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Strategic substitutabilities versus strategic complementarities

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Expectational Coordination in a class of Economic Models : Strategic Substitutabilities versus Strategic Complementarities *

Roger Guesnerie[†] Pedro Jara-Moroni[‡]

Abstract

We consider an economic model that features : 1. a continuum of agents 2. an aggregate state of the world over which agents have an infinitesimal influence. We first propose a review, based on work by Jara-Moroni (2007), of the connections between the *eductive viewpoint* that puts emphasis on *Strongly Rational Expectations equilibrium* and the standard game-theoretical rationalizability concepts. We explore the scope and limits of this connection depending on whether standard rationalizability versus point-rationalizability, or the local versus the global viewpoint, are concerned. In particular, we define and characterize the set of *Point-Rationalizable States* and prove its convexity. Also, we clarify the role of the heterogeneity of beliefs in general contexts of expectational coordination (see Evans and Guesnerie (2005)). Then, as in the case of strategic complementarities the study of some *best response* mapping is a key to the analysis, in the case of *unambiguous* strategic substitutabilities the study of some second iterate, and of the corresponding two-period cycles, allows to describe the point-rationalizable states. We provide application in microeconomic and macroeconomic contexts.

1 Introduction

Our purpose in this paper is then twofold.

First, we attempt to bring in a similar light, the standard game theoretical viewpoint of coordination on rationalizable solutions and the related viewpoint adopted in the study of expectational coordination in economic contexts, as for example in Guesnerie (1992, 2002), Evans and Guesnerie (1993, 2003, 2005). In this work, as well as in most related work on expectational coordination in economic contexts, (Morris and Shin (1998), Chamley (1999, 2004)) as well as in the theory of crisis, economic agents are *non-atomic*, in the sense that they are too small to have a significant influence on the economic system, and the *eductive* reasoning that governs the evaluation of expectational stability refers to game-theoretical rationalizability ideas. Our aim of linking the "economic" and the "game-theoretical" views brings us to adopt the framework of a game with a continuum of agents and aggregators in the sense used by Rath (1992). Relying in particular on Jara-Moroni (2007), we show the precise connections between the game-theoretical concepts of rationalizability, point-rationalizability and the *economic* concepts of *eductive stability*. We stress the convexity properties of the different sets of *rationalizable outcomes* that follow, in the continuum game, from Liapounov like theorems. We establish the connections between the concepts of IE-Stability, the different concepts of Strong Rationality as well as between their local counterparts that allow to select locally *eductively stable* equilibria.

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Second, relying on this framework, we focus attention on two classes of economic problems. In the first one, aggregate strategic complementarities, we reassess and strengthen well known game-theoretical results concerning equilibria and rationalizable solutions. The second class of models has, on the contrary, aggregate strategic substitutabilities. All expectational properties obtained in the strategic complementarity case are shown to have counterparts here. In particular, the set of *Rationalizable states* is precisely located from *cycles of order 2* associated with the system. With differentiability assumptions we get stronger results and simple sufficient conditions assuring the existence of a global Strongly Rational Expectations or a unique rationalizable solution. Applications are given for example using the general equilibrium model of Guesnerie (2001a).

The paper proceeds as follows. In Section 2, we introduce games with a continuum of players and we relate it to a class of economic models with a continuum of agents. We show how this setting may be viewed from a game theoretical point of view and we introduce the concepts of Economic equilibrium and Nash Equilibrium. In section 3 we formulate Rationalizability in this context. We introduce first the concepts of Point-Rationalizable Strategies. We present an economic version of Rationalizability introducing Point-Rationalizable States and Rationalizable States and we relate these concepts to the game theoretical ones. In section 4 we address the economic concepts of Iterative Expectational Stability and Eductive Stability using the tools defined in section 3. Then in Section 5 we successively focus attention on aggregate models with Strategic Complementarities or Strategic Substitutabilities. Our general results here are tightened when we examine the differentiable version of the model. In Section 7 we conclude.

2 The canonical model and concepts.

This Section aims at making a careful connection between the underlying game theoretical concepts and the economic application which they are solicited for. We present first a game theoretical framework that underlies the standard economic model with a continuum of agents, presented afterwards. We then introduce and compare the parallel tools used for the analysis.

2.1 Games with a continuum of players

We consider a *game with a continuum of players*. Non atomic games with continuum of players where first introduced by Schmeidler (1973). In these games the set of players is the measure space $(I, \mathcal{I}, \lambda)$, where I is the unit interval of $\mathbb{R}$, $I \equiv [0,1]$, and λ is the lebesgue measure. Each player chooses a strategy $s(i) \in S(i)$ and we take $S(i) \subseteq \mathbb{R}^n$. Strategy profiles in this setting are identified with integrable selections[1] of the set valued[2] mapping $i \rightrightarrows S(i)$. For simplicity, we will assume that all the players have the same compact strategy set $S(i) \equiv S \subset \mathbb{R}^n_+$. As a consequence, since S is compact, the set of meaningful strategy profiles is the set of measurable functions from I to S [3] noted from now on S^I.

In a game, players have payoff functions that depend on their own strategy and the complete profile of strategies of the player $\pi(i, \cdot, \cdot) : S \times S^I \to \mathbb{R}$. In our particular framework these functions depend, for each player, on his own strategy and an average of the strategies of all the other players. To obtain this average we use the integral of the strategy profile, $\int_I s(i)\,\mathrm{di}$. This implies that all the relevant information about the actions of the opponents is summarized by the values of the integrals, which are points in the set [4]

$$\mathcal{A} \equiv \int_I S(i)\,\mathrm{di}.$$

[1]A selection is a function $\mathbf{s} : I \to \mathbb{R}^n$ such that $s(i) \in S(i)$.

[2]We use the notation $\rightrightarrows$ for set valued mappings (also referred to as correspondences), and $\to$ for functions.

[3]Equivalently, the set of measurable selections of the constant set valued mapping $i \rightrightarrows S$.

[4]Following Aumann (1965) we define for a correspondence $F : I \rightrightarrows \mathbb{R}^n$ its' integral, $\int_I F(i)\ \mathrm{di}$, as:

$$\int_I F(i)\ \mathrm{di} := \left\{ x \in \mathbb{R}^n \ : \ x = \int_I f(i)\ \mathrm{di} \text{ and } f \text{ is an integrable selection of } F \right\}$$

Hypothesis over the correspondence $i \rightrightarrows S(i)$ that assure that the set $\mathcal{A}$ is well defined can be found in Aumann (1965) or in Chapter 14 of Rockafellar and Wets (1998). In this case we get that $\mathcal{A}$ is a convex set (Aumann 1965). Moreover, since $S(i) \equiv S$ we have that [5]

$$\mathcal{A} \quad \equiv \quad \text{co}\,\{S\}\,. \tag{2.1}$$

Pay-offs $\pi(i,\,\cdot\,,\,\cdot\,)$ in this setting are evaluated from an auxiliary utility function $u(i,\,\cdot\,,\,\cdot\,)$: $S \times \text{co}\,\{S\} \to \mathbb{R}$ such that:

$$\pi(i,y,\mathbf{s}) \equiv u\Big(i,y,\int_I s(i)\;\text{di}\Big) \tag{2.2}$$

We assume:

C : *For all agent $i \in I$, $u(i,\,\cdot\,,\,\cdot\,)$ is continuous.*

HM : *The mapping that associates to each agent a utility function [6] is measurable.*

C is standard and does not deserve special comments. **HM** is technical but in a sense natural in this setting. Adopting both assumptions on utility functions put us in the framework of Rath (1992). We begin by giving a definition of Nash Equilibrium in this setting.

Definition 2.1. A (pure strategy) Nash Equilibrium of a game is a strategy profile $\mathbf{s}^* \in S^I$ such that:

$$\forall\, y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i)\;\text{di}\Big) \geq u\Big(i,y,\int s^*(i)\;\text{di}\Big)\,, \quad \forall\, i \in I\;\lambda\text{-a.e.} \tag{2.3}$$

It is useful to use the best reply correspondence $\text{Br}(i,\,\cdot\,) : S^I \rightrightarrows S$ defined as:

$$\text{Br}(i,\mathbf{s}) \quad := \quad \text{argmax}_{y\in S}\,\pi(i,y,\mathbf{s})\,. \tag{2.4}$$

The correspondence $\text{Br}(i,\,\cdot\,)$ describes the optimal response set for player $i \in I$ facing a strategy profile $\mathbf{s}$.

In our setting, and considering the auxiliary function $u(i,\,\cdot\,,\,\cdot\,)$, we can define as well the optimal strategy correspondence $B(i,\,\cdot\,) : \mathcal{A} \rightrightarrows S$ as the correspondence which associates to each point $a \in \mathcal{A}$ the set:

$$B(i,a) \quad := \quad \text{argmax}_{y\in S}\,\{u(i,y,a)\}\,. \tag{2.5}$$

Since, in this setting, $a = \int_I s(i)\;\text{di}$, then $\text{Br}(i,\mathbf{s}) = B(i,a)$, it follows that a Nash equilibrium is a strategy profile $\mathbf{s}^* \in S^I$ such that, $\forall\, i \in I$ λ-a.e., $s^*(i) \in \text{Br}(i,\mathbf{s}^*)$, or equivalently, $s^*(i) \in B\big(i, \int s^*(i)\,\text{di}\big)$ (see Proposition 2.4 below).

Under the previously mentioned hypothesis Rath shows that for every such game there exists a Nash Equilibrium.

Theorem 2.2 (Rath 1992)**.** *The above game has a (pure strategy) Nash Equilibrium.*

The proof of the Theorem is based on the Kakutani fixed point Theorem applied to what we call later the Cobweb Mapping, defined in (3.3). Indeed, a fixed point of such a correspondence determines an equilibrium of the game (see Proposition 2.4 below as well).

[5] Where $\text{co}\,\{X\}$ stands for the convex hull of a set X (see Rath (1992)).

[6] The set of functions for assumption **HM** is the set of real valued continuous functions defined on $S \times \text{co}\,\{S\}$ endowed with the sup norm topology.

2.2 Economic System with a continuum of agents

We address now a class of stylized economic models in which there is a large number of small agents $i \in I$. In this economic system, there is an aggregate variable or signal that represents the *state* of the system. We call $\mathcal{A} \subseteq \mathbb{R}^K$ the set of all possible states of the economic system. Interaction of agents occurs through an aggregation operator, A, that to each strategy profile $\mathbf{s}$ associates a state of the model $a = A(\mathbf{s})$ in the set of states $\mathcal{A}$. The key feature of the system is that no agent can affect unilaterally the state of the system. That is, a change of the actions of only one, or a *small* group of agents, does not modify the value of the state of the system.

These features are those captured with the non-atomic game-theoretical framework described in the previous subsection. The so-called economic system is then naturally imbedded onto the just defined game with a continuum of players when we use as the aggregation operator A the integral[7] of the strategy profile $\mathbf{s}$:

$$A(\mathbf{s}) \equiv \int_I s(i)\,\mathrm{di}.$$

so that the state set $\mathcal{A}$ is $\mathrm{co}\,\{S\}$ (see equation 2.1 and the comments therein). This assures that $\mathcal{A}$ is a nonempty convex compact subset of $\mathbb{R}^n$ (i.e. $K = n$) (Aumann 1965).

The variable $a \in \mathcal{A}$, that represents the state of the system, determines, along with each agents' own action, his payoff. For each agent $i \in I$ then, we use the payoff function $u(i,\cdot,\cdot) : S \times \mathcal{A} \to \mathbb{R}$ introduced in (2.2). Agents act to maximize this payoff function.

In a situation where agents act in ignorance of the actions taken by *the others* or, for what matters, of the value of the state of the system, they have to rely on forecasts. That is, their actions must be a best response to some subjective probability distribution over the space of aggregate data $\mathcal{A}$. Mathematically, actions have to be elements of the set of points that maximize expected utility, where the expectation is taken with respect to this subjective probability. We can consider then the best reply to forecasts correspondence $\mathbb{B}(i,\cdot) : \mathcal{P}(\mathcal{A}) \rightrightarrows S$ defined by:

$$\mathbb{B}(i,\mu) \quad := \quad \mathrm{argmax}_{y\in S}\, \mathbb{E}_\mu\left[u(i,y,a)\right] \tag{2.6}$$

where $\mu \in \mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{A})$ is the space of probability measures over $\mathcal{A}$. Since the utility functions are continuous, problems (2.5) and (2.6) are well defined and have always a solution, so consequently the mappings $B(i,\cdot)$ and $\mathbb{B}(i,\cdot)$ take non-empty compact values for all $a \in \mathcal{A}$. Clearly $B(i,a) \equiv \mathbb{B}(i,\delta_a)$, where δ_a is the Dirac measure concentrated in a.

An equilibrium of this system is a state a^* generated by actions of the agents that are optimal reactions to this state. We denote $\Gamma(a) = \int_I B(i,a)\ \mathrm{di}$.

Definition 2.3. An *equilibrium* is a point $a^* \in \mathcal{A}$ such that:

$$a^* \in \Gamma(a^*) \equiv \int_I B(i,a^*)\,\mathrm{di} \equiv \int_I \mathbb{B}(i,\delta_{a^*})\,\mathrm{di} \tag{2.7}$$

Assumptions **C** and **HM** assure that the integrals in Definition 2.3 are well defined [8]. The equilibrium conditions in (2.7) are standard description of self fulfilling forecasts. That is, in an equilibrium a^*, agents must have a self-fulfilling point forecast (Dirac measures) over a^*, i.e with the economic terminology, a perfect foresight equilibrium (see Guesnerie (1992)).

It is unsurprising that an *equilibrium* as defined in (2.7) has as a counterpart in the game-theoretical approach a *Nash Equilibrium* of the underlying game as defined in (2.3). Precisely :

[7]The aggregation operator can as well be the integral of the strategy profile with respect to any measure $\bar{\lambda}$ that is absolutely continuous with respect to the lebesgue measure, or the composition of this result with a continuous function. That is,

$$A(\mathbf{s}) \equiv G\Big(\int_I s(i)\,f(i)\ \mathrm{di}\Big)$$

where $G : \int_I S(i)\ \mathrm{d}\bar{\lambda}(i) \to \mathcal{A}$ is a continuous function and f is the density of the measure $\bar{\lambda}$ with respect to the lebesgue measure. However not all the results in this work remain true if we choose such a setting.

[8]See Lemma A.1 in the appendix.

Proposition 2.4. *For every (pure strategy) Nash Equilibrium* $\boldsymbol{s}^*$ *of the system's underlying game, there exists a unique equilibrium* a^* *given by* $a^* := A(\boldsymbol{s}^*)$ *and if* a^* *is an equilibrium of the system, then* $\exists\ \boldsymbol{s}^* \in S^I$ *that is a Nash Equilibrium of the underlying game.*

Proof. Indeed, if a^* satisfies (2.7), then there exists an integrable strategy profile $\mathbf{s}^*$ such that $s^*(i) \in B(i, a^*)$ and $A(\mathbf{s}^*) = a^*$. That is $s^*(i) \in B\big(i, \int_I s^*(i)\ \mathrm{di}\big)$, or equivalently

$$\forall\ y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i)\ \ \mathrm{di}\Big) \geq u\Big(i, y, \int s^*(i)\ \ \mathrm{di}\Big)\,, \quad \forall\ i \in I\ \lambda\text{-a.e.}$$

Conversely, if

$$\forall\ y \in S, \quad u\Big(i, s^*(i)\,, \int s^*(i)\ \ \mathrm{di}\Big) \geq u\Big(i, y, \int s^*(i)\ \ \mathrm{di}\Big)\,, \quad \forall\ i \in I\ \lambda\text{-a.e.} \tag{2.8}$$

then $s^*(i) \in \mathrm{Br}(i, \mathbf{s}) \equiv B\big(i, \int s^*(i)\ \ \mathrm{di}\big)\ \forall\ i \in I\ \lambda$-a.e.. Defining $a^* := \int_I s^*(i)\ \ \mathrm{di}$ we get that $a^* \in \int_I B(i, a^*)\,\mathrm{di}$.

■

We will refer equivalently then, to equilibria as points $a^* \in \mathcal{A}$, representing *economic equilibria*, and $\mathbf{s}^* \in S^I$, as Nash Equilibria of the underlying game.

Theorem 2.5. *The stylized economic model has an equilibrium.*

Proof. It is the consequence of the proof of Theorem 2.2 and is related to the previous Proposition.

■

Example 1. Variant of Muth's (1961) model presented in Guesnerie (1992). In this example there is a group of farmers indexed by the unit interval. Farmers decide a positive production quantity $q(i)$ and get as payoff income from sales minus the cost of production: $pq(i) - C_i(q(i))$, where p is the price at which the good is sold. The price is obtained from the inverse demand (or price) function, evaluated in total aggregate production Q. We see that this model fits our framework.

We already said that the set of agents is the unit interval $I = [0, 1]$ and we endow it with the lebesgue measure. Strategies are production quantities, so strategy profiles are functions from the set of agents to the positive line $\mathbb{R}_+$ (i.e. $n = 1$), $\mathbf{q} : I \to \mathbb{R}_+$. The aggregate variable in this case is aggregate production. Agents can calculate their payoff by knowing aggregate production through the price function and deciding their production. So the aggregate state space is the positive line as well, $\mathbb{R}_+$ (i.e. $K = 1 = n$). The payoff of an agent is income from sales minus cost of production, the utility function is then $u(i, q, Q) = P(Q)\,q - C_i(q)$. Where $P : \mathbb{R}_+ \to \mathbb{R}_+$ is an inverse demand (or price) function that, given a quantity of good, gives the price at which this quantity is sold. If we suppose that P is bounded and attains the value 0 from a certain q_{max} on, then we get that the aggregate state set $\mathcal{A}$ is equal to the set of strategies $S(i) \equiv S$, and both are the interval $[0, q_{max}]$. The aggregation operator, the integral of the production profile $\mathbf{q}$, gives aggregate production $Q = \int_I q(i)\ \ \mathrm{di}$.

On this example we can make the observation that the state of he game could be chosen to be the price instead of aggregate production. This is not always the case if we want to obtain the properties stated further on in this work. However, since this example is one-dimensional, it is the case that most of the properties herein presented are passed on from the aggregate production set to the price set.

We are interested now on the plausibility of the equilibrium forecasts, or equivalently to the assessment of the strength of expectational coordination described here. Our assessment relies on the concepts of Rationalizability (Bernheim 1984; Pearce 1984) or on the derived concepts, in our economic framework, of Strong Rationality (Guesnerie 1992). In the next two sections then, we exploit the game-theoretical viewpoint to asses Rationalizability in the economic context.

3 Rationalizability and the "eductive learning viewpoint".

3.1 Rationalizability : the game viewpoint.

Rationalizability is associated with the work of Bernheim (1984) and Pearce (1984). The set of Rationalizable Strategy profiles were defined and characterized in the context of games with a finite number of players, continuous utility functions and compact strategy spaces. It has been argued that Rationalizable strategy profiles are profiles that can not be discarded as outcomes of the game based on the premises of rationality of players, independence of decision making and common knowledge (see Tan and da Costa Werlang (1988)).

First, agents only use strategies that are best responses to their forecasts and so strategies in S that are never best response will never be used; second, agents know that other agents are rational and so know that the others will not use the strategies that are not best responses and so each agent may find that some of his remaining strategies may no longer be best responses, since each agent knows that all agents know, etc. . This process continues ad-infinitum. The set of Rationalizable solutions is such that it is a *fixed point* of the elimination process, and it is the maximal set that has such a property (Basu and Weibull 1991).

Rationalizability has been studied in games with finite number of players. In a game with a continuum of agents, the analysis has to be adapted. Following Jara-Moroni (2007), and coming to our setting, in a game-theoretical perspective, the recursive process of elimination of non best responses, when agents have point expectations, is associated with the mapping $Pr : \mathcal{P}\big(S^I\big) \to \mathcal{P}\big(S^I\big)$ which to each subset $H \subseteq S^I$ associates the set $Pr(H)$ defined by:

$$Pr(H) := \big\{ \mathbf{s} \in S^I \ : \ \mathbf{s} \text{ is a measurable selection of } i \ \rightrightarrows \ \mathrm{Br}(i,H) \big\} \,. \tag{3.1}$$

The operator Pr represents the process under which we obtain strategy profiles that are constructed as the reactions of agents to strategy profiles contained in the set $H \subseteq S^I$. If it is known that the outcome of the game is in a subset $H \subseteq S^I$, with point expectations, the strategies of agent $i \in I$ are restricted to the set $\mathrm{Br}(i,H) \equiv \bigcup_{\mathbf{s}\in H} \mathrm{Br}(i,\mathbf{s})$ and so actual strategy profiles must be measurable selections of the set valued mapping $i \ \rightrightarrows \ \mathrm{Br}(i,H)$. It has to be kept in mind that strategies of different agents in a strategy profile in $Pr(H)$ can be the reactions to (possibly) different strategy profiles in H.

We then define :

Definition 3.1. The set of Point-Rationalizable[9] Strategy Profiles is the maximal subset $H \subseteq S^I$ that satisfies:

$$H \ \equiv \ Pr(H)\,. \tag{3.2}$$

and we note it $\mathbb{P}_S$.

Rationalizable Strategies should be obtained from a similar exercise but considering forecasts as probability measures over the set of strategies of the opponents. Loosely speaking each player should consider a profile of probability measures (his forecasts over each of his opponents play) and maximize some expected utility, expectation taken over an induced probability measure over the set of strategy profiles. A difficulty in a context with continuum of players, relates with the continuity or measurability properties that must be attributed to subjective beliefs, as a function of the agent's name. There is no straightforward solution in any case. However, in our framework it is possible to bypass this difficulty. We present in the next section the concepts of Rationalizable States and Point-Rationalizable States, where forecasts and the process of elimination are now taken over the set of states $\mathcal{A}$.

[9]Following Bernheim (1984) we refer as Point-Rationalizability to the case of forecasts as points in the set of strategies or states and plain Rationalizability to the case of forecasts as probability distributions over the corresponding set.

3.2 Rationalizability : an "economic" viewpoint.

Before going to the rationalizability, it is useful to describe the Cobweb mapping, which we will refer to sometimes later as the Iterative Expectational process.

3.2.1 Cobweb Mapping and Equilibrium

Given the optimal strategy correspondence, $B(i,\cdot)$, defined in (2.5) we can define the *cobweb mapping*[10] $\Gamma : \mathcal{A} \rightrightarrows \mathcal{A}$:

$$\Gamma(a) := \int_I B(i,a)\ \mathrm{di} \tag{3.3}$$

This correspondence represents the actual possible states of the model when all agents react to the same state $a \in \mathcal{A}$. Following Definition 2.3 we see that the equilibria of the economic system are identified with the fixed points of the cobweb mapping.

Definition 3.2. The set of *Aggregate Cobweb Tâtonnement Outcomes*, $\mathbb{C}_{\mathcal{A}}$, is defined by:

$$\mathbb{C}_{\mathcal{A}} := \bigcap_{t\geq 0} \Gamma^t(\mathcal{A})$$

where Γ^t is the tth iterate[11] of the correspondence Γ.

From the proof of Theorem 2.2 (see Rath (1992)) we get that in our framework the cobweb mapping Γ is upper semi continuous as a set valued mapping, with non-empty, compact and convex values $\Gamma(a)$.

3.2.2 State Rationalizability

Below we present the mathematical formulation of *Point-Rationalizable States* and *Rationalizable States*, and explore the relation between Point-Rationalizability and Rationalizability in our context. We aim at clarifying the different perspectives on equilibrium stability and the connections between the notions of local and global Strong Rationality (Section 3.2.3)). For the proofs of the results herein stated and a more detailed treatment the reader is referred to Jara-Moroni (2007).

Analogously to what is done in subsection 3.1, given the optimal strategy correspondence defined in equation (2.5) we can define the process of non reachable or non generated states, considering forecasts as points in the set of states, as follows:

$$\tilde{Pr}(X) := \int_I B(i,X)\,\mathrm{di} \tag{3.4}$$

This is, if initially agents' common knowledge about the actual state of the model is a subset $X \subseteq \mathcal{A}$ we have that forecasts are constrained by X. Then, if expectations are restricted to point-expectations, agents deduce that the possible actions of each agent $i \in I$ are in the set $B(i,X) := \bigcup_{a\in X} B(i,a)$. Since all agents know this, each agent can only discard the strategy profiles $s \in S^I$ that are not a selection of the mappings that assign each agent to the these sets. Finally, they would conclude that the actual state outcome will be restricted to the set obtained as the integral of this set valued mapping.

Definition 3.3. The set of *Point-Rationalizable States* is the maximal subset $X \subseteq \mathcal{A}$ that satisfies the condition:

$$X \equiv \tilde{Pr}(X)$$

and we note it $\mathbb{P}_{\mathcal{A}}$.

[10]The name cobweb mapping comes from the familiar cobweb tâtonnement although in this general context the process of iterations of this mapping may not necessarily have a cobweb-like graphic representation.

[11]This is:

$$\Gamma^0 := \mathcal{A} \quad \Gamma^t := \Gamma(\Gamma^t(\mathcal{A}))$$

We define similarly the set of Rationalizable States. The difference between Rationalizability and Point-Rationalizability is that in Rationalizability forecasts are no longer constrained to points in the set of outcomes. To assess Rationalizability we consider the correspondence $\mathbb{B}(i,\cdot):\mathcal{P}(\mathcal{A})\rightrightarrows S$ defined in (2.6). The process of elimination of non expected-utility-maximizers is described with the mapping $\tilde{R}:\mathcal{B}(\mathcal{A})\to\mathcal{P}(\mathcal{A})$:

$$\tilde{R}(X) := \int_I \mathbb{B}(i,\mathcal{P}(X))\,\mathrm{di} \tag{3.5}$$

If it is common knowledge that the actual state is restricted to a borel subset $X\subseteq\mathcal{A}$, then agents will use strategies only in the set $\mathbb{B}(i,\mathcal{P}(X)) := \cup_{\mu\in\mathcal{P}(X)}\mathbb{B}(i,\mu)$ where $\mathcal{P}(X)$ stands for the set of probability measures whose support is contained in X. Forecasts of agents can not give positive weight to points that do not belong to X. Strategy profiles then will be selections of the correspondence $i\rightrightarrows\mathbb{B}(i,\mathcal{P}(X))$. The state of the system will be the integral of one of these selections.

Definition 3.4. The set of *Rationalizable States* is the maximal subset $X\subseteq\mathcal{A}$ that satisfies:

$$\tilde{R}(X)\equiv X \tag{3.6}$$

and we note it $\mathbb{R}_{\mathcal{A}}$.

The difference between $\tilde{Pr}$ and $\tilde{R}$ is that the second operator considers expected utility maximizers and so for a given borel set $X\subseteq\mathcal{A}$ we have $\tilde{Pr}(X)\subseteq\tilde{R}(X)$. We get directly the result in Proposition 4.1 below.

Bypassing the game-theoretical difficulties occurring in games with a continuum of players, the states set approach provides a substitute for the Rationalizability concept.

3.2.3 Rationalizability : the game versus the economic viewpoint

Rationalizability in the context of the games with continuum of players that we are considering is studied in Jara-Moroni (2007). Therein it is proved that, in our context, the set of Point-Rationalizable pure Strategies is paired with the set of Point-Rationalizable States; moreover, in the context of the original model of Schmeidler[12] these sets are also paired with the set of Rationalizable (pure) Strategies. We state the result that is pertinent to our framework.

Proposition 3.5. *We have:*

$$\mathbb{P}_S \equiv \left\{\, \boldsymbol{s}\in S^I \;:\; \boldsymbol{s} \text{ is a measurable selection of } i \rightrightarrows B(i,\mathbb{P}_{\mathcal{A}})\right\} \tag{3.7}$$

$$\mathbb{P}_{\mathcal{A}} \equiv \left\{\, a\in\mathcal{A} \;:\; a=\int_I s(i)\,\mathrm{di} \text{ and } \boldsymbol{s} \text{ is a measurable function in } \mathbb{P}_S \right\}. \tag{3.8}$$

Equations (3.7) and (3.8) stress the equivalence for point-rationalizability between the state approach and the strategic approach in games with continuum of players : the sets of point-rationalizable states can be obtained from the set of point-rationalizable strategies and vice versa. In (3.7) we see that the strategy profiles in $\mathbb{P}_S$ are profiles of best responses to $\mathbb{P}_{\mathcal{A}}$. Conversely in (3.8) we get that the points in $\mathbb{P}_{\mathcal{A}}$ are obtained as integrals of the profiles in $\mathbb{P}_S$.

We will make use of Proposition 3.6 below, which provides, in the continuum of agents framework, a key technical property of the set of Point-Rationalizable States.

Proposition 3.6. *The set of Point-Rationalizable States can be computed as*

$$\mathbb{P}_{\mathcal{A}} \equiv \bigcap_{t=0}^{\infty} \tilde{Pr}^{\,t}(\mathcal{A})$$

The set $\mathbb{P}_{\mathcal{A}}$, indeed obtains as the outcome of the iterative elimination of unreachable states.

[12]The functions $u(i,\cdot,\cdot)$ are defined on a **finite** strategy set S and depend on the integral of a **mixed** strategy profile.

4 Rationalizable outcomes, Equilibria and Stability

4.1 The global viewpoint.

We denote by $\mathbb{E} \subseteq \mathcal{A}$, the set of equilibria of the economic system. The inclusions below are unsurprising, in the sense that they reflect the decreasing strength of the expectational coordination hypothesis, when going from equilibria to Aggregate Cournot outcomes, then to Point-Rationalizable States, and finally to Rationalizable States.

Proposition 4.1. *We have:*

$$\mathbb{E} \subseteq \mathbb{C}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{A}} \subseteq \mathbb{R}_{\mathcal{A}}$$

The first inclusion is direct since fixed points of Γ are obtained as integrals of selections of the best response correspondence $i \rightrightarrows B(i, a^*)$ and so will not be eliminated during the process that characterizes the set $\mathbb{C}_{\mathcal{A}}$. We can obtain the two last inclusions of Proposition 4.1 noting that if a set satisfies $X \subseteq \tilde{Pr}(X)$ then it is contained in $\mathbb{P}_{\mathcal{A}}$ and equivalently if it satisfies $X \subseteq \tilde{R}(X)$ then it is contained in $\mathbb{R}_{\mathcal{A}}$. Then, the second inclusion is obtained from the fact that each point in $\mathbb{C}_{\mathcal{A}}$, as a singleton, satisfies $\{a^*\} \subseteq \tilde{Pr}(\{a^*\})$ and the third inclusion is true because the set $\mathbb{P}_{\mathcal{A}}$ satisfies $\mathbb{P}_{\mathcal{A}} \subseteq \tilde{R}(\mathbb{P}_{\mathcal{A}})$.

An important corollary of Proposition 3.6 is that the set of Point-Rationalizable States is convex. This is a specific and nice property of our setting with a continuum of agents.

Theorem 4.2.
The set of Point-Rationalizable States is well defined, non-empty, convex and compact.
The set of Rationalizable States is non-empty and convex.

Proof. The properties are obtained from the convexity of each of the sets that are involved in the intersection in the characterization of $\mathbb{P}_{\mathcal{A}}$ in Proposition 3.6. That is, $\mathbb{P}_{\mathcal{A}}$ is the intersection of a nested family of non-empty, compact, convex sets. Non-emptiness of $\mathbb{P}_{\mathcal{A}}$ is guaranteed by Proposition 3.6 along with Theorem 2.2, since an equilibrium would never be eliminated, and so there exists a point $a^* \in \mathcal{A}$ that belongs to every set $\tilde{Pr}^t(\mathcal{A})$. Proposition 4.1 implies the property for $\mathbb{R}_{\mathcal{A}}$, while its' convexity obtains from the definition.

■

In Evans and Guesnerie (1993), two stability concepts of Rational Expectations Equilibria are compared: Iterative Expectational Stability, based on the convergence of iterations of forecasts; and Strong Rationality, based on the uniqueness of the Rationalizable Outcomes (Guesnerie 1992) of an economic model. In what follows, we define these two concepts following the terminology of Guesnerie and Evans and Guesnerie, for our setting.

Definition 4.3. An equilibrium a^* is said to be *Globally Iterative Expectationaly Stable* if $\forall$ $a^0 \in V$ any sequence $a^t \in \Gamma(a^{t-1})$ satisfies $\lim_{t\to\infty} a^t = a^*$ $(= \mathbb{E})$.

The terminology of Iterative Expectational Stability is adopted from the literature on expectational stability in dynamical systems (Evans and Guesnerie (1993, 2003, 2005)). It captures the idea that virtual coordination processes converge globally, under the implicit assumption that agents have *homogenous deterministic expectations.*

Definition 4.4. The equilibrium state a^* is (globally) *Strongly Point Rational* if

$$\mathbb{P}_{\mathcal{A}} \equiv \{a^*\} \ \ (= \mathbb{E}).$$

The idea is now hat virtual coordination processes converge globally, under the implicit assumption that agents have *heterogenous and deterministic expectations.*

Definition 4.5. The equilibrium state a^* is (globally) *Strongly Rational* if

$$\mathbb{R}_{\mathcal{A}} \equiv \{a^*\} \ \ (= \mathbb{E}).$$

Eductive coordination then obtains when agents have *heterogenous and stochastic expectations.*

Strong Rationality and Strong Point Rationality can be related to heterogeneous beliefs of agents. Both concepts refer to heterogenous forecasts of agents, (even if these agents were homogeneous (have the same utility function)). With Strong rationality, forecasts are based on stochastic expectations, when with Strong Point Rationalizability, we restrict attention to point expectations. When we turn to Iterative Expectational Stability (IE-Stability), we drop the possibility of heterogeneity of forecasts. The iterative process associated with IE-Stability is based on iterations of the cobweb mapping Γ which describe agents reactions to the same point forecast over the set of states.

It is straightforward that these concepts are increasingly demanding : Strong Rationality implies Strong Point Rationalizability that implies Iterative Expectational Stability.

We turn now to the local version of these concepts.

4.2 The local viewpoint.

We now give the local version of the above stability concepts.

Again, the definition of (local) IE-Stability (Lucas 1978; DeCanio 1979), stated below is similar to the one given in Evans and Guesnerie (1993)

Definition 4.6. An equilibrium a^* is said to be *Locally Iterative Expectationaly Stable* if there is a neighborhood $V \ni a^*$ such that $\forall\ a^0 \in V$ any sequence $a^t \in \Gamma\big(a^{t-1}\big)$ satisfies $\lim_{t\to\infty} a^t = a^*$.

Definition 4.7. An equilibrium state a^* is *Locally Strongly Point Rational* if there exists a neighborhood $V \ni a^*$ such that the process governed by $\tilde{Pr}$ started at V generates a nested family that leads to a^*. This is, $\forall\ t \geq 1$,

$$\tilde{Pr}^t(V) \subset \tilde{Pr}^{t-1}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{Pr}^t(V) \equiv \{a^*\}\,.$$

Definition 4.8. An equilibrium state a^* is *Locally Strongly Rational* (Guesnerie 1992) if there exists a neighborhood $V \ni a^*$ such that the eductive process governed by $\tilde{R}$ started at V generates a nested family that leads to a^*. This is, $\forall\ t \geq 1$,

$$\tilde{R}^t(V) \subset \tilde{R}^{t-1}(V)$$

and

$$\bigcap_{t\geq 0} \tilde{R}^t(V) \equiv \{a^*\}\,.$$

The connections between the concepts stressed in the next Proposition, are straightforward.

Proposition 4.9. *We have:*

(i) *a^* is (Locally) Strongly Rational $\Longrightarrow$ a^* is (Locally) IE-Stable.*

(ii) *a^* is Locally Strongly Rational $\Longrightarrow$ a^* is Locally Strongly Point Rational.*

A sufficient condition for the converse to be true is that there exist a neighborhood V of a^* such that for almost all $i \in I$, for any borel subset $X \subseteq V$:

$$\mathbb{B}(i, \mathcal{P}(X)) \subseteq \mathrm{co}\,\{B(i, X)\} \tag{4.1}$$

The proof of this and of the following Proposition are relegated to the appendix.

At a first glance the hypothesis in the second part of Proposition 4.9 appears to be very restrictive, however it involves only local properties of the agents' utility functions. It intuitively states that given a restriction on common knowledge (subsets of the set V), when agents evaluate all the possible actions to take when facing probability forecasts with support "close" to the equilibrium, these actions are somehow "not to far" or "surrounded" by the set of actions that could be taken when facing point forecasts ($\mathbb{B}(i,\mu) \subseteq \text{co}\{B(i,X)\}$ if $\text{supp}(\mu) \subseteq X$). The assumption is true in most applications and standard assumptions over utility functions imply it.

Condition (4.1) relates the individual reactions of agents facing non degenerate subjective forecasts, with their reactions when facing point (dirac) forecasts. A different approach can be overtaken when comparing the aggregate reaction of the system to common knowledge on the restriction of the possible outcomes. In this approach we are interested on the convergence of the process generated by point forecasts. If this convergence is sufficiently fast, then we say that the equilibrium is *Strictly* Locally Point Rational, and we may get that this convergence speed, drags the eductive process to the equilibrium as well.

For a positive number $\alpha > 0$ and a set $V \subseteq \mathcal{A}$ that contains a unique equilibrium a^* we will denote by V_α the set:

$$V_\alpha := \{x \in \mathcal{A} \ : \ x = \alpha(v - a^*)\,,\ v \in V\}$$

Definition 4.10. We say that an equilibrium state a^* is *Strictly Locally Point Rational* if it is Locally Strongly Point Rational and there is a number $\bar{k} < 1$ such that, $\forall\ 0 < \alpha \geq 1$,

$$\sup_{v \in \tilde{Pr}(V_\alpha)} \| \ v - a^* \| < \bar{k} \sup_{v' \in V_\alpha} \| \ v' - a^* \| \,.$$

Strict Locally Point Rationality assesses the idea of fast convergence of the point forecast process. Under this property, we have that $\tilde{Pr}(V) \subset V_{\bar{k}}$, with $\bar{k} < 1$, and so $\tilde{Pr}^t(V) \subset V_{\bar{k}^t}$.

Proposition 4.11. *If the utility functions are twice continuously differentiable, $a^* \in \text{int}\,\mathcal{A}$, $\mathbb{B}(i,\mu)$ is single valued for all μ with support in a neighborhood of a^* and $Du_{ss}(s,a)$ is non singular in an open set $V \ni a^*$, then*

$$a^* \textit{ is Locally Strongly Rational} \quad \Longleftrightarrow \quad a^* \textit{ is Strictly Locally Point Rational.}$$

The idea of the proposition is that if the process governed by point forecasts is sufficiently fast, then, although the eductive process may be slower, it is anyhow dragged to the equilibrium state. This is, the eductive process may converge at a lower rate, it can not escape the force of $\tilde{Pr}$.

5 Economic games with strategic complementarities or substitutabilities.

5.1 Economic games with strategic complementarities.

In this section we want to study the consequences over expectational coordination and eductive stability of the presence of *Strategic Complementarities* in our Economic System with a Continuum of Agents. We will say that the economic system presents Strategic Complementarities if the individual best response mappings of the underlying game are increasing for each $i \in I$. That is, if we consider the general payoff functions $\pi(i, \cdot\,, \cdot\,) : S \times S^I \to \mathbb{R}$, the usual product order in $\mathbb{R}^n$ over S and the order $\geq_{S^I}$ defined by $\mathbf{s} \geq_{S^I} \mathbf{s}'$ if and only if $s(i) \geq s(i)'$ for almost all $i \in I$, over S^I, then we would like the mappings $\text{Br}(i, \cdot\,) : S^I \rightrightarrows S$ defined in (2.4) to be increasing for the induced set ordering in S. That is, if $\mathbf{s} \geq_{S^I} \mathbf{s}'$ then $\text{Br}(i, \mathbf{s}) \succeq \text{Br}(i, \mathbf{s}')$.

The most classical representation of complementarity in games is the theory of *supermodular games* as studied in Milgrom and Roberts (1990) and Vives (1990) (see as well Topkis (1998)). In a supermodular game, a normal form game with a finite number of players is embedded within a lattice structure.

A normal form game $\mathcal{G} := \left\langle I, (S_i)_{i \in I}\,, (\pi_i(\,\cdot\,,\,\cdot\,))_{i \in I} \right\rangle$ is supermodular if $\forall\ i \in I$:

1.A S_i is a complete lattice.

2.A $\pi_i(s_i, s_{-i})$ is order upper semi-continuous in s_i and order continuous in s_{-i}, with finite upper bound.

3.A $\pi_i(\,\cdot\,, s_{-i})$ is supermodular on s_i for all $s_{-i} \in S_{-i}$

4.A $\pi_i(s_i, s_{-i})$ has increasing differences in s_i and s_{-i}

We will understand strategic complementarity then, as supermodularity of the underlying game. Supermodularity (and of course submodularity as in the next section) could be studied in the context of games with continuum of agents with a broad generality using the strategic approach (using for instance the tools available from Riesz spaces). However, our present concern suggests to focus on the set of states and introduce strategic complementarities ideas directly in this framework. This last assertion is not at all superfluous since it is the fact that we work with a continuous of agents that allows to focus on forecasts over the set of aggregate states. Since agents can not affect the state of the system, all agents have forecasts over the same set, namely the set of states $\mathcal{A}$. This would not be possible in the context of *small* game since then the forecast of different agents would be in different sets, namely the set of aggregate values of *the others* which could well be a different set for each agent. Another difficulty is passing from strategies to states in terms of complementarity. Part of the work to be presented focuses on the possibility of inheritance by the state approach of the properties of complementarity (and substitutability in the next section). An important result related with this issue is treated in Lemma A.2 in the appendix.

Our objective will then be to understand the consequences of the assumptions introduced on all the sets under scrutiny (equilibria, Cournot outcomes, Point-Rationalizable States, Rationalizable States).

Let us proceed as suggested and make, in the economic setting, the following assumptions over the strategy set S and the utility functions $u(i, \cdot\,, \cdot\,)$.

1.B S is the product of n compact intervals in $\mathbb{R}_+$.

2.B $u(i, \cdot\,, a)$ is supermodular for all $a \in \mathcal{A}$ and all $i \in I$.

3.B $\forall\, i \in I$, the function $u(i, y, a)$ has increasing differences in y and a. That is, $\forall\, y,\, y' \in S$, such that $y \geq y'$ and $\forall\, a,\, a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i, y, a) - u(i, y', a) \geq u(i, y, a') - u(i, y', a') \tag{5.1}$$

Assumption 2.B is straightforward. Assumption 1.B implies that the set of strategies is a complete lattice in $\mathbb{R}^n$. Since in our model we already assumed that the utility functions $u(i, \cdot\,, \cdot\,)$ are continuous, we obtain that in particular the functions $\pi(i, \cdot\,, \cdot\,)$ satisfy 2.A (endowing S^I with the weak topology for instance, this is of no relevance for what follows). Finally, if we look at $\mathcal{A}$ and S^I as ordered sets (with the product order of $\mathbb{R}^n$ in $\mathcal{A}$ and $\geq_{S^I}$ in S^I), we see that the aggregation mapping $A : S^I \to \mathcal{A}$ is increasing, and so assumption 3.B implies 4.A.

Proposition 5.1. *Under assumptions 1.B through 3.B, the mappings $B(i, \cdot\,)$ are increasing in a in the set $\mathcal{A}$, and the sets $B(i, a)$ are complete sublattices of S.*

Proof. The first property is a consequence of Theorem 2 in Milgrom and Roberts (1990) and the second part we apply Theorem 2.8.1 in Topkis (1998) considering the constant correspondence $S_a \equiv S\ \forall\, a \in \mathcal{A}$ ■

Definition 5.2. We name $\mathcal{G}$, an economic system such that S and $u(i, \cdot\,, \cdot\,)$ satisfy assumptions 1.B through 3.B.

One implication of our setting is that since S is a convex complete lattice, then $\mathcal{A} \equiv \text{co}\,\{S\} \equiv S$ is as well a complete lattice. From now on we will refer to the supermodular setting as $\mathcal{G}$.

Proposition 5.3. *In $\mathcal{G}$ the correspondence Γ is increasing and $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$.*

Recall that the set of equilibria is $\mathbb{E} \subseteq \mathcal{A}$ and this is the set of fixed points of Γ. For a correspondence $F : \mathcal{A} \rightrightarrows \mathcal{A}$, we will denote the set of fixed points of F as E_F. Consequently $\mathbb{E} \equiv E_\Gamma$. We see now that under assumptions 1.B through 3.B we get the hypothesis of Proposition 5.4.

Proposition 5.4. *If $\mathcal{A}$ is a complete lattice, Γ is increasing, $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$, then $\mathbb{E} \neq \emptyset$ is a complete lattice.*

Proof. As a consequence of the Theorem 2.5.1 in Topkis (1998) E_Γ is a non-empty complete lattice. ■

In the previous proposition we have an existence result, but what is most important is that the set of equilibria has a complete lattice structure. In particular we know that there exist points $\underline{a}^* \in \mathcal{A}$ and $\bar{a}^* \in \mathcal{A}$ (that could be the same point) such that if $a^* \in \mathbb{E}$ is an equilibrium, then $\underline{a}^* \leq a^* \leq \bar{a}^*$.

The previous results tell us that when the economic system's underlying game is supermodular and since the aggregate mapping is monotone (in this case increasing), then we can apply Proposition 5.4 and work in a finite dimensional setting (the set $\mathcal{A}$) rather than infinite dimensional. We state this as a formal result in the next proposition.

Proposition 5.5. *In $\mathcal{G}$ we have*

$$\mathbb{P}_{\mathcal{A}} \subseteq \left[\inf_{E_\Gamma} \{E_\Gamma\}, \sup_{E_\Gamma} \{E_\Gamma\}\right]$$

and $\inf_{E_\Gamma} \{E_\Gamma\}$ and $\sup_{E_\Gamma} \{E_\Gamma\}$ are equilibria.

The proof is relegated to the appendix. The intuitive interpretation of the proof is as follows. Originally, agents know that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. Since the actual state is in the image through $\tilde{Pr}$ of $\mathcal{A}$, the monotonicity properties of the forecasts to state mappings allow agents to deduce that the actual state will be in fact greater than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and smaller than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, it suffices to consider the cases where all the agents having the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$. The eductive procedure then can be secluded on each iteration, only with iterations of Γ. Since Γ is increasing, we get an increasing sequence that starts at $\underline{a}^0$ and a deceasing sequence that starts at $\bar{a}^0$. These sequence converge and upper semi continuity of Γ implies that their limits are fixed points of Γ.

There are three key features to keep in mind, that lead to the conclusion. First, the fact that there exists a set $\mathcal{A}$ that, being a complete lattice and having as a subset the whole image of the mapping A, allows the eductive process to be initiated. Second, monotonic structure of the model implies that it suffices to use Γ to seclude, in each step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the structure of the model allow the process to converge. Now that we have proved this result for the Point-Rationalizable set, we can use the proof of Proposition 5.5 to get the same conclusion for the set of Rationalizable States. For this we use the following Lemma.

Lemma 5.6. *In $\mathcal{G}$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\ a \in \text{supp}(\mu)$. Then $\forall\ i \in I$*

$$B(i, a') \preceq \mathbb{B}(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\ a \in \text{supp}(\mu)$. Then $\forall\ i \in I$

$$B(i, a') \succeq \mathbb{B}(i, \mu)\,.$$

This is, if the forecast of an agent has support on points that are larger than a point $a' \in \mathcal{A}$, then his optimal strategy set is larger than the optimal strategy associated to a' (for the induced set ordering) and analogously for the second statement.

Proof. Observe first that supermodularity of $u(i,\,\cdot\,,a)$ is preserved[13] when we take expectation on a.

Now consider $y' \in B(i,a')$ and $y \in \mathbb{B}(i,\mu)$ we show that $\min\{y,y'\} \in B(i,a')$ and $\max\{y,y'\} \in \mathbb{B}(i,\mu)$. Since $y' \in B(i,a')$ we have that:

$$0 \leq u(i,y',a') - u(i,\min\{y,y'\},a')\,.$$

Increasing differences of $u(i,y,a)$ in (y,a) implies that $\forall\ a \in \mathrm{supp}(\mu)$,

$$u(i,y',a') - u(i,\min\{y,y'\},a') \leq u(i,y',a) - u(i,\min\{y,y'\},a)$$

and so if on the right hand side we take expectation with respect to μ we get

$$u(i,y',a') - u(i,\min\{y,y'\},a') \leq \mathbb{E}_\mu\left[u(i,y',a)\right] - \mathbb{E}_\mu\left[u(i,\min\{y,y'\},a)\right].$$

Supermodularity of $u(i,\,\cdot\,,a)$ implies that

$$\mathbb{E}_\mu\left[u(i,y',a)\right] - \mathbb{E}_\mu\left[u(i,\min\{y,y'\},a)\right] \leq \mathbb{E}_\mu\left[u(i,\max\{y,y'\},a)\right] - \mathbb{E}_\mu\left[u(i,y,a)\right]$$

and the last term is less or equal to 0 since $y \in \mathbb{B}(i,\mu)$.

All these inequalities together imply that $\max\{y,y'\} \in \mathbb{B}(i,\mu)$ and $\min\{y,y'\} \in B(i,a')$

The second statement is proved analogously.

■

The fact that the points $\sup_{E_\Gamma}\{E_\Gamma\}$ and $\inf_{E_\Gamma}\{E_\Gamma\}$ are equilibria, implies that they are Point-Rationalizable and Rationalizable states and so Proposition 5.5 states that the interval $\left[\inf_{E_\Gamma}\{E_\Gamma\},\sup_{E_\Gamma}\{E_\Gamma\}\right]$ is the smallest interval that contains the set $\mathbb{P}_\mathcal{A}$. Considering Lemma 5.6 and the proof of Proposition 5.5 we get that this same interval contains tightly the set $\mathbb{R}_\mathcal{A}$.

Theorem 5.7. *In the economic system with Strategic Complementarities we have:*

(i) *The set of equilibria $\mathbb{E} \subseteq \mathcal{A}$ is complete lattice.*

(ii) *There exist a greatest equilibrium and a smallest equilibrium, that is $\exists\ \underline{a}^* \in \mathbb{E}$ and $\bar{a}^* \in \mathbb{E}$ such that $\forall\ a^* \in \mathbb{E}$, $\underline{a}^* \leq a^* \leq \bar{a}^*$.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex sets, tightly contained in the interval $[\underline{a}^*,\bar{a}^*]$. That is,*

$$\mathbb{P}_\mathcal{A} \quad\subseteq\quad \mathbb{R}_\mathcal{A} \quad\subseteq\quad \{\underline{a}^*\} + \mathbb{R}^n_+ \bigcap \{\bar{a}^*\} - \mathbb{R}^n_+$$

and $\bar{a}^* \in \mathbb{P}_\mathcal{A}$ and $\underline{a}^* \in \mathbb{P}_\mathcal{A}$.

Proof. Using Lemma 5.6 in the proof of Proposition 5.5 we can see that $\tilde{R}^t(\mathcal{A}) \subseteq [\underline{a}^t,\bar{a}^t]$ and so we get the result.

■

Convexity of $\mathbb{P}_\mathcal{A}$ implies the convex hull of $\mathbb{E}$ is contained in $\mathbb{P}_\mathcal{A}$, in particular the segment

$$\{\ a \in \mathcal{A}\ :\ a = \alpha\underline{a}^* + (1-\alpha)\,\bar{a}^* \quad \alpha \in [0,1]\} \ \subseteq\ \mathbb{P}_\mathcal{A} \ \subseteq\ \mathbb{R}_\mathcal{A}$$

Let us also note :

[13]If $u(i,\,\cdot\,,a)$ is supermodular, then for $s,s' \in S$, we have for each $a \in \mathcal{A}$:

$$u(i,\min\{s,s'\},a) + u(i,\max\{s,s'\},a) - \left(u(i,s,a) + u(i,s',a)\right) \geq 0$$

Taking expectation we get the result.

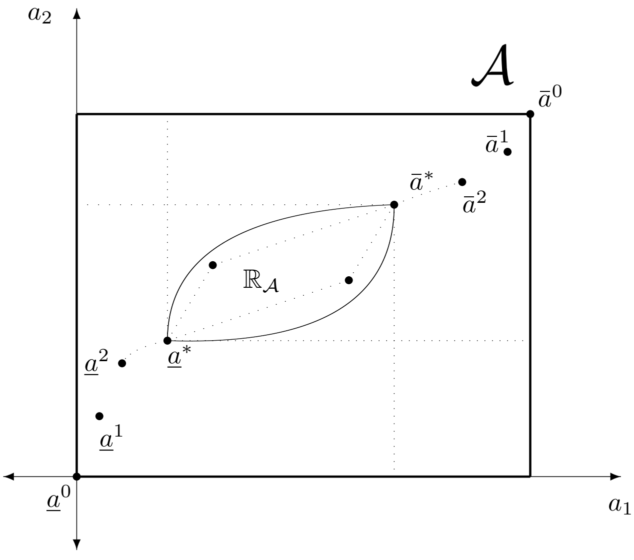

Figure 1: Strategic Complementarities for $\mathcal{A} \subset \mathbb{R}^2$ with four equilibria.

Corollary 5.8. *If in $\mathcal{G}$ Γ has a unique fixed point a^*, then*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}} \equiv \{a^*\}\,.$$

Our results are unsurprising. In the context of an economic game with a continuum of agents, they mimic, in an expected way, the standards results obtained in a game-theoretical framework with a finite number of agents and strategic complementarities. Additional convexity properties reflect the use of a continuum setting.

We see from Theorem 5.7 and Corollary 5.8, that under the presence of strategic complementarity, uniqueness of equilibrium implies the success of the elimination of unreasonable states. The unique equilibrium is then Strongly Rational and this stability is global.

Corollary 5.9. *In $\mathcal{G}$, the three following statements are equivalent:*

(i) *an equilibrium a^* is Strongly Rational.*

(ii) *an equilibrium a^* is IE-Stable.*

(iii) *there exists a unique equilibrium a^*.*

Proof. From the definitions of both concepts of stability we see that Strong Rationality implies IE-Stability. The relevant part of the corollary is then that under Strategic Complementarities, we have the inverse. If a^* is IE-Stable, then from the proof of Proposition 5.5 we see that the sequences $\{\underline{a}^t\}_{t\geq 0}$ and $\{\bar{a}^t\}_{t\geq 0}$ must both converge to a^* and so we get that a^* is *Eductively Stable*.

■

This last statement may be interpreted as the fact that in the present setting, heterogeneity of expectations does not play any role in expectational coordination. This is a very special feature of expectational coordination as argued in Evans and Guesnerie (1993). Surprisingly enough, a similar feature appears in the next class of models under consideration.

5.2 Economic games with Strategic Substitutabilities

We turn to the case of *Strategic Substitutabilities*. We will say that the economic system presents Strategic Substitutabilities if the individual best response mappings of the underlying game are decreasing for each $i \in I$. That is, if $\mathbf{s} \geq_{S^I} \mathbf{s}'$ then $\mathrm{Br}(i,\mathbf{s}) \preceq \mathrm{Br}(i,\mathbf{s}')$. Where $\mathrm{Br}(i,\cdot)$ and $\geq_{S^I}$ are the same as in section 5.1.

To study the consequences of embedding our model in a setting of strategic substitutabilities we use the same structure as in the previous section except that we replace assumption 3.B with assumption 3.B' below.

1.B S is the product of n compact intervals in $\mathbb{R}_+$.

2.B $u(i,\cdot,a)$ is supermodular for all $a \in \mathcal{A}$

3.B' $u(i,y,a)$ has decreasing differences in y and a. That is, $\forall\ y,\ y' \in S$, such that $y \geq y'$ and $\forall\ a,\ a' \in \mathcal{A}$ such that $a \geq a'$:

$$u(i,y,a) - u(i,y',a) \leq u(i,y,a') - u(i,y',a') \tag{5.2}$$

Assumptions 1.B through 3.B' turn the underlying game of our model into a *submodular game*[14] with a continuum of agents. The relevant difference with the previous section is that now the monotonicity of the mapping A along with assumption 3.B' implies that the best response mappings are decreasing on the strategy profiles.

Example 2. : This example is the model described in Guesnerie (2001b). There is a production sector of L sectors indexed by l, with N_l firms in each sector. Firms hire workers at a fixed wage w and sell at price p_l. Firm i's in sector l supply function is denoted $S_i^l(p_l/w)$.

The following Propositions are the counterparts of Propositions 5.1 and 5.3.

Proposition 5.10. *Under assumptions 1.B, 2.B and 3.B', the mappings $B(i,\cdot)$ are decreasing in a in the set $\mathcal{A}$, and the sets $B(i,a)$ are complete sublattices of S.*

Definition 5.11. We name $\mathcal{G}'$, an economic system such that S and $u(i,\cdot,\cdot)$ satisfy assumptions 1.B, 2.B, and 3.B'.

Proposition 5.12. *In $\mathcal{G}$ the correspondence Γ is decreasing and $\Gamma(a)$ is subcomplete for each $a \in \mathcal{A}$.*

We denote Γ^2 for the second iterate of the cobweb mapping, that is $\Gamma^2 : \mathcal{A} \rightrightarrows \mathcal{A}$, $\Gamma^2(a) := \cup_{a' \in \Gamma(a)} \Gamma(a')$.

Corollary 5.13. *In $\mathcal{G}$ the correspondence Γ^2 is increasing and $\Gamma^2(a)$ is subcomplete for each $a \in \mathcal{A}$.*

Proof. Is a consequence of Γ being decreasing. ∎

The correspondence Γ^2 will be our main tool for the case of strategic substitutabilities. This is because, in the general context, the fixed points of Γ^2 are point-rationalizable just as the fixed points of Γ are. Actually, it is direct to see that the fixed points of any iteration of the mapping Γ are as well point-rationalizable. The relevance of strategic substitutabilities is that under their presence it suffices to use the second iterate of the cobweb mapping to seclude the set of point-rationalizable states. Using Proposition 5.4 we get that under assumptions 1.B, 2.B and 3.B', the set of fixed points of Γ^2, E_{Γ^2}, shares the properties that the set of equilibria $\mathbb{E}$ had under strategic complementarities.

Proposition 5.14. *The set of fixed points of Γ^2, E_{Γ^2} is a non empty complete lattice.*

Proof. Apply Proposition 5.4 to Γ^2. ∎

[14] A submodular game is a game under assumptions 1.A to 3.A with assumption 4.A replaced by assumption 4.A': the payoff functions $\pi_i(s_i, s_{-i})$ have decreasing differences in (s_i, s_{-i})

The relevance of Proposition 5.14 is that, as in the case of strategic complementarities, under strategic substitutabilities it is possible to seclude the set of Point-Rationalizable States into a tight compact interval. This interval is now obtained from the complete lattice structure of the set of fixed points of Γ^2, which can be viewed, in a multi-period context, as cycles of order 2 of the system.

Proposition 5.15. *In $\mathcal{G}'$ we have*

$$\mathbb{P}_{\mathcal{A}} \subseteq \left[\inf_{E_{\Gamma^2}} \{E_{\Gamma^2}\}, \sup_{E_{\Gamma^2}} \{E_{\Gamma^2}\} \right]$$

and $\inf_{E_{\Gamma^2}} \{E_{\Gamma^2}\}$ *and* $\sup_{E_{\Gamma^2}} \{E_{\Gamma^2}\}$ *are point-rationalizable.*

The proof is relegated to the appendix. Keeping in mind the proof of Proposition 5.5, we can follow the idea of the proof of Proposition 5.15. As usual, common knowledge says that the state of the system will be greater than $\inf \mathcal{A}$ and smaller than $\sup \mathcal{A}$. In first order basis then, the actual state is known to be in the image through $\tilde{Pr}$ of $\mathcal{A}$. Since now the cobweb mapping is decreasing, the structure of the model allows the agents to deduce that the actual state will be in fact smaller than the image through Γ of the constant forecast $\underline{a}^0 = \inf \mathcal{A}$ and greater than the image through Γ of the constant forecast $\bar{a}^0 = \sup \mathcal{A}$. That is, again it suffices to consider the cases where all the agents having the same forecasts $\inf \mathcal{A}$ and $\sup \mathcal{A}$ and this will give $\underline{a}^1$, associated to $\bar{a}^0$, and $\bar{a}^1$, associated to $\underline{a}^0$. However, now we have a difference with the strategic complementarities case. In the previous section the iterations started in the lower bound of the state set were lower bounds of the iterations of the eductive process. As we see, this is not the case anymore. Nevertheless, here is where the second iterate of Γ gains relevance. In a second order basis, once we have $\underline{a}^1$ and $\bar{a}^1$ obtained as above, we can now consider the images through Γ of these points and we get new points $\bar{a}^2$, from $\underline{a}^1$, and $\underline{a}^2$, from $\bar{a}^1$, that are respectively upper and lower bounds of the second step of the eductive process. This is, in two steps we obtain that the iterations started at the upper (resp. lower) bound of the states set is un upper (resp. lower) bound of the second step of the eductive process. Moreover, the sequences obtained by the second iterates are increasing when started at $\underline{a}^0$ and decreasing when started at $\bar{a}^0$. The complete lattice structure of $\mathcal{A}$ again implies the convergence of the monotone sequences while Γ^2 inherits upper semi continuity from Γ. This implies that the limits of the sequences are fixed points of Γ^2.

The three key features that lead to the conclusion are analogous to the strategic complementarity case. First, $\mathcal{A}$ is a complete lattice that has as a subset its' image through the function A and thus allows the eductive process to be initiated. Second, monotonic structure of the model implies that it now suffices to use Γ^2 to seclude, every second step, the set obtained from the eductive process into a compact interval. Third, continuity properties of the utility functions and the monotonic structure of the model allow the process to converge.

Note that, also as in the case of strategic complementarities, since the limits of the interval in Proposition 5.15 are point-rationalizable, this is the smallest interval that contains the set of point-rationalizable states.

Adapting the proof of Lemma 5.6 to the decreasing differences case, we obtain its' counterpart for the strategic substitutabilities case stated below.

Lemma 5.16. *In $\mathcal{G}'$, for $a' \in \mathcal{A}$ and $\mu \in \mathcal{P}(\mathcal{A})$, if $a' \leq a$, $\forall\ a \in \text{supp}(\mu)$. Then $\forall\ i \in I$*

$$B(i, a') \succeq \mathbb{B}(i, \mu)\,,$$

equivalently, if $a' \geq a$, $\forall\ a \in \text{supp}(\mu)$. Then $\forall\ i \in I$

$$B(i, a') \preceq \mathbb{B}(i, \mu)\,.$$

We are now able to state the main result of the strategic substitutabilities case.

Theorem 5.17. *In the economic system with Strategic Substitutabilities we have:*

(i) *There exists at least one equilibrium* a^**.*

(ii) *There exist greatest and a smallest rationalizable strategies, that is* $\exists\ \underline{a} \in \mathbb{R}_{\mathcal{A}}$ *and* $\bar{a} \in \mathbb{R}_{\mathcal{A}}$ *such that* $\forall\ a \in \mathbb{R}_{\mathcal{A}}$*,* $\underline{a} \leq a \leq \bar{a}$*, where* $\underline{a}$ *and* $\bar{a}$ *are cycles of order 2 of the Cobweb mapping.*

(iii) *The sets of Rationalizable and Point-Rationalizable States are convex.*

(iv) *The sets of Rationalizable and Point-Rationalizable States are tightly contained in the interval* $[\underline{a}, \bar{a}]$*. That is,*

$$\mathbb{P}_{\mathcal{A}} \quad \subseteq \quad \mathbb{R}_{\mathcal{A}} \quad \subseteq \quad \{\underline{a}\} + \mathbb{R}^n_+ \bigcap \{\bar{a}\} - \mathbb{R}^n_+$$

and $\bar{a} \in \mathbb{P}_{\mathcal{A}}$ *and* $\underline{a} \in \mathbb{P}_{\mathcal{A}}$*.*

Proof. Using Lemma 5.16 in the proof of Proposition 5.15 we can see that $\tilde{R}^{2t}(\mathcal{A}) \subseteq \left[\underline{a}^{2t}, \bar{a}^{2t}\right]$ and so we get the two first results.

The last assertion is a consequence of the general setting of Rath (1992). Theorem 2.2 gives the existence of equilibrium.

■

Summing up, we have that in the case of Strategic Substitutabilities we can still use the correspondence Γ (through its' second iterate) to seclude to an interval the sets of Point-Rationalizable and Rationalizable States. This inclusion is tight since the boundaries of this interval are in fact Point-rationalizable States.

Corollary 5.18. *If in* $\mathcal{G}'$*,* Γ^2 *has a unique fixed point* a^**, then*

$$\mathbb{R}_{\mathcal{A}} \equiv \mathbb{P}_{\mathcal{A}} \equiv \{a^*\}\,.$$

Proof. Observe that both limits of the interval presented in Theorem 5.17, $\underline{a}$ and $\bar{a}$, are fixed points of Γ^2. Hence the result.

■

As opposed to the case of strategic complementarities, the optimistic equivalence result of Corollary 5.9 can not be directly obtained in the setting of strategic substitutabilities. If the sequences $\bar{b}^t$ and $\underline{b}^t$ defined in the proof of Proposition 5.15 converge to the same point, i.e. $\underline{b}^* = \bar{b}^* = a^*$, then a^* is the unique equilibrium of the system, it is strongly rational and IE-stable. However, under strategic substitutabilities there could well be a unique equilibrium that is not necessarily strongly rational. Think of the case of $\mathcal{A} \subset \mathbb{R}$, where a continuous decreasing function Γ has unique fixed point, that could well be part of a bigger set of Point-Rationalizable States (see figure 2).

Corollary 5.19. *The following statements are equivalent.*

(i) *an equilibrium* a^* *is Strongly Rational.*

(ii) *an equilibrium* a^* *is IE-Stable.*

Again heterogeneity of expectations in a sense does not matter, for evaluating the quality of expectational coordination. However, here uniqueness of equilibrium does not assure its' global stability. We recover the intuitions stated in Guesnerie (2005).

6 The differentiable case.

Here, we add an assumption concerning the cobweb mapping Γ:

H1 $\Gamma : \mathcal{A} \to \mathcal{A}$ is a $\mathcal{C}^1$-differentiable function.

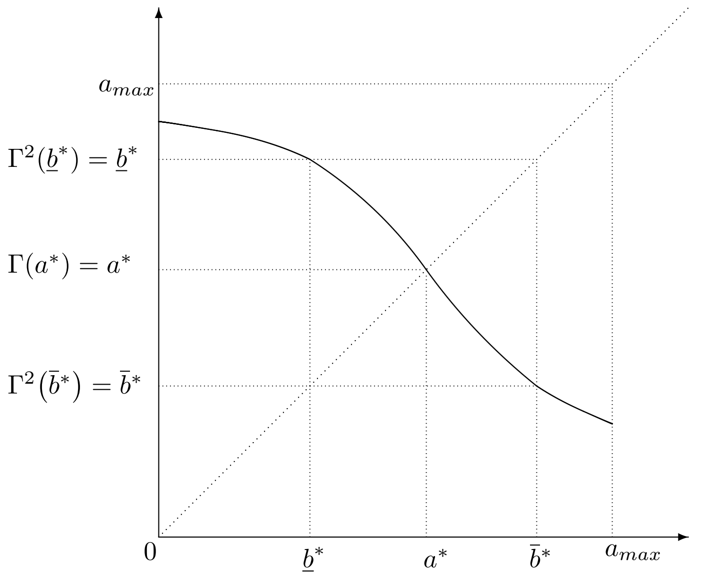

Figure 2: Strategic substitutes for $\mathcal{A} \equiv [0, a_{max}] \subset \mathbb{R}$. There exists a unique equilibrium and multiple fixed points for Γ^2

Remark 6.1. Note that from the definition of Γ, the vector-field $(a - \Gamma(a))$ points outwards on $\mathcal{A}$: formally, this means that if $p(a)$ is a supporting price vector at a boundary point of $\mathcal{A}$ $(p(a) \cdot \mathcal{A} \leq 0)$, then $p(a) \cdot (a - \Gamma(a)) \geq 0$. When, as in most applications $\mathcal{A}$ is the product of intervals for example $[0, M_h]$, this means $\Gamma_h(a) \geq 0$, whenever $a_h = 0$, and $M_h - \Gamma_h(a) \geq 0$, whenever $a_h = M_h$.

The jacobian of the function Γ, $\partial\Gamma$, can be obtained from the first order conditions of problem (2.5) along with (3.3).

$$\partial\Gamma(a) = \int_I \partial B(i, a) \ \mathrm{di}$$

where $\partial B(i, a)$ is the jacobian of the optimal strategy (now) function. This jacobian is equal to:

$$\partial B(i, a) \equiv -[Du_{ss}(i, B(i, a), a)]^{-1} Du_{sa}(i, B(i, a), a) \tag{6.1}$$

where $Du_{ss}(i, B(i, a), a)$ is the matrix of second derivatives with respect to s of the utility functions and $Du_{sa}(i, B(i, a), a)$ is the matrix of cross second derivatives, at the point $(B(i, a), a)$.

6.1 The strategic complementarities case.

Under assumptions 1.B to 3.B, along with $\mathcal{C}^2$ differentiability of the functions $u(i, \cdot, \cdot)$, we get from (6.1) that the matrices $\partial B(i, a)$ are positive[15], and consequently so is $\partial\Gamma(a)$.

From the properties of positive matrices are well known. When there exists a positive vector x, such that $Ax < x$, the matrix A is said *productive* : its eigenvalue of highest modulus is positive and smaller than one. When a is one-dimensional, the condition says that the slope of Γ is smaller than 1.

In this special case, as well as in our more general framework, the condition has the flavor that actions do not react too wildly to expectations..

In this case, we obtain :

[15]It is a well know fact that increasing differences implies positive cross derivatives on $Du_{sa}(i, B(i, a), a)$ and it can be proved that for a supermodular function the matrix $-[Du_{ss}(i, \cdot, \cdot)]^{-1}$ is positive at $(B(i, a), a)$

Theorem 6.2 (Uniqueness)**.** *If* $\forall\ a \in \mathcal{A}$*,* $\partial\Gamma(a)$ *is a productive matrix, then there exists a unique Strongly Rational Equilibrium.*

Proof.

Compute in any equilibrium a^* the sign of $\det[I - \partial\Gamma(a^*)]$. If $\partial\Gamma(a^*)$ is productive, its eigenvalue of highest modulus is real positive and smaller than 1. Hence the real eigenvalues of $[I - \partial\Gamma(a^*)]$ are all positive[16]. It follows that the sign of $\det[I - \partial\Gamma(a^*)]$ is the sign of the characteristics polynomial $\det\{[I - \partial\Gamma(a^*)] - \lambda I\}$ for $\lambda \to -\infty$, i.e is plus. The index of $\varphi(a) = a - \Gamma(a)$ is then +1. The Poincaré-Hopf theorem for vector fields pointing inwards implies that the sum of indices must be equal to +1, hence the conclusion of uniqueness. Strong Rationality follows from Corollary 5.9.

■

Our assumptions also have consequences for *eductive* stability.

Theorem 6.3 (Expectational coordination)**.** *If sign of* $\det[I - \partial\Gamma(a^*)]$ *is* $+$*, for some equilibrium* a^**, then* a^* *is locally eductively stable.*

Proof.

Take as initial local hypothetical CK neighborhood $\left\{\{a_-\} + \mathbb{R}^n_+\right\} \cap \left\{\{a_+\} - \mathbb{R}^n_+\right\}$ where $a_- < a^*$, $a_+ > a^*$, both being close to a^*.

The general argument of Proposition 5.5 works.

■

The above statements generalize in a reasonable way the intuitive findings easily obtainable from the one-dimensional model.

6.2 The strategic substitutabilities case.

Let us go to the Strategic Substitutabilities case. We maintain the previous boundary assumptions.

When passing from 3.B to 3.B' we get that now the matrix $\partial\Gamma$ has negative[17] entries. And $I - \partial\Gamma(a)$ is a positive matrix. Again, it has only positive eigenvalues, whenever the positive eigenvalue of highest modulus of -$\partial\Gamma$ is smaller than 1.

Theorem 6.4. *Let us assume that* $\forall\ a_1,\ a_2 \in \mathcal{A}$*,* $\partial\Gamma(a_1)\,\partial\Gamma(a_2)$ *is productive,*

1- There exists a unique equilibrium.

2- It is globally Strongly Rational.

Proof.

The assumption implies that $\forall a \in \mathcal{A}$, $-\partial\Gamma(a)$ is productive.

Hence $I - \partial\Gamma$ is a positive matrix, and whenever the positive eigenvalue of highest modulus of $-\partial\Gamma$ is smaller than 1, it has only positive eigenvalues. Then its determinant is positive.

Then the above Poincaré-Hopf argument applies to the first and second iterate of Γ.

It follows that there exists a unique equilibrium and no two-cycle.

Then, Theorem 5.17 applies.

■

Also, as above one can show that the *productive* condition when it holds in one equilibrium ensures local strong rationality.

[16] It has at least one real eigenvalue, associated with the eigenvalue of highest modulus of $\Gamma(a^*)$.

[17] Since the matrix $Du_{sa}(i, B(i,a), a)$ has only non-positive entries under strategic substitutabilities (see note 15).

7 Comments and Conclusions

The Rational Expectations Hypothesis has been subject of scrutiny in recent years through the assessment of Expectational Coordination. Although the terminology is still fluctuating, the ideas behind what we call here *Strong Rationality* or *Eductive Stability* have been at the heart of the study of diverse macroeconomic and microeconomic models of standard markets with one or several goods, see Guesnerie, models of information transmission (Desgranges (2000), Desgranges and Heinemann (2005), Ben Porath-Heifetz).

In this work we aimed to address the subject of *eductive stability* with broad generality. We have presented a stylized framework that encompasses a significant class of economic models. We have made the connection between what may be called the *economic* viewpoint and a now standard line of research in game-theory: games with a continuous of players. The paper has assessed the connections between a number of tools with game-theoretical flavor which are available for analyzing the expectational *stability* or *plausibility* of equilibria in a so-called *economic* context with non-atomic agents. The presence of an aggregate variable in the model allowed for us to go back and forth between the economic and game-theoretical point of view, making the connection between the different approaches.

We have exhibited properties of what we called the set of Rationalizable and Point-Rationalizable States. The Rationalizable set is proved to be non-empty and convex as the set of Point-Rationalizable States, with this last set also being compact.

In this context, when the economy is dominated by strategic complementarities, we have derived results that reformulate the classical game-theoretical findings of Milgrom and Roberts and Vives.

In the opposite polar case of strategic substitutabilities, using the properties of the second iterate of the Cobweb mapping, we have exhibited results that parallel the first ones, while stressing however striking differences. For example, when in the strategic complementarities, uniqueness triggers all expectational stability criteria, this is no longer the case with strategic substitutabilities : uniqueness does not imply *expectational stability*, whatever the exact sense given to the assertion. Related remarks apply for local uniqueness that has different implications for local stability in the two cases under examination. We give then simple and appealing conditions implying uniqueness of equilibria and stability in the sense of Strong Rationality, although in this case the former does not imply the latter.

In all cases, it turns out that the *eductive process* that allows to obtain Point-Rationalizable and (locally) Strong Rationalizable States can be achieved tightly with the *iterative expectations process* or "Cobweb tâtonnement " used to explain Iterative Expectational Stability. In both circumstances, one may argue that heterogeneity of expectations makes no difference for expectational coordination. This is a most significant feature of these situations that strikingly contrast the general case studied in Evans and Guesnerie (2005). Many economic models that fit our framework, such as the one associated with the analysis of expectational stability in a class of general dynamical systems (Evans and Guesnerie 2005) have neither strategic complementarities nor substitutabilities. The complexity of the findings that has increased when going from the first case to the second one, will still increase. In this sense, we hope that these results provide a useful benchmark for a deeper understanding of the role of the heterogeneity of beliefs in expectational coordination. The beginning of the road map drawn from this paper should help to continue the route.

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A Technical Lemmas

Lemma A.1. *Under assumptions **C** and **HM**, for a closed set $X \subseteq \mathcal{A}$ the correspondence $i \rightrightarrows B(i, X)$ is measurable and has compact values.*

Proof.

We show first that the mapping $\mathrm{G} : I \rightrightarrows \mathcal{A} \times S$, that associates with each agent $i \in I$ the graph of the best response mapping $B(i, \cdot)$, $\mathrm{G}(i) := \operatorname{gph} B(i, \cdot)$, is measurable.

Take a closed set $C \subseteq \mathcal{A} \times S$. We need to prove that the set

$$\mathrm{G}^{-1}(C) \equiv \{i \in I \;:\; C \cap \operatorname{gph} B(i, \cdot) \neq \emptyset\}$$

is measurable. Consider the subset $U \subseteq \mathcal{U}_{S\times\mathcal{A}}$ defined by:

$$U := \{\, g \in \mathcal{U}_{S\times\mathcal{A}} \;:\; \exists\,(a, s) \in C \text{ such that } g(s, a) \geq g(y, a)\ \forall\, y \in S\}$$

note that $\mathbf{u}^{-1}(U) \equiv \mathrm{G}^{-1}(C)$ and so, from the measurability assumption over $\mathbf{u}$, it suffices to prove that U is closed. That is, we have to show that for any sequence $\{g^\nu\}_{\nu\in\mathbb{N}} \subset U$, such that $g^\nu \to g^*$ uniformly $g^* \in U$.

Since the functions g^ν are finite and continuous in $S \times \mathcal{A}$, from Weierstrass' Theorem g^* is continuous and so it belongs to $\mathcal{U}_{S\times\mathcal{A}}$. Moreover, g^ν converges continuously to g^*, that is, for any convergent sequence (a^ν, s^ν) with limit (a^*, s^*), the sequence $g^\nu(s^\nu, a^\nu)$ converges to $g^*(s^*, a^*)$. Indeed, consider any $\varepsilon > 0$. By the continuity of g^* there exists $\bar{\nu}_1 \in \mathbb{N}$ such that $\forall\ \nu > \bar{\nu}_1$,

$$|g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \frac{\varepsilon}{2}.$$

From the uniform convergence of g^ν we get that there exists $\bar{\nu}_2 \in \mathbb{N}$ such that,

$$|g^\nu(s, a) - g^*(s, a)| < \frac{\varepsilon}{2} \text{ for all } \nu \geq \bar{\nu}_2 \text{ and } \forall\,(s, a) \in S \times \mathcal{A},$$

in particular this is true for all the elements of the sequence of points. We get then that $\forall\ \nu \geq \max\{\bar{\nu}_1, \bar{\nu}_2\}$,

$$|g^\nu(s^\nu, a^\nu) - g^*(s^*, a^*)| \leq |g^\nu(s^\nu, a^\nu) - g^*(s^\nu, a^\nu)| + |g^*(s^\nu, a^\nu) - g^*(s^*, a^*)| < \varepsilon.$$

We have to show then that there exists a point $(a, s) \in C$ such that $g^*(s, a) \geq g^*(y, a)\ \forall\ y \in S$. Since $g^\nu \in U$, we have for each $\nu \in \mathbb{N}$, points $(a^\nu, s^\nu) \in C$ such that $g^\nu(s^\nu, a^\nu) \geq g^\nu(y, a^\nu)\ \forall\ y \in S$. Let $(a^*, s^*) \in C$ be the limit of a convergent subsequence of $\{(a^\nu, s^\nu)\}_{\nu\in\mathbb{N}}$, which without loss of generality we can take to be the same sequence. We see that (a^*, s^*) is the point we are looking for since for a fixed $y \in S$, continuous convergence implies that in the limit

$$g^*(s^*, a^*) \geq g^*(y, a^*)\,.$$

We conclude then that $g^* \in U$. Thus, U is closed and since $\mathbf{u}$ is a measurable mapping, $\mathbf{u}^{-1}(U)$ is measurable.

With this in mind, consider a closed set $X \subseteq \mathcal{A}$ and the mapping $i \rightrightarrows B(i,X)$. Applying Theorem 14.3 in Rockafellar and Wets (1998) to the constant mapping $i \rightrightarrows X$ along with G above, we get that $i \rightrightarrows B(i,X)$ is measurable and has closed values (hence compact since S is compact).

■

Lemma A.2. *If $S \subset \mathbb{R}^n$ is a complete lattice for the product order in $\mathbb{R}^n$, then for a measurable correspondence $F : I \rightrightarrows S$ with nonempty, closed and subcomplete values, the functions $\underline{\mathbf{s}} : I \to S$ and $\bar{\mathbf{s}} : I \to S$, defined by*

$$\underline{s}(i) := \inf_S F(i)\,, \tag{A.1}$$

$$\bar{s}(i) := \sup_S F(i)\,, \tag{A.2}$$

are measurable selections of F.

Proof.
Since $F(i)$ is subcomplete, $\underline{s}(i)$ and $\bar{s}(i)$ belong to $F(i)$. We have to prove that $\underline{\mathbf{s}}$ and $\bar{\mathbf{s}}$ are measurable.

Since F is measurable, it has a Castaing representation. That is, there exists a countable family of measurable functions $\mathbf{s}^\nu : I \to \mathbb{R}^n$, $\nu \in \mathbb{N}$, such that $s^\nu(i) \in F(i)$ and,

$$F(i) \equiv \mathrm{cl}\{\{s^\nu(i) \ : \ \nu \in \mathbb{N}\}\}\,. \tag{A.3}$$

For $\underline{\mathbf{s}}$, consider then for each $\nu \in \mathbb{N}$ the set valued mappings $F^\nu : I \rightrightarrows \mathbb{R}^n$, defined by [18]

$$F^\nu(i) := F(i) \bigcap \;]-\infty, s^\nu(i)]$$

Since F is measurable and closed valued, and we can write $]-\infty, s^\nu(i)] = s^\nu(i) - \mathbb{R}^n_+$ which is as well measurable and closed valued, the correspondences F^ν are measurable and closed valued [19].

Note that $\forall\ \nu \in \mathbb{N}$, $\underline{s}(i) \in F^\nu(i)$. Defining the closed valued correspondence $\underline{F} : I \rightrightarrows \mathbb{R}^n$:

$$\underline{F}(i) := \bigcap_{\nu \in \mathbb{N}} F^\nu(i)$$

we get then that $\underline{s}(i) \in \underline{F}(i)$. The correspondence $\underline{F}$ is as well measurable [19].

We now prove that actually $\underline{F}(i) \equiv \{\underline{s}(i)\}$, which completes the proof. Indeed, suppose that $y \in \underline{F}(i)$. Then, by definition of $\underline{F}$, $y \in F(i)$ and $y \leq s^\nu(i)$, $\forall\ \nu \in \mathbb{N}$. From equality (A.3) we get that any point in $F(i)$ can be obtained as the limit of a subsequence of $\{s^\nu(i) \ : \ \nu \in \mathbb{N}\}$, so in the limit the inequality is maintained, this is $\forall\ s \in F(i)$, $y \leq s$. That is, y is a lower bound for $F(i)$. This implies, by the definition of $\inf_S F(i)$, that $y \leq \inf_S F(i)$, but $y \in F(i)$, so $\inf_S F(i) \leq y$. Thus, $y \leq \underline{s}(i) = \inf_S F(i) \leq y$.

Analogous arguments applied to the mapping $\bar{F} : I \rightrightarrows \mathbb{R}^n$:

$$\bar{F}(i) := F(i) \bigcap \left(\bigcap_{\nu \in \mathbb{N}} [s^\nu(i)\,, +\infty\,[\right)$$

prove the statement for $\bar{\mathbf{s}}$.

■

[18] The *interval* $]-\infty, x]$ is the set of points of $\mathbb{R}^n$ that are smaller than $x \in \mathbb{R}^n$, similarly $[x, +\infty[$ is the set of points in $\mathbb{R}^n$ that are greater than x.

[19] See Proposition 14.11 in Rockafellar and Wets (1998)

B Proof of Proposition 4.9

Proof.

For (i): note that

$$\Gamma(a) \equiv \int_I B(i,a)\,\mathrm{di} \equiv \begin{cases} \tilde{Pr}(\{a\}) \\ \int_I \mathbb{B}(i,\delta_{a^*})\,\mathrm{di} \equiv \tilde{R}(\{a\}) \end{cases}$$

and use Proposition 4.1.

For (ii) from Proposition 4.1 we see that we only need to prove that under condition 4.1:

$$a^* \text{ is Locally Strongly Point Rational} \quad \Longrightarrow \quad a^* \text{ is Locally Strongly Rational.}$$

For a subset $X \subseteq \mathcal{A}$ call $\mathbb{P}(X) := \bigcap_{t\geq 0} \tilde{Pr}^t(X)$ and note that if $\mathbb{P}(X) \equiv \{a^*\}$ then $\forall\ X' \subseteq X$, $\mathbb{P}(X') \equiv \{a^*\}$.

Take V, the neighborhood of the Proposition. For a borel subset $X \subseteq V$ the hypothesis implies that the integral of $i \rightrightarrows \mathbb{B}(i, \mathcal{P}(X))$ is contained in the integral of $i \rightrightarrows \operatorname{co}\{B(i,X)\}$. From Aumann (1965) we know that:

$$\int_I \operatorname{co}\{B(i,X)\}\,\mathrm{di} \equiv \int_I B(i,X)\,\mathrm{di}$$

and so

$$\tilde{R}(X) \equiv \int_I \mathbb{B}(i,\mathcal{P}(X))\,\mathrm{di} \subseteq \int_I B(i,X)\,\mathrm{di} \equiv \tilde{Pr}(X) \tag{B.1}$$

If a^* is Locally Strongly Point Rational then there exists a neighborhood V' such that $\mathbb{P}(V') = \{a^*\}$. So now take an open ball of radius $\varepsilon > 0$ around a^* that is contained in both V and V'. To ensure that the process for probability forecasts is well defined, we can take the closed ball of radius $\varepsilon/2$, $\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)$, that is strictly contained in the previous ball and of course in the intersection of both neighborhoods. In particular, we have that $\mathbb{P}\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big) \equiv \{a^*\}$ and that $\tilde{R}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big)$ is well defined and closed for all $t \geq 1$. The last assertion, along with (B.1), imply that for all $t \geq 1$ $\tilde{R}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big) \equiv \tilde{Pr}^t\big(\mathrm{B}\big(a^*, \frac{\varepsilon}{2}\big)\big)$. We conclude that,

$$\bigcap_{t\geq 0} \tilde{R}^t\Big(\mathrm{B}\Big(a^*, \frac{\varepsilon}{2}\Big)\Big) \equiv \mathbb{P}\Big(\mathrm{B}\Big(a^*, \frac{\varepsilon}{2}\Big)\Big) \equiv \{a^*\}$$

■

C Proof of Proposition 4.11

Proof.

We give the proof for the case where all the agents have the same utility function $u : S \times \mathcal{A} \to \mathbb{R}$.

Consider then a convex neighborhood V of a^* and the space of probability measures $\mathcal{P}(V)$. Take a probability measure with finite support, μ, in this space, this is $\mu = \sum_{l=1}^{L} \mu_l \delta_{a_l}$, with $\{a_l\}_{l=1}^{L} \subset V$. For this measure, under the differentiability hypothesis, we can prove that if the support of μ, $\{a_1, \ldots, a_L\}$, is contained in a ball [20] $\mathrm{B}(a^*, \varepsilon_1)$, then

$$\| \mathbb{B}(\mu) - B(\mathbb{E}_\mu[a])\| < \varepsilon_1^2.$$

[20] Since $\mathcal{A}$ is compact V is bounded.

Since $\mathbb{E}_\mu[a] \in V$ we get that $B(\mathbb{E}_\mu[a]) \in B(V)$. Using a density argument we may conclude that $\mathbb{B}(\mu)$ is "close" to $B(V) \subseteq \text{co}\{B(V)\} \equiv \tilde{Pr}(V)$ for any measure in $\mathcal{P}(V)$. We can take then $\varepsilon_1 > 0$ small, related to the neighborhood V, such that,

$$\tilde{R}(V) \subset \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \tag{C.1}$$

From the hypothesis we get that we can choose a number $\bar{k} < k' < 1$ such that the following inclusions hold:

$$\tilde{Pr}(V) \subset V_{\bar{k}} \subset V_{k'} \subset V \tag{C.2}$$

$$\tilde{R}(V) \subset \tilde{Pr}(V) + \mathrm{B}\big(0, \varepsilon_1^2\big) \subset V_{k'} \tag{C.3}$$

Moreover, taking the second iterate of $\tilde{R}$ starting at V, using (C.3) and (C.1) on $V_{k'}$,

$$\tilde{R}^2(V) \subset \tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big)$$

where ε_2 depends on k'. However it can be chosen in such a way that the following inclusions hold. Using (C.2) we get

$$\begin{aligned}\tilde{Pr}(V_{k'}) + \mathrm{B}\big(0, \varepsilon_2^2\big) &\subset V_{\bar{k}k'} + \mathrm{B}\big(0, \varepsilon_2^2\big)\\ &\subset V_{k'^2}.\end{aligned}$$

We have then,

$$\tilde{R}^2(V) \subset V_{k'^2}$$

Using the same argument, choosing ε_t related to the powers of k', ${k'}^{t-1}$, we get that for all t,

$$\tilde{R}^t(V) \subset \tilde{Pr}(V_{k'^{t-1}}) + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{\bar{k}k'^{t-1}} + \mathrm{B}\big(0, \varepsilon_t^2\big) \subset V_{k'^t}$$

We conclude then that the eductive process converges to the equilibrium a^*.

■

D Proof of Proposition 5.5

Proof.

From Propositions 5.4 and 5.3 we get that E_Γ is non empty and has a greatest and a smallest element.

Following the structure of the proof of Theorem 5 in Milgrom and Roberts we prove that $\tilde{Pr}^t(\mathcal{A})$ is contained in some interval $[\underline{a}^t, \bar{a}^t]$ and that the sequences $\underline{a}^t$ and $\bar{a}^t$ satisfy $\underline{a}^t \to \inf_{E_\Gamma}\{E_\Gamma\}$ and $\bar{a}^t \to \sup_{E_\Gamma}\{E_\Gamma\}$.

Define $\underline{a}^0$ and $\underline{a}^t$ as:

$$\underline{a}^0 \quad := \quad \inf \mathcal{A} \tag{D.1}$$

$$\underline{a}^t \quad := \quad \inf_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big) \qquad \forall\, t \geq 1 \tag{D.2}$$

- $\tilde{Pr}^t(\mathcal{A}) \subseteq [\underline{a}^t, +\infty[$

  Clearly it is true for $t = 0$.

  Suppose that it is true for $t \geq 0$. That is, $\underline{a}^t \leq a \;\forall\, a \in \tilde{Pr}^t(\mathcal{A})$. Since $B(i, \cdot)$ is increasing, we get that $B(i, \underline{a}^t) \preceq B(i, a) \;\forall\, a \in \tilde{Pr}^t(\mathcal{A})$. In particular $\forall\, y \in B(i, a)$ and $\forall\, a \in \tilde{Pr}^t(\mathcal{A})$, we have $\inf_S B(i, \underline{a}^t) \leq y$. From Lemma A.2, the correspondence $i \rightrightarrows \inf_S B(i, \underline{a}^t)$ is measurable. This implies that for any measurable selection $\mathbf{s} \in S^I$ of $i \rightrightarrows B\Big(i, \tilde{Pr}^t(\mathcal{A})\Big)$,

$$\int \inf_S B\big(i, \underline{a}^t\big)\,\mathrm{di} \leq \int s(i)\,\mathrm{di}. \tag{D.3}$$

Since $B(i, \underline{a}^t)$ is subcomplete, $\inf_S B(i, \underline{a}^t) \in B(i, \underline{a}^t)$ and so we get that:

$$\begin{aligned}\inf_{\mathcal{A}} \Gamma\big(\underline{a}^t\big) &\equiv \inf_{\mathcal{A}} \big\{\; b \in \mathcal{A} \;:\; \exists\, \mathbf{s} \text{ measurable selection of } i \;\rightrightarrows\; B\big(i, \underline{a}^t\big) \text{ such that, } b = A(\mathbf{s})\big\} \\ &\leq \int \inf_S B\big(i, \underline{a}^t\big)\end{aligned} \tag{D.4}$$

We conclude then that

$$\underline{a}^{t+1} \equiv \inf_{\mathcal{A}} \Gamma\big(\underline{a}^t\big) \leq \int \inf_S B\big(i, \underline{a}^t\big) \leq a \qquad \forall\, a \in \tilde{Pr}^{t+1}(\mathcal{A})\,.$$

The equality is the definition of $\underline{a}^{t+1}$, the first inequality comes from (D.4) and the second one is obtained from (D.3) and the definition of $\tilde{Pr}$.

- The sequence is increasing:

  By definition of $\underline{a}^0$, $\underline{a}^0 \leq \underline{a}^1$. Suppose that $\underline{a}^{t-1} \leq \underline{a}^t$, then from Lemma 2.4.2 in Topkis (1998), $\underline{a}^t \equiv \inf_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big) \leq \inf_{\mathcal{A}} \Gamma(\underline{a}^t) \equiv \underline{a}^{t+1}$.

- The sequence has a limit and $\lim_{t\to+\infty} \underline{a}^t$ is a fixed point of Γ:

  Since the sequence is increasing and $\mathcal{A}$ is a complete lattice, it has a limit $\underline{a}^*$. Furthermore, since Γ is subcomplete, upper semi-continuity of Γ implies that $\underline{a}^* \in \Gamma(\underline{a}^*)$.

- $\underline{a}^* \equiv \inf_{E_\Gamma} \{E_\Gamma\}$:

  According to the previous demonstration, since the fixed points of Γ are in the set $\mathbb{P}_{\mathcal{A}}$, all fixed points must be in $[\underline{a}^*, +\infty[$ and so $\underline{a}^*$ is the smallest fixed point.

Defining $\bar{a}^0$ and $\bar{a}^t$ as:

$$\bar{a}^0 \;:=\; \sup \mathcal{A} \tag{D.5}$$

$$\bar{a}^t \;:=\; \sup_{\mathcal{A}} \Gamma\big(\bar{a}^{t-1}\big) \qquad \forall\, t \geq 1 \tag{D.6}$$

In an analogous way we obtain that $\mathbb{P}_{\mathcal{A}} \subseteq\ \ ]-\infty, \bar{a}^*]$, with $\bar{a}^*$ being the greatest fixed point of Γ. ■

E Proof of Proposition 5.15

Proof.
Following the proof of Proposition 5.5, consider the sequence $\{\underline{a}^t\}_{t=0}^{\infty}$ therein defined, but let us change the definition of $\underline{a}^t$ when t is odd to:

$$\underline{a}^t := \sup_{\mathcal{A}} \Gamma\big(\underline{a}^{t-1}\big)\,.$$

By the definition of $\underline{a}^0$, we know that $\forall\, a \in \mathcal{A}$, $a \geq \underline{a}^0$. Since the mappings $B(i, \cdot\,)$ are decreasing we have $B\big(i, \underline{a}^0\big) \succeq B(i, a)$ $\forall\, a \in \mathcal{A}$ and in particular

$$\sup_S B\big(i, \underline{a}^0\big) \geq y \quad \forall\, y \in B(i, a)\ \ \forall\, a \in \mathcal{A}$$

Since $B\big(i, \underline{a}^0\big)$ is subcomplete $\sup_S B\big(i, \underline{a}^0\big) \in B\big(i, \underline{a}^0\big)$ and from Lemma A.2 the function $i \rightarrow \sup_S B\big(i, \underline{a}^0\big)$ is measurable, so $\int \sup_S B\big(i, \underline{a}^0\big) \in \Gamma\big(\underline{a}^0\big)$, thus

$$\sup_{\mathcal{A}} \Gamma\big(\underline{a}^0\big) \geq \int \sup_S B\big(i, \underline{a}^0\big)\,\mathrm{di} \geq \int s(i)\,\mathrm{di}$$

for any measurable selection $\mathbf{s}$ of $i \rightrightarrows B(i,\mathcal{A})$. That is $\underline{a}^1 \geq a \ \forall \ a \in \tilde{Pr}^1(\mathcal{A})$; or equivalently,

$$\tilde{Pr}^1(\mathcal{A}) \subseteq \left]-\infty, \underline{a}^1\right].$$

A similar argument leads to conclude that $\tilde{Pr}^2(\mathcal{A}) \subseteq \left[\underline{a}^2, +\infty\right[$.

Let us define then the sequence $\underline{b}^t := \underline{a}^{2t}$, $t \geq 0$. This sequence satisfies:

1. $\tilde{Pr}^{2t} \subseteq \left[\underline{b}^t, +\infty\right[$. This can be obtained as above by induction over t.
2. $\left\{\underline{b}^t\right\}_{t\geq 0}$ is increasing.

As before, we get that $\left\{\underline{b}^t\right\}_{t\geq 0}$ has a limit $\underline{b}^*$. Since Γ is u.s.c. and $\mathcal{A}$ is compact, we obtain that the second iterate of Γ, Γ^2 is as well u.s.c.. Moreover, from Proposition 5.13, we get that $\underline{b}^t \in \Gamma^2\left(\underline{b}^{t-1}\right)$. This implies that $\underline{b}^*$ is a fixed point of Γ^2 and so it is a point-rationalizable state. Consequently we get

1. $\mathbb{P}_{\mathcal{A}} \subseteq [\underline{b}^*, +\infty[$
2. $\underline{b}^* \in \Gamma^2\left(\underline{b}^*\right)$ and $\underline{b}^*$ is a point-rationalizable state.

Considering the analogous sequence to obtain the upper bound for $\mathbb{P}_{\mathcal{A}}$:

$$\begin{array}{lcll} \bar{a}^0 & := & \sup \mathcal{A} & \\ \bar{a}^t & := & \inf\limits_{\mathcal{A}} \Gamma\left(\bar{a}^{t-1}\right) & \text{when } t \text{ is odd} \\ \bar{a}^t & := & \sup\limits_{\mathcal{A}} \Gamma\left(\bar{a}^{t-1}\right) & \text{when } t \text{ is even} \end{array} \tag{E.1}$$

We generate a decreasing sequence $\left\{\bar{b}^t\right\}_{t\geq 0}$ defined by $\bar{b}^t := \bar{a}^{2t}$, $t \geq 0$, who's limit $\bar{b}^*$, is a point-rationalizable state and an upper bound for $\mathbb{P}_{\mathcal{A}}$, that is:

1. $\mathbb{P}_{\mathcal{A}} \subseteq \left]-\infty, \bar{b}^*\right]$
2. $\bar{b}^* \in \Gamma^2\left(\bar{b}^*\right)$. Which implies that $\bar{b}^*$ is a point-rationalizable state.

As a summary, we get:

$$\mathbb{P}_{\mathcal{A}} \subseteq \bigcap_{t\geq 0} \tilde{Pr}^t(\mathcal{A}) \subseteq \bigcap_{t\geq 0} \tilde{Pr}^{2t}(\mathcal{A}) \subseteq \left[\underline{b}^*, \bar{b}^*\right] \tag{E.2}$$

■