm{d}\theta,$$

or equivalently,

$$\left[S(c^*, \theta) - \lambda - m(\theta)S'_\theta(c^*, \theta)\right] f(\theta) \, \mathrm{d}v \, \mathrm{d}\theta.$$

Using the expressions of $S(c^*, \theta)$ and $S'_\theta(c^*, \theta)$ given in (22) and (23), one obtains

$$S(c^*, \theta) - \lambda - m(\theta)S'_\theta(c^*, \theta) = \lambda\phi(c^*, \theta).$$

This shows that $\phi(c^*, \theta)$ governs the change in the total collected tax that results from introducing small perturbations on the after-tax income of low type agents, while exploiting incentives to reduce the transfers received by high type agents.

7.3 Application to Generalized Weibull

In the expression of $\phi(c^*, \theta)$ given in Proposition 4, the term into the first bracket is the derivative in after-tax income c of the contribution of type θ to the deterministic objective $V(\theta^{\inf})$ given in (16). This contribution is concave in c. In addition, if (17) is satisfied for all θ, its global maximizer $c^{**}(\theta)$ is increasing in θ. Therefore $c^{**}(\theta) < c^*$ for low types. It follows that $\phi(c, \theta)$ evaluated at $c = c^*$ is negative for such types. This shape makes it difficult to get socially useful income randomization: in the presence of some income noise, the variance $v(\theta)$ has to be positive for low types, and so all the terms $\phi(c^*, \theta)v(\theta)$ that appear in the sum in Proposition 4 will be non positive for low types. This is however possible with our Weibull specification.

We consider

$$v(\theta) = \begin{cases} v > 0 & \text{for } \theta \leq \bar{\theta}^*, \\ 0 & \text{otherwise.} \end{cases}$$

In the case of the generalized Weibull distribution with parameters $a = 0.9$, $b = 0.01$ and $s = 5$, the function $\phi(c^*, \theta)f(\theta)$ evaluated at $c^* = 1$ is single-peaked in θ. It is negative for low and high types, but it reaches a positive value at its maximum. The sum

$$\int_{\theta^{\inf}}^{\bar{\theta}^*} \phi(c^*, \theta) \, \mathrm{d}F(\theta)$$

is positive for all values of the threshold $\bar{\theta}^*$ between 5/2 and 20. Thus, by Proposition 4, offering after-tax income lotteries with positive variance v to types below $\bar{\theta}^*$ chosen in this interval and a deterministic contract to the remaining types improves upon the deterministic optimum.

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A The deterministic optimum

The government is assumed to design a deterministic menu $(c(\theta), y(\theta))$ subject to the feasibility constraint (2) and the incentive constraints

$$\ln(c(\theta)+\theta) - y(\theta) \geq \ln(c(\tau)+\theta) - y(\tau)$$

for all θ and τ. The incentive constraints are satisfied if and only if

$$y'(\theta) = \frac{c'(\theta)}{c(\theta)+\theta} \Leftrightarrow V'(\theta) = \frac{1}{c(\theta)+\theta} \tag{25}$$

and

$$c'(\theta) \leq 0 \tag{26}$$

for all θ.

From (25), indirect utility is increasing with θ, and so type $\theta^{\inf}$ gets the lowest utility. A Rawlsian government thus chooses to maximize $V(\theta^{\inf})$ subject to (2), (25) and (26). Using (2) at equality and (25) the indirect utility of type θ is

$$V(\theta) = V(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} \frac{1}{c(z)+z}\,\mathrm{d}z, \tag{27}$$

where $V(\theta^{\inf})$ is given in (16).

The optimal schedule is an after-tax income $(c(\theta))$ that maximizes $V(\theta^{\inf})$ subject to the monotonicity condition $c'(\theta) \leq 0$ for all θ. The associated before-tax income then is $y(\theta) = \ln(c(\theta)+\theta) - V(\theta)$.

Let $c^{**}(\theta)$ be the (interior) after-tax income that maximizes the contribution

$$W(c(\theta),\theta) = \ln(c(\theta)+\theta) - c(\theta) - \frac{m(\theta)}{c(\theta)+\theta}$$

of type θ to the social surplus in (16). That is,

$$c^{**}(\theta) + \theta = \frac{1}{2}(1+\sqrt{1+4m(\theta)}). \tag{28}$$

The monotonicity condition (26) binds for all types if $c^{**}(\theta)$ is always increasing. Using (28), this is equivalent to (17).

The use of utility (15) yields a single tax bracket case at the optimum. Indeed, by Theorem 6.1 in Fleming and Rishel (1975), the after-tax income is continuous provided that the deterministic social program does not display multiple maximizers for some θ. Since $W(c,\theta)$ is concave in c, the program has at most one maximizer on an interval of types such that the second order monotonicity $c'(\theta) \leq 0$ does not bind at the deterministic optimum $(c'(\theta) < 0)$. Consider now an interval of types $[\theta_1, \theta_2]$ such that $c'(\theta) = 0$. For such types, $c(\theta) = \bar{c}$ such that

$$c(\theta_1) = c(\theta_2) = \bar{c}$$

satisfies

$$\int_{\theta_1}^{\theta_2} \left[\frac{1}{\bar{c}+\theta} - 1 + \frac{m(\theta)}{(\bar{c}+\theta)^2}\right] \mathrm{d}F(\theta) = 0$$

Use again $W(c,\theta)$ concave in c, so that the integrand is decreasing in $\bar{c}$, to conclude that there is a unique solution $\bar{c}$.

With single tax bracket, the objective (16) simplifies to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c+\theta) - c - \frac{m(\theta)}{c+\theta}\right] \mathrm{d}F(\theta) \tag{29}$$

An integration by parts yields

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \ln(c+\theta)\,\mathrm{d}F(\theta) = \ln(c+\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{m(\theta)}{c+\theta}\,\mathrm{d}F(\theta),$$

that is, $\ln(c+\theta^{\inf})-c$. The expressions of c^* and V^* given in the text follow.

B Proof of Proposition 4

We first compute the utility of type $\theta^{\inf}$ for a menu of lotteries that satisfies feasibility

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [c^* + \lambda\mu(\theta) - y(\theta)]\ \mathrm{d}F(\theta) \leq 0 \tag{30}$$

and the first-order necessary conditions (21) for the incentive constraints to be met. Replacing $y(\theta)$ with $u(c^*,\theta)+\beta\lambda u'_c(c^*,\theta)+S(c^*,\theta)v(\theta)-V(\theta)$ into (30) at equality, and using

$$V(\theta) = V(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} V'(z)\,\mathrm{d}z,$$

yields

$$V(\theta^{\inf}) = -\int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - u(c^*,\theta) - \beta\lambda u'_c(c^*,\theta) - S(c^*,\theta)v(\theta) + \int_{\theta^{\inf}}^{\theta} V'(z)\,\mathrm{d}z\right] \mathrm{d}F(\theta).$$

An integration by parts gives

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \int_{\theta^{\inf}}^{\theta} V'(z)\,\mathrm{d}z\ \ \mathrm{d}F(\theta) = \int_{\theta^{\inf}}^{\theta^{\sup}} m(\theta)V'(\theta)\,\mathrm{d}F(\theta).$$

And, using the expression of the social objective V^*, we get

$$V(\theta^{\inf}) - V^* = \int_{\theta^{\inf}}^{\theta^{\sup}} \left[\frac{m(\theta)}{c^* + \theta} - \lambda\mu(\theta) + \beta\lambda u'_c(c^*, \theta) + S(c^*, \theta)v(\theta) - m(\theta)V'(\theta)\right] \mathrm{d}F(\theta).$$

Finally, using (19), and the expressions of $S(c^*, \theta)$ and $S'_\theta(c^*, \theta)$, the change in utility $V'(\theta)$ in (24) and the characterization of c^* as the consumption maximizing the concave objective (29), we get

$$V(\theta^{\inf}) - V^* = \lambda \int_{\theta^{\inf}}^{\theta^{\sup}} \phi(c^*, \theta)v(\theta)\ \mathrm{d}F(\theta),$$

where $\phi(c, \theta)$ is given in Proposition 4.