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Coordination on Saddle-Path Solutions: the Eductive Viewpoint - Linear Multivariate Models*

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Abstract

We examine local strong rationality (LSR) in multivariate models with both forward-looking expectations and predetermined variables. Given hypothetical common knowledge restrictions that the dynamics will be close to those of a specified minimal state variable solution, we obtain eductive stability conditions for the solution to be LSR. In the saddlepoint stable case the saddle-path solution is LSR provided the model is structurally homogeneous across agents. However, the eductive stability conditions are strictly more demanding when heterogeneity is present, as can be expected in multisectoral models. Heterogeneity is thus a potentially important source of instability even in the saddlepoint stable case.

Key words: Coordination, structural heterogeneity, strong rationality, eductive stability, multisectoral models.

JEL classification: C72, C62.

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1 Introduction

This paper comes as a continuation of a previous paper entitled "Coordination on saddle-path solutions : the eductive viewpoint – linear univariate models", by G. Evans and R. Guesnerie (2003). The purpose of both papers is to revisit the justifications of the saddle-path stable solution by taking the somewhat more basic perspective of "eductive learning," which refers to considerations that have a game-theoretical flavour and explicitly refer to Common Knowledge considerations.[1]

Specifically, the viewpoint we take, the "Strong Rationality viewpoint"[2], proceeds as follows. We start from restrictions on the possible paths of the system, which themselves reflect restrictions on individual strategies. These restrictions, tentatively supposed to be Common Knowledge (from now on CK) trigger a mental process, which, when rationality is itself "commonly known," mimics the process of determination of rationalizable strategies (from the initial set of restricted strategies). When such a process converges to the candidate equilibrium, the equilibrium is said to be *Strongly Rational*. Actually, as in the following, the CK initial restrictions will always be taken locally, so that we shall only be concerned with a weaker variant of the test that selects *Locally* Strongly Rational Equilibria. The question treated in this paper, as well as in the companion paper, can then be more compactly reformulated: when is it the case that the *saddle-path stable solution* of a dynamical system *is* a good candidate for expectational coordination, in the sense just introduced of being *Locally Strongly Rational*, for restrictions to be made precise?

At this stage, two different points are in order.

First, it is useful to stress the relevance of the question: economic modelling routinely assumes that saddle-path stable solutions, or stable manifold solutions, provide the appropriate "rational expectations solutions" even when they compete with many others. Convenience pleads in favour of such a practice, but it is not as such a fully convincing intellectual argument. Determinacy considerations, which point out that such solutions are "locally isolated" rational expectations equilibria, even when the broader concept of

[1] A related but distinct approach is "adaptive learning," e.g. Evans and Honkapohja (2001). For a comparison of eductive and adaptive learning see Evans (2001).

[2] This could be called the "local unique rationalizability" viewpoint, in the terminology of Bernheim (1984) or Pearce (1984), or the "local dominance solvability"viewpoint in the terminology of Farqhason (1969) and Moulin (1979).

sunspot equilibria is envisaged, provide better intellectual arguments for serious foundations but do not exhaust the question. The present approach proposes an alternative, and in our view more basic, view on the problem.

As the reader will easily guess, our methodology to approach the question, as briefly sketched above, is almost meaningless if we refer to standard reduced forms of dynamical systems. In order to make sense of the question we raise concerning Common Knowledge, we must, as we did in Evans-Guesnerie (1993) in a different context, imbed the model in a framework where agents and their strategies are well defined. This is indeed what we did in the previous paper and we repeat this set-up here in next Section.

2 The framework.

2.1 Dynamic expectations models

We are interested in models of the following kind:

$$Q(y_{t-1}, y_t, y^e_{t+1}) = O,$$

where t is a time index, y is a finite dimensional vector, and Q is a temporary equilibrium map that relates y_t to its lagged values and to expectations. The quantity y^e_{t+1} denotes the expectation of y_{t+1} formed by agents at time t. In this formulation we assume that agents are able to observe y_t when forming their expectations or, if not, that they can condition their actions on the values y_t that are realized.

We need to be more precise on the strategic aspects of the coordination problem. To do so we will adopt a very simple strategic interpretation of the model which makes explicit the decision theoretic aspects of the model and the aggregation of these decisions into a temporary equilibrium map.

2.2 Strategic expectations model.

2.2.1 The basic structure

We thus embed the dynamic model in a dynamic game, along lines that are somewhat similar to those of Evans-Guesnerie (1993). We assume that, at each period t, there exists a continuum of agents, a part of whose strategies are not reactive to expectations (in an OLG context, these are the agents, who are at the last period of their lives), and a part of which "react to expectations". The latter agents are denoted ω_t and belong to a convex segment of R, endowed with Lebesgue measure $d\omega_t$. It is assumed that an agent of period t is different from any other agent of period $t^{'}, t^{'} \neq t$. [3] More precisely, agent ω_t has a (possibly indirect) utility function that depends upon

1) his own strategy $s(\omega_t)$,

2) sufficient statistics of the strategies played by others i.e. on $y_t = F(\Pi_{\omega_t}\{s(\omega_t)\}, *)$, where F in turn depends first, upon the strategies of all agents who at time t react to expectations, and second, upon $(*)$, which is here supposed to be sufficient statistics of the strategies played by those who do not react to expectations, and that includes but is not necessarily identified with – see below – y_{t-1},

3) finally upon the sufficient statistics for time $t+1$, as perceived at time t: i.e. on $y_{t+1}(\omega_t)$, which *may be random* and, now directly, upon the sufficient statistics y_{t-1}.

We assume that the strategies played at time t can be made conditional on the equilibrium value of the of the t sufficient statistics y_t. Now, let $(\bullet)$ denote both (the product of) y_{t-1} and the probability distribution of the random variable $\tilde{y}_{t+1}(\omega_t)$, (the random subjective forecasts held by ω_t of y_{t+1}). Let then $G(\omega_t, y_t, \bullet)$ be the best response function of agent ω_t. Under these assumptions, the sufficient statistics for the strategies of agents who do not react to expectations is $(*) = (y_{t-1}, y_t)$.

The equilibrium equations at time t are written:

$$y_t = F\left[\Pi_{\omega_t}\left\{G(\omega_t, y_t, y_{t-1}, \tilde{y}_{t+1}(\omega_t))\right\}, y_{t-1}, y_t\right]. \tag{1}$$

Note that when all agents have the same point expectations denoted y^e_{t+1}, the equilibrium equations determine what we called earlier the temporary equilibrium mapping

$$Q(y_{t-1}, y_t, y^e_{t+1}) = y_t - F\left[\Pi_{\omega_t}\left\{G(\omega_t, y_t, y_{t-1}, y^e_{t+1})\right\}, y_{t-1}, y_t\right].$$

[3] This means either that each agent is "physically" different or that the agents have strategies that are independent from period to period. In an OLG interpretation of the model, each agent lives for two periods but only reacts to expectations in the first period of his life.

2.2.2 Linearization

The right hand side of (1) is a rather complex term, but under regularity assumptions[4], it has, through two different channels, derivatives with respect to y_t, and with respect to y_{t-1}. Also assuming that all $\tilde{y}_{t+1}$ have a very small common support "around" some given y^e_{t+1}, decision theory suggests that G, to the first order, depends on the expectation[5] of the random variable $\tilde{y}_{t+1}(\omega_t)$ that is denoted $y^e_{t+1}(\omega_t)$ (and is close to y^e_{t+1})

Taking into account the previous remark, the heterogeneity of expectations across agents, and assuming again the existence of adequate derivatives, it is reasonable to linearize (1), around any initially given situation, denoted (0), as follows[6]:

$$y_t = U(0)y_t + V(0)y_{t-1} + \int W(0,\omega_t)y^e_{t+1}(\omega_t)d\omega_t,$$

where $y_t, y_{t-1}, y^e_{t+1}(\omega_t)$ now denote small deviations from the initial values of y_t, y_{t-1}, y^e_{t+1}, and $U(0), V(0), W(0,\omega_t)$ are $n \times n$ square matrices.

Such a linearization is valid everywhere, but we will consider it only around a steady state of the system. Hereafter, y_t, y_{t-1}, etc., denote deviations from the steady state and $U(0), V(0), W(0,\omega_t)$ are simply $U, V, W(\omega_t)$.

Supposing $I - U$ is invertible, we have

$$y_t = ((I-U)^{-1}V)y_{t-1} + (I-U)^{-1}\int W(\omega_t)y^e_{t+1}(\omega_t)d\omega_t.$$

When expectations are homogenous, $y^e_{t+1}(\omega_t) = y^e_{t+1}$, the system becomes

$$y_t = By^e_{t+1} + Dy_{t-1}, \text{ with } B = (I-U)^{-1}W, \text{ where } W = \int W(\omega_t)d\omega_t. \quad (2)$$

With the new notation, assuming W invertible, the initial system can also be written

$$y_t = Dy_{t-1} + BW^{-1}\int W(\omega_t)y^e_{t+1}(\omega_t)d\omega_t.$$

[4] For a less sketchy discussion, see Evans-Guesnerie (1993), p.637.

[5] This could be formalized along lines similar to those taken in Chiappori-Guesnerie (1989), who also argue that the property is general in economic models that adopt the Bayesian view of uncertainty.

[6] As in Guesnerie (2002), this can be viewed as an "axiom", whose field of validity is very large.

Putting $Z(\omega_t) = W^{-1}W(\omega_t)$, we rewrite this as

$$y_t = Dy_{t-1} + B \int Z(\omega_t) y^e_{t+1}(\omega_t) d\omega_t, \quad (3)$$

where $\int Z(\omega_t)d\omega_t = I$. This will be the basic equation of our study. We assume that (3) holds for $t = 1, 2, 3, \ldots$, and that initial conditions y_0 are given.

3 An Economic Example

To illustrate our results we develop a two-sector version of the overlapping generations model with production that was introduced and analyzed by Reichlin (1986). In Reichlin's model there is a single perishable output ("corn"), which can either be consumed or set aside as capital ("seed corn") for use in production the following period. Capital is combined with labor according to a Leontief fixed coefficients technology to produce output and the capital is fully used up in production. We develop a two-sector competitive version of this model in which there is trade in goods but no labor or capital flow permitted between sectors. That is, in the version we set forth here (other formulations would of course be possible), labor is immobile and agents can only invest in capital in their own sector. However, there is trade in goods, and households in both sectors consume both goods.

For ease of presentation we begin with the perfect foresight version. Population, which is normalized to one in each sector, is stationary and composed of identical consumers living for two periods. Households work when young and consume when old. For convenience we assume that all agents in both sectors have the same utility functions. In each sector $i = 1, 2$, the household problem is thus

$$\begin{aligned} & \max \ u(c^1_{t+1}, c^2_{t+1}) - v(l^i_t), \\ \text{subject to } N^i_t &= W^i_t l^i_t \text{ and } c^1_{t+1} + p_{t+1}c^2_{t+1} = N^i_t R^i_{t+1}, \end{aligned}$$

where l^i_t is labor supply, c^i_{t+1} is the consumption of sector i goods, N^i_t is investment in sector i capital carried into the following period, W^i_t is the wage rate in sector i units, R^i_{t+1} is the real interest factor in sector i units, and p_{t+1} is the relative price of sector 2 goods in terms of sector 1 goods in period $t + 1$. We make the standard assumptions that $v'(l) > 0, v''(l) > 0$

with $\lim_{l\to\infty} v'(l) = +\infty$ and $\lim_{l\to 0} v'(l) = 0$, and $u(c^1, c^2)$ is assumed to be concave with positive first partial derivatives. It is convenient to additionally require that $u(c^1, c^2)$ is homogeneous of degree one so that its first derivatives depend only on the ratio c^1/c^2. The first-order conditions for the household are therefore the Euler equations

$$v'(l_t^i) = W_t^i R_{t+1}^i u_i \left(c_{t+1}^1/c_{t+1}^2\right), \tag{4}$$

where $u_i\left(c^1/c^2\right) \equiv \frac{\partial}{\partial c^i} u(c^1, c^2)$ and the static condition

$$p_{t+1} = u_2\left(c_{t+1}^1/c_{t+1}^2\right)/u_1\left(c_{t+1}^1/c_{t+1}^2\right).$$

Firms produce output under conditions of perfect competition. In each sector output is given by $x_t^i = \min(\alpha_i N_{t-1}^i, \beta_i l_t^i)$, where $\alpha_i, \beta_i > 0$. Profit maximization gives

$$x_t^i = \alpha_i N_{t-1}^i = \beta_i l_t^i,$$

and goods market equilibrium implies that $x_t^i = c_t^i + N_t^i$. It follows that

$$l_t^i = (\alpha_i/\beta_i) N_{t-1}^i \text{ and } c_{t+1}^i = \alpha_i N_t^i - N_{t+1}^i$$

Finally, from the zero profit condition

$$W_t^i l_t^i + R_t^i N_{t-1}^i = x_t^i,$$

together with $N_t^i = W_t^i l_t^i$ and $x_t^i = \alpha_i N_{t-1}^i$, we obtain

$$R_{t+1}^i = \alpha_i - \frac{N_{t+1}^i}{N_t^i}.$$

Substituting the preceding relationships into the Euler equation we arrive at the equation that specifies the perfect foresight dynamics, namely

$$V\left(\frac{\alpha_i}{\beta_i} N_{t-1}^i\right) = \left(\alpha_i N_t^i - N_{t+1}^i\right) u_i \left(\frac{\alpha_1 N_t^1 - N_{t+1}^1}{\alpha_2 N_t^2 - N_{t+1}^2}\right) \tag{5}$$

for $i = 1, 2$, where

$$V(z) \equiv z v'(z),$$

where $v'(z)$ denotes the derivative of $v(z)$. Note that $V(z), V'(z) > 0$ for all $z > 0$. Existence of a steady state requires $\alpha_1, \alpha_2 > 1$. When linearized this yields a perfect foresight dynamic equation of the form $y_t = B y_{t+1} + D y_{t-1}$

where $y_t = (N_t^1, N_t^2)'$ with variables expressed as deviations from steady state values.

The strategic form of the model is obtained by dropping perfect foresight and allowing for heterogeneous expectations across individual agents. The Euler equation (4) becomes

$$v'(l_t^i(\omega_t^i)) = W_t^i R_{t+1}^{i,e} u_i \left(c_{t+1}^{1,e}(\omega_t^i)/c_{t+1}^{2,e}(\omega_t^i) \right),$$

where ω_t^i denotes an agent in sector i, $l_t^i(\omega_t^i)$ and $c_{t+1}^{j,e}(\omega_t^i)$ are the agent's labor supply and consumption demands and a superscript e denotes the expectation of a variable. Because every agent in every period will equate its marginal rate of substitution between the two goods to the relative price, we have $c_{t+1}^1(\omega_t^i)/c_{t+1}^2(\omega_t^i) = c_{t+1}^1/c_{t+1}^2$, where now variables without ω_t^i denote aggregate quantities. Furthermore the aggregate relationships $N_t^i = W_t^i l_t^i$, $l_t^i = (\alpha_i/\beta_i)N_{t-1}^i$, $c_{t+1}^i = \alpha_i N_t^i - N_{t+1}^i$ and $R_{t+1}^i = \alpha_i - N_{t+1}^i/N_t^i$ continue to hold. It follows that the individual Euler equations, specifying labor supplies, can thus be rewritten as

$$l_t^i(\omega_t^i) = (v')^{-1} \left\{ W_t^i \left(\alpha_i - \frac{N_{t+1}^{i,e}}{N_t^i} \right) u_i \left(\frac{\alpha_1 N_t^1 - N_{t+1}^{1,e}(\omega_t^i)}{\alpha_2 N_t^2 - N_{t+1}^{2,e}(\omega_t^i)} \right) \right\}.$$

In equilibrium we have $l_t^i = \int l_t^i(\omega_t^i) d\omega_t^i$, so that using again the above aggregate relationships we arrive at

$$\frac{\alpha_i}{\beta_i} N_{t-1}^i = \int (v')^{-1} \left\{ \frac{\beta_i}{\alpha_i N_{t-1}^i} \left(\alpha_i N_t^i - N_{t+1}^{i,e} \right) u_i \left(\frac{\alpha_1 N_t^1 - N_{t+1}^{1,e}(\omega_t^i)}{\alpha_2 N_t^2 - N_{t+1}^{2,e}(\omega_t^i)} \right) \right\} d\omega_t^i,$$

for $i = 1, 2$, which fits the framework of Section 2 with

$$y_t = (N_t^1, N_t^2)' \text{ and } y_{t+1}^e(\omega_t) = (N_{t+1}^{1,e}(\omega_t^i), N_{t+1}^{2,e}(\omega_t^i))'.$$

Linearization around a steady state then yields a reduced form that can be written as (3).

The above formulation allows for heterogeneous expectations within each sector, but it is revealing to consider the special case in which expectations *within* each sector are homogeneous. In this case the linearization (3) reduces to

$$y_t = D y_{t-1} + B \left\{ Z(1) y_{t+1}^e(1) + Z(2) y_{t+1}^e(2) \right\} \tag{6}$$

where now $y^e_{t+1}(i) = (N^{1,e}_{t+1}(i), N^{2,e}_{t+1}(i))'$. Here $N^{j,e}_{t+1}(i)$ denotes the expectations held at time t by agents in sector i concerning the future capital expenditure in sector j. Despite the simplification of assuming homogeneous within-sector expectations, this formulation of the model retains the key feature of intersectoral expectational heterogeneity combined with differential impacts of these expectations. Furthermore, and this is a point we stress below, it is apparent from the form of the nonlinear model that $Z(1)$ and $Z(2)$ are not identical (or proportional), even in the symmetric case in which $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

4 Perfect Foresight.

We begin our analysis of the linearized model (3) with a discussion of the perfect foresight solutions.

4.1 Perfect Foresight Paths.

A Perfect Foresight path is a sequence of n-dimensional vectors, y_t, $t = 1, ... + \infty$, starting from y_0, and such that :

$$y_t = By_{t+1} + Dy_{t-1}. \tag{7}$$

We begin with a review of the standard methodology of the study of such paths. We assume throughout that B is nonsingular.

Defining $X(t) = \left\{ \begin{array}{c} y_t \\ y_{t-1} \end{array} \right\}$, we write

$$X(t+1) = \Phi X(t),$$

where Φ is the $2n \times 2n$ matrix

$$\Phi = \left[\begin{array}{cc} B^{-1} & -B^{-1}D \\ I & O \end{array} \right].$$

Such a matrix has $2n$ eigenvalues, λ_i, $i = 1, ...2n$, associated with eigenvectors of the form $\left\{ \begin{array}{c} \lambda_i x_i \\ x_i \end{array} \right\}$. We will now remind the reader of the solutions to this dynamical system.

Associated with the equilibrium point 0 of the dynamical system is a stable linear manifold, generated by all eigenvectors associated with eigenvalues of modulus strictly smaller than one and an unstable linear manifold generated by all eigenvectors associated with eigenvalues of modulus strictly greater than one. Throughout the paper we assume that Φ is diagonalizable in the set of complex matrices and that it has no eigenvalue with modulus equal to one.

We further assume that we are in the so-called "saddle-path" case in which the stable manifold has dimension n and is in "general position." It follows that for any given initial y_0 there is a unique y_1 on the stable manifold and a trajectory converging to the equilibrium. It also follows that any other trajectory starting from y_0 has at least one component going to infinity.

Thus we have exactly n eigenvalues of modulus strictly smaller than one. We rank the eigenvalues in the order of increasing modulus, so that $i \leq j, \Leftrightarrow |\lambda_i| \leq |\lambda_j|$. It is well known that when Φ is "semisimple", i.e. diagonalizable in the set of complex matrices, it has a real factorization of the form

$$\Phi = P\Lambda P^{-1},$$

where Λ is a $2n \times 2n$ block diagonal matrix, in which each block is either a single element λ_j, when λ_j is real, or is a 2×2 block $\begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$ corresponding to non-real eigenvalues $\lambda_j = a_j \pm ib_j$. We order the elements and blocks in terms of increasing eigenvalue modulus. The nonsingular $2n \times 2n$ matrix P has columns given by the coordinates of the eigenvectors in the canonical basis of R^{2n} if the corresponding eigenvector is real. In the case of nonreal eigenvalues the corresponding pair of columns of P are given by the coordinates of the imaginary and real parts, respectively, of the corresponding eigenvectors.

With this factorization we can obtain

$$X(t+1) = P\Lambda^t P^{-1} X(1).$$

Partitioning P, and calling $P^{-1}X(1) = \left\{ \begin{matrix} I_1 \\ I_2 \end{matrix} \right\}$, the dynamics of the system can be written, with straightforward notation,

$$\left\{ \begin{matrix} y_{t+1} \\ y_t \end{matrix} \right\} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} \Lambda_1^t & 0 \\ 0 & \Lambda_2^t \end{pmatrix} \left\{ \begin{matrix} I_1 \\ I_2 \end{matrix} \right\}.$$

Here the submatrices P_{ij} and Λ_i are $n \times n$ and note that $P_{11} = P_{21}\Lambda_1$, and $P_{12} = P_{22}\Lambda_2$. It follows, in particular, that

$$y_t = P_{21}\Lambda_1^t I_1 + P_{22}\Lambda_2^t I_2. \tag{8}$$

We assume that P_{21} is nonsingular.

Let X^S denote the (n dimensional) subspace of R^{2n} generated by the eigenvectors associated with the n eigenvalues of smallest modulus, $\lambda_1, ...\lambda_n$. X^S is a solution subspace, i.e. $X(t-1) \in X^S$ and $X(t) = \Phi X(t-1)$ implies $X(t) \in X^S$. A vector $X(t) = (y_t', y_{t-1}')'$ belongs to X^S if and only if, in the basis of eigenvectors, it can be written as $\left\{ \begin{array}{c} \eta \\ 0 \end{array} \right\}$, i.e. in the canonical basis it is of the form $\left\{ \begin{array}{c} P_{11}\eta \\ P_{21}\eta \end{array} \right\}$. Hence $X(t) \in X^S$ if and only if $y_t = P_{11}(P_{21})^{-1}y_{t-1}$, i.e.

$$y_t = Sy_{t-1}, \text{ where } S = P_{21}\Lambda_1(P_{21})^{-1}. \tag{9}$$

This solution corresponds to (8) with $I_2 = 0$, i.e. $y_t = P_{21}\Lambda_1^t I_1$.

We have shown the following:

Proposition 1 *A saddle-path solution* $(y_0, y_1, \ldots, y_t, \ldots)$ *satisfies* $y_t = S_* y_{t-1}$, *where* $S_* = P_{21}\Lambda_1(P_{21})^{-1}$ *and where* P_{21} *and* Λ_1 *are the matrices just defined.*

The Proposition describes the unique nonexplosive solution in the "saddle-point stable case," for given initial y_0. Any other perfect foresight solution satisfies (8) with $P_{22}\Lambda_2^t I_2 \neq 0$, which implies that at least one component of y_t tends to $\pm\infty$ as $t \to \infty$.

We conclude this Section with a parenthetical digression. Economists have long been interested in solutions of the dynamical system that are of the form $y_t = Sy_{t-1}$. These are called by McCallum (1983) "minimal state variable" solutions and more recently by Gauthier "equilibrium extended growth rate" solutions. If such a solution exists then from (7) we have

$$y_t = By_{t+1} + DS^{-1}y_t,$$

provided S is nonsingular. Thus $B^{-1}(I - DS^{-1})y_t = y_{t+1}$ and

$$S = B^{-1}(I - DS^{-1}), \tag{10}$$

which can be rewritten as the matrix quadratic equation

$$S^2 - B^{-1}S + B^{-1}D = O,$$

so that minimal state variable solutions are solutions of this equation.

Obviously, in our case, S_* is a solution of this equation, but others can also be constructed based on the diagonalization of Φ. In the saddle-point case S_* uniquely delivers a nonexplosive solution, but outside the saddle-point case there can be a multiplicity of such solutions. The Appendix describes how the set of minimal state variable solutions can be obtained, whether or not the saddle-point case holds.

5 Eductive Learning.

5.1 Iterative Expectational Stability.

In developing the Strong Rationality conditions we will examine their connection to the conditions for Iterative Expectational Stability (or IE-stability). The analysis focuses on minimal state variables solutions, i.e. perfect foresight solutions of the form $y_t = \bar{S}y_{t-1}$ for all t. IE-Stability can be viewed as a process, taking place in virtual or notional time τ, that works as follows (see, for example, Evans (1985)). Economic agents posit a conjectured or "perceived" law of motion, consistent with a minimal state variable solution, in which y_t evolves in accordance with some arbitrary fixed coefficient matrix S_τ. That is, all agents believe that $y_t = S_\tau y_{t-1}$ for all t, where S_τ is some fixed matrix. From this PLM (perceived law of motion) S_τ, one can obtain the actual law of motion and show that the actual dynamics take the same form, but with a fixed coefficient matrix $T(S_\tau)$, which is in general different from S_τ. IE-stability then considers the iterative revision, in notional time τ, given by $S_{\tau+1} = T(S_\tau)$. If this sequence converges to a fixed point $\bar{S} = T(\bar{S})$, from all initial S in a neighborhood of $\bar{S}$, then we say that $\bar{S}$ is locally IE-stable (or LIE-stable). The sequence $S_{\tau+1} = T(S_\tau)$ can be thought of as a stylized notional time learning rule in which the PLM coefficient matrices used to make forecasts are updated to the actual coefficient matrices implied by those forecasts.

More specifically, for the case at hand, consider forecasts

$$y^e_{t+1} = S_\tau y_t, \forall t.$$

Inserting these (homogeneous) expectations into the model (2) the actual dynamics between $t-1$ and t will be $y_t = Dy_{t-1} + BS_\tau y_t$, so that

$$y_t = (I - BS_\tau)^{-1} Dy_{t-1}, \forall t,$$

provided $I - BS_\tau$ is invertible. Thus constant PLM coefficients S_τ would lead to constant actual coefficients $T(S_\tau) = (I - BS_\tau)^{-1}D$. The IE-Stability (or "IE-learning") dynamics are therefore given by

$$S_{\tau+1} = (I - BS_\tau)^{-1}D. \tag{11}$$

Fixed points of the IE learning process $S = (I - BS)^{-1}D$ satisfy (10).

We are now going to prove the following:

Proposition 2 *Assume that we are in the saddle-point case $|\lambda_n| < 1 < |\lambda_{n+1}|$ with solution $y_t = S_* y_{t-1}$, where $S_* = P_{21}\Lambda_1 P_{21}^{-1}$. Then this solution is LIE-Stable.*

One possible proof of this is would be to show that all eigenvalues of the linear mapping tangent (at S_*) to the mapping $S \to (I - BS)^{-1}D$ are smaller that one in modulus. However, a more interesting proof obtains by making use of the concept of perfect foresight extended growth rates proposed by Gauthier (2002, 2003), which is defined as follows.

Suppose that

$$y_t = S_t y_{t-1}$$

for some given $n \times n$ invertible matrix S_t. Then the perfect foresight "follower" of y_t is the y_{t+1} that satisfies, assuming B invertible,

$$y_{t+1} = B^{-1}(I - DS_t^{-1})y_t.$$

Writing $y_{t+1} = S_{t+1}y_t$ it follows that

$$S_{t+1} = B^{-1}(I - DS_t^{-1}). \tag{12}$$

Choosing S_1 arbitrarily, (12) generates an infinite sequence of matrices, S_t, $t = 1, 2, 3, \ldots$ with the property that for arbitrary y_0 the sequence $y_1 = S_1 y_0, \ldots, y_t = S_t y_{t-1}, \ldots$ is a perfect foresight path. Following Gauthier, one may call the sequence S_t a sequence of "extended growth rates,"[7] or an "EGR sequence." A limit point S of a sequence of extended growth rates must satisfy (10).

Comparing the dynamics (11) of IE-stability with the EGR dynamics (12) we immediately see that they are governed by inverse mappings. A fixed point is therefore locally a sink under IE dynamics if and only if it is a source under EGR dynamics. This observation immediately yields:

[7] cf the one dimensional case for the terminology.

Lemma 3 *S_* is LIE stable if and only if it is locally determinate under EGR dynamics.*

Note that this fact, here obvious, obtains under more general models and is stressed by Gauthier as a general "equivalence principle." We are now in a position to complete the proof of Proposition 2.

Proof. From the Lemma it is enough to show that the equilibrium S_* is locally determinate, i.e. locally divergent, under the EGR dynamics. Assume the contrary. Then in every neighborhood of S_* there exists initial $S_2 \neq S_*$ such that $S_t \to S_*$ under the perfect foresight dynamics (12). Then take $y_0 \neq 0$ and $y_1 \neq S_* y_0$, so that $(y_1', y_0')'$ does not belong to the stable subspace. The sequence $y_2 = S_2 y_1, \ldots, y_t = \left(\prod_{i=2}^t S_i\right) y_1, \ldots$ is a perfect foresight sequence in states y_t. But $\prod_{i=2}^t S_i \to 0$ as $t \to \infty$, since $S_t \to S_*$ and all eigenvalues of S_* are smaller than one in modulus. Hence $y_t \to 0$. This is a contradiction since we know that there does not exist a perfect foresight sequence in states that starts outside the stable subspace and converges to zero. *Q.E.D.*

LIE-stability plays a role in the theory of strong rationality, to which we now turn.

5.2 Strong Rationality.

5.2.1 Preliminaries on "eductive learning"

We now develop the eductive learning argument and the criterion, called in Guesnerie (1992) Strong Rationality, under which eductive learning will lead to coordination on an equilibrium path. We therefore return to the "strategic reduced form" of the model (3) developed in Section 2.2.2 and reproduced here for convenience:

$$y_t = D y_{t-1} + B \int Z(\omega_t) y_{t+1}^e(\omega_t) d\omega_t.$$

We apply the strong rationality test to the saddle-point solution S_*, but it could also be applied to any $\bar{S}$ that satisfies $\bar{S} = (I - B\bar{S})^{-1} D$.

We develop the argument as follows.

Our subsequent analysis assumes that beliefs of the agents are of the form $y_{t+1} = S_t y_t$. In such a setting the subjective expected value $y_{t+1}^e(\omega_t)$ of agent ω_t can be viewed as $S(\omega_t) y_t$, where $S(\omega_t)$ is the "expected" matrix S that agent ω_t's subjective probability distribution on an appropriate set of matrices $V(\bar{S})$ generates. Then, for this state of beliefs the value of y_t is given by

$$y_t = Dy_{t-1} + \left(B \int Z(\omega_t) S(\omega_t) d\omega_t \right) y_t.$$

This is our basic relationship, which we rewrite :

$$y_t = \left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right)^{-1} Dy_{t-1}$$

We will consider a neighborhood $V(\bar{S})$ of the form:

$$\left\| S - \bar{S} \right\| \leq \epsilon,$$

where $\|.\|$ is a Euclidean vector norm (discussed further below) on R^{n^2}, which is identified with the space of $n \times n$ matrices.

Next consider the map

$$G : \prod_{\omega_t} (\Delta(S(\omega_t))) \rightarrow \Delta(\left[I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right]^{-1} D),$$

where Δ denotes deviations from $\bar{S}$, e.g. $\Delta(S(\omega_t)) = S(\omega_t) - \bar{S}$.

We can now introduce our definition of local strong rationality.

Definition 4 *$\bar{S}$ is said to be LSR (Locally Strongly Rational) if there exists $\eta > 0, 0 < \mu < 1$ and a Euclidean vector norm $\|.\|$ on R^{n^2} such that for all $0 < \epsilon < \eta$, $\|\Delta(S(\omega_t))\| \leq \epsilon$ for all ω_t implies*

$$\left\| G \left(\prod_{\omega_t} \Delta(S(\omega_t)) \right) \right\| < \mu\epsilon.$$

We remark that each choice of basis for R^{n^2} yields a different vector norm, namely the Euclidean norm computed using the coordinates of the vector in that basis.

Applied to S_*, the definition of LSR states that if beliefs of the agents, concerning the law of motion of the system, are close enough to the saddle path beliefs, then the actual law of motion is even closer. We also say then that the equilibrium is "eductively stable."

Alternatively, we can refer to the standard procedure of starting from a Common Knowledge restriction, i.e. to assume first:

CK Assumption: *It is CK that, for all t, $y_t = S_t y_{t-1}$ with $S_t \in V(\bar{S})$, where V is a small (enough) neighborhood of $\bar{S}$.*

Then:

Lemma 5 *If $\bar{S}$ is LSR in the sense of the previous definition, and if the above CK assumption holds for $V(\bar{S}) = [S : \left\|S - \bar{S}\right\| \leq \epsilon]$, for some $\epsilon < \eta$, then it is CK that $S = \bar{S}$*

Proof. Given the CK assumption $\|\Delta S_t\| = \left\|S_t - \bar{S}\right\| \leq \epsilon$ the expectations of all agents ω_t must satisfy $\|\Delta(S(\omega_t)\| \leq \epsilon$ for all t. LSR according to our definition implies that it is CK, after a first step of "mental process" that $\|\Delta S_t\| \leq \mu\epsilon$ for all t, which tightens the CK assumption The above process can then be iterated and, after n stages it is CK that $\|\Delta S_t\| \leq \mu^n\epsilon$ for all t. Since this holds for all positive integers n it follows that it is CK that $\|\Delta S_t\| = 0$ for all t and hence that $y_t = \bar{S}y_{t-1}$

Hence the definition of LSR given here is closely connected to previous definitions that directly refer to a CK restriction. It is indeed quite similar, although not exactly identical: for further discussion, the reader will refer to Guesnerie (2002)

5.2.2 Characterizing Local Strong Rationality

The next step of our investigation relies on the study of the linear approximation to the above map G. We call this linear map Γ, and write it:

$$\Gamma : \prod_{\omega_t} (\Delta(S(\omega_t))) \rightarrow \int L(\omega_t)\Delta(S(\omega_t))d\omega_t$$

where $L(\omega_t)$ is a linear mapping from R^{n^2} into R^{n^2}, given explicitly later below.

Hence $\left\|\Delta(\left[I - B\int Z(\omega_t)S(\omega_t)d\omega_t\right]^{-1} D)\right\|$ is approximately equal to

$$\left\|\int L(\omega_t)\Delta(S(\omega_t))d\omega_t\right\| \leq \int \|L(\omega_t)\Delta(S(\omega_t)\| \, d\omega_t.$$

Let $\rho(\omega_t) = \|L(\omega_t)\|$ be the norm induced[8] (on linear mappings from R^{n^2}*to* R^{n^2}) by the initial vector norm (on R^{n^2}). Then

$$\|\Delta(S(\omega_t))\| = \left\|(S(\omega_t)) - \bar{S}\right\| \leq \epsilon \ \forall\omega_t$$

[8] For the theory of matrix norms, see Horn and Johnson (1985). We shall refer here only to induced matrix norms. Given a vector norm $|.|$, the matrix norm $\|.\|$ induced by the vector norm is defined as follows: $\|A\| = \max_{|x|=1} |Ax|$

implies that to a first approximation

$$\left\| \Delta(\left[I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right]^{-1} D) \right\| \leq (\int \rho(\omega_t) d\omega_t) \epsilon.$$

We can also define $\|\Gamma\|$ as the norm of the linear mapping $\Gamma\left(\Pi_{\omega_t}\left(\Delta(S(\omega_t))\right)\right)$ induced by the norm on $\Pi_{\omega_t}\left(\Delta(S(\omega_t))\right)$, which we take to be $\sup_{\omega_t} \|\Delta(S(\omega_t))\|$. Then

$$\|\Gamma\| \leq \int \rho(\omega_t) d\omega_t.$$

Consider now the homogenous expectations case and introduce the map g (from R^{n^2} to R^{n^2})

$$g : S \rightarrow (I - BS)^{-1} D.$$

Call γ the linear map tangent, at $\bar{S}$, to the map g.

Note, that, since $\int Z(\omega_t) d\omega_t = I$ whenever $(S(\omega_t) - \bar{S}) = \Delta S$ independently of ω_t, the map Γ, acting on $\Pi_{\omega_t}(\Delta S\ )$, takes the same value as the map γ acting on ΔS.

In other words,

$$\gamma = \int L(\omega_t) d\omega_t$$

Hence, it must be the case that

$$\|\gamma\| \leq \|\Gamma\|,$$

where $\|\gamma\|$ is the norm induced by the vector norm in R^{n^2} previously introduced.

Finally, this implies, to a first order approximation, that

$$\|\gamma\| \epsilon \leq \sup_{\|\Delta(S(\omega_t)\| \leq \epsilon} \left\| \Delta(\left[I - B \int Z(\omega_t) S(\omega_t) d\omega_t \right]^{-1} D) \right\| \leq \left(\int \rho(\omega_t) d\omega_t \right) \epsilon$$

It follows that the condition $\int \rho(\omega_t) d\omega_t < 1$ is sufficient for LSR of $\bar{S}$.

We can translate this analysis into a formal proposition that will provide a basic reference for further reflection.

Theorem 6 *i) LSR $\Longrightarrow$ LIE-Stability*
ii) If agents are homogeneous, LSR is identical to LIE-Stability
iii) A sufficient condition for LSR is

$$\int \rho(\omega_t) d\omega_t < 1,$$

where $\rho(\omega_t)$ is the norm of $L(\omega_t)$, induced by a vector norm on R^{n^2}, and where $L(\omega_t)$ describes the approximate change on the aggregate state variable, triggered by a change in expected EGR of agent ω_t.

Proof. i) $\|\gamma\|$ is greater than the modulus of the eigenvalue of maximal modulus of γ, and $\|\gamma\| \leq \|\Gamma\|$, which is smaller than one because of LSR. That is enough for LIE-Stability, which requires that the maximal modulus of γ be less than one.[9]

ii) Take $Z(\omega_t) = \sigma(\omega_t)I, \sigma(\omega_t) \geq 0$, and $\int \sigma(\omega_t) d\omega_t = 1$. Then Γ "coincides" with γ, and $\|\gamma\|$ can be made arbitrarily close to the maximal modulus of γ, for an appropriate choice of norms, (See Horn and Johnson (1991)).

iii) already shown. *Q.E.D.*

We remark that this proposition implies the corollary that, in the saddle-point case, if agents are homogeneous then the equilibrium S_* is LSR since it is LIE-Stable by Proposition 2. In fact, as can be seen from the proof of ii), some (mild) deviations from complete homogeneity leave this result intact.

5.2.3 Further characterization

In the Proposition, iii) captures the idea of heterogeneity: it is powerful but abstract. In order to make it more intuitive, we have to specialize the statement by choosing some special vector norms in R^{n^2}.

First Specialization:

Consider the mapping γ, the derivative of $(I - BS)^{-1}D$, at S_*. This can be written, in matrix form[10]:

$$\gamma : \Delta S \to (I - BS_*)^{-1} B(\Delta S)(I - BS_*)^{-1} D = S_* D^{-1} B(\Delta S) S_*.$$

[9] The result is a priori obvious, but it is worth deriving from our inequalities.

[10] Taking the differential of $(I - BS)^{-1}(I - BS) = I$ it follows that the differential of $(I - BS)^{-1}$ is given by $d(I - BS)^{-1} = (I - BS)^{-1} B(dS)(I - BS)^{-1}$. See Magnus and Neudecker (1988) for these and related matrix results.

or, after "vectorization,"

$$\gamma : \mathrm{vec}\,\Delta S \to (S'_* \otimes S_* D^{-1} B)(\mathrm{vec}\,\Delta S)$$

where $\otimes$ designates the Kronecker product[11], S' is the transpose of S and $\mathrm{vec}(\Delta S)$ denotes the vector obtained by stacking in order the columns of ΔS. We have here used the relationship $\mathrm{vec}(ABC) = (C' \otimes A)\,\mathrm{vec}(B)$.

Similarly the mappings $L(\omega_t)$ can be written in matrix form as[12]

$$L(\omega_t) : \Delta S \to S_* D^{-1} B Z(\omega_t)(\Delta S) S_*, \text{ or}$$

$$L(\omega_t) : \mathrm{vec}\,\Delta S \to \left(S'_* \otimes \left(S_* D^{-1} B Z(\omega_t)\right)\right)(\mathrm{vec}\,\Delta S).$$

Again, one can check that :

$$\gamma = \int L(\omega_t) d\omega_t$$

Now, take as vector norm for R^{n^2}, the Euclidean norm in the eigenvector basis of γ.[13] For this vector norm, the induced matrix norm for $S'_* \otimes S_* D^{-1} B$ is the modulus of its eigenvalue of highest modulus.

Now $\rho(\omega_t)$ is the norm of the matrix $S'_* \otimes (S_* D^{-1} B Z(\omega_t))$ induced by the just defined Euclidean norm.(It must be at least as large as the modulus of the eigenvalue of highest modulus of the considered matrix).

Hence, the next Theorem specializes conditions i) and iii) of Theorem 6 :

Theorem 7 *A sufficient condition for the saddle-path solution, S_*, to be LSR is that:*

$$\int \rho(\omega_t) d\omega_t < 1$$

where $\rho(\omega_t)$ is the norm of the matrix $S'_* \otimes (S_* D^{-1} B)(Z(\omega_t))$ induced by the Euclidean norm of the eigenvector basis of the matrix $(S'_* \otimes S_* D^{-1} B)$.

---

[11] If A is an $m \times n$ matrix with elements a_{ij} and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ matrix obtained by replacing each element a_{ij} of A with the $p \times q$ block $a_{ij} B$.

[12] The differential of the map $\left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t\right)^{-1} D$ at $S(\omega_t) = \bar{S}\ \forall \omega_t$ is $d\left(I - B \int Z(\omega_t) S(\omega_t) d\omega_t\right)^{-1} D = (I - B\bar{S})^{-1} B \left(d\left(\int Z(\omega_t) S(\omega_t) d\omega_t\right)\right)(I - B\bar{S})^{-1} D$
$= \bar{S} D^{-1} B \left(d\left(\int Z(\omega_t) S(\omega_t) d\omega_t\right)\right) \bar{S} = \int \bar{S} D^{-1} B Z(\omega_t)(dS(\omega_t)) \bar{S} d\omega_t$
$= \int L(\omega_t)(dS(\omega_t)) d\omega_t$, with $L(\omega_t)$ given as stated and $\bar{S} = S_*$.

[13] which exists if γ is semisimple, as we are assuming.

A necessary condition is that

$$\alpha \leq 1,$$

where α is the modulus of the eigenvalue of highest modulus of the matrix $(S'_* \otimes S_* D^{-1} B)$

Note that although this theorem has been stated for saddle-path solutions, this result (and the next theorem) apply generally to minimal state variable solutions ("extended growth rate" solutions), i.e. to solutions of the form $y_t = \bar{S} y_{t-1}$, where $\bar{S}$ satisfies (10).

The next Theorem gives alternative sufficient conditions (the proof is similar):

Theorem 8 *A sufficient condition for the saddle-path stable solution S_*, to be LSR is that:*

$$\int \rho(\omega_t) d\omega_t < 1$$

where $\rho(\omega_t)$ is the norm of the matrix $S'_* \otimes (S_* D^{-1} B)(Z(\omega_t))$ induced by any vector norm, and in particular the standard Euclidean norm, on R^{n^2}.

In other words, alternative sufficient conditions may be obtained by alternative choices of norms. This may prove useful in applications.

We show, in particular that the just stated theorem, applied to the one dimensional case that we have studied earlier (Evans-Guesnerie (2003)), gives the sharpest[14] statement that can be then obtained :

Corollary 9 *Let us consider the one dimensional version of our problem ($n = 1$). Then a sufficient condition for the saddle-path solution to be LSR is*

$$-1/[2(\Omega - 1)] < BD < 1/[2(\Omega + 1)]$$

where $\Omega = Z^+ - Z^-$, with $Z^+ = \int_{\omega_t / Z(\omega_t) > 0} Z(\omega_t) d\omega_t$, $Z^- = \int_{\omega_t / Z(\omega_t) < 0} Z(\omega_t) d\omega_t$

Proof. The norm of the (now one dimensional) matrix of Theorem 4 is $\int (S^*)^2 |D|^{-1} |B| |Z(\omega_t)| d\omega_t$. But, here S^* satisfies $B(S^*)^2 - S^* + D = 0$, so that the condition becomes $|D| |B| (1 - BS^*)^{-2} \Omega < 1$. This is the intermediate inequality from which we obtain the above condition (see Evans-Guesnerie (2003), footnote 14).

[14] It is sharpest in the sense that it is then a necessary and sufficient condition.

5.3 Discussion

The results of the previous section provide a defense, from basic principles, of the saddle path solution in the "saddle-point stable" case. When there are homogeneous agents, or if the degree of structural heterogeneity is not too large, we have shown that the saddle-path solution is always LSR, so that a process of eductive reasoning, beginning with a local CK restriction leads ineluctably to coordination on this solution. Theorems 6 and 7 show that when there is sufficient heterogeneity the saddle-point solution will no longer invariably be LSR, and we provide convenient sufficient conditions for LSR of the saddle-point solution. Thus these results highlight the importance of heterogeneity, which might be overlooked in "representative agent" or reduced form models.

In fact, coming back on the differences between LIE stability and LSR, which coincide in the "representative agent" case, we see now that in the case of heterogeneity they differ strictly. More precisely, considering the linear map $\Gamma : \prod_{\omega_t} (\Delta(S(\omega_t))) \rightarrow \int L(\omega_t)\Delta(S(\omega_t))d\omega$, which allows us to check LSR, and the map γ that is obtained by replacing $\Delta(S(\omega_t)$ with $\Delta(S)$ and which determines LIE, and assuming that both $\gamma = \int L(\omega_t)d\omega_t$ and (almost) all $L(\omega_t)$ have full rank, we can state:

Proposition 10 *LSR is always strictly more demanding than LIE, unless (except for a subset of measure zero of ω_t) all the linear maps $L(\omega_t)$ are proportional. In the latter case agents can be said to be "essentially identical".*

Proof. We will only give here an incomplete proof with only "two" agents ω_t (as in the case of the specific illustrative model introduced above). Extending the argument to a finite number of agents is straightforward; going to the continuum requires more formal care.

In the case being considered, the assertion will hold if one can prove that if A and B are two linear maps, here from R^{n^2} into R^{n^2}, and if S is a closed convex set with smooth boundary, here containing zero, then:

$AS + BS = [y : y = Ax_1 + Bx_2, x_1 \in S, x_2 \in S]$ *strictly contains* $(A + B)S = [y : y = Ax + Bx, x \in S]$.

The fact that $AS + BS \supseteq (A + B)S$ is obvious. Assume now that the inclusion is not strict so that $AS + BS = (A + B)S$. Consider $x \in FrS$, where FrS denotes the frontier of S, and consider the tangent hyperplane to S in x, denoted Tgx.

Ax is on the (smooth) frontier of the (convex) set AS, with a tangent hyperplane $ATgx$. Bx is on the (smooth) frontier of the (convex) set BS, with a tangent hyperplane $BTgx$. $(A+B)x$ is on the (smooth) frontier of the (convex) set $(A+B)S$.

Our assumption implies that $Fr(AS+BS) = Fr((A+B)S)$, so that $(A+B)x$ is on the frontier of $(AS+BS)$. However, from the standard theory of addition of convex sets, we know that latter fact is possible only if AS and BS are associated with the same tangent hyperplane in Ax and Bx, i.e. if $ATgx = BTgx$.

The argument just made holds for all x on FrS. In order to conclude it remains only to note that as x varies, Tgx can be any hyperplane of the initial space, so that $ATgx = BTgx$ is possible only if $A = \mu B, \mu \neq 0$. It is finally easy to show, using the intuition of the one dimensional case, that $\mu > 0$. *Q.E.D.*

The above statement, together with the one dimensional result of the previous Corollary, provides our sharpest abstract illustrations of the destabilizing effect of heterogeneity.

Example: We provide a numerical illustration using our economic example of Section 3. Suppose that $\alpha_1 = \alpha_2 = \alpha > 1$ and $\beta_1 = \beta_2 = \beta$, that disutility of labor is $v(l) = (1/\xi)l^{\xi}$, where $\xi > 1$, and that consumption utility is

$$u(c^1, c^2) = 2^{-1/(\theta-1)} \left(\left(c^1\right)^{(\theta-1)/\theta} + \left(c^2\right)^{(\theta-1)/\theta} \right)^{\theta/(\theta-1)},$$

where $\theta > 0$ and $\theta \neq 1$. One can compute that at the steady state

$$B = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad D = \frac{(\alpha-1)\theta\xi}{\alpha(\theta-1)} \begin{pmatrix} 1-(2\theta)^{-1} & -(2\theta)^{-1} \\ -(2\theta)^{-1} & 1-(2\theta)^{-1} \end{pmatrix}.$$

If expectations are heterogeneous across sectors but are homogeneous within sectors then the linearized system is given by (6) with

$$Z(1) = \frac{\theta}{\theta-1} \begin{pmatrix} \left(1-(2\theta)^{-1}\right)^2 & (2\theta)^{-1}\left(1-(2\theta)^{-1}\right) \\ -(2\theta)^{-1}\left(1-(2\theta)^{-1}\right) & -(2\theta)^{-2} \end{pmatrix}$$

and $Z(2) = I - Z(1)$. As $\theta \to \infty$, i.e. the case of perfect substitutes, the model reduces to a pair of independent one-sector Reichlin models, but for finite θ the sectors are interdependent.

As a numerical example, suppose that $\alpha = 1.1, \xi = 1.2$ and $\theta = 1.7$. There is a minimal state variable solution approximately equal to $\bar{S} = \begin{pmatrix} 0.284 & -0.161 \\ -0.161 & 0.284 \end{pmatrix}$. Both roots of $\bar{S}$ are inside the unit circle. This solution is LIE-stable and indeed for the map γ one can compute that, approximately, $\alpha = 0.679$. However, this solution is not eductively stable: Let $T = \left[\begin{array}{cc} L(1) & L(2) \end{array} \right]$ where $L(i) = \bar{S}' \otimes \left(\bar{S}D^{-1}BZ(i)\right)$ for $i = 1, 2$. The eigenvalues of $T'T$ are positive and real with the largest root approximately equal to $r = 1.589$. Consider any Euclidean vector norm and let $\tilde{s}' = (\tilde{s}_1, \tilde{s}_2)$ denote an eigenvector of $T'T$ corresponding to r with $\|\tilde{s}\| = 1$. Then $\|L(1)\tilde{s}_1 + L(2)\tilde{s}_2\|^2 = \|T\tilde{s}\|^2 = \tilde{s}'T'T\tilde{s} = r\|\tilde{s}\|^2 = r > 1$. Since $\|\tilde{s}_1\|, \|\tilde{s}_2\| \leq 1$ it follows that the solution $\bar{S}$ is not LSR even though it is LIE-stable.

The class of models we have considered, multivariate one-step ahead one-step memory multivariate models, is quite general, but it is of course not fully general.[15] In particular, altering the information assumptions so that not all time t information is available, when time t decisions are taken, can lead to more restrictive LSR conditions that may not be met in the saddle-point case, even with homogeneous agents, as our earlier work has shown (Guesnerie (1992), Evans and Guesnerie (1993, 2003)).

The multivariate framework of the current paper serves, however, to emphasize a crucial aspect of heterogeneity that can impede coordination of expectations. As our simple economic example illustrates, multisectoral models will typically exhibit a dependence of the economic decisions in each sector on expected future economic activity in both the same and in other sectors. Furthermore, because of the interconnection of *current* economic variables, the expectations of agents in other sectors will also matter. In consequence economic activity in each sector will typically depend on the expectations of agents in all sectors about future economic activity in all sectors.

We have seen that if expectations of agents are heterogenous in terms of effects then this can impede coordination on rational expectations. Our economic illustration shows how differential impacts are likely to arise even if the economic structure is symmetric. Intersectoral and intrasectoral differences in production technologies and preferences can be expected to magnify structural heterogeneity, increasing further the problem of rational agents coordinating on "rational expectations."

[15] An important generalization will be to consider the model (3) with B singular.

6 Conclusions

In the saddle-path stable case, the unique non-explosive perfect foresight solution provides a natural focus of attention for economic theorists, but there remain deep questions about how this solution would be attained by economic agents. The eductive approach examines this issue from the viewpoint of full rationality: would rational agents necessarily coordinate on the saddle-point solution if they knew the correct model, knew that other agents knew the correct model and knew that other agents were rational?

In addressing this question we provide our agents with strong additional common knowledge restrictions designed to facilitate this coordination: specifically we assume that it is common knowledge that the state dynamics, in every subsequent period, will be close to those followed by the perfect foresight path. The economy is said to be Locally Strongly Rational, or eductively stable, if these (hypothetical) restrictions are sufficient to imply common knowledge of the perfect foresight path itself.

Our characterization provides alternative sufficient conditions for LSR of a minimal state variable solution. Iterative expectational stability provides simple necessary conditions, and these will be satisfied by the saddle-point solution in the saddle-point stable case. However, we have shown that these conditions are not in general sufficient. Our analysis has emphasized in particular the potential role of heterogeneity in destabilizing the economy. When expectations of different agents have heterogeneous impacts on the economy, as is natural even in symmetric multisectoral models, the required conditions for eductive stability are tightened because of the potential interaction of this structural heterogeneity with heterogeneous expectations. Sufficient structural heterogeneity can therefore render rational coordination of expectations impossible even in the saddle-point stable case.

Appendix: Minimal State Variable Solutions

Consider a set of n vectors of R^n, $\left(. \quad z_i, \quad . \quad z_j, \ .\right)$ where the n vectors $z_i, z_j, ...$ are associated with a subset K of n of the $2n$ eigenvalues of Φ, provided that if a nonreal eigenvalue is included in K then so is its complex conjugate. If $\lambda_j \in K$ is real then z_j is taken to be the n-dimensional restriction of the corresponding eigenvector $\left\{ \begin{array}{c} \lambda_j x_j \\ x_j \end{array} \right\}$, i.e. $z_j = x_j$. If $\lambda_j, \lambda_{j+1} \in K$ are not real and equal $a_j \pm ib_j$ the eigenvectors take the same form but with $x_j, x_{j+1} = u_j \pm iv_j$. In this case the corresponding vectors z_j, z_{j+1} are taken to be v_j and u_j. Consider the case where the n vectors under consideration form a basis of R^n. The matrix, denoted S_K, which transforms z_j to $\lambda_j z_j$ for real $\lambda_j \in K$ and analogously for nonreal conjugate members of K, is a fixed point of (10). Indeed, in the basis consisting of the z_j, for $\lambda_j \in K$, a vector $\left\{ \begin{array}{c} y_t \\ y_{t-1} \end{array} \right\} = \left\{ \begin{array}{c} \Lambda\alpha \\ \alpha \end{array} \right\}$ is transformed into $y_{t+1} = \Lambda^2\alpha$, so that $y_{t+1} = \Lambda y_t$.

In the canonical basis of R^n,

$$S_K = P_K \Lambda_K P_K^{-1},$$

where we have now factored $\Phi = P\Lambda P^{-1}$ as

$$P = \left(\begin{array}{cc} P_K\Lambda_K & P_L\Lambda_L \\ P_K & P_L \end{array} \right) \text{ and } \Lambda = \left(\begin{array}{cc} \Lambda_K & 0 \\ 0 & \Lambda_L \end{array} \right)$$

The solution of the previous section, $S_* = P_{21}\Lambda_1 P_{21}^{-1}$, corresponds to the choice $\Lambda_K = \Lambda_1$. We remark that there can be up to C_n^{2n} distinct solutions S_K, with exactly C_n^{2n} such solutions when all roots are real and all subsets of n vectors $\left(. \quad z_i, \quad . \quad z_j, \ .\right)$ yield linearly independent sets.

It is straightforward to verify algebraically that, in the canonical basis of R^n, $S_K = P_K \Lambda_K P_K^{-1}$ satisfies the fixed point equation (10). This follows immediately from partitioned matrix multiplication of the equation $\Phi P = P\Lambda$, using the above partition. Therefore $y_t = S_K y_{t-1}$ is a solution for any initial condition y_0. The converse can also be shown, i.e. every S that provides a solution of the form $y_t = Sy_{t-1}$ for every initial condition y_0 can be expressed as $P_K \Lambda_K P_K^{-1}$.

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