Weak redistribution and certainty equivalent domination*

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Abstract

We assess optimal deterministic nonlinear income taxation in a Mirrlees economy with a continuum of risk-averse agents whose utilities are quasilinear in labor. A weak redistributive motive makes random taxes socially dominated by the deterministic policy where after-tax income lotteries are replaced with their certainty equivalent.

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1 Introduction

It is known that the optimal menu offered to risk averse agents may involve lotteries in the presence of asymmetric information. In public finance lotteries take the form of random taxes yielding random after-tax income, such as in Weiss (1976), Stiglitz (1981), Stiglitz (1982) or Brito, Hamilton, Slutsky, and Stiglitz (1995). The main economic intuition is straightforward. Suppose that the government would like to redistribute income to low-skilled in a population of risk-averse workers. Redistribution is potentially limited if the government observes neither skill nor the exact amount of labor, as high-skilled then might reduce labor effort to enjoy higher transfers. Introducing randomness in the after-tax income designed for low-skilled is detrimental to their welfare, but this also expands the scope of possible redistribution by discouraging high-skilled from relaxing effort.

A deterministic optimum obtains if the welfare cost incurred by those facing noise overcomes the gain from expanded scope of redistribution. This is more likely to happen if high-skilled do not suffer much from income noise. Hellwig (2007) indeed shows that a unweighted utilitarian (Benthamite) government should rely on deterministic redistribution if risk aversion decreases with labor productivity, i.e., risk aversion is higher for low than high skilled.

This paper explores the more general case of a weighted utilitarian government. The impact of relaxing the assumption of equal weighting of every agent in the social welfare function is a priori ambiguous. On the one hand, the suffering of the less well-off part of the population that faces random noise is magnified; one may think for instance of the utility cost of random allocation in social housing. On the other hand, however, greater redistribution desires put more pressure on incentives, as bundles designed for the poor become more desirable. The potential social welfare gains from discouraging the rich to mimic the poor are therefore magnified.

We analyze the welfare impact of removing lotteries in the redistribution policy by switching to the menu of certainty equivalents associated with these lotteries. We find that maintaining incentive compatibility in the menu of certainty equivalents puts strong limits on the social welfare gains from switching to the deterministic menu. Eventually the deterministic policy improves upon the menu of lotteries in the case of weak redistribution motives, with a social welfare function close enough to the Benthamite pattern. Sharper redistribution motives instead reduce the likelihood that the certainty equivalents dominate random redistribution.

The paper proceeds as follows. Our setup is described in Section 2. Section 3 provides conditions for incentive compatibility of menus of lotteries. Section 4 compares lotteries and the associated certainty equivalents. Incentive compatibility of the menu of certainty equivalents is analyzed in Section 5. Section 6 concludes.

2 General framework

The general setup is as in Gauthier and Laroque (2023). A government wants to redistribute income between a continuum of agents in a population of total unit size. Every agent is indexed by her type θ, a real parameter taking values in Θ. The type has cdf $F : \Theta \to [0, 1]$ associated with positive probability density function $f : \Theta \to \mathbb{R}_{++}$. The preferences of a type θ agent are represented by the quasilinear utility function

$$u(c, \theta) - y \tag{1}$$

when she earns before-tax income y and pays $y - c$ as tax. The after-tax income c is also her consumption. Earning y requires providing an effort, hence the disutility cost. The function u is increasing, differentiable everywhere in c and θ, and strictly concave in c. It satisfies the Spence-Mirrlees condition that the cross-derivative $u''_{c\theta}(c, \theta)$ is negative for all (c, θ).

The government offers a menu $(\tilde{c}(\theta), \tilde{y}(\theta))$ of after and before-tax income lotteries. The menu is feasible if aggregate consumption falls below aggregate production,

$$\int_{\Theta} \mathbb{E}\left[\tilde{c}(\theta) - \tilde{y}(\theta)\right] \, \mathrm{d}F(\theta) \leq 0. \tag{2}$$

If θ is private information to the agent, the government must also ensure that every agent chooses the income pair designed for her. This is satisfied if the incentive constraints

$$\mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right] \geq \mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)] \tag{3}$$

hold for all (θ, τ) in $\Theta \times \Theta$.

The social welfare objective is

$$\int_{\Theta} \tilde{V}(\theta) \, \mathrm{d}G(\theta), \tag{4}$$

where $\tilde{V}(\theta) = \mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right]$ is the indirect utility of type θ. The social weights embodied in $G(\cdot)$ are non-negative and normalized so that they sum up to 1. Hellwig (2007) considers a unweighted utilitarian government where every agent is valued equally, $F(\theta) = G(\theta)$ for all θ. We allow for general utilitarian preferences where the utility of type θ can be assigned any given (positive) weight in the objective.

An optimal redistribution policy is a menu of lotteries that maximizes the objective (4) subject to the feasibility constraint (2) and the incentive constraints (3).

A deterministic policy consists of degenerated lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ yielding the sure outcome $(c(\theta), y(\theta))$. In view of the quasilinear utility (1), replacing the lottery $\tilde{y}(\theta)$ with the sure outcome $\mathbb{E}[\tilde{y}(\theta)]$ affects neither the constraints (2) and (3) nor the objective (4). Any social gain from a deterministic policy must therefore come from making the after-tax certain.

3 Dealing with incentives

Suppose that the lottery $\tilde{c}(\theta)$ designed for a type θ agent is such that its owner receives an after-tax income smaller than c with probability $H(c, \theta)$, $c \in [c^{\text{inf}}, c^{\text{sup}}]$, associated with density $h(c, \theta)$. We restrict our attention to menus of lotteries such that $H(c, \theta)$ is continuously differentiable in θ, and we denote by $H'_\theta(c, \theta)$ its partial derivative in θ and $h'_\theta(c, \theta)$ the partial derivative of the associated density.

Lemma 1. *The incentive constraints (3) associated with a menu of lotteries $(\tilde{c}(\theta), y(\theta))$ are satisfied only if*

$$\tilde{V}'(\theta) = \mathbb{E}\left[u'_\theta(\tilde{c}(\tau), \theta)\right] \tag{5}$$

and

$$\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau), \theta)\right] \geq 0 \tag{6}$$

for all θ and $\tau = \theta$. These conditions are sufficient for incentive compatibility if (6) holds true for all θ and τ.

Proof. The proof reproduces standard arguments used in the case of deterministic contracts; see, e.g., Section 2.3 in Salanié (2017). The incentive constraints (3) can be rewritten as

$$\theta = \arg\max_{\tau} \mathbb{E}[u(\tilde{c}(\tau), \theta)] - y(\tau)$$

for all θ. This requires that the truthful report $\tau = \theta$ is a local extremum of the utility $\mathbb{E}[u(\tilde{c}(\tau), \theta)] - y(\tau)$, i.e.,

$$\frac{\partial}{\partial \tau}\mathbb{E}[u(\tilde{c}(\tau), \theta)] - y(\tau) = 0 \tag{7}$$

at $\tau = \theta$. Using the envelope theorem, this is equivalent to (5).

In addition, truthful reporting $\tau = \theta$ must be a local maximizer of the utility. This is the case if $\mathbb{E}[u(\tilde{c}(\tau), \theta)] - y(\tau)$ is locally concave in τ at the extremum $\tau = \theta$. If (7) holds at $\tau = \theta$ for all θ, we have

$$\begin{aligned}\frac{\partial^2}{\partial \tau^2}\left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - y(\tau)\right) &= -\frac{\partial^2}{\partial \tau \partial \theta}\left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - y(\tau)\right) \\ &= -\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau), \theta)\right]\end{aligned}$$

at $\tau = \theta$. Local concavity thus reads as (6).

Conditions (5) and (6) are necessary and sufficient for $\mathbb{E}[u(\tilde{c}(\tau), \theta)] - y(\tau)$ to stand below $\mathbb{E}[u(\tilde{c}(\theta), \theta)] - y(\theta)$ for all τ close to θ. They do not ensure that truthful reporting is a global maximum. A sufficient condition for a global maximum obtains by observing that, using (7) with $\theta = \tau$,

$$\frac{\partial}{\partial \tau}\left(\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - y(\tau)\right) = \int_\tau^\theta \frac{\partial}{\partial \tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau), z)\right] \, \mathrm{d}z.$$

If (6) holds true for all τ and θ, then the right-hand side of this equality has the same sign as $\theta - \tau$, which implies that $\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] - y(\tau)$ is single-peaked in τ with a global maximum attained at $\tau = \theta$. ■

The inequality (6) relates to agents preferences and properties of the lotteries under scrutiny. This seems at odds with to the familiar case of deterministic menus, where under the Spence-Mirrlees condition the first-order approach is valid if and only if consumption is non-increasing with type θ. Actually Lemma 2 shows that for a special class of menus incentive compatibility obtains if and only if (3) is met.

Lemma 2. *Suppose that $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and θ_2, $\theta_1 < \theta_2$. Then inequality (6) holds true for all τ and θ. Therefore incentive compatibility obtains if and only if (5) is met.*

Proof. For the class of lotteries under scrutiny, we have

$$\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_{\theta}\left(\tilde{c}(\tau),\theta\right)\right] = \int\limits_{c^{\inf}}^{c^{\sup}} u'_{\theta}\left(c,\theta\right)\mathrm{d}H'_{\theta}(c,\tau).$$

Using the integration by parts formula yields

$$\int\limits_{c^{\inf}}^{c^{\sup}} u'_{\theta}\left(c,\theta\right)\mathrm{d}H'_{\theta}(c,\tau) = \left[u'_{\theta}\left(c,\theta\right)H'_{\theta}(c,\tau)\right]_{c^{\inf}}^{c^{\sup}} - \int\limits_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c,\theta\right)H'_{\theta}(c,\tau)\,\mathrm{d}c.$$

Since $H(c^{\inf},\tau) = 0$ and $H(c^{\sup},\tau) = 1$ for all τ, we have $H'_{\theta}(c^{\inf},\tau) = H'_{\theta}(c^{\sup},\tau) = 0$ for all τ. It follows that

$$\int\limits_{c^{\inf}}^{c^{\sup}} u'_{\theta}\left(c,\theta\right)\mathrm{d}H'_{\theta}(c,\tau) = -\int\limits_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c,\theta\right)H'_{\theta}(c,\tau)\,\mathrm{d}c.$$

The Spence-Mirrlees condition $u''_{\theta c}\left(c,\theta\right) < 0$ for all (c,θ) implies that (6) holds true for all τ and θ if $H'_{\theta}(c,\theta) \geq 0$ for all (c,θ), i.e., $H(c,\theta_1) \leq H(c,\theta_2)$ for all c and $\theta_1 \leq \theta_2$. This corresponds to the case where $H(c,\theta_1)$ first-order stochastically dominates $H(c,\theta_2)$. ■

Lemma 2 provides us with a natural generalization of the familiar monotonicity condition for incentive compatibility in a deterministic environment. Under the Spence-Mirrlees condition, the monotonicity of the deterministic consumption with type is replaced with a stochastic dominance ranking of lotteries. Namely, higher types face higher probabilities of receiving low amounts of consumption. Note, however, that first-order stochastic dominance is sufficient, but not necessary for local incentive compatibility.

4 Certainty equivalent domination

Suppose that we switch from a feasible and incentive compatible menu of lotteries $(\tilde{c}(\theta), y(\theta))$ to the deterministic menu where type θ instead gets the after-tax income certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ with probability 1 and produces $y(\theta) - \delta(\theta)$ for some deterministic before-tax income adjustment

$\delta(\theta)$. The certainty equivalent $\mathbb{C}(\tilde{c},\theta)$ of type θ when facing lottery $\tilde{c}$ is the sure consumption such that

$$u\left(\mathbb{C}(\tilde{c},\theta),\theta\right)=\mathbb{E}\left[u\left(\tilde{c},\theta\right)\right]. \tag{8}$$

The associated risk premium is $\pi(\tilde{c},\theta)=\mathbb{E}\left[\tilde{c}\right]-\mathbb{C}(\tilde{c}(\tau),\theta)$.

In the class of lotteries under scrutiny, the certainty equivalent $\mathbb{C}(\tilde{c}(\tau),\theta)$ is continuously differentiable in τ and θ, for all τ and θ in $\Theta\times\Theta$. We denote its first derivatives in τ and θ by $\mathbb{C}'_\tau(\tilde{c}(\tau),\theta)$ and $\mathbb{C}'_\theta(\tilde{c}(\tau),\theta)$, respectively. Our assumptions on u imply that $\mathbb{C}(\tilde{c}(\tau),\theta)$ is continuously differentiable in θ. The behavior of the certainty equivalent with τ is restricted, however, as it requires that any given after-tax income is received by neighboring types with neighboring probabilities. This leaves open the possibility that deterministic, but non-differentiable menus perform better than lotteries, a case that the present setup does not cover.

For such lotteries, we have:

Proposition 1. *Consider a feasible and incentive compatible menu of lotteries $(\tilde{c}(\theta),y(\theta))$. There is a deterministic menu where every type θ gets the certainty equivalent $(\mathbb{C}(\tilde{c}(\theta),\theta))$ that is feasible, incentive compatible, and improves upon lotteries if and only if:*

1. *the inequality*

$$\int_{\theta^{\inf}}^{\theta^{\sup}}\left[\pi(\tilde{c}(\theta),\theta)-\frac{G(\theta)-F(\theta)}{f(\theta)}\pi'_\theta(\tilde{c}(\theta),\theta)u'_c(\mathbb{C}(\tilde{c}(\theta),\theta),\theta)\right]\mathrm{d}F(\theta)>0 \tag{9}$$

   is met.

2. $\mathbb{C}(\tilde{c}(\theta),\theta)$ *is non-increasing in θ.*

A proof is in Appendix A. An intuition for the inequality (9) obtains from the simpler reform that removes randomness for just a small part of the population, i.e., we replace $\tilde{c}(\theta)$ with $\mathbb{C}(\tilde{c}(\theta),\theta)$ for all θ between some $\underline{\theta}$ and $\overline{\theta}=\underline{\theta}+\mathrm{d}\theta$, $\mathrm{d}\theta>0$ small. The after-tax income, random or not, is unchanged outside the interval $[\underline{\theta},\overline{\theta}]$.

Within this interval, incentives call for a before-tax income adjustment. The utility from consumption obtained by θ when mimicking $\tau\in\left[\underline{\theta},\overline{\theta}\right]$ is $\mathbb{E}\left[u(\tilde{c}(\tau),\theta)\right]=u(\mathbb{C}(\tilde{c}(\tau),\theta),\theta)$ before the reform, and $u(\mathbb{C}(\tilde{c}(\tau),\tau),\theta)$ after the reform. The reform contributes to relax the incentive constraint involving these two types if $\mathbb{C}(\tilde{c}(\tau), \theta) > \mathbb{C}(\tilde{c}(\tau), \tau) \simeq (\theta - \tau)\mathbb{C}'_\theta(\tilde{c}(\tau), \tau) > 0$ for θ close to τ. The before-tax income adjustment that compensates for the change in utility from consumption thus obeys

$$\delta'(\theta) = -\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c \left(\mathbb{C}(\tilde{c}(\theta), \theta), \theta\right).$$

Since only the before-tax income can change for types outside $[\underline{\theta}, \overline{\theta}]$, incentive compatibility requires that $\delta(\theta)$ is some uniform amount $\underline{\delta}$ for all $\theta \leq \underline{\theta}$, and $\overline{\delta}$ for all $\theta \geq \overline{\theta}$. Relying on the approximation $\bar{\delta} \simeq \underline{\delta} + \delta'(\underline{\theta})\, \mathrm{d}\theta$, we have

$$\bar{\delta} - \underline{\delta} \simeq -\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c \left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right)\, \mathrm{d}\theta. \tag{10}$$

The adjustments $\underline{\delta}$ and $\bar{\delta}$ follow from (10) and the feasibility constraint that total resources

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[y(\theta) - \delta(\theta)\right]\, \mathrm{d}F(\theta)$$

must finance consumption

$$\int_{\theta^{\inf}}^{\underline{\theta}} \mathbb{E}\left[\tilde{c}(\theta)\right]\, \mathrm{d}F(\theta) + \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \mathbb{C}(\tilde{c}(\theta), \theta)\, \mathrm{d}F(\theta) + \int_{\underline{\theta}+\mathrm{d}\theta}^{\theta^{\sup}} \mathbb{E}\left[\tilde{c}(\theta)\right]\, \mathrm{d}F(\theta).$$

Replacing $\mathbb{C}(\tilde{c}(\theta), \theta)$ with $\mathbb{E}\left[\tilde{c}(\theta)\right] - \pi(\tilde{c}(\theta), \theta)$ and using (2) at equality, we get

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta)\, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \pi(\tilde{c}(\theta), \theta)\, \mathrm{d}F(\theta). \tag{11}$$

Hence the total amount of before-tax income resources created by the reform equals the aggregate risk premium of agents who no longer face income risk. For $\mathrm{d}\theta$ close to 0, we have

$$\underline{\delta} F(\underline{\theta}) + \bar{\delta}\left(1 - F(\underline{\theta})\right) \simeq \pi(\tilde{c}(\underline{\theta}), \underline{\theta}) f(\underline{\theta})\, \mathrm{d}\theta, \tag{12}$$

where the left-hand side uses $F(\bar{\theta}) \simeq F(\underline{\theta}) + f(\underline{\theta})\mathrm{d}\theta$ and neglects the second-order term $(\bar{\delta} - \underline{\delta}) f(\underline{\theta})\, \mathrm{d}\theta$.

The system formed by (10) and (12) gives the $\underline{\delta}$ and $\bar{\delta}$ adjustments consistent with feasibility and incentive compatibility at the outcome of the reform,

$$\underline{\delta} \simeq \left[\pi(\tilde{c}(\underline{\theta}),\underline{\theta})f(\underline{\theta}) + (1 - F(\underline{\theta}))\,\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta}\right)\right]\,\mathrm{d}\theta,$$

and

$$\bar{\delta} \simeq \left[\pi(\tilde{c}(\underline{\theta}),\underline{\theta})f(\underline{\theta}) - F(\underline{\theta})\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta}\right)\right]\,\mathrm{d}\theta.$$

The total change in before-tax income resources $\pi(\tilde{c}(\underline{\theta}),\underline{\theta})f(\underline{\theta})\,\mathrm{d}\theta$ in (12) is positive. Therefore the deterministic menu always improves upon the menu of lotteries in the absence of redistributive concerns.

If redistributive concerns matter, the reform improves social welfare if $\underline{\delta}G(\underline{\theta}) + \bar{\delta}\,(1 - G(\underline{\theta})) > 0$, which is

$$\pi(\tilde{c}(\underline{\theta}),\underline{\theta})f(\underline{\theta}) + (G(\underline{\theta}) - F(\underline{\theta}))\,\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c\left(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta}\right) > 0.$$

Using $\mathbb{C}'_\theta(\tilde{c}(\theta),\theta) = -\pi'_\theta(\tilde{c}(\theta),\theta)$, this is the expression that appears in (9).

The inequality (9) validates the intuitive idea that an economy consisting of agents who display high risk aversions ($\pi(\tilde{c}(\theta),\theta)$ is high) tends to be immune from socially beneficial income tax randomizations.

However the shape of the distribution of risk aversion in the population matter as well. Indeed (9) is always satisfied if the risk premium is identical across agents, $\pi'_\theta(\tilde{c},\theta) = 0$ for all $\tilde{c}$ and θ. This happens if taxpayers have the same preferences, $u(c,\theta)$ does not depend on θ. This can also accommodate preference heterogeneity. For instance, in the multiplicative formulation $u(c,\theta) = \theta v(c)$, the certainty equivalent of a lottery $\tilde{c}$ is defined by $v(\mathbb{C}(\tilde{c},\theta)) = \mathbb{E}\left[v(\tilde{c})\right]$, and so it does not depend on type θ.

To address the role played by redistribution social tastes, observe first that (9) is satisfied independently of risk aversions for unweighted utilitarian social preferences ($G(\theta) = F(\theta)$ for all θ). Weak redistribution motives thus tend to favor deterministic taxation tools. It also holds for weighted utilitarian social preferences if $\left[G(\theta) - F(\theta)\right]\pi'_\theta(\tilde{c},\theta)$ is non-positive for all types, i.e., if the socially favored agents display a higher risk aversion (captured by a higher risk premium). In this configuration, the optimal deterministic redistribution policy dominates random policies that have the optimal deterministic menu as certainty equivalent, a property in line with Hellwig (2007). Equivalently, the optimal redistribution policy, if involving random taxes while (9) is satisfied, must be associated with a certainty equivalent menu which fails to meet the monotonicity conditions for incentive compatibility; otherwise there would exist a feasible incentive compatible deterministic menu improving upon the optimal deterministic policy, a contradiction.

Our Spence-Mirrlees assumption makes the marginal utility of consumption decreasing with θ. This suggests to interpret low types as more likely poor, since they put a high value on consumption. It is usually argued that the poor display higher risk aversions than the rich. If redistribution favors the poor, $G(\theta) \geq F(\theta)$ for all θ and $\pi'_\theta(\tilde{c}, \theta) \leq 0$, and so (9) is met. The deterministic menu, if incentive compatible, performs better than the lotteries and all redistribution should be made deterministically.

As will be shown in Section 5, however, the most relevant case for (9) instead is the one where $[G(\theta) - F(\theta)]\, \pi'_\theta(\tilde{c}, \theta)$ is positive. Incentive considerations indeed point to $\pi'_\theta(\tilde{c}, \theta) \geq 0$. Then (9) is no longer necessarily satisfied. The conflict between redistribution to the poor, captured by $G(\theta) - F(\theta) > 0$, and incentives weakens the case for deterministic redistribution.

CARA-Gaussian example. In the CARA-Gaussian case, (9) is satisfied for a high enough level of risk aversion independently of the shape of $G(\theta) - F(\theta)$, i.e., whatever the social desires for redistribution. Type θ agents have CARA preferences

$$u(c, \theta) = -\frac{1}{\theta} \exp(-\theta c),$$

with θ her absolute risk aversion coefficient. They face Gaussian after-tax income lotteries $(\tilde{c}(\theta))$ with mean $m(\tilde{c}(\theta))$ and variance $v(\tilde{c}(\theta)) > 0$. Since

$$\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] = -\frac{1}{\theta} \exp\left[-\theta \left(m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau)) \right)\right],$$

the certainty equivalent of lottery $\tilde{c}(\tau)$ for a type θ agent is

$$\mathbb{C}(\tilde{c}(\tau), \theta) = m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau)),$$

which is assumed to be positive, and the risk premium

$$\pi(\tilde{c}(\tau), \theta) = \frac{\theta}{2} v(\tilde{c}(\tau)).$$

Hence inequality (9) is equivalent to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} v(\theta) \left[\theta - (G(\theta) - F(\theta)) \exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right)\right] \mathrm{d}\theta > 0.$$

Since both $G(\theta) - F(\theta) \leq 1$ and $\exp(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)) \leq 1$ for all θ (the after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-negative), we have $(G(\theta) - F(\theta)) \exp(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)) \leq 1$ for all θ. Hence, for $\theta^{\inf} \geq 1$, every term in the sum in (9) is non-negative, and so (9) is met. □

Although (9) suggests that the certainty equivalent menu performs better than the lottery, we must be careful when drawing such a conclusion. Indeed Condition 2 in Proposition 1 requires that the lotteries are associated with lower certainty equivalents for higher types. It is not clear whether this requirement is consistent with incentive compatibility of the initial menu of lotteries. This is what we study in the next Section.

5 Monotone certainty equivalents

If $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$, then any given risk averse agent θ prefers $\tilde{c}(\theta_1)$, so $\mathbb{C}(\tilde{c}(\theta_1), \theta) \geq \mathbb{C}(\tilde{c}(\theta_2), \theta)$ for all θ. This ranking of lotteries in terms of stochastic dominance does not imply monotonicity of the certainty equivalents, which instead requires a more demanding comparison between $\mathbb{C}(\tilde{c}(\theta_1), \theta_1)$ and $\mathbb{C}(\tilde{c}(\theta_2), \theta_2)$, i.e., how different types of agents value these two lotteries. Lemma 2 thus is not directly useful to assess Condition 2 in Proposition 1: incentive compatibility of the lotteries does not guarantee incentive compatibility of the certainty equivalents.

To delineate the more demanding circumstances where the menu of certainty equivalents meets the incentive constraints, observe that differentiation of (8) in θ yields

$$\frac{\mathrm{d}\mathbb{C}(\tilde{c}(\theta), \theta)}{\mathrm{d}\theta} \leq 0$$

$$\Leftrightarrow \mathbb{E}\left[u'_\theta\left(\tilde{c}(\theta), \theta\right)\right] - u'_\theta(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) + \int_{c^{\inf}}^{c^{\sup}} u(c, \theta) h'_\theta(c, \theta)\, \mathrm{d}c \leq 0. \qquad (13)$$

Recall that $\mathbb{C}(\tilde{c}(\theta), \theta) = \mathbb{E}[\tilde{c}(\theta)] - \pi(\tilde{c}(\theta), \theta)$ with $\pi(\tilde{c}(\theta), \theta)$ a positive risk premium. The Spence-Mirrlees condition $u''_{c\theta}(c, \theta) < 0$ yields

$$u'_\theta(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) > u'_\theta(\mathbb{E}[\tilde{c}(\theta)], \theta).$$

Therefore,

$$\mathbb{E}\left[u'_\theta\left(\tilde{c}(\theta), \theta\right)\right] - u'_\theta(\mathbb{C}(\tilde{c}(\theta), \theta), \theta) < \mathbb{E}\left[u'_\theta\left(\tilde{c}(\theta), \theta\right)\right] - u'_\theta(\mathbb{E}[\tilde{c}(\theta)], \theta).$$

It follows that $\mathbb{E}\left[u'_\theta\left(\tilde{c}(\theta),\theta\right)\right] - u'_\theta(\mathbb{C}(\tilde{c}(\theta),\theta),\theta) \leq 0$ if

$$\mathbb{E}\left[u'_\theta\left(\tilde{c}(\theta),\theta\right)\right] \leq u'_\theta(\mathbb{E}\left[\tilde{c}(\theta)\right],\theta),$$

a condition that is satisfied if $u'_\theta(c,\theta)$ is concave in c. In this case the difference between the first two terms in (13) is negative.

Following the insights in Lemma 2, we now account for stochastic dominance properties. Suppose that the initial menu of lotteries is such that the lotteries designed for lower types first-order stochastically dominate those designed for higher types, $H(c,\theta+\mathrm{d}\theta) \leq H(c,\theta)$ for all c, θ and $\mathrm{d}\theta < 0$. For neighboring types, $\mathrm{d}\theta \simeq 0$, this inequality reads

$$H'_\theta(c,\theta) \geq 0.$$

To exploit this property, we use the integration by parts formula to rewrite

$$\int_{c^{\inf}}^{c^{\sup}} u(c,\theta)h'_\theta(c,\theta)\,\mathrm{d}c = -\int_{c^{\inf}}^{c^{\sup}} u'_c(c,\theta)H'_\theta(c,\theta)\,\mathrm{d}c,$$

Utility is increasing with consumption, so that this sum is negative. It follows that:

Lemma 3. *Let $\mathbb{C}(\tilde{c},\theta)$ be the type θ certainty equivalent consumption associated with lottery $\tilde{c}$. The certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ is non-increasing in θ if*

1. *$\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and θ_2, with $\theta_1 < \theta_2$.*
2. *$u'_\theta(c,\theta)$ is concave in c.*

Condition 1 relates to the initial menu of lotteries. Lotteries designed for low types dominate those designed for high types: any given agent prefers the lotteries designed for low types to those designed for high types. In view of Lemma 2, circumstances for incentive compatibility of the initial menu of lotteries are in line with those for incentive compatibility of the associated menu of certainty equivalents. Keeping with the interpretation of low types as being the poor, incentives tend to be preserved if the preferred bundles are designed for the less well-off part of the population.

Condition 2 instead is on individual preferences. It allows us to get a better understanding of the status of the inequality (9) in Proposition 1.

If one refers to concavity of the utility function u in c as a measure of risk aversion, then the monotonicity properties of the certainty equivalent needed for implementing this deterministic menu obtain if higher types display a higher risk aversion. That is, $u''_{cc}(c, \theta)$, which takes negative values in the presence of risk aversion, is decreasing with θ. Condition 2 thus fails to be satisfied in the empirically plausible case where the poor, rather than the rich, display the greatest risk aversions. This suggests that incentive compatibility of the menu of lotteries tends to be inconsistent with incentive compatibility of the menu of the certainty equivalents. In practice, the menu of certainty equivalents likely violates second-order monotonicity condition for incentive compatibility.

6 Conclusion

We have provided necessary and sufficient conditions for incentive compatibility of a menu of consumption or after-tax income lotteries. If the marginal utility of consumption is decreasing with type, the first-order approach can be applied if the lotteries designed for low types first-order stochastically dominate those designed for high types. In our setup low types give more importance to consumption than high types. Low types may accordingly be considered as the less well-off part of the population. When one switches to the deterministic menu where every consumption lottery is replaced with the associated certainty equivalent consumption, incentive compatibility is compromised if the poor display the greatest risk aversions, a case that is usually considered as the most relevant in practice. Redistributing consumption in a deterministic way is costly as this makes incentive requirements more difficult to meet, but this also yields more tax resources coming from the confiscated risk premia. As a result of this conflict, deterministic redistribution through certainty equivalents dominates random redistribution for weak enough redistribution motives in high enough risk-averse populations.

Although the menu of certainty equivalents provides us with a natural benchmark, other deterministic menus could improve upon the menu of lotteries. Hence it may be that redistribution should be made deterministically while the certainty equivalents are dominated or fail to meet incentive compatibility. The role played by bunching in the deterministic optimum is examined in Gauthier and Laroque (2023). On the other hand, incentive compatibility of menus of lotteries may obtain even though they fail to meet the ranking of first-order stochastic dominance. It is not clear how such menus can then be consistent with preservation of incentives in certainty equivalents. This opens new room for optimal random menus.

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A Proof of Proposition 1

The switch to the menu of certainty equivalent incomes leads to a change in social welfare (4) equal to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [u(\mathbb{C}(\tilde{c}(\theta),\theta),\theta) + \delta(\theta))]\,\mathrm{d}G(\theta) - \int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\,[u(\tilde{c}(\theta),\theta)]\,\mathrm{d}G(\theta).$$

Using (8), this change reduces to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta)\ \mathrm{d}G(\theta). \tag{14}$$

Social welfare improves if agents with high social valuations enjoy a reduction in their before-tax income.

The before-tax income adjustments $(\delta(\theta))$ must meet feasibility (2) and incentive compatibility (3). The incentive constraints associated with the final deterministic schedule are

$$\theta = \arg\max_{\tau} u(\mathbb{C}(\tilde{c}(\tau),\tau),\theta) - y(\tau) + \delta(\tau) \tag{15}$$

for all θ. The differentiability assumption made on $\mathbb{C}(\tilde{c}(\tau),\theta)$ and the fact that type $\tau = \theta$ solves the maximization program in (15) imply that the before-tax income $y(\tau)+\delta(\tau)$ is also continuously differentiable (see Guesnerie and Laffont (1984), Theorem 1). Incentive compatibility thus requires

$$[\mathbb{C}'_{\tau}(\tilde{c}(\tau),\tau) + \mathbb{C}'_{\theta}(\tilde{c}(\tau),\tau)]\,u'_c(\mathbb{C}(\tilde{c}(\tau),\tau),\theta) - y'(\tau) + \delta'(\tau) = 0 \tag{16}$$

for all θ and τ, $\tau = \theta$. The incentive constraints for the initial random menu $(\tilde{c}(\theta), y(\theta))$,

$$\theta = \arg\max_{\tau} \mathbb{E}\,[u(\tilde{c}(\tau),\theta)] - y(\tau) = \arg\max_{\tau} u(\mathbb{C}(\tilde{c}(\tau),\theta),\theta) - y(\tau)$$

for all θ, require $\mathbb{C}'_{\tau}(\tilde{c}(\tau),\theta)u'_c(\mathbb{C}(\tilde{c}(\tau),\theta),\theta) - y'(\tau) = 0$ for all θ and τ, $\tau = \theta$. Hence (16) simplifies to

$$\delta'(\theta) = -\mathbb{C}'_{\theta}(\tilde{c}(\theta),\theta)u'_c(\mathbb{C}(\tilde{c}(\theta),\theta),\theta) \tag{17}$$

for all θ. By summation over types we obtain

$$\delta(\theta) = \delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta} \mathbb{C}'_\theta(\tilde{c}(z), z) u'_c(\mathbb{C}(\tilde{c}(z), z), z)\, \mathrm{d}z \tag{18}$$

for all θ.

The feasibility constraint (2) at equality gives the value of $\delta(\theta^{\inf})$. The derivation is as follows. After replacing in (2) the sure income $\mathbb{C}(\tilde{c}(\theta), \theta)$ with the difference $\mathbb{E}[\tilde{c}(\theta)] - \pi(\tilde{c}(\theta), \theta)$, the feasibility constraint takes the form

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta) - \mathbb{E}[\tilde{c}(\theta)] + \pi(\tilde{c}(\theta), \theta)]\ \mathrm{d}F(\theta) = 0.$$

Since the initial random menu $(\tilde{c}(\theta), y(\theta))$ also meets (2), this equality simplifies to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [\pi(\tilde{c}(\theta), \theta) - \delta(\theta)]\ \mathrm{d}F(\theta) = 0.$$

The change in before-tax income resources equals the total risk premium that the government can extract from the risk-averse agents when it removes risk. Using (18) to relate $\delta(\theta)$ to $\delta(\theta^{\inf})$, and the identity $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$ finally gives

$$\delta(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left[\pi(\tilde{c}(\theta), \theta) - \frac{1 - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\right]\ \mathrm{d}F(\theta).$$

We are now in a position to write the change in social welfare from a reform replacing lotteries with certainty equivalent incomes. Reintroducing the expression of $\delta(\theta)$ found in (18) into (14) and using the integration by parts formula, the change in social welfare rewrites

$$\delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - G(\theta)}{f(\theta)} \mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\, \mathrm{d}F(\theta). \tag{19}$$

The expression of $\delta(\theta^{\inf})$ derived above yields the inequality stated in Condition 1. Indeed replacing the random menu with the deterministic one yields a social welfare improvement if and only if (14) is positive. The inequality in Proposition 1 obtains after using the expression of $\delta(\theta)$ and $\delta(\theta^{\text{inf}})$ in (19).

To prove the statement in Condition 2, observe that the monotonicity condition on $\mathbb{C}(\tilde{c}(\theta), \theta)$ is necessary for incentive compatibility. Under the Spence-Mirrlees condition, it also ensures that the incentive constraints hold for every type θ and every report τ. This concludes the proof.