Random taxation and redistribution

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Abstract

We assess optimal random nonlinear income taxation in a Mirrlees economy with a continuum of risk-averse agents agents whose utilities are quasilinear in labor. A weak redistributive motive makes random taxes socially dominated by the deterministic policy where after-tax income lotteries are replaced with their certainty equivalent. In a parametric example, a strong redistributive motive instead makes lotteries locally dominate the optimal deterministic menu. In this example, the downward pattern of incentives prevailing in the deterministic case is reversed to an upward pattern in the stochastic case.

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1 Introduction

Solutions to optimization programs that we encounter in economics may involve randomness because of failures of convexity assumptions. Such failures often are due to the incentive constraints used in the presence of informational asymmetries. The allocation designed for a given agent then influences both her actual utility and the utility that would be obtained by those who stand ready to mimic this agent. The same allocation thus appears in the two sides of incentive constraints, implying nonconvexities. Randomizing over allocations located on the frontier of the set delimited by the constraints, if allowed, leads to allocations on the convex hull of the constraints of the deterministic program. Nonconvexities make that the two constraint sets, deterministic or random, differ and so open the possibility that some allocations in the convex hull improve upon the best deterministic alternative.[1]

Early contributions to public finance in Weiss (1976), Stiglitz (1981), Stiglitz (1982) or Brito, Hamilton, Slutsky, and Stiglitz (1995) have exploited this feature to show that deterministic redistribution sometimes is socially dominated. In the recent studies of Lang (2017), Ederer, Holden, and Meyer (2018) or Lang (2021), randomization is viewed as a way to contain gaming social rules by making incentive schemes opaque; courts may opt for vague standards and allow for legal uncertainty to discourage firms to undertake socially detrimental strategies, and health authorities may prefer that hospitals are not fully aware of the exact reimbursement system to prevent them from selecting healthy patients in order to get a higher volume-based reimbursement.[2]

The main theoretical argument developed in this literature is especially insightful in Hellwig (2007) and Miller, Wagner, and Zeckauser (2010). Suppose that the government would like to redistribute income from high to low skill in a population of risk-averse workers. Redistribution is potentially limited if skills remain private information to workers, as high skilled may be willing to mimic low skilled by reducing labor effort. Introducing randomness in the income designed for low skilled is detrimental to their welfare, but this also expands the scope of possible redistribution by relaxing incentive constraints.[3]

[1]Such failures may also be due to the convexity of individual objectives, rather than non-convexity of the set of constraints. For instance, Hines and Keen (2021) use the convexity of profit with respect to price of a firm under perfect competition to show that tax uncertainty may be optimal.

[2]See also Pavlov (2011), Gauthier and Laroque (2014), Pycia and Unver (2015) or Gauthier and Laroque (2017) for a more mathematical approach.

[3]Similar arguments are also developed in industrial organization, following Gjesdal (1982). The revenue maximizing rule of a monopolist selling multiple goods to a single buyer may involve lotteries when the amount that buyers are willing to pay is privately known; see, e.g., Thanassoulis (2004), Manelli and Vincent (2010), Hart and Reny (2015) and Rochet and Thanassoulis (2019). In the context of optimal insurance design, Jeleva and Villeneuve (2004) derives conditions for insurers to offer random insurance contracts where reimbursements in a given state are made conditional to the realization of some other random circumstances. In a different vein, Bianchi and Jehiel (2015) shows that a principal may gain by

A deterministic optimum obtains if the direct welfare cost bearing on those who actually face noise overcomes the gain from the expanded scope of redistribution. This is more likely to happen if low skilled are strongly risk averse while high skilled do not suffer much from the noise, since then noise does not deter much the high skilled from mimicking the low skilled workers. Hellwig (2007) indeed shows that random redistribution is socially useless if risk aversion decreases with labor productivity. The property is established in a Mirrlees optimal taxation model with a continuum of skill types of workers, each one with general individual preferences on many consumption goods. However social preferences are supposed to fit an unweighted Utilitarian criterion.

The role played by social redistributive concerns is not entirely clear. On the one hand, a high social valuation put on low skilled tends to reinforce the argument in Hellwig (2007) as it magnifies the valuation by the society of the suffering of these agents when they face randomness in taxes and transfers. But, on the other hand, it also affects the reference scheme of deterministic redistribution where risk attitudes are evaluated and to which random redistribution is compared.

In this paper, we reexamine the issue of social usefulness of random redistribution in the Mirrlees model. We simplify the exposition compared to Hellwig (2007) by assuming that individual preferences are quasilinear in labor, as in Lollivier and Rochet (1983) or Weymark (1987). This allows us to deal with a more general redistributive stance obeying a weighted Utilitarian objective, the Rawlsian limit case included.

We obtain a necessary and sufficient condition for improving upon a redistributive policy where income is random by proposing instead the certainty equivalent of the random income. The condition shows that random redistribution is socially useless in the unweighted Utilitarian case. Random redistribution is also dominated if the government puts a higher social weight on the agents with the highest risk premia, which yields a natural generalization of Hellwig (2007).

But the positive correlation between social valuation and risk aversion is not necessary for random redistribution to be socially useless. This is because the level of risk aversion matters on top of its correlation with social valuation. A high risk aversion enables the government to extract more resources from risk averse agents when certainty equivalent incomes replace lotteries. Random redistribution is found useless in a population of agents who are highly risk averse, provided that the government does not value too much low risk aversion types. A weak redistribution motive favors deterministic policies.

The certainty equivalent allows us to shed light on the role played by the shape of the redistributive stance in yielding a deterministic optimal taxation rule. However, other deterministic policies may possibly improve upon random taxes and transfers. We show

keeping agents uninformed about some aspects of the environment, i.e., to offer a menu of contracts that does not remove all sources of uncertainty.

that one cannot improve upon the optimal deterministic redistribution policy by relying on tax randomization if the deterministic policy involves no bunching, that is the incentive constraints do not restrict the government to offer the same pair of before and after-tax income to a pool of different types of agents.

Optimal income randomizations thus need high social redistribution concerns and bunching in the deterministic optimum. We exploit these two features to build a parametric example where the optimal deterministic policy of a Rawlsian government is socially dominated by a random policy where the lotteries consist of small local randomizations around the deterministic optimum. To the best of our knowledge, this is the first example of useful randomization in the Mirrlees optimal taxation model with a continuum of types of workers.

The example reveals a new channel through which noise operates in the redistribution process. The gain from the relaxation of the incentive constraints works through a reversal of the incentive pattern. The reversal of incentives works as follows. If the government restricts to a deterministic policy, then incentives go downward, with every type envying the neighboring smaller types. The high social desire to redistribute to low skilled leads to bunching taking the form of an egalitarian policy where income is equalized between all the agents. The familiar downward pattern is reversed when noise is introduced, and so risk aversion matters: the direction of incentives now is upward, with every type envying neighboring higher types. Local randomizations thus allow for a full alignment of incentives and social preferences. Eventually the social gain from this alignment is enough to overcome the cost from the income noise bearing on risk averse agents.

The paper proceeds as follows. Our setup is described in Section 2. Section 3 studies when a deterministic menu of certainty equivalent is socially better than the original menu of lotteries; some intuition relying on a marginal argument is given in Section 4. The role played by bunching in the deterministic optimum is presented in Section 5 and exploited in the parametric example in Section 6.

2 General framework

A government wants to redistribute income between a continuum of agents in a population of total unit size. Every agent is indexed by her type θ, a real parameter taking values in $\Theta = \left[\theta^{\inf}, \theta^{\sup}\right]$. The type has cumulative distribution function $F : \Theta \to [0, 1]$ associated with positive probability density function $f : \Theta \to \mathbb{R}_{++}$. The preferences of a type θ agent are represented by the quasilinear utility function

$$u(c, \theta) - y \tag{1}$$

when she earns a before-tax income y and pays the tax $y - c$. The before-tax income y requires providing a labor effort, hence the disutility cost. The after-tax income c is also her consumption. The function u is increasing, strictly concave in c and differentiable in θ. It satisfies the Spence-Mirrlees condition that the cross-derivative $u''_{c\theta}(c, \theta)$ keeps a constant sign for all (c, θ). Without loss of generality the cross-derivative is taken as negative.

The government offers a menu of contracts $(\tilde{c}(\theta), \tilde{y}(\theta))$ that consists of after and before-tax income lotteries. The menu is feasible if the aggregate consumption falls below the aggregate production,

$$\int_{\Theta} \mathbb{E}\left[\tilde{c}(\theta) - \tilde{y}(\theta)\right] \mathrm{d}F(\theta) \leq 0. \tag{2}$$

If θ is private information to the agent, the government must also ensure that every agent chooses the contract designed for her. This is satisfied if the incentive constraints

$$\mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right] \geq \mathbb{E}[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)] \tag{3}$$

hold true for all (θ, τ) in $\Theta \times \Theta$.

An optimal redistribution policy is a menu of lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ that maximizes the social welfare objective

$$\int_{\Theta} \tilde{V}(\theta)\, \mathrm{d}G(\theta), \tag{4}$$

where $\tilde{V}(\theta) = \mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{y}(\theta)\right]$ represents the indirect utility of type θ. The maximization is made subject to the feasibility constraint (2) and the incentive constraints (3).

The social weights embodied in $G(\cdot)$ are non-negative and normalized so that they sum up to 1 in the population. The government values equally each agent if $F(\theta) = G(\theta)$ for all θ, which corresponds to unweighted Utilitarian social preferences. Our paper allows for more general Utilitarian preferences where the utility of type θ can be assigned any given (positive) weight in the social welfare objective.

3 Certainty equivalent domination

We first are interested in conditions ensuring a deterministic optimal redistribution policy. A deterministic policy consists of degenerated lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ yielding the outcome $(c(\theta), y(\theta))$ with probability 1. In view of the quasilinear form of utility (1), there is no loss in considering nonrandom before-tax income profiles $(y(\theta))$ only. Indeed for every θ replacing the lottery $\tilde{y}(\theta)$ with the sure outcome $\mathbb{E}[\tilde{y}(\theta)]$ affects neither the constraints (2) and (3) nor the objective (4).

Any social gain from a deterministic policy must therefore come from making the after-tax certain. Here we consider the switch from a feasible and incentive compatible menu of lotteries to the deterministic menu where type θ instead gets her after-tax income certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ with probability 1 and produces $y(\theta)-\delta(\theta)$ for some well-chosen differentiable $\delta(\theta)$. The certainty equivalent after-tax income $\mathbb{C}(\tilde{c},\theta)$ of type θ when she faces the lottery $\tilde{c}$ is the sure consumption such that

$$u\left(\mathbb{C}(\tilde{c},\theta),\theta\right)=\mathbb{E}\left[u\left(\tilde{c},\theta\right)\right].$$

It is associated with the risk premium $\pi(\tilde{c},\theta)=\mathbb{E}\left[\tilde{c}\right]-\mathbb{C}(\tilde{c},\theta)$.

The switch to certainty equivalent incomes implies a change in social welfare (4) equal to

$$\int_{\theta^{\inf}}^{\theta^{\sup}}\left[u(\mathbb{C}(\tilde{c}(\theta),\theta),\theta)-y(\theta)+\delta(\theta))\right]\mathrm{d}G(\theta)-\int_{\theta^{\inf}}^{\theta^{\sup}}\left[\mathbb{E}\left[u(\tilde{c}(\theta),\theta)\right]-y(\theta)\right]\mathrm{d}G(\theta).$$

Using the characterization of the certainty equivalent, this change reduces to

$$\int_{\theta^{\inf}}^{\theta^{\sup}}\delta(\theta)\ \mathrm{d}G(\theta). \tag{5}$$

That is, social welfare improves if agents with high social valuations enjoy a reduction in their before-tax income.

To assess whether the government may find socially profitable to appeal to the certainty equivalent after-tax incomes associated with the initial random menu, we need to express the before-tax income adjustment $\delta(\theta)$ as a function of certainty equivalent incomes. To this aim we use the fact that the final deterministic menu of after-tax certainty equivalent incomes must fit the feasibility constraint (2) and incentive compatibility requirements (3).

We first deal with incentive compatibility. When faced with the deterministic income tax schedule, the incentive constraints can be written

$$\theta=\arg\max_{\tau}u(\mathbb{C}(\tilde{c}(\tau),\tau),\theta)-y(\tau)+\delta(\tau)$$

for all θ and τ. They require that the necessary first-order conditions

$$\left[\mathbb{C}'_{\tau}(\tilde{c}(\tau),\tau)+\mathbb{C}'_{\theta}(\tilde{c}(\tau),\tau)\right]u'_{c}(\mathbb{C}(\tilde{c}(\tau),\tau),\theta)-y'(\tau)+\delta'(\tau)=0 \tag{6}$$

are satisfied for all θ when evaluated at $\tau=\theta$. Our assumption that the initial random menu $(\tilde{c}(\theta),y(\theta))$ satisfies the incentive constraints,

$$\theta=\arg\max_{\tau}\mathbb{E}\left[u(\tilde{c}(\tau),\theta)\right]-y(\tau)=\arg\max_{\tau}u(\mathbb{C}(\tilde{c}(\tau),\theta),\theta)-y(\tau)$$

for all θ and τ, allows us to simplify (6) as

$$\mathbb{C}'_\tau(\tilde{c}(\tau), \theta) u'_c(\mathbb{C}(\tilde{c}(\tau), \theta), \theta) - y'(\tau) = 0 \tag{7}$$

when evaluated at $\tau = \theta$. Thus (6) can be written

$$\delta'(\theta) = -\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta). \tag{8}$$

Finally we sum (8) over types to obtain

$$\delta(\theta) = \delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta} \mathbb{C}'_\theta(\tilde{c}(z), z) u'_c(\mathbb{C}(\tilde{c}(z), z), z)\, \mathrm{d}z \tag{9}$$

for all θ.

We now exploit the feasibility constraint (2) at equality to get the value of $\delta(\theta^{\inf})$. We replace in (2) the sure after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ with the difference $\mathbb{E}[\tilde{c}(\theta)] - \pi(\tilde{c}(\theta), \theta)$, so that the feasibility constraint takes the form

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta) - \mathbb{E}[\tilde{c}(\theta)] + \pi(\tilde{c}(\theta), \theta)]\ \mathrm{d}F(\theta) = 0.$$

Since the initial random menu $(\tilde{c}(\theta), y(\theta))$ also meets (2), this equality becomes

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [\pi(\tilde{c}(\theta), \theta) - \delta(\theta)]\ \mathrm{d}F(\theta) = 0.$$

The change in before-tax income resources thus equals the total risk premium that the government can extract from the risk-averse agents when it removes risk. Using the expression of $\delta(\theta)$ obtained in (9) and the identity $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) = -\pi'_\theta(\tilde{c}(\theta), \theta)$ gives

$$\delta(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \pi(\tilde{c}(\theta), \theta)\, \mathrm{d}F(\theta) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - F(\theta)}{f(\theta)} \pi'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\, \mathrm{d}F(\theta).$$

We are now in a position to write the change in social welfare from a reform replacing the initial lotteries with their certainty equivalent incomes. Reintroducing the expression of $\delta(\theta)$ found in (9) into (5) and using the integration by parts formula, the change in social welfare rewrites

$$\delta(\theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - G(\theta)}{f(\theta)} \mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c(\mathbb{C}(\tilde{c}(\theta), \theta), \theta)\, \mathrm{d}F(\theta). \tag{10}$$

The expression of $\delta(\theta^{\inf})$ yields the following result:

Proposition 1. *Consider a feasible and incentive compatible menu of lotteries $(\tilde{c}(\theta), y(\theta))$. Suppose that the change in social welfare (5) is positive, i.e.,*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\pi(\tilde{c}(\theta),\theta) - \frac{G(\theta)-F(\theta)}{f(\theta)}\pi'_\theta(\tilde{c}(\theta),\theta)u'_c(\mathbb{C}(\tilde{c}(\theta),\theta),\theta)\right] \mathrm{d}F(\theta) > 0. \tag{11}$$

If $\mathbb{C}(\tilde{c}(\theta),\theta)$ is non-increasing in θ, then the deterministic menu where every type θ gets the sure outcome $\mathbb{C}(\tilde{c}(\theta),\theta)$ satisfies both feasibility and incentive compatibility, and it yields a higher social welfare than $(\tilde{c}(\theta), y(\theta))$.

Argument. Replacing the random menu with the deterministic one yields a social welfare improvement if and only if (5) is positive. The inequality in Proposition 1 directly obtains after using the expression of $\delta^{\inf}$ in (10). Under the Spence-Mirrlees condition, the monotonicity condition on $\mathbb{C}(\tilde{c}(\theta),\theta)$ ensures that the incentive constraints hold for every type θ and every report τ. Properties of incentive compatible menus are given in Appendix A. Corollary A1 in this Appendix gives sufficient conditions for the certainty equivalent after-tax income to decrease with θ. ■

Condition (11) shows how agents' risk aversion interacts with the social tastes for redistribution embodied in the weighting $G(\theta)$ to favor deterministic redistribution policies.

First, it validates the intuitive idea that an economy consisting of agents who display high risk aversions ($\pi(\tilde{c}(\theta),\theta)$ is high) tends to be immune from socially beneficial income tax randomizations. This may be seen as a partial converse of Ederer, Holden, and Meyer (2018) where the best redistribution policy involves income randomness whenever taxpayers have low risk aversions.

But the magnitude of risk aversion must be weighed against the shape of the distribution of risk aversion in the population. Indeed the inequality (11) is satisfied regardless of the level of risk aversion if the risk premium is the same for each type, $\pi'_\theta(\tilde{c},\theta) = 0$ for all $\tilde{c}$ and θ. That is, violations of (11), which make the menu of lotteries socially better than its certainty equivalent counterpart, require enough heterogeneity in agents' risk aversions. Of course, taxpayers display the same risk aversion if they have the same preferences, $u(c,\theta)$ does not depend on θ. But a uniform risk aversion is also consistent with some heterogeneity in preferences. For instance, in the multiplicative utility formulation $u(c,\theta) = \theta v(c)$ used by e.g., Lollivier and Rochet (1983), the certainty equivalent is defined by $v(\mathbb{C}(\tilde{c},\theta)) = \mathbb{E}[v(\tilde{c})]$, which is type independent.

An important insight from Proposition 1 is that the exploitation of heterogeneity in risk aversion to justify using random perturbations into the tax system can be done only if there is some social redistribution concerns. In the case of unweighted Utilitarian social preferences, i.e., every agent is weighted equally in the social welfare objective ($G(\theta) = F(\theta)$ for all θ), the inequality (11) is always satisfied and so the optimal redistribution policy is deterministic.

In the weighted Utilitarian case, (11) is met if $[G(\theta) - F(\theta)]\, \pi'_\theta(\tilde{c}, \theta)$ is non-positive for all types, i.e., if the socially favored agents display a higher risk aversion (captured by a higher risk premium). This finding has a very similar flavor as those found by Hellwig (2007) for more general individual preferences than our quasilinear utility specification, but with no redistribution motive.

Still, a positive correlation between the social weight in $G(\theta)$ and risk aversion appears as not necessary for relying on deterministic redistribution.[4] We exploit this feature in the parametric example in Section 6 where a high (Rawlsian) redistributive stance yields a non-deterministic optimal policy.

CARA-Gaussian example. Type θ agents have CARA preferences

$$u(c, \theta) = -\frac{1}{\theta} \exp(-\theta c),$$

and they face Gaussian after-tax income lotteries $(\tilde{c}(\theta))$ with mean $m(\tilde{c}(\theta))$ and variance $v(\tilde{c}(\theta)) > 0$. Then (11) is satisfied independently of the social tastes for redistribution embodied in $G(\theta)$. To show this property, recall that

$$\mathbb{E}\left[u(\tilde{c}(\tau), \theta)\right] = -\frac{1}{\theta} \exp\left[-\theta \left(m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau))\right)\right].$$

This yields the certainty equivalent

$$\mathbb{C}(\tilde{c}(\tau), \theta) = m(\tilde{c}(\tau)) - \frac{\theta}{2} v(\tilde{c}(\tau))$$

and the risk premium

$$\pi(\tilde{c}(\tau), \theta) = \frac{\theta}{2} v(\tilde{c}(\tau)).$$

Hence inequality (11) is equivalent to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} v(\theta) \left[\theta - (G(\theta) - F(\theta)) \exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right)\right] \mathrm{d}\theta > 0.$$

Since both $G(\theta) - F(\theta) \leq 1$ and $\exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right) \leq 1$ for all θ (the after-tax income $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-negative), we have $(G(\theta) - F(\theta)) \exp\left(-\theta \mathbb{C}(\tilde{c}(\theta), \theta)\right) \leq 1$ for all θ. For $\theta^{\inf} \geq 1$, every term in the sum is non-negative. ■

[4] Proposition 3 restricts to menus where the certainty equivalent $\mathbb{C}(\tilde{c}(\theta), \theta)$ is non-increasing with θ. This is necessary for incentive compatibility of the deterministic menu. This is consistent with any sign of the partial derivative $\mathbb{C}'_\theta(\tilde{c}(\theta), \theta)$.

4 A marginal interpretation

The role played by redistribution may be better seen by considering a tax reform that removes after-tax income randomness for just a small part of the population. We replace the lottery $\tilde{c}(\theta)$ with the certainty equivalent $\mathbb{C}(\tilde{c}(\theta), \theta)$ for every type in the interval $[\underline{\theta}, \overline{\theta}]$, $\overline{\theta} = \underline{\theta} + \mathrm{d}\theta$ for positive small $\mathrm{d}\theta$, while it remains unchanged for types outside this interval.

Within the interval concerned by the reform, incentives call for a before-tax income adjustment that obeys (8),

$$\delta'(\theta) = -\mathbb{C}'_\theta(\tilde{c}(\theta), \theta) u'_c \left(\mathbb{C}(\tilde{c}(\theta), \theta), \theta\right).$$

Since the contracts offered to types outside this interval only change by the before-tax income adjustment, incentive compatibility requires that $\delta(\theta)$ is some uniform amount $\underline{\delta}$ for every $\theta \leq \underline{\theta}$, and $\bar{\delta}$ for every $\theta \geq \overline{\theta}$. Since $\bar{\delta} \simeq \underline{\delta} + \delta'(\underline{\theta})\,\mathrm{d}\theta$, these two before-tax income adjustments are linked by

$$\bar{\delta} \simeq \underline{\delta} - \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c \left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right) \mathrm{d}\theta. \tag{12}$$

The before-tax income adjustments $\underline{\delta}$ and $\bar{\delta}$ follow from (12) and the feasibility constraint at the outcome of the reform. The total resources

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - \delta(\theta)]\; \mathrm{d}F(\theta)$$

must finance the total amount of redistributed consumption

$$\int_{\theta^{\inf}}^{\underline{\theta}} \mathbb{E}\left[\tilde{c}(\theta)\right]\; \mathrm{d}F(\theta) + \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \mathbb{C}(\tilde{c}(\theta), \theta)\, \mathrm{d}F(\theta) + \int_{\underline{\theta}+\mathrm{d}\theta}^{\theta^{\sup}} \mathbb{E}\left[\tilde{c}(\theta)\right]\; \mathrm{d}F(\theta).$$

Replacing $\mathbb{C}(\tilde{c}(\theta), \theta)$ with $\mathbb{E}\left[\tilde{c}(\theta)\right] - \pi(\tilde{c}(\theta), \theta)$ and using (2) at equality, we get

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \delta(\theta)\, \mathrm{d}F(\theta) = \int_{\underline{\theta}}^{\underline{\theta}+\mathrm{d}\theta} \pi(\tilde{c}(\theta), \theta)\, \mathrm{d}F(\theta). \tag{13}$$

For $\mathrm{d}\theta$ close to 0, we have

$$\underline{\delta} F(\underline{\theta}) + \bar{\delta}\,(1 - F(\underline{\theta})) \simeq \pi(\underline{\theta}) f(\underline{\theta})\, \mathrm{d}\theta. \tag{14}$$

The system formed by (12) and (14) gives the two before-tax income adjustments $\underline{\delta}$ and $\bar{\delta}$ consistent with both feasibility and incentive compatibility at the outcome of the reform. Substituting $\bar{\delta}$ from (12) into (14) yields

$$\underline{\delta} \simeq \left[\pi(\underline{\theta}) f(\underline{\theta}) + (1 - F(\underline{\theta}))\, \mathbb{C}'_\theta(\tilde{c}(\underline{\theta}), \underline{\theta}) u'_c \left(\mathbb{C}(\tilde{c}(\underline{\theta}), \underline{\theta}), \underline{\theta}\right)\right] \mathrm{d}\theta,$$

and one can finally get $\bar{\delta}$ from (12),

$$\bar{\delta} \simeq [\pi(\underline{\theta})f(\underline{\theta}) - F(\underline{\theta})\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta})]\,\mathrm{d}\theta.$$

The total change in before-tax income resources $\pi(\underline{\theta})f(\underline{\theta})\,\mathrm{d}\theta$ in (14) is positive; in the absence of redistributive concerns, the deterministic menu always improves upon the menu of lotteries. Instead, if redistributive concerns matter, the reform improves social welfare if $\underline{\delta}G(\underline{\theta}) + \bar{\delta}(1 - G(\underline{\theta})) > 0$, or equivalently, using $\mathrm{d}\theta > 0$,

$$\pi(\underline{\theta})f(\underline{\theta}) + (G(\underline{\theta}) - F(\underline{\theta}))\,\mathbb{C}'_\theta(\tilde{c}(\underline{\theta}),\underline{\theta})u'_c(\mathbb{C}(\tilde{c}(\underline{\theta}),\underline{\theta}),\underline{\theta}) > 0.$$

This is the expression that is summed up in (11), with $\mathbb{C}'_\theta(\tilde{c}(\theta),\theta) = -\pi'_\theta(\tilde{c}(\theta),\theta)$.

An interpretation proceeds as follows. Consider a type θ agent mimicking another type τ within the interval concerned by the reform, $\tau \in [\underline{\theta}, \underline{\theta} + \mathrm{d}\theta]$. Some lottery $\tilde{c}(\tau)$ is designed for type τ before the reform, while this type receives the certainty equivalent income $\mathbb{C}(\tilde{c}(\tau),\tau)$ at the outcome of the reform. It follows that the expected utility from consumption of type θ when mimicking type τ is $\mathbb{E}[u(\tilde{c}(\tau),\theta)]$ before the reform, and $u(\mathbb{C}(\tilde{c}(\tau),\tau),\theta)$ after the reform. The reform relaxes the incentive constraint involving these two types only if $\mathbb{E}[u(\tilde{c}(\tau),\theta)] > u(\mathbb{C}(\tilde{c}(\tau),\tau),\theta)$. Since $\mathbb{E}[u(\tilde{c}(\tau),\theta)] = u(\mathbb{C}(\tilde{c}(\tau),\theta),\theta)$, this is equivalent to $\mathbb{C}(\tilde{c}(\tau),\theta) > \mathbb{C}(\tilde{c}(\tau),\tau)$, which is $(\theta - \tau)\mathbb{C}'_\theta(\tilde{c}(\tau),\tau) > 0$ for θ close to τ.

Suppose now that small type agents display a higher risk aversion, $\mathbb{C}'_\theta(\tilde{c},\theta) > 0$. In this case, the reform relaxes the incentive constraint only if $\theta > \tau$, i.e., the downward pattern for incentives prevails. Then the reform enables the government to transfer before-tax income resources from high to low types. Indeed (4) shows that $\bar{\delta} < \underline{\delta}$ if $\mathbb{C}'_\theta(\tilde{c}(\tau),\tau) > 0$, so that there is a greater reduction of before-tax income for low types. A high social valuation put on low types ($G(\theta) \geq F(\theta)$) makes it more likely that high types envy low ones, and the switch to deterministic redistribution is socially useful.

If, instead, the government values high types ($G(\theta) \leq F(\theta)$), then it becomes more plausible that incentives imply high types envying low ones. Incentives force the government to leave more resources to low social valuation agents, which yields a loss to the society. This loss must be balanced against the social gain due to the extra resources coming from the extraction of the risk premium of agents concerned by the reform. In this case, a strong redistribution motive makes the move to the certainty equivalent undesirable.

5 Deterministic policies and bunching

So far, we have analyzed whether certainty equivalents improve upon lotteries. We now reverse the perspective and use instead as reference the income menu in the optimal deterministic redistribution policy. We give a sufficient condition for this menu to be also optimal within the larger class of menus of lotteries.

Following Hellwig (2007) an optimal 'relaxed' redistribution policy is a menu $(\tilde{c(\theta)}, y(\theta))$ maximizing the social objective (4) subject to the feasibility constraint (2) and the necessary first-order conditions for a truthful report,

$$\tilde{V}'(\theta) = \mathbb{E}\left[u'_\theta(\tilde{c}(\theta), \theta)\right],$$

for all θ. If the optimal relaxed redistribution policy meets the incentive constraints (3), then it is also the optimal policy. Otherwise the necessary first-order conditions for a truthful report are not sufficient for incentive compatibility and some bunching of types must occur.

Summing up over types the first-order conditions yields

$$\tilde{V}(\theta) = \tilde{V}(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} \mathbb{E}\left[u'_\theta(\tilde{c}(z), z)\right] \, \mathrm{d}z.$$

The feasibility constraint (2) with $y(\theta) = \mathbb{E}\left[u(\tilde{c}(\theta), \theta)\right] - \tilde{V}(\theta)$ gives the indirect utility of the least type $\theta^{\inf}$, after using the integration by parts formula,

$$\tilde{V}(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left(\mathbb{E}\left[u(\tilde{c}(\theta), \theta) - \tilde{c}(\theta)\right] - \frac{1 - F(\theta)}{f(\theta)} \mathbb{E}\left[u'_\theta(\tilde{c}(\theta), \theta)\right] \right) \, \mathrm{d}F(\theta).$$

The menu $(\tilde{c}(\theta))$ of after-tax income lotteries in an optimal relaxed redistribution program thus maximizes

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \tilde{V}(\theta) \, \mathrm{d}G(\theta) = \int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\left[W(\tilde{c}(\theta), \theta)\right] \, \mathrm{d}F(\theta) \tag{15}$$

where

$$W(c, \theta) = u(c, \theta) - c - \frac{G(\theta) - F(\theta)}{f(\theta)} u'_\theta(c, \theta)$$

represents the virtual contribution of type θ to social welfare when this type faces an after-tax income c with certainty.

Suppose that the optimal deterministic redistribution policy involves no bunching. This policy consists of a menu $(c^*(\theta))$ where every type θ gets for sure the after-tax income

$$c^*(\theta) = \arg\max_c W(c, \theta).$$

Since then $W(c^*(\theta), \theta) \geq W(c, \theta)$ for all (c, θ), we have $W(c^*(\theta), \theta) \geq \mathbb{E}\left[W(\tilde{c}, \theta)\right]$ for all $\tilde{c}$ and θ. In particular the inequality holds true if for all θ we set $\tilde{c}$ equal to the lottery $\tilde{c}(\theta)$ that maximizes (15). This yields:

Lemma 1. *A random redistribution policy is socially useless if the optimal deterministic redistribution policy involves no bunching.*

Proof. The argument given in the text leads to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \mathbb{E}\left[W(\tilde{c}(\theta),\theta)\right] \mathrm{d}F(\theta) \leq \int_{\theta^{\inf}}^{\theta^{\sup}} W(c^*(\theta),\theta)\,\mathrm{d}F(\theta).$$

for every menu $(\tilde{c}(\theta))$. To conclude the proof, observe that social welfare in an optimal redistribution policy cannot be greater than in an optimal relaxed policy. ■

Lemma 1 says that socially useful randomness in the redistribution policy cannot obtain if the incentive constraints associated with a deterministic redistribution policy do require an equal treatment of different types of agents.

6 An example of useful randomization

The paper provided us with two main insights on the social usefulness of redistribution policies with random after-tax incomes: Proposition 1 suggests that optimal income randomizations need a high redistributive motive, while Lemma 1 shows that the optimal deterministic redistribution policy must involve bunching. We exploit these two features to build a parametric example where random redistribution increases social welfare.

In this example, a Rawlsian government would like to favor the lowest type $\theta^{\inf}$ of agents only, but incentives require a uniform treatment of every agent in the optimal deterministic policy. Income randomization allows the government to relax the incentive constraints and to discriminate.

The utility of type θ, $\theta \in \left[\theta^{\inf}, \theta^{\sup}\right]$, $\theta^{\inf} = 0$, is

$$u(c,\theta) - y = \ln(c+\theta) - y. \tag{16}$$

Her indirect utility is $V(\theta) = \ln(c(\theta)+\theta) - y(\theta)$ when she faces the pair $(c(\theta), y(\theta))$ with certainty. The corresponding incentive constraints

$$V(\theta) \geq V(\tau) + \ln(c(\tau)+\theta) - \ln(c(\tau)+\tau)$$

show that the informational rent $\ln(c(\tau)+\theta) - \ln(c(\tau)+\tau)$ is positive for $\tau < \theta$. That is, type θ agents contemplate mimicking type τ, $\tau < \theta$, which fits the downward incentive pattern often encountered in the literature. Using the envelope theorem, the necessary first-order conditions for a truthful report are

$$V'(\theta) = \frac{1}{c(\theta)+\theta}$$

for all θ. Indirect utility must increase with type. As a result, the Rawlsian government only considers the welfare of the least type $\theta^{\inf}$ but it has to leave higher rents to higher types for incentive reasons. Otherwise incentive compatibility would fail.

Using (15) in Section 5, the optimal deterministic redistribution policy maximizes

$$V(\theta^{\inf}) = \int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c(\theta)+\theta) - c(\theta) - \frac{1-F(\theta)}{f(\theta)} \frac{1}{c(\theta)+\theta} \right] \mathrm{d}F(\theta) \tag{17}$$

subject to the second-order monotonicity condition for a truthful report

$$c'(\theta) \leq 0 \tag{18}$$

for all θ.

In all the sequel, we assume that the Mills ratio

$$m(\theta) = \frac{1-F(\theta)}{f(\theta)}$$

satisfies

$$m'(\theta) - [1+4m(\theta)]^{1/2} > 0, \tag{19}$$

for all θ. The most common distributions display a decreasing Mills ratio ($m'(\theta) < 0$ for all θ) and so they cannot meet (19). Still the inequality holds for all but a small negligible subset of high types if θ is distributed according to a variant of Weibull distribution.

Generalized Weibull distribution. Given two real positive shape parameters a and b, and a real positive scale parameter $s > 0$, θ is distributed according to a Generalized Weibull distribution if

$$1 - F(\theta) = \exp(1 - (1 + s\theta^b)^a)$$

for all $\theta > 0$. For $a < 1$ and $b \leq 1$ the associated hazard rate is monotone decreasing and therefore $m(\theta)$ is increasing for all $\theta > 0$. See Dimitrakopoulou, Adamidis, and Loukas (2007) for properties of this distribution. For $a = 0.9$, $b = 0.01$ and $s = 5$ the left-hand side of the inequality (19) is decreasing in θ, from positive to negative values. There is therefore a threshold $\bar{\theta}$ such that the inequality (19) is satisfied for all $\theta < \bar{\theta}$. With the specific values chosen for the parameters a, b and s, the threshold $\bar{\theta}$ is equal to 6.39. At the threshold $F(\bar{\theta}) = 98.4\%$, i.e., (19) is satisfied for all types of agents but a small negligible subset consisting of the highest types. ■

The inequality (19) implies that the after-tax income maximizing pointwise $V(\theta^{\inf})$ is increasing with θ. The optimal 'relaxed' deterministic policy thus violates the monotonicity requirement (18). The optimal deterministic redistribution policy involves full equalization of after-tax incomes.[5] Every type θ earns the same before-tax-income, $y(\theta) = y^*$, and gets the same after-tax income, $c^*(\theta) = c^*$. Feasibility reads as $y^* = c^*$, so that eventually no income can be redistributed. In this situation, the utility of the least type equals $V(\theta^{\inf}) = \ln(c^* + \theta^{\inf}) - c^*$. It is maximized by setting $c^* = 1 - \theta^{\inf} = 1$ (since $\theta^{\inf} = 0$), which yields social welfare $V(\theta^{\inf}) = -1$.

6.1 Incentives and local tax randomization

We now introduce small local randomizations taking the form of income lotteries $\tilde{c}(\theta) = c^* + \tilde{\varepsilon}(\theta)$ where realizations of $\tilde{\varepsilon}(\theta)$ stand close to 0. The mean and variance of the random perturbation $\tilde{\varepsilon}(\theta)$ are parametrized so that $\mathbb{E}[\tilde{\varepsilon}(\theta)] = \lambda\mu(\theta)$ and $\mathrm{var}[\tilde{\varepsilon}(\theta)] = \lambda v(\theta)$, with λ a positive real number close to 0, and $\mu(\theta)$ and $v(\theta) \geq 0$ arbitrary bounded real numbers.

Relying on lotteries allows the government to take advantage of agents' risk aversions. The (second-order Taylor expansion of the) expected utility of type θ when she chooses the lottery $\tilde{c}(\tau)$ designed for some type τ writes

$$\mathbb{E}[u(c^* + \tilde{\varepsilon}(\tau), \theta)] \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta)\left(\mu(\tau) - \frac{A(c^*, \theta)}{2} v(\tau)\right), \tag{20}$$

where

$$A(c, \theta) = -\frac{u''_{cc}(c, \theta)}{u'_c(c, \theta)} = \frac{1}{c + \theta} > 0$$

is the coefficient of absolute risk aversion of type θ. Note that, given the after-tax income c, lower types display a higher risk aversion, captured by a higher coefficient of risk aversion. In particular, the socially favored type $\theta^{\inf}$ is the most risk averse in the population.

The analysis of incentive compatibility simplifies drastically if one restricts attention to the special case where the mean and the variance of the perturbation are linearly related. Here we set

$$\mu(\theta) = v(\theta) \tag{21}$$

for all θ. For this class of lotteries, the expected utility in (20) becomes

$$\mathbb{E}[u(c^* + \tilde{\varepsilon}(\tau), \theta)] \simeq u(c^*, \theta) + \lambda u'_c(c^*, \theta)\left(1 - \frac{A(c^*, \theta)}{2}\right) v(\tau),$$

so that the indirect utility $\tilde{V}(\theta) = \mathbb{E}[u(c^* + \tilde{\varepsilon}(\theta), \theta)] - y(\theta)$ of type θ can be expressed as

$$\tilde{V}(\theta) \simeq u(c^*, \theta) + U(\theta), \tag{22}$$

where

$$U(\theta) = S(c^*, \theta) v(\theta) - y(\theta) \tag{23}$$

[5] See Appendix B for a detailed derivation of the optimal deterministic menu.

and

$$S(c,\theta) = \lambda u_c'(c,\theta)\left(1 - \frac{A(c,\theta)}{2}\right) = \frac{\lambda}{c+\theta}\left(1 - \frac{1}{2}\frac{1}{c+\theta}\right). \tag{24}$$

The incentive constraints (3) then become

$$\begin{aligned} U(\theta) = S(c^*,\theta)v(\theta) - y(\theta) &\geq S(c^*,\theta)v(\tau) - y(\tau) \\ &= U(\tau) + (S(c^*,\theta) - S(c^*,\tau))v(\tau). \end{aligned} \tag{25}$$

for all τ and θ. Local income tax randomizations within the class of perturbations (21) give rise to incentive constraints which have the familiar quasilinear textbook shape.

They reveal a surprizing feature about the incentive pattern. Since $S(c^*,\theta)$ is decreasing with θ, the rent given to type θ agents, $S(c^*,\theta) - S(c^*,\tau)$, is positive for $\theta < \tau$. In this example, the introduction of income lotteries leads to a reversal of incentives compared to the case of a deterministic policy: low types contemplate mimicking high types, i.e., the incentive pattern goes upward rather than downward. This is valuable for the government as incentives are now aligned with social preferences, but noise comes at a cost for risk averse agents. Actually the trade-off between the alignment of incentives and the cost from income noise is especially sharp in this example: the following result shows that the lowest type $\theta^{\inf}$ must face the highest income volatility.

Lemma 2. *Consider a menu where the government uses random perturbations $\tilde{\varepsilon}(\theta)$ to the optimal deterministic after-tax income c^* with mean $\lambda\mu(\theta)$ and variance $\lambda v(\theta)$ for some $\lambda \geq 0$ close to 0. Assume that $\mu(\theta) = v(\theta)$ for all θ. The incentive constraints (3) are satisfied if and only if*

$$U'(\theta) = S_\theta'(c^*,\theta)v(\theta)$$

and $v'(\theta) \leq 0$ for all θ.

Proof. The argument relies on the standard methodology developed by Myerson (1981). The necessary first-order condition for a local truthful report is $U'(\theta) = S_\theta'(c^*,\theta)v(\theta)$ for all θ. The sufficient second-order condition is $S_\theta'(c^*,\theta)v'(\theta) \geq 0$ for all θ. If

$$\int_\tau^\theta S_\theta'(c^*,z)v'(\tau)\,\mathrm{d}z$$

has the same sign as $\theta - \tau$, then the conditions for local incentive compatibility are sufficient for (3) to hold for all admissible τ and θ. In our case,

$$S_\theta'(c,\theta) = -\frac{\lambda}{(c+\theta)^2}\left(1 - \frac{1}{c+\theta}\right) \leq 0 \tag{26}$$

when evaluated at $c = c^* = 1$ (since $\theta \geq 0$). This concludes the proof. ■

In Lemma 2 the slope $U'(\theta)$, derived from necessary conditions for local incentive compatibility, is proportional to λ. For a small noise (λ close to 0), $\tilde{V}'(\theta) = u'_\theta(c^*, \theta) + U'(\theta) \simeq u'_\theta(c^*, \theta) > 0$. This shows that type $\theta^{\inf}$ remains the one considered by the Rawlsian planner after introducing income perturbations.

The requirement that the variance of the income noise decreases with type is a second-order condition for local incentive compatibility. Thus, by (21), the least type $\theta^{\inf}$ receives the highest income transfer but faces the highest income volatility. As the least type has the highest coefficient of risk aversion, it is not clear at this stage whether the Rawlsian planner can gain from noisy redistribution.

6.2 Welfare improving tax randomization

In order to assess the social usefulness of random redistribution, we compute the utility $\tilde{V}(\theta^{\inf})$ of the least type of agents in the presence of a small after-tax income noise, and we compare it with their utility $V(\theta^{\inf}) = -1$ evaluated at the optimal deterministic optimum. The detailed computation of $\tilde{V}(\theta^{\inf})$ is given in Appendix C. It leads to the following result:

Proposition 2. *Consider a menu of lotteries $(\mu(\theta), v(\theta))$ such that $\mu(\theta) = v(\theta)$ for all θ. Let $v(\theta)$ be non-increasing bounded from above. The random menu improves upon the deterministic optimum if and only if*

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \phi(c^*, \theta) v(\theta) \, \mathrm{d}F(\theta) > 0,$$

where

$$\phi(c, \theta) = \left(\frac{1}{c + \theta} - 1 + \frac{m(\theta)}{(c + \theta)^2} \right) - \frac{1}{(c + \theta)^2} \left(\frac{1}{2} + \frac{m(\theta)}{(c + \theta)} \right).$$

An intuition for this condition obtains by considering the total amount of collected tax

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [y(\theta) - c^* - \lambda\mu(\theta)] \, \mathrm{d}F(\theta).$$

Using the definition of $U(\theta) = S(c^*, \theta)v(\theta) - y(\theta)$ and our modeling restriction (21) that the average transfer $\mu(\theta)$ to type θ coincides with its variance $v(\theta)$, this tax can be rewritten as

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [S(c^*, \theta)v(\theta) - U(\theta) - c^* - \lambda v(\theta)] \, \mathrm{d}F(\theta).$$

Consider a small reform that takes the form of an increase $\mathrm{d}v > 0$ in the variance of the after-tax income for types between θ and $\theta + \mathrm{d}\theta$, $\mathrm{d}\theta$ close to 0.

Let us first focus on the change in collected tax abstracting from the informational rent adjustment needed to meet incentive requirements once the government introduces the income random perturbations. Then $U(\theta)$ is maintained fixed at its initial level. Every agent concerned by the reform, whose type locates between θ and $\theta + \mathrm{d}\theta$ does react by providing a higher labor effort, which leads to an increase in the before-tax income equal to $S(c^*, \theta)\,\mathrm{d}v$. The government takes advantage of the noise to increase the average transfer of every such agents by an amount $\lambda\,\mathrm{d}v$. Overall the change in the total collected tax equals $[S(c^*, \theta) - \lambda]\, f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta$. Using the expression of $S(c^*, \theta)$ given in (24) and $\lambda > 0$, we have

$$S(c^*, \theta) < \lambda \Leftrightarrow \frac{1}{c^* + \theta}\left(1 - \frac{1}{2}\frac{1}{c^* + \theta}\right) < 1,$$

a condition that is always satisfied since $c^* = 1$ and $\theta \geq 0$. Hence, given $U(\theta)$, the reform yields a lower amount of collected taxes.

Introducing noise affects $U(\theta)$ through incentives. By Lemma 2, $U'(\theta) = S'_\theta(c^*, \theta)v(\theta)$ changes by $\mathrm{d}U'(\theta) = S'_\theta(c^*, \theta)\,\mathrm{d}v$, which yields a change in the rent $\mathrm{d}U = \mathrm{d}U'(\theta)\,\mathrm{d}\theta = S'_\theta(c^*, \theta)\,\mathrm{d}v\,\mathrm{d}\theta$ given to all types above $\theta + \mathrm{d}\theta$. The total collected tax thus changes by $-(1 - F(\theta))S'_\theta(c^*, \theta)\,\mathrm{d}v\,\mathrm{d}\theta$. Since $S'_\theta \leq 0$ this corresponds to additional tax resources.

Finally the total change in tax induced by the introduction of a small noise on the after-tax income is

$$[S(c^*, \theta) - \lambda]\, f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta - (1 - F(\theta))S'_\theta(c^*, \theta)\,\mathrm{d}v\,\mathrm{d}\theta,$$

or equivalently,

$$[S(c^*, \theta) - \lambda - m(\theta)S'_\theta(c^*, \theta)]\, f(\theta)\,\mathrm{d}v\,\mathrm{d}\theta.$$

Using the expressions of $S(c^*, \theta)$ and $S'_\theta(c^*, \theta)$ given in (24) and (26), the term into brackets actually is $\lambda\phi(c^*, \theta)$. This shows that $\phi(c^*, \theta)$ governs the change in the total collected tax due to small random perturbations on the after-tax income. The effect balances the lower taxes collected from type θ agents and the social gain from the lower rents given to high types above θ.

6.3 Application to Generalized Weibull

The whole sum in Proposition 2 can reach a positive value with the generalized Weibull specification. In this case, it is optimal to rely on random redistribution. For some threshold type $\bar{\theta}^* \geq \theta^{\inf}$, let

$$v(\theta) = \begin{cases} v > 0 & \text{for } \theta \leq \bar{\theta}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Choose parameters of the generalized Weibull distribution $a = 0.9$, $b = 0.01$ and $s = 5$, the function $\phi(c^*, \theta) f(\theta)$ evaluated at $c^* = 1$ is single-peaked in θ. It is negative for low and high types, but it is positive at its maximum. The sum

$$\int_{\theta^{\inf}}^{\bar{\theta}^*} \phi(c^*, \theta) \; \mathrm{d}F(\theta)$$

is found positive for every value of the threshold $\bar{\theta}^*$ between 5/2 and 20. Thus, by Proposition 2, offering after-tax income lotteries with positive variance v to every type below $\bar{\theta}^*$ chosen in this interval and a deterministic after-tax income to the remaining types improves upon the deterministic optimum.

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A Incentives in the presence of lotteries

Suppose that the lottery $\tilde{c}(\theta)$ designed for a type θ agent is such that its owner receives an after-tax income smaller than c with probability $H(c,\theta)$, $c \in [c^{\inf}, c^{\sup}]$ and $H(c,\theta)$ differentiable in θ for all (c,θ).

Lemma A1. *Consider a menu where the after-tax income lottery $\tilde{c}(\theta)$ is designed for a type θ agent. The incentive constraints (3) are satisfied only if*

$$\tilde{V}'(\theta) = \mathbb{E}\left[u'_\theta(\tilde{c}(\tau),\theta)\right] \tag{27}$$

and

$$\frac{\partial}{\partial\tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau),\theta)\right] \geq 0 \tag{28}$$

for all θ and $\tau = \theta$. These conditions are sufficient for incentive compatibility if (28) holds true for all θ and τ.

Proof. The proof reproduces standard arguments used in the case of deterministic contracts; see, e.g., Section 2.3 in Salanié (2017). The incentive constraints (3) can be rewritten as

$$\theta = \arg\max_{\tau} \mathbb{E}[u(\tilde{c}(\tau),\theta) - \tilde{y}(\tau)]$$

for all θ. This requires that the truthful report $\tau = \theta$ is a local extremum of the utility $\mathbb{E}[u(\tilde{c}(\tau),\theta) - \tilde{y}(\tau)]$, i.e.,

$$\frac{\partial}{\partial\tau}\mathbb{E}[u(\tilde{c}(\tau),\theta) - \tilde{y}(\tau)] = 0 \tag{29}$$

at $\tau = \theta$. Using the envelope theorem, this is equivalent to (27).

In addition, truthful reporting $\tau = \theta$ must be a local maximizer of the utility. This is the case if $\mathbb{E}[u(\tilde{c}(\tau),\theta) - \tilde{y}(\tau)]$ is locally concave in τ at the extremum $\tau = \theta$. If (29) holds at $\tau = \theta$ for all θ, we have

$$\begin{aligned}\frac{\partial^2}{\partial\tau^2}\left(\mathbb{E}\left[u(\tilde{c}(\tau),\theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right]\right) &= -\frac{\partial^2}{\partial\tau\partial\theta}\left(\mathbb{E}\left[u(\tilde{c}(\tau),\theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right]\right)\\ &= -\frac{\partial}{\partial\tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau),\theta)\right]\end{aligned}$$

at $\tau = \theta$. Local concavity thus reads as (28).

Conditions (27) and (28) are necessary and sufficient for $\mathbb{E}[u(\tilde{c}(\tau),\theta) - \tilde{y}(\tau)]$ to stand below $\mathbb{E}[u(\tilde{c}(\theta),\theta) - \tilde{y}(\theta)]$ for all τ close to θ. They do not ensure that truthful reporting is a global maximum for the utility. A sufficient condition for a global maximum obtains by observing that, using (29) with $\theta = \tau$,

$$\frac{\partial}{\partial\tau}\left(\mathbb{E}\left[u(\tilde{c}(\tau),\theta)\right] - \mathbb{E}\left[\tilde{y}(\tau)\right]\right) = \int_{\tau}^{\theta}\frac{\partial}{\partial\tau}\mathbb{E}\left[u'_\theta(\tilde{c}(\tau),z)\right]dz.$$

If (28) holds true for all τ and θ, then the right-hand side of this equality has the same sign as $\theta - \tau$, which implies that $\mathbb{E}\left[u(\tilde{c}(\tau), \theta) - \tilde{y}(\tau)\right]$ is single-peaked in τ with a global maximum attained at $\tau = \theta$. ■

In the case where the redistribution policy only involves deterministic contracts, every type θ is offered some bundle $(c(\theta), y(\theta))$ with certainty and gets utility

$$V(\theta) = u(c(\theta), \theta) - y(\theta).$$

Appealing to the Spence-Mirrlees condition, the second-order condition (28) for local incentive compatibility rewrites

$$\frac{\partial}{\partial \tau} u'_\theta(c(\tau), \theta) = u''_{c\theta}\left(c(\tau), \theta\right) c'(\tau) \geq 0 \Leftrightarrow c'(\tau) \leq 0, \tag{30}$$

for all τ. The incentive constraints corresponding to a deterministic menu where the after-tax income is non-increasing are thus satisfied if and only if (27) holds true, which is $V'(\theta) = u'_\theta(c(\theta), \theta)$ for all θ.

Consider a menu of lotteries $(\tilde{c}(\theta), \tilde{y}(\theta))$ satisfying incentive compatibility (3). We are interested into menus of such lotteries where incentive compatibility is transmitted to menus of the associated certainty equivalent incomes.

Lemma A2. *Suppose that $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and $\theta_2 > \theta_1$. Then inequality (28) holds true for all τ and θ. Furthermore, $\mathbb{C}(\tilde{c}(\theta_1), \theta) \geq \mathbb{C}(\tilde{c}(\theta_2), \theta)$ for all θ.*

Proof. We have

$$\frac{\partial}{\partial \tau}\mathbb{E}\left[u'_\theta\left(\tilde{c}(\tau), \theta\right)\right] = \int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau).$$

An integration by parts yields

$$\int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau) = \left[u'_\theta\left(c, \theta\right) H'_\theta(c, \tau)\right]_{c^{\inf}}^{c^{\sup}} - \int_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c, \theta\right) H'_\theta(c, \tau)\,\mathrm{d}c.$$

Since $H(c^{\inf}, \tau) = 0$ and $H(c^{\sup}, \tau) = 1$ for all τ, we have $H'_\theta(c^{\inf}, \tau) = H'_\theta(c^{\sup}, \tau) = 0$ for all τ. It follows that

$$\int_{c^{\inf}}^{c^{\sup}} u'_\theta\left(c, \theta\right) \mathrm{d}H'_\theta(c, \tau) = -\int_{c^{\inf}}^{c^{\sup}} u''_{\theta c}\left(c, \theta\right) H'_\theta(c, \tau)\,\mathrm{d}c.$$

The Spence-Mirrlees condition $u''_{\theta c}(c,\theta) < 0$ for all (c,θ) implies that (28) holds true for all τ and θ if $H'_\theta(c,\theta) \geq 0$ for all (c,θ), i.e., $H(c,\theta_1) \leq H(c,\theta_2)$ for all c, θ_1 and $\theta_2 \geq \theta_1$. This corresponds to the case where $H(c,\theta_1)$ first-order stochastically dominates $H(c,\theta_2)$. Since $u(c,\theta)$ is increasing with c, the expected utility of any type θ is greater with the lottery $H(c,\theta_1)$ than with $H(c,\theta_2)$. This yields the ranking of certainty equivalent incomes announced in the lemma. ■

Lemma A2 provides us with a natural generalization of (30) for incentive compatibility in a stochastic environment. Under the Spence-Mirrlees condition, the monotonicity of the deterministic consumption with type is replaced with a first-order stochastic dominance ranking of lotteries.

The sufficient condition for local incentive compatibility of the deterministic menu $(\mathbb{C}(\tilde{c}(\theta),\theta))$ relies on the monotonicity properties of $\mathbb{C}(\tilde{c}(\theta),\theta)$ in θ. The lemma shows that any given agent θ will find the certainty equivalent associated with the first-order stochastic dominant lottery $\tilde{c}(\theta_1)$ more attractive, but it brings no information on how different types would value different certainty equivalent incomes. Lemma A3 takes a first step on this issue by looking at preferences of different types of agents facing the same lottery.

Lemma A3. *If $u'''_{\theta cc}(c,\theta) \leq 0$ for all (c,θ), then $\mathbb{C}(\tilde{c},\theta_1) > \mathbb{C}(\tilde{c},\theta_2)$ for all $\tilde{c}$ and any two types θ_1 and $\theta_2 > \theta_1$.*

Proof. The risk premium $\pi(\tilde{c},\theta)$ is positive since $u(c,\theta)$ is strictly concave in c. It is such that $\mathbb{E}[u(\tilde{c},\theta)] = u(\mathbb{E}[\tilde{c}] - \pi(\tilde{c},\theta),\theta)$ for every θ. By differentiation, we have

$$\pi'_\theta(\tilde{c},\theta)u'_c\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) = u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) - \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right].$$

Note that the derivative $\pi'_\theta(\tilde{c},\theta)$ is well-defined since $u'_c(c,\theta)$ is non-zero. With $\pi(\tilde{c},\theta) > 0$, the Spence-Mirrlees condition $u''_{\theta c}(c,\theta) < 0$ for all (c,θ) gives

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) > u'_\theta\left(\mathbb{E}\left[\tilde{c}\right],\theta\right).$$

If $u'_\theta(c,\theta)$ is concave in c, i.e., $u'''_{\theta cc}(c,\theta) \leq 0$ for all (c,θ), then Jensen's inequality leads to

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right],\theta\right) \geq \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right].$$

Therefore

$$u'_\theta\left(\mathbb{E}\left[\tilde{c}\right] - \pi(\tilde{c},\theta),\theta\right) - \mathbb{E}\left[u'_\theta\left(\tilde{c},\theta\right)\right] > 0.$$

The result follows from $u'_c(c,\theta) > 0$ for all (c,θ) and $\mathbb{C}'_\theta(\tilde{c},\theta) = -\pi'_\theta(\tilde{c},\theta)$. ■

The negative sign of the third cross-derivative in Lemma A3 says that the second derivative of the utility with respect to consumption, which captures the curvature of the utility function, is decreasing with type. This second derivative is negative for risk-averse agents. The negative sign of $u'''_{\theta cc}(c,\theta)$ thus implies that higher type agents display higher risk aversions. It follows that a given lottery yields more utility to low types (low risk aversion) than high types.

The combination of Lemma A2 and A3 shows that the sign of this derivative also matters for the implementation of a menu of deterministic contracts consisting of certainty equivalent incomes.

Corollary A1. *Suppose that $\tilde{c}(\theta_1)$ first-order stochastically dominates $\tilde{c}(\theta_2)$ for any two types θ_1 and $\theta_2 > \theta_2$. If $u'''_{\theta cc}(c,\theta) \leq 0$* for all (c,θ), *then $\mathbb{C}(\tilde{c}(\theta),\theta)$ is non-increasing with θ.*

Proof. This directly follows from Lemma A2 and A3. ■

Consider a reform that replaces a menu of lotteries $(\tilde{c}(\theta),\tilde{y}(\theta))$ with a menu where the certainty equivalent $\mathbb{C}(\tilde{c}(\theta),\theta)$ instead is given to type θ with probability 1. For the class of lotteries considered in Corollary 1, if $u''_{cc}(c,\theta)$ is decreasing with θ, then incentive compatibility at the outcome of the reform obtains if and only if the first-order conditions (27) hold true in the final (deterministic) situation, i.e., $V'(\theta) = u'_\theta(\mathbb{C}(\tilde{c}(\theta),\theta),\theta)$ for all θ.

B Parametric example: deterministic optimum

The government chooses a deterministic menu $(c(\theta),y(\theta))$ subject to the feasibility constraint (2) and the incentive constraints

$$\ln(c(\theta)+\theta) - y(\theta) \geq \ln(c(\tau)+\theta) - y(\tau)$$

for all θ and τ. The incentive constraints are satisfied if and only if

$$y'(\theta) = \frac{c'(\theta)}{c(\theta)+\theta} \Leftrightarrow V'(\theta) = \frac{1}{c(\theta)+\theta} \tag{31}$$

and

$$c'(\theta) \leq 0 \tag{32}$$

for all θ.

From (31), indirect utility is increasing with θ, and so the Rawlsian government maximizes $V(\theta^{\inf})$ subject to (2), (31) and (32). Using (2) at equality and (31) the indirect utility of type θ is

$$V(\theta) = V(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} \frac{1}{c(z)+z}\,\mathrm{d}z, \tag{33}$$

where $V(\theta^{\inf})$ is given in (17).

The optimal after-tax income menu $(c(\theta))$ maximizes $V(\theta^{\inf})$ subject to the monotonicity condition $c'(\theta) \leq 0$ for all θ. The associated before-tax income then is $y(\theta) = \ln(c(\theta)+\theta) - V(\theta)$.

Let $c^{**}(\theta)$ be the (interior) after-tax income that maximizes the contribution

$$W(c(\theta), \theta) = \ln(c(\theta) + \theta) - c(\theta) - \frac{m(\theta)}{c(\theta) + \theta}$$

of type θ to the social surplus in (17). That is,

$$c^{**}(\theta) + \theta = \frac{1}{2}(1 + \sqrt{1 + 4m(\theta)}). \tag{34}$$

The monotonicity condition (32) binds for all types if $c^{**}(\theta)$ is always increasing. Using (34), this is equivalent to (19).

By Theorem 6.1 in Fleming and Rishel (1975) there is a single tax bracket at the deterministic optimum. Indeed the after-tax income is continuous provided that the deterministic social program does not display multiple maximizers for some θ. Since $W(c, \theta)$ is concave in c, the program has at most one maximizer on an interval of types such that the second order monotonicity $c'(\theta) \leq 0$ does not bind at the deterministic optimum ($c'(\theta) < 0$). Consider now an interval of types $[\theta_1, \theta_2]$ such that $c'(\theta) = 0$. For such types, $c(\theta) = \bar{c}$ such that

$$c(\theta_1) = c(\theta_2) = \bar{c}$$

satisfies

$$\int_{\theta_1}^{\theta_2} \left[\frac{1}{\bar{c} + \theta} - 1 + \frac{m(\theta)}{(\bar{c} + \theta)^2} \right] \mathrm{d}F(\theta) = 0$$

Use again $W(c, \theta)$ concave in c, so that the integrand is decreasing in $\bar{c}$, to conclude that there is a unique solution $\bar{c}$.

With single tax bracket, the objective (17) simplifies to

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\ln(c + \theta) - c - \frac{m(\theta)}{c + \theta} \right] \mathrm{d}F(\theta) \tag{35}$$

An integration by parts yields

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \ln(c + \theta)\, \mathrm{d}F(\theta) = \ln(c + \theta^{\inf}) + \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{m(\theta)}{c + \theta}\, \mathrm{d}F(\theta).$$

Therefore the objective (35) equals $\ln(c + \theta^{\inf}) - c$. The after-tax income that maximizes this expression is $c^* = 1$ and the corresponding objective is $V(\theta^{\inf}) = -1$.

C Parametric example: computation of $\tilde{V}(\theta^{\inf})$

We first express the before-tax income of type θ as a function of her indirect utility $\tilde{V}(\theta)$. From (22), we have

$$y(\theta) = u(c^*, \theta) + S(c^*, \theta)v(\theta) - \tilde{V}(\theta).$$

The expression of the indirect utility $\tilde{V}(\theta)$ obtains from the first-order necessary condition in Lemma 2 for incentive compatibility when agents face local randomization of their after-tax income, $U'(\theta) = S'_\theta(c^*, \theta)v(\theta)$ for all θ, which yields

$$U(\theta) = U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\mathrm{d}z.$$

Hence, using (22),

$$\tilde{V}(\theta) = u(c^*, \theta) + U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\mathrm{d}z.$$

The feasibility constraint (2) reads

$$\int_{\theta^{\inf}}^{\theta^{\sup}} [c^* + \lambda\mu(\theta) - y(\theta)]\, \mathrm{d}F(\theta) = 0$$

$$\Leftrightarrow \int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + U(\theta^{\inf}) + \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\,\mathrm{d}z\right] \mathrm{d}F(\theta) = 0.$$

Using the integration by parts formula,

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \int_{\theta^{\inf}}^{\theta} S'_\theta(c^*, z)v(z)\,\mathrm{d}z\,\mathrm{d}F(\theta) = \int_{\theta^{\inf}}^{\theta^{\sup}} \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta)\,\mathrm{d}F(\theta),$$

the feasibility constraint allows us to get

$$U(\theta^{\inf}) = -\int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta)\right] \mathrm{d}F(\theta).$$

Finally (22) gives us the indirect utility $\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) + U(\theta^{\inf})$ of the least type

$$\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} \left[c^* + \lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta)\right] \mathrm{d}F(\theta).$$

In the absence of income noise, $\mu(\theta) = v(\theta) = 0$ for all θ, we have

$$\tilde{V}(\theta^{\inf}) = u(c^*, \theta^{\inf}) - \int_{\theta^{\inf}}^{\theta^{\sup}} c^* \, \mathrm{d}F(\theta) = u(c^*, \theta^{\inf}) - c^* = V(\theta^{\inf}).$$

Therefore local income tax randomizations do improve upon the deterministic optimum, $\tilde{V}(\theta^{\inf}) > V(\theta^{\inf})$, if and only if

$$\int_{\theta^{\inf}}^{\theta^{\sup}} \left[\lambda\mu(\theta) - S(c^*, \theta)v(\theta) + \frac{1 - F(\theta)}{f(\theta)} S'_\theta(c^*, \theta)v(\theta) \right] \mathrm{d}F(\theta) < 0.$$

After replacing $S(c^*, \theta)$ and $S'_\theta(c^*, \theta)$ with their expressions in (24) and in the proof of Lemma 2, and using the restriction $\mu(\theta) = v(\theta)$ for all θ, this inequality rewrites as in Proposition 2.