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Macroeconomic and monetary policies from the "eductive" viewpoint*

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Abstract

The "eductive" viewpoint provides a theoretically sophisticated analysis as well as an intuitively plausible shortcut to the study of expectational coordination in economic models. From the review of expectational criteria in a class of dynamical models of macroeconomic theory, the paper shows how such an "eductive" viewpoint completes and deepens rather than contradicts standard analysis. It however argues that the "eductive" approach, when correctly implemented, challenges the conditions of learning in infinite-horizon models with infinitely-lived agents. In particular, in a simple monetary model adopting such a framework, Taylor rules may be stabilizing, in the demanding sense under scrutiny, but only within a small window for the reaction coefficient.

Résumé : Le point de vue dit "divinatoire" fournit à la fois une alternative théoriquement élaborée et un raccourci intuitivement plausible à l'étude de la coordination des anticipations. Il conduit à approfondir l'analyse standard, comme le montre, à partir d'une revue de familles de modèles dynamiques utilisés en macroéconomie, l'examen

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et la comparaison des critères habituellement utilisés. Cependant, le point de vue, lorsqu'il est convenablement compris, conduit à un réexamen profond de l'apprentissage dans les modèles à horizon infini peuplés d'agents à durée de vie infinie. En particulier dans un modèle simple, suceptible d'éclairer l'analyse des politiques monétaires, les régles de Taylor ne sont stabilisantes, au sens exigeant requis ici, que lorsque les coefficients de réaction sont dans une fenêtre étroite.

1 Introduction.

The "quality" of coordination of expectations, a key issue for monetary policy, obtains from different, but interrelated, channels : both the "credibility" of the Central Bank intervention and the ability of decentralized agents to coordinate on a dynamical equilibrium matter. For both purposes, it is important to understand how agents learn. Indeed, many studies on monetary policy have focused attention on learning processes involving "evolutive", real time learning rules (adaptive learning rules, etc. . .).

The "eductive" viewpoint as illustrated in many references of this bibliography and in my 2005 MIT Press book partly abstracts from the real time dimension of learning, with the aim of more directly capturing the systems' characteristics that are coordination-friendly. The paper first presents the philosophy of analysis of expectational coordination underlying the just called "eductive" viewpoint. Giving a synthetical flavour of the "eductive" viewpoint is a pre-requisite to the confrontation of the methods that this viewpoint suggests with those actually adopted in most present studies of learning in the context of macroeconomic and monetary policy. Such a confrontation rests on the review of existing learning results in the context of dynamical systems, the main present field of applications of the "eductive" method to macroeconomics[1]. Such applications however, have not born most directly on monetary policy issues. Then, following the review, the paper explores the differences for standard monetary policy analysis between the

[1]See in particular, Guesnerie R. (2001) “"Short run expectational coordination: Fixed versus flexible wages." Quarterly Journal of Economics, p. 1115,1147, Evans G., R. Guesnerie (2005) "Coordination on saddle path solutions: the eductive viewpoint, 2 - Linear multivariate models, Journal of Economic Theory, 2005, p.202-229.

traditional viewpoint and this competing viewpoint. This exploration is tentative, although it seems potentially promising

The paper will proceed as follows:

- It will recall the logic of the “eductive” viewpoint and stress differences as well as complementarities with the “evolutive” viewpoint. (Section 2)
- It will review results that allow to compare the most standard expectational criteria and the "eductive" criterion. It will first introduce four more or less standard criteria in the framework of a one-dimensional (Section 3) and then multi-dimensional (Section 4) simple dynamical system. The comparison with the "eductive" viewpoint taken in this paper will be finalized in Section 5. The analysis will emphasize the role of heterogeneity of expectations and will suggest that the chosen alternative view, when applied to an overlapping generation framework, leads to complete and deepen rather than contradict the conclusions of more standard approaches
- However, the "eductive" analysis of a simple cashless economy undertaken in Section 6, will stress, in an infinite-horizon model with infinitely-lived agents, conditions for expectational coordination strikingly different from the classical ones. *In particular, the "eductive" evaluation of the stabilizing performance of the Taylor rule suggest that its reaction coefficient to inflation has to be severely restricted.*

2 Expectational stability : the "eductive viewpoint".

The notion of ”eductively stable” equilibrium or ”strongly rational equilibrium” relies on considerations that have game-theoretical underpinnings, and refer to ”rationalizability”, ”dominance solvability”, ”Common Knowledge” ideas. These concepts serve to provide a ”high tech” justification of the expectational stability criteria that are proposed. Emphasis is first put on this ”high tech” approach for proposing global stability concepts that have a clearly ”eductive” flavour (2-A). Now, the local transposition of the global ideas allows to stress, besides the previous ”high tech” justification, a ”low tech”, more intuitive, interpretation (2-B). Comments are finally offered on the connections between the ”eductive” viewpoint and the standard ”evolutive” learning viewpoint (2-C).

2.1 Global "eductive" stability.

We are in a world populated of rational economic agents, (in all the following, I shall assume that these agents are infinitesimal and modelled as a continuum). The agents know the logic of the collective economic interactions (the underlying model). Both the rationality of the agents and the model are Common Knowledge. The state of the system is denoted E and belongs to some subset $\mathcal{E}$ of some vector space. .

Note that E can be a number, (the value of an equilibrium price or a growth rate), a vector (of equilibrium prices,...), a function (an equilibrium demand function), or an infinite trajectory of states, or a probability distribution.

Let us elaborate on that by taking a few examples.

In the variant of the Muth model considered in Guesnerie (1992), E is a number, the market clearing price to-morrow on the wheat market. The agents are farmers whose crop's profitability will depend on the wheat price. They know the model in the sense that they understand how the market price depends on the total amount of wheat available to-morrow : the market clearing price, as a function of total crop, is determined from the inverse of some demand function which is known. All this, (Bayesian) rationality and "the model", is known by the agents and it is known that it is known, and it is known that it is known that it is known,.... and, with straightforward notation, (it is known)^{p}for any p.(i.e it is Common Knowledge, from now on CK).In general equilibrium models, (Guesnerie (2001), (2002)), Ghosal (2006)), E is a vector (price vector or quantity vector). In models focusing on the transmission of information through prices, (Desgranges (2002), Desgranges-Heinemann (2005), Desgranges-Geoffard-Guesnerie (2003)), E is a function, a function that relates the non-noisy part of excess demand to the asset price. In infinite horizon models E is an infinite trajectory consisting, at each date t, either of a number or of a vector, describing the state of the system at this date. Introducing uncertainty in the partial equilibrium, general equilibrium, intertemporal models just recalled lead to substitute for E a probablity distribution over the set of finitelor infinite dimensional vectors previously considered.

Now, let us come to equilibria on which we focus on, that are, in standard terminology, rational expectations or perfect foresight equilibria . Emphasizing the expectational aspects of the problem, we view an equilibrium of the system as a state E^*, such that if everybody believes that it prevails, it does prevail. (Note, from our previous introductory remarks, that E^* is such that the assertion : "it is CK that $E = E^*$ " is meaningful).

We say that E^* is "eductively" stable of "strongly rational" iif Assertion A implies Assertion B, (given that Bayesian rationality and the model are CK).

Assertion A : It is CK that $E \in \mathcal{E}$.

Assertion B : it is CK that $E = E^*$.

The mental process that leads from Assertion A to Assertion B is the following.

1- As every body knows that E $\in \mathcal{E}$, everybody knows that everybody limits its responses to actions that best responses to some probability distributions over $\mathcal{E}$. It follows that everybody knows that the state of the system will be in $\mathcal{E}(1) \subset \mathcal{E}$

2- If $\mathcal{E}(1)$ is a proper subset of $\mathcal{E}$, the mental process goes on as in step 1, but based now on $\mathcal{E}(1)$ instead of $\mathcal{E}$.

3- etc...

We then have a decreasing sequence $\mathcal{E}(n) \subset \mathcal{E}(n-1) \subset .... \subset \mathcal{E}(1) \subset \mathcal{E}$. When the sequence converges to E^*, the equilibrium is "strongly rational" or "eductively" stable. When it is not the case, the limit set is the set of rationalizable equilibria of the model. (See Guesnerie-Jara-Moroni (2007)).

Global "eductive" stability is clearly very demanding, although it can be shown to hold under plausible economic conditions in a variety of models, either in partial equilibrium (Guesnerie (1992)) or general equilibrium (Guesnerie (2001)) standard markets contexts, in finance models of transmission of information through prices (Desgranges-Geoffard-Guesnerie (2002)), or in general settings involving strategic complementarities or substitutabilities (Guesnerie-Jara-Moroni(2007).

2.2 Local "eductive" Stability

Local "eductive stability" may be defined through the same 'high tech" or hyperrationality view (2B-1). However, the local criterion has also a very intuitive (and "low tech" and in a sense boundedly rational, interpretation (2B-2).

2.2.1 Local "eductive" stability as a CK statement.

We say that E^* is locally "eductively" stable or locally "strongly rational" iif one can find some *non trivial* neighbourhood of E^*, $V(E^*)$ such that Assertion A implies assertion B.

Assertion A : It is CK that $E \in V(E^*)$

Assertion B : it is CK that $E = E^*$.

Hypothetically, the state of the system is assumed to be in some non-trivial neighbourhood of E^* and this "hypothetically CK" assumption implies CK of E^*.

In other words, the deletion of non-best responses, starts under the assumption that the state of the system is close to the equilibrium state. In that sense, the viewpoint refers to the same "hyper-rationality" view as referred to before. However, the statement can be read in a simpler way.

2.2.2 Local "eductive" stability as a common sense requirement.

It seems intuitively plausible to define local expectational stability as follows : there exists a non trivial neighbourhood of the equilibrium such that, if everybody believes that the state of the system is in this neighbouhood, *it is necesarily the case, whatever the specific form taken by eveybody's belief, that the state is in the stressed neighbouhood*. Intuitively, the absence of such a neighbourhood signals some tendancy to instability : there can be facts falsifying any universally shared conjecture on the set of possible states, unless this set reduces to the equilibrium itself.

Naturally, it is easy to check, and left to the reader, that the failure of getting local "expectational stability" in the precise sense defined above is (roughly) equivalent to a failure of the just stressed local intuitive requirement.

2.3 "Eductive" versus "evolutive" learning stability.

There is an informal argument, due to Milgrom-Roberts (1990), according to which, in a system that repeats itself, non best responses to existing observations will be deleted after a while, initiating a "real time" counterpart of the "notional" time deletion of non-best responses that underlies "eductive" reasoning. Let us focus here on the connections between local "eductive stability" and the local convergence of standard "evolutive" learning rules. What local "eductive stability", as just defined, does involve is that once, for whatever reasons, the (possibly stochastic) beliefs of the agents will be trapped in $V(E^*)$, they will remain in $V(E^*)$, whenever updating satisfies natural requirements that are met in particular by Bayesian updating rules. Although it is not quite enough to be sure that any "evolutive" learning rule will converge, it is the case that, in many settings, one can show that local "eductive" stability involves that every "reasonable" evolutive real time learning rule converges asymptotically (see Guesnerie (2002), Gauthier-Guesnerie (2005)). Furthermore, it should be clear that the failure to find a set $V(E^*)$ for which the the equilibrium is locally strongly rational, signals a tendancy for "reasonable" states of beliefs, close to the equilibrium, and then likely to be reachable with some "reasonable" evolutive updating process, to be triggered away in some cases, a fact that threatens the convergence of the corresponding learning rule[2].

Hence, our very abstract and hyper-rational criterion, provides a short-cut for understanding the difficulties of expectational coordination, without entering into the business of specifying the real time, bounded rationality considerations that may matter. Naturally, the "eductive" criterion is in general more demanding than most fully specified "evolutive" learning rules one can think of (as strongly suggested by the just sketched argument and precisely shown in the previously referred contributions).

The connection is however less clearcut than just suggested in cases of models with "extrinsic uncertainty". Then, the equilibrium is a probability distribution, a state of the system in the sense of the word taken here is a probability distribution. An observation is not an observation on the state in our sense, but an information on the state in the standard sense of the word. "Evolutive" and "eductive learning may then differ significantly..

3 Expectational criteria in infinite horizon models : one-dimensional state variable.

Models used for monetary policy generally adopt an infinite horizon approach. This section and the following review existing results on "eductive" stability in infinite horizon models. They are based on Gauthier (2003),

[2] And certainly forbids a strong form of "monotonic" convergence.

Evans-Guesnerie (2003, 2005), Gauthier-Guesnerie (2005). The review will allow us later to improve the comparison of the game-theoretically oriented viewpoint stressed in this paper with the standard macroeconomic approach to the problem as reported in Evans-Honkappohja (2001).

We start by focusing attention on one-dimensional one step-forward memory one models.

3.1 The model

Consider a model in which the one-dimensional state of the system today is determined from its value yesterday and its expected value tomorrow, according to the linear (for the sake of simplicity) equation :

$$\gamma E\left[x(t+1) \mid I_t\right] + x(t) + \delta x(t-1) = 0.$$

where x is a one-dimensional variable γ and δ are real parameters $(\gamma, \delta \neq 0)$.[3].

A *perfect foresight trajectory* is a sequence $(x(t), t \geq -1)$ such that

$$\gamma x(t+1) + x(t) + \delta x(t-1) = 0. \tag{1}$$

in any period $t \geq 0$, given the initial condition $x(-1)$.

Assume that the equation $g_1 = -\gamma g_1^2 - \delta$ has only two real solutions λ_1 and λ_2 (which arises if and only if $1 - \delta\gamma \geq 0$) with different moduli (with $|\lambda_1| < |\lambda_2|$ by definition). Therefore, given an initial condition $x(-1)$, although there are "many" perfect foresight solutions, there are two perfect foresight solutions having the simple form :

$$\begin{aligned} x(t) &= \lambda_1 x(t-1). \\ \text{and } x(t) &= \lambda_2 x(t-1). \end{aligned}$$

They are called *constant growth rates* solutions.

The steady state sequence $(x(t) = 0, t \geq -1)$ is a perfect foresight equilibrium if and only the initial state $x(-1)$ equals 0. The steady state is a sink if $|\lambda_2| < 1$, a saddle if $|\lambda_1| < 1 < |\lambda_2|$, or a source if $|\lambda_1| > 1$. We focus

[3] Such dynamics obtain in particular from linearized versions of overlapping generations models with production, at least for particular technologies (Reichlin (1986)), or infinite horizon models with a cash-in-advance constraint (Woodford (1988)).

attention here on the saddle case. In this case, the solution $x(t) = \lambda_1 x(t-1)$, is generally called the *saddle-path.* It has been stressed for a long time by economists as the "focal" solution, on the basis of arguments that refer to expectational plausibility. We review, first, the standard expectational criteria that are used and confirm that the saddle-path solution fits them.

3.2 The standard expectational criteria.

3.2.1 Determinacy.

The first criterion is determinacy. Determinacy means that the equilibrium under consideration is "locally isolated". In our infinite horizon setting, determinacy has to be viewed as a property of trajectories : a trajectory $(x(t), t \geq -1)$ is determinate if there is no other equilibrium trajectory $(x'(t), t \geq -1)$ that is "close" to it. This calls for a reflection about the notion of proximity of trajectories, i.e on the choice of a topology. Yet, the choice of the suitable topology is open. The most natural candidate is the C_0 topology, according to which two different trajectories $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$ are said to be close whenever $|x(t) - x'(t)| < \varepsilon$, for any $\varepsilon > 0$ arbitrarily small, and any date $t \geq -1$. In fact, with such a concept of determinacy, the saddle-path solution, along which $x(t) = \lambda_1 x(t-1)$ when $|\lambda_1| < 1 < |\lambda_2|$, is the only solution to be locally isolated, i.e determinate, in the C_0 topology.

Growth rates determinacy. In the present context of models with memory, a saddle-path solution is characterized by a constant *growth rate* of the state variable. This suggests that determinacy should be applied in terms of growth rates, in which case closedness of two trajectories $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$ would require that the ratio $x(t)/x(t-1)$ be close to $x'(t)/x'(t-1)$ in each period $t \geq 0$. This is an ingredient of a kind of C_1 topology, as advocated by Evans and Guesnerie (2003). In this topology, two trajectories $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$ are said to be close whenever both the levels $x(t)$ and $x'(t)$ are close, and the ratios $x(t)/x(t-1)$ and $x'(t)/x'(t-1)$ are close, in any period.

As stressed for example by Gauthier (2002), the examination of proximity in terms of growth rates leads to the examination of the dynamics with perfect foresight in terms of growth rates.

Define : $$g(t) = x(t)/x(t-1),$$

For any $x(t-1)$ and any $t \geq 0$, then the perfect foresight dynamics implies :

$$x(t) = -\left[\gamma g\left(t+1\right) g\left(t\right) + \delta\right] x(t-1).$$

Or

$$g\left(t\right) = -\left[\gamma g\left(t+1\right) g\left(t\right) + \delta\right] \tag{2}$$

Associated with the initial perfect foresight dynamics1, is then a *perfect foresight dynamics of growth rates*. The growth factor $g\left(t\right)$ is determined at date t from the "correct" forecast of the next growth factor $g\left(t+1\right)$. This new dynamics 2is non-linear, and it has a one-step forward looking structure, without predetermined variable.

We have then reassessed the problem in terms of one-dimensional one-step forward looking models which are more familiar

3.2.2 Sunspots on growth rates

Maintaining the focus on growth rates, let us now define a concept of sunspot equilibrium, in the neighborhood of a constant growth rate solution. Suppose that agents a priori believe that the growth factor is perfectly correlated with sunspots.

Namely, if the sunspot event is $s = 1, 2$ at date t, they a priori believe that $g(t) = g(s)$, that is

$$x(t) = g(s)x(t-1).$$

Thus, their common expected growth forecast is :

$$E\left[x(t+1) \mid I_t\right] = \pi(s,1)g(1)x(t) + \pi(s,2)g(2)x(t),$$

where $\pi(s,1)$ and $\pi(s,2)$ are the sunspot transition probabilities.

As shown by Desgranges and Gauthier (2003), this consistency condition is written :

$$g(s) = -\left[\gamma\left[\pi(s,1)g(1) + \pi(s,2)g(2)\right] g(s) + \delta\right] \tag{3}$$

When $g(1) \neq g(2)$,the formula defines a sunspot equilibrium on the growth rate, as soon as the stochastic dynamics of growth rates is extended[4] as

$$g(t) = -\gamma E[g(t+1) \mid I_t] \, g(t) - \delta.$$

3.2.3 Evolutive learning on growth rates.

It makes sense to learn growth rates from past observations. Agents then update their forecast of the next period growth rates from the observation of past or present actual rates.

Reasonable learning rules in the sense of Guesnerie (2002), Gauthier-Guesnerie (2005) consist of *adaptive* learning rules that are *able to "detect cycles of order two"*.

3.2.4 Iterative Expectational Stability (IE Stability)

We shall refer here to IE-stability criterion [5](see Evans (1985), de Canio, (1978). Lucas (1979), for early studies), and apply it to conjectures on growth rates

Let agents a priori believe that the law of motion of the economy is given by:

$$x(t) = g(\tau)x(t-1),$$

where $g(\tau)$ denotes the conjectured growth rate at step τ in some mental reasoning process. Then, they expect the next state variable to be $g(\tau)x(t)$, so that the actual value is $x(t) = -\delta x(t-1)/(\gamma g(\tau)+1)$. Assume that all the agents understand that the actual growth factor is $-\delta/(\gamma g(\tau)+1)$ when their initial guess is $g(\tau)$, they should revise their guess as

$$g(\tau+1) = -\frac{\delta}{\gamma g(\tau)+1}. \tag{4}$$

By definition, IE-stability obtains whenever the sequence $(g(\tau), \tau \geq 0)$ converges (towards one of its fixed point), a fact that is interpreted as reflecting

[4] Clearly this equivalence relies on special assumptions about linearity and certainty equivalence.

[5] This concept differs from the more usual concept of Differential Expectational stability (see Evans-Honkappohja (2001))

the success of some mental process of learning (leading to the constant growth rate associated with the considered fixed point). It is easy to check that this dynamics is the time mirror of the perfect foresight dynamics of growth rate: then, a fixed point λ_1 or λ_2 is locally IE-stable if and only if it is locally unstable in the previous growth rates dynamics, that is, in this dynamics, locally determinate.

This is, within a simple model, a somewhat careful reminder of the four possible (and more or less standard) viewpoints on "expectational stability". We want later to compare such viewpoints with the so called "eductive viewpoint" emphasized here. The next statemnt makes this comparison much easier : it turns out that here these a priori different approaches of the problem select the same solutions.

3.2.5 An equivalence theorem on standard "expectational criteria"

Proposition 3-1. Equivalence principle in one-step forward, memory one, one-dimensional linear systems.

Consider a one-step forward looking model (with one lagged predetermined variable, where $\gamma, \delta \neq 0$). Assume that we are in the saddle case. Then the following four statements are equivalent:

1. A constant growth rate solution is locally determinate in the perfect foresight growth rate dynamics and equivalently here is determinate in the C1 topology of trajectories.

2. A constant growth rate solution is locally immune to (stationary) sunspots on growth rates.

3. For any a priori given "reasonable" learning rules bearing on growth rates, a constant growth rate solution is locally asymptotically stable.

4. A constant growth rate solution is locally IE stable.

In particular, a saddle-path solution which meets requirement 1, meets all the others.The argument of Guesnerie (2002), gathers earlier findings. For example the fact that "reasonable" learning processes converge relies on a definition of "reasonableness" integrating the suggestions of Grandmont-Laroque (1991) and the results of Guesnerie-Woodford (1991).

We postpone to Section 5 the discussion of the comparison of the just analysed criteria with the "eductive" viewpoint on learning , the analysis of which requires the introduction of some game-theoretical flesh into the model.

4 Standard Expectational criteria in infinite horizon models : the multi-dimensional case

While keeping with one-step forward looking linear models with memory one, we now turn to the case of a multidimensional state variable.

4.1 The framework

The dynamics of the multidimensional linear one-step forward looking economy with one predetermined variable, is now governed by :

$$\mathbf{G}E\left(\mathbf{x}(t+1) \mid I_t\right) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{0},$$

where $\mathbf{x}$ is a $n \times 1$ dimensional vector, $\mathbf{G}$ and $\mathbf{D}$ are two $n \times n$ matrices, and $\mathbf{0}$ is the $n \times 1$ zero vector. A *perfect foresight equilibrium* is a sequence $(\mathbf{x}(t), t \geq 0)$ associated with the initial condition $\mathbf{x}(-1)$, and such that :

$$\mathbf{G}\mathbf{x}(t+1) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{0} \tag{5}$$

The dynamics with perfect foresight is governed by the $2n$ eigenvalues λ_i $(i = 1, ..., 2n)$ of the following matrix (the companion matrix associated with the recursive equation)

$$\mathbf{A} = \begin{pmatrix} -\mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{D} \\ \mathbf{I}_n & (\mathbf{0}) \end{pmatrix},$$

where $(\mathbf{0})$ is the n-dimensional zero matrix.

In what follows, we shall be interested in the perfect foresight dynamics restricted to a n-dimensional eigensubspace, and especially the one spanned by the eigenvectors associated with the n roots of lowest modulus. Let assume that the eigenvalues are distinct and define $|\lambda_i| < |\lambda_j|$ whenever $i < j$ $(i, j = 1, ..., 2n)$ and focus attention on the *generalized saddle case*, where $|\lambda_n| < 1 < |\lambda_{n+1}|$.

Let $\mathbf{u}_i$ denote the eigenvector associated with λ_i $(i = 1, ..., 2n)$. Since all the eigenvalues are distinct, the n eigenvectors form a basis of the subspace associated with $\lambda_1, ..., \lambda_n$. Let:

$$\mathbf{u}_i = \begin{pmatrix} \tilde{\mathbf{v}}_i \\ \mathbf{v}_i \end{pmatrix}$$

where $\mathbf{v}_i$ and $\tilde{\mathbf{v}}_i$ are of dimension n. It is straightforward to check that if $\mathbf{u}_i$ is an eigenvector, then $\tilde{\mathbf{v}}_i = \lambda_i \mathbf{v}_i$.

Hence, if we pick up some $\mathbf{x}(0)$, and if the n-dimensional subspace generated by $(\mathbf{u}_1, ..., \mathbf{u}_n)$ is in "general position", we can find a single $\mathbf{x}(1)$ such that $(\mathbf{x}(0), \mathbf{x}(1)) = \sum a_i \mathbf{u}_i$ is in the subspace and generate a sequence $(\mathbf{x}(t), t \geq 0), (\mathbf{x}(1), x(2)) = \sum a_i \lambda_i \mathbf{u}_i, ...)$ following the just defined dynamics 5. This generates a solution, which is converging in the saddle-path case.

The methodology proposed for constructing "constant growth rates" solution in the previous Section can be replicated to obtain what is called *minimum order solutions*. Assume that

$$\mathbf{x}(t) = \mathbf{B}\mathbf{x}(t-1) \tag{6}$$

in every period t, and for any n-dimensional vector $\mathbf{x}(t-1)$ ($\mathbf{B}$ is an $n.n$ matrix). Also, $\mathbf{x}(t+1) = \mathbf{B}\mathbf{x}(t)$. Thus, it must be the case that

$$\mathbf{B} = -(\mathbf{G}\mathbf{B} + \mathbf{I}_n)^{-1}\mathbf{D}$$

or equivalently

$$(\mathbf{G}\mathbf{B} + \mathbf{I}_n)\mathbf{B} + \mathbf{D} = \mathbf{0} \tag{7}$$

A matrix $\bar{\mathbf{B}}$ satisfying this equation[6] is a minimum order solution in the sense of Mc Callum (1983). Gauthier (2002) calls it a *stationary extended growth rate*. In view of the analysis of constant growth rates solutions made in the previous section, we use this latter terminology : we then focus attention now on the expectational stability of "extended growth rates".

[6] It is shown in Evans and Guesnerie (2005) that $\bar{\mathbf{B}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, where $\mathbf{\Lambda}$ is a $n \times n$ diagonal matrix whose iith entry is λ_i $(i = 1, \ldots, n)$ and $\mathbf{V}$ is the associated matrix of eigenvectors. In what follows, we focus attention on the saddle case, where $|\lambda_n| < 1 < |\lambda_{n+1}|$.

4.2 The expectational plausibility of Extended Growth Rates (EGR) solutions according to standard criteria.

We will concentrate on three of the above criteria : determinacy, immunity to sunspots, and IE-stability.

Determinacy is viewed through a dynamics of *perfect foresight of extended growth rates* that extends the dynamics of growth rates previously introduced. Consider for every t, $\mathbf{B}(t)$ a n-dimensional matrix whose ijth entry is equal to $b_{ij}(t)$ and $\mathbf{x}(t) = \mathbf{B}(t)\mathbf{x}(t-1)$. Such a matrix will be called, in line with the just introduced terminology of "stationary extended growth rates", an "extended growth rate".

Assume that such a relationship holds whatever t, so that $\mathbf{x}(t+1) = \mathbf{B}(t+1)\mathbf{x}(t)$;then the dynamics with perfect foresight of the endogenous state variable $\mathbf{x}(t)$ implies :

$$\mathbf{GB}(t+1)\mathbf{x}(t) + \mathbf{x}(t) + \mathbf{Dx}(t-1) = \mathbf{0}$$

i.e

$$\mathbf{x}(t) = -(\mathbf{GB}(t+1) + \mathbf{I}_n)^{-1}\mathbf{Dx}(t-1), \tag{8}$$

provided that $\mathbf{GB}(t+1) + \mathbf{I}_n$ is a n-dimensional regular matrix.

Then, a perfect foresight dynamics of such matrices $\mathbf{B}(t)$ may be associated with a sequence of matrices $(\mathbf{B}(t), t \geq 0)$ such that :

$$\mathbf{B}(t) = -(\mathbf{GB}(t+1) + \mathbf{I}_n)^{-1}\mathbf{D} \Leftrightarrow (\mathbf{GB}(t+1) + \mathbf{I}_n)\mathbf{B}(t) + \mathbf{D} = \mathbf{0}. \tag{9}$$

This defines *the extended growth rates perfect foresight dynamics.*

Its fixed points are the stationary matrices $\bar{\mathbf{B}}$ such that $\mathbf{B}(t) = \bar{\mathbf{B}}$, whatever t. They are solutions of 7.

Determinacy of the "stationary extended growth rate" associated with the matrix $\bar{\mathbf{B}}$, is standardly defined as the fact that (the infinite trajectory with constant extended growth rate) $\bar{\mathbf{B}}$ is locally isolated, i.e that there does not exist a sequence $\mathbf{B}(t)$ of perfect foresight extended growth rates converging to $\bar{\mathbf{B}}$. From now on, we refer to an "extended growth rate" as an *EGR.*

A sunspot equilibrium on extended growth rates, in the spirit of previous section, is a situation in which the whole matrix $\mathbf{B}(t)$ that links $\mathbf{x}(t)$ to $\mathbf{x}(t-1)$

is perfectly correlated with sunspots. If sunspot event is s ($s = 1, 2$) at date t, so that $E\left(\mathbf{x}(t+1) \mid s\right) = \left[\pi(s,1)\mathbf{B}(1) + \pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s)\mathbf{x}(t-1)$.

$$\mathbf{x}(t) = -\left[\mathbf{G}\left[\pi(s,1)\mathbf{B}(1) + \pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s) + \mathbf{D}\right]\mathbf{x}(t-1).$$

In a sunspot equilibrium, the a priori belief that $\mathbf{B}(t) = \mathbf{B}(s)$ is selfulfilling whatever $x(t-1)$, so that :

$$\mathbf{B}(s) = -\left[\mathbf{G}\left[\pi(s,1)\mathbf{B}(1) + \pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s) + \mathbf{D}\right].$$

It remains for us to examine the stability properties of the (virtual time) learning dynamics associated with the IE-stability criterion. At virtual time τ of the learning process, agents believe that, whatever t:

$$\mathbf{x}(t) = \mathbf{B}(\tau)\mathbf{x}(t-1),$$

where $\mathbf{B}(\tau)$ is the τth estimate of the n-dimensional matrix $\mathbf{B}$.

Their forecasts are accordingly:

$$E\left(\mathbf{x}_{t+1} \mid I_t\right) = \mathbf{B}(\tau)\mathbf{x}_t.$$

The actual dynamics is obtained by reintroducing forecasts into the temporary equilibrium map :

$$\mathbf{GB}(\tau)\mathbf{x}_t + \mathbf{x}_t + \mathbf{D}\mathbf{x}_{t-1} = \mathbf{o} \Leftrightarrow \mathbf{x}_\tau = -(\mathbf{GB}(\tau) + \mathbf{I}_n)^{-1}\mathbf{D}\mathbf{x}_{\tau-1}.$$

As a result, the dynamics with learning is written:

$$\mathbf{B}(\tau+1) = -(\mathbf{GB}(\tau) + \mathbf{I}_n)^{-1}\mathbf{D}. \tag{10}$$

A stationary *EGR* $\bar{\mathbf{B}}$ is a fixed point of the above dynamics. It is locally IE-stable if and only if the dynamics is converging when $\mathbf{B}(0)$ is close enough to $\bar{\mathbf{B}}$.

4.3 The dynamic equivalence principle

We can state the following proposition :

Proposition 4-1. Equivalence principle in one-step forward, memory one, multi-dimensional linear systems.

Consider a stationary EGR.

The following three statements are equivalent:

1. The EGR *solution is determinate in the perfect foresight extended growth rates dynamics.*

2. *The* EGR *solution is immune to sunspots, that is, there are no neighbour local sunspot equilibria on extended growth rates with finite support, as defined above.*

3. *The* EGR *solution is locally IE-stable.*

In particular, the saddle-path like solution (that exists when the n smallest eigenvalues of A have modulus less than 1, the $(n+1)th$ having modulus greater than 1) meets all these conditions.

The statement is proved in Gauthier-Guesnerie (2005)[7].

Clearly, the flavour of this statement is very close to that of the statement obtained in the one dimensional case. Note however, that the connection between "evolutive" learning and "eductive" learning is now more intricate. The performance of adaptive learning processes bearing on the multi-dimensional object "extended growth rates" is less easy to assess than in the one-dimensional situation of the previous section : part 3 of Proposition 3-1 has no counterpart here.

5 Eductive learning in dynamical models.

5.1 The underlying strategic framework.

The discussion of the basic viewpoint of "eductive learning" requires that some game-theoretical flesh be given to the dynamical models under scrutiny. In other words, we need to imbed the dynamic model in a dynamic game. We present, for the sake of completeness, the construct proposed in Evans-Guesnerie (2003), a construct that explicitely refers to an OLG context.

At each period t, there exists a continuum of agents. A part of the agents "react to expectations", another part uses strategies which are not reactive to expectations (in an OLG context, these are the agents, who are at the last period of their lives)[8]. The former are denoted ω_t and belong to a convex

[7] The equivalence of 1 and 3 follows easily from the above definitions and sketch of analysis. The reader will convince himself that the equivalence with 2 is plausible.

[8] It is assumed that an agent of period t is different from any other agent of period $t', t' \neq t$.This means either that each agent is "physically" different or that the agents

segment of R, endowed with Lebesgue measure $d\omega_t$. More precisely, agent ω_t has a (possibly indirect) utility function that depends upon:

1) his own strategy $s(\omega_t)$,

2) sufficient statistics of the strategies played by others i.e. upon $y_t = F(\Pi_{\omega_t}\{s(\omega_t)\}, *)$, where F in turn depends first, upon the strategies of all agents who at time t react to expectations, and second, upon $(*)$, which is here supposed to represent sufficient statistics of the strategies played by those who do not react to expectations, and that includes but is not necessarily identified with y_{t-1},

3) finally upon the sufficient statistics for time $t+1$, as perceived at time t, i.e. upon $y_{t+1}(\omega_t)$, which *may be random* and also, now directly, upon the $t-1$ sufficient statistics y_{t-1}.

We assume that the strategies played at time t can be made conditional on the equilibrium value of the t sufficient statistics y_t. Now, let $(\bullet)$ denote both (the product of) y_{t-1} and the probability distribution of the random variable $\tilde{y}_{t+1}(\omega_t)$, (the random subjective forecasts held by ω_t of y_{t+1}). Let then $G(\omega_t, y_t, \bullet)$ be the best response function of agent ω_t. Under these assumptions, the sufficient statistics for the strategies of agents who do not react to expectations is $(*) = (y_{t-1}, y_t)$.

The equilibrium equations at time t are written:

$$y_t = F\left[\Pi_{\omega_t}\{G(\omega_t, y_t, y_{t-1}, \tilde{y}_{t+1}(\omega_t))\}, y_{t-1}, y_t\right]. \quad (11)$$

Note that when all agents have the same point expectations denoted y^e_{t+1}, the equilibrium equations determine what is called the temporary equilibrium mapping

$$Q(y_{t-1}, y_t, y^e_{t+1}) = y_t - F\left[\Pi_{\omega_t}\left\{G(\omega_t, y_t, y_{t-1}, y^e_{t+1})\right\}, y_{t-1}, y_t\right].$$

Also assuming that all $\tilde{y}_{t+1}$ have a very small common support "around" some given y^e_{t+1}, decision theory suggests that G, to the first order, depends on the expectation of the random variable $\tilde{y}_{t+1}(\omega_t)$ that is denoted $y^e_{t+1}(\omega_t)$ (and is close to y^e_{t+1}), we are able to linearize (11), around any initially given situation, denoted (0), as follows:

$$y_t = U(0)y_t + V(0)y_{t-1} + \int W(0, \omega_t)y^e_{t+1}(\omega_t)d\omega_t,$$

have strategies that are independent from period to period. In an OLG interpretation of the model, each agent lives for two periods but only reacts to expectations in the first period of his life.

where $y_t, y_{t-1}, y^e_{t+1}(\omega_t)$ now denote small deviations from the initial values of y_t, y_{t-1}, y_{t+1}, and $U(0), V(0), W(0, \omega_t)$ are $n \times n$ square matrices.

If such a linearization is considered only around a steady state of the system, y_t, y_{t-1}, etc., will denote deviations from the steady state and $U(0), V(0), W(0, \omega_t)$ are simply $U, V, W(\omega_t)$.

Adding an invertibility assumption, we get two reduced forms :

- the standard temporary equilibrium reduced form, associated with homogenous expectations, $(y^e_{t+1}(\omega_t) = y^e_{t+1},)$, is :

$$y_t = By^e_{t+1} + Dy_{t-1}, \tag{12}$$

- the "stochastic beliefs" reduced form is :

$$y_t = Dy_{t-1} + B \int Z(\omega_t) y^e_{t+1}(\omega_t) d\omega_t, \tag{13}$$

where $\int Z(\omega_t) d\omega_t = I$.

"Eductive" Stability will be analysed from this latter reduced form13.

5.2 "Eductive Stability"

5.2.1 One-dimensional setting.

>From the above analysis, it seems natural to make beliefs indexed with growth rates (as underlined in Evans- Guesnerie (2003), beliefs on the proximity of trajectories in the C_0 sense have not enough grip on the agents'actions).

Hence, the hypothetical Common Knowledge assumption, to be taken into account, concerns growth rates, (the C_1 topology).

(Hypothetical) **CK Assumption.** The growth rate of the system is between $\lambda_1 - \epsilon$ and $\lambda_1 + \epsilon$

Such an assumption on growth rates triggers a mental process that, in successful cases, progressively reinforces the initial restriction and converges towards the solution. The mental process takes into account the variety of beliefs associated with the initial restriction. Common beliefs with point expectations are then a particular case, and it is intuitively easy to guess that convergence of the general mental process under consideration implies convergence of the special process under examination when studying IE-stability. This is stressed as such : IE-stability is a necessary condition of eductive stability (Evans and Guesnerie (2003)). It then follows, from the above "equivalence theorem" (Proposition 3-1) of Section 3 :

Proposition 5-1 :

If a constant growth rate solution is locally "eductively stable" or "locally strongly rational" then it is determinate in growth rates, locally IE stable, locally immune to susnpots, and attracts all reasonable evolutive learning rules.

Hence "Eductive Stability" is more demanding in general than all the previous equivalent criteria. The fact that it is strictly more demanding is shown in the Evans-Guesnerie (2003)'s paper, although it becomes equally demanding when some behavioural homogeneity condition is introduced.

5.2.2 Multi-dimensional setting

In a natural way, the hypothetical Common Knowledge assumption, to be taken into account has to bear on extended growth rates.

(Hypothetical) **CK Assumption.** The extended growth rate of the system B belongs to $V(\overline{B})$, where $V(\overline{B})$is a neighbourhood in the space of matrices (that has to be defined with respect to some distance, normally evaluated from some matrix norm)

As we said earlier, if Common Knowledge of $B \in V(\overline{B})$. $\Rightarrow B = \overline{B}$, then the solution is localy "eductively" stable or locally Strongly Rational.

As in the one-dimensional case, one can show, using now Proposition 4-1.

Proposition 5-2 :

If a *stationnary extended growth rate solution is locally "eductively stable" or "locally strongly rational" then it is determinate, locally IE stable, locally immune to susnpots.*

Again, "Eductive Stability" is more demanding in general than all the standard and, as stressed earlier, equivalent criteria. The reason is that it takes into account

1- the stochastic nature of beliefs,

2- the heterogeneity of beliefs.

Both dimensions are explicitely neglected in the Iterative Expectational stability construct, and implicitely in the other equivalent constructs. In fact, as soon as local "eductive" stability is concerned, the results of Guesnerie-Jara-Moroni (2007), although obtained in a different context make clear that, in a local context, point-expectations and stochastic expectations do not make so much difference. Hence, locally at least, the key differences between

Strong Rationality and and standard expectational stability criteria would come from the heterogeneity of expectations.

5.2.3 Standard expectational coordination approaches and the "eductive" viewpoint : a tentative conclusion.

Let us note first that our attempt at comparing the standard expectational coordination criteria, determinacy, absence of neighbour sunspot equilibria, IE-stability, has been limited to the above class of models. An exhaustive attempt would have to extend the class of models under scrutiny in different directions.

- Introduce uncertainty (intrinsic uncertainty) in the models of previous sections. The analysis should extend, with some technical difficulties, the appropriate objects under scrutiny being then respectively, probability distributions on growth rates and extended growth rates. It is reasonable to conjecture that the above equivalence proposition of Section 4 would have a close counterpart in the new setting.

- Introduce longer memory lags and/or more forward looking perceptions. The theory seems applicable although the concept of "extended growth rate" becomes more intricate (Gauthier (2004)).

The next set of remarks brings us back to the models used in monetary theory (starting for example from Sargent-Wallace (1975)). A number of these models have a structure analogous to the ones examined here, although they often involve intrinsic uncertainty. This suggest two provisional conclusions that will be put under scrutiny in the next section.

1- The standard criterion used in monetary theory for assessing expectational coordination, local determinacy, is less demanding than the "eductive" criterion. This can be seen, within the present perspective, as the reflection of a neglect of a dimension of heterogeneity of expectations that is present in the problem.

2- However, the connections between the "evolutive viewpoint" and the "eductive" one are less clearcut than in our prototype model. Differences have two sources :

- The theoretical connection between the two types of learning is less well established in the multidimensional case, that often obtains in monetary models of the new Keynesian type, than in the one-dimensional one. (Proposition 3-1-3 has no counterpart in Proposition 4-1)

- In a noisy system, agents do not observe, at each step, a "state" of the system, as defined in our construct, i.e a probability distribution, but a random realisation drawn from this probability distribution. Learning rules, aiming at being efficient, have to react slowly to new information. Intuitively, IE stability and consequently eductive stability will be more demanding local criteria than success of, necessarily slow, evolutive learning.

However, the above analysis and its just suggested provisional conclusions implicitely refer *to a truly overlapping generations framework.* The equations from which the expectational coordination aspects of monetary policy are most often examined are indeed of the "overlapping form" but come *from infinite horizon models.* Their interpretation, within the framework of an "eductive" analysis should hence be different. We will stress this, sometimes considerable, difference in the next and final Section.

6 "Eductive stability' in a cashless economy.

The objective here is to introduce very simple versions or models that are used for the discussion of monetary policy and of the Central Bank policy. Indeed attention will be focused here on a simple model of a cashless economy, in the sense of Woodford (2003).

6.1 The model, and the standard viewpoint.

Let us consider an economy populated by a continuum of identical agents, living for ever. Each agent α [9]receive $\overline{y}$ units of a perishable good at every period. There is money and the good has a money price P_t at each period,

The agents have an identical utility function

$$U = \sum \beta^t u(C_t)$$

where $u(C_t)$ will be taken as iso-elastic

$$u(C_t) = [1/(1-\sigma)](C_t)^{(1-\sigma)}$$

.First order conditions are :

$$(1 + i_t) = (1/\beta)[u'(C_{t+1})/u'(C_t)](P_t/P_{t+1})^{-1} = (1/\beta)(P_{t+1}/P_t)[\frac{C_t}{C_{t+1}}]^\sigma$$

[9]Although we shall keep in mind the continuum interpretation, the reasoning will formally refer to a representative consumer, leaving aside the notation α.

where i_t is the nominal interest rate.

The Central bank decides on a nominal interest rate according to a Wicksellian rule. We assume that this rule takes the form

$$i_t^m = \phi(P_t/P_{t-1})$$

where ϕ is increasing.

We assume also that the targeted inflation rate is $\Pi^* > \beta$ so that

$$1 + \phi(\Pi^*) = \Pi^*/\beta.$$

The money price at time 0 is denoted P_0^*. The targeted price path is

$$P_t^* = P_0^*(\Pi^*)^t$$

.

Indeed, the economy is considered as starting at time 1.

We note that the path $P_t = P_t^*, C_t(\alpha) = \overline{y}, t = 1, 2, ... + \infty$, defines a Rational Expectations, here a Perfect Foresight, Equilibrium, associated with a nominal interest rate $\phi(\Pi^*) = (\Pi^*/\beta) - 1$.

Is this equilibrium **determinate** ? It should be noted that, since all agents are similar and face the same conditions in any equilibrium, any equilibrium has to meet $C_t(\alpha) = \overline{y}$. It follows that any other (perfect foresight) equilibrium $\{P_t'\}$ has to meet :

$$(1 + \phi(P_t'/P_{t-1}'))\beta = (P_{t+1}'/P_t')$$

which can be written, using π_t as the inflation rate :

$$(1 + \phi(\pi_t)) = \pi_{t+1}.$$

Any equilibrium close to the stationary equilibrium Π^* would satisfy (with straigthforward notation) :

$$\phi'(*)\beta(\delta\pi_t) = (\delta\pi_{t+1})$$

an equation incompatible with the proximity of the new equilibrium trajectory to the steady state trajectory, as soon as $\phi'(*)\beta > 1$.

In other words, if $\phi'(*) > (1/\beta)$, which is the form taken here by the Taylor rule, then the equilibrium is locally determinate.

Note that :

- The above sketched argument does not demonstrate, as such, that, outside the neighbourhood under consideration, there are no other perfect foresight equilibria, although the one under scrutiny is the only stationary one

- If we accept to view the equations as coming from an OLG framework, we would argue that the equilibrium is locally IE-Stable, or even here locally "eductively" stable. Indeed, assume that a) it is initially CK that inflation will remain for ever in a neighbourhood of Π^*. Take into account the fact that b) : it is CK that a (general) departure in inflation expectation of $\delta\pi_{t+1}$ involves a departure in period t inflation of $\delta\pi_t = \frac{1}{\beta\phi'}\delta\pi_{t+1}$ Assertions a) and b) together do imply that the steady inflation state $*$ is CK. In other words, the equilibrium * is locally "eductively stable"[10].

However, assertion b) which is a core element of the construction in an OLG framework, makes no full sense here, where what happens today does not only depend on the expectations concerning tomorrow, but necessarily on the whole trajectory of beliefs of the agents. To put it in another way, the fact that the to-morrow (period $(t+1)$) inflation expectation is π_{t+1}has no final bite on what the equilibrium price may be today in period t. Indeed, demand of an agent at period t, as seen from period 1 is :

$$C_t(\alpha) = C_1(\alpha)\left[\beta^{(t-1/\sigma)}\Pi_1^{t-1}[(1+i_s)(P_s/P_{s+1})]^{1/\sigma}\right]$$

At period t,agent α may be viewed as determining its demand as follows :

- Take $C_t(\alpha)$ as a starting parameter and compute the infinite sequence

$$C_{t+\tau}(\alpha) = C_t(\alpha)\left[\beta^{(\tau-1)/\sigma}\Pi_t^{t+\tau-1}[(1+i_s)(P_s/P_{s+1})]^{1/\sigma}\right].$$

- Then choose $C_t(\alpha)$ so that it meets the consumer's discounted intertemporal budget constraint.

Clearly, such a computation has to be fed by the whole agents' beliefs over the period and not only by their beliefs over the next period ! In other words, the connection between t and $t+1$ emphasized above for the analysis of "eductive stability" only captures one intermediate step of the

[10]Strictly speaking, the sketched argument only shows that the equilibrium $*$ is locally IE-stable. The fact that agents are identical here is more than needed to insure that heterogeneity of beliefs does not matter, so that IE stability implies Eductive Stability.

choice procedure and not, as it would do in a true OLG framework, the whole story.

The right question is then the following : assume that it is the case that hypothetically it is CK that π_s is close to $\Pi_s^* = \Pi^*$, then is it the case that the equilibrium is CK ? We answer this question in the next Section.

6.2 "Eductive" Stability in the infinite horizon cashless economy : preliminaries.

Let us consider the world at time 1.

Let us assume that, at the margin of the stationnary equilibrium, where the **real interest rate** is r^*, a small departure $dr_s, s = 1, ...,$is expected by all the agents. It does not matter, at this stage, whether such a departure comes from a(n) (expected) change in nominal interest rate or an expected change in inflation. We ask the question : given these changes in beliefs, what is the new first period equilibrium ?

Naturally, consumption will not change in period 1, the only adjustment variable is the first period interest rate that will become $r^* + dr_1$. What will be the equilibrium dr_1 ?

The answer is given by Lemma 1:

Lemma 1 *The new equilibrium real interest rate is, to the first order approximation, $r^* + dr_1$,with :*

$$dr_1 = -\frac{\beta}{(1-\beta))}(dr_2)$$

Proof. *Consider the first order conditions :*

$$C_t(\alpha) = C_1(\alpha)\left[\beta^{((t-1)/\sigma)}\Pi_1^{t-1}[(1+r_s)^{1/\sigma}\right]$$

Take the Log :

$$LogC_t = LogC_1 + ((t-1)/\sigma)Log\beta + (1/\sigma)\sum_1^{t-1} Log(1+r_s)$$

so that, approximately, in the neighbourhood of the stationary equilibrium with consumtion C^*, and interest rate r^*, (with : $\beta(1+r^*) = 1$)

$$\frac{dC_t}{C*} = \frac{dC_1}{C*} + \frac{\beta}{\sigma}(\sum_{s=1}^{t-1} dr_s)$$

Let us single out the adjustment variable dr_1 :

$$\frac{dC_t}{C*} = \frac{dC_1}{C*} + \frac{\beta}{\sigma}dr_1 + \frac{\beta}{\sigma}(\sum_{s=2}^{t-1} dr_s)$$

We now make a key remark : the expected price change only induces a second order welfare change for the consumer. As is known from consumption theory[11]*, the welfare change, obtains, to the first order approximation, as the inner product of the price change and of the market exchange vector (the difference between the consumption and the endowment vector). As this latter vector is zero, the result obtains. Now, the above finding implies that :*

$$\sum_{1}^{+\infty} \beta^{t-1}(\frac{dC_t}{C*}) = 0$$

Let us compute the above expression :

$$\sum_{1}^{+\infty} \beta^{t-1}(\frac{dC_t}{C*}) = (\frac{1}{1-\beta})(\frac{dC_1}{C*}) + \frac{1}{\sigma}[\sum_{2}^{+\infty} \beta^t(dr_1 + (\sum_{s=2}^{t-1} dr_s))]$$

In the case $dr_s = dr_2, \forall s,$we have :

$$\sum_{1}^{+\infty} \beta^{t-1}(\frac{dC_t}{C*}) = (\frac{1}{1-\beta})(\frac{dC_1}{C*}) + \frac{1}{\sigma}[\sum_{2}^{+\infty} \beta^t[(dr_1) + \sum_{3}^{+\infty}(t-2)\beta^t(dr_2)]$$

Which, as $\sum_2^{+\infty} \beta^t = \beta^2/(1-\beta)$, $\sum_3^{+\infty}(t-2)\beta^t = \beta^3/(1-\beta)^2$, implies :

$$(\frac{dC_1}{C*}) = -\frac{\beta^2}{\sigma}(dr_1) - (\beta^3/\sigma)(1-\beta))(dr_2)$$

As in equilibrium $dC_1 = 0$, the result follows. ■

[11] The fact that we are in an infinite comodity setting does not modify the part of the theory we are soliciting.

6.3 "Eductive Stability" ; the core analysis.

As explained above, we implicitely assume that the model, as well as rationality, are CK. Also the monetary rule of the Central Bank (ϕ) is credibly committed and hence believed.

The initial CK restriction. As argued above the initial CK restriction has to be a hypothetical restriction on the state of the system. Here the state of the system is entirely defined, once the monetary rule is adopted, by the sequence of inflation rates. As the equilibrium inflation rate is Π^*, a natural local restriction on beliefs is that the inflation rate is between $[\Pi^* - \epsilon, \Pi^* + \epsilon]$.

The question is then : does such a belief trigger a collective mental process leading to the general conclusion that $*$ will emerge ?

The process under discussion takes place in period 1.

In order to understand this process, we now look at the following question : what will happen if in period 1, all agents believe that future inflation will be for ever $\Pi^* + \epsilon$?

The expected price path will then be $P_t^{'} = P_1(\Pi^* + \epsilon)^{t-1}, t = 2, ... + \infty$

The expected real interest rate between t and $t+1, t \geq 2$ will be :

$$\frac{(1 + \varphi(\Pi^* + \epsilon))}{\Pi^* + \epsilon}$$

i.e will approximately differ from $r^*$by :

$$(1/(\Pi^*)^2)[\varphi'\Pi^* - (1 + \varphi)]\epsilon$$

i.e

$$1/(\Pi^*)[\varphi' - 1/\beta]\epsilon$$

At period one, I assume that agents make plans contingent on the interest rate (they submit a demand curve). Then, their conditional inference of the nominal interest rate is : $\varphi(P_1/P_0^*)$.

Concerning their inference of the next period price P_2, [12], $P_2 = P_1(\Pi^* + \epsilon)$.

Hence the expected real interest rate is

$$[1 + \varphi(P_1/P_0^*)]/(\Pi^* + \epsilon)$$

[12] A different assumption on beliefs would be to see the expected price path as : $P_t^{'} = P_0^*(\Pi^* + \epsilon)^t, t = 2, ... + \infty$

so that in period 1, $P_2^e = P_0^*(\Pi^* + \epsilon)^2$.

This leads to slightly different results.

i.e, approximately, when writing the first period inflation rate $(P_1/P_0^*) = (\Pi^* - \epsilon')$,

:

$$(\varphi'\epsilon'/\Pi^*) - (1+\varphi)/(\Pi^*)^2\epsilon = (1/\Pi^*)(\varphi'\epsilon' - (1/\beta)\epsilon)$$

Putting $v = \varphi'$, we get the next lemma :

Lemma 2 *Under the just considered state of beliefs, the first period inflation rate is* $(\Pi^* - \epsilon')$,*where*

$$v\epsilon' = [(1/\beta) - (\frac{\beta}{(1-\beta))})(v - 1/\beta)]\epsilon$$

Proof. We apply the above formula

$$dr_1 = -\frac{\beta}{(1-\beta))}(dr_2)$$

,with

$$dr_2 = 1/(\Pi^*)[\varphi' - 1/\beta]\epsilon$$

and

$$dr_1 = (1/\Pi^*)(\varphi'\epsilon' - (1/\beta)\epsilon)$$

With $\varphi' = v$,we have :

$$v\epsilon' = [(1/\beta) - (\frac{\beta}{(1-\beta))})(v - 1/\beta)]\epsilon$$

■

We are now in a position to give our main result.

Proposition 6.

A necessary condition for "strong rationality" of the equilibrium is $(1/\beta) \leq v \leq (1/\beta)[(1/(2\beta-1)]$.

Reminding $1 + r^* = 1/\beta$,*the condition can also be written,* $(1+r^*) \leq v \leq \frac{(1+r^*)^2}{(1-r^*)}$

Proof. For "eductive stability" it must be the case that the initial belief is not self defeating. For that it must be the case that

$$-1 \leq [(1/\beta v) - (\frac{\beta}{(1-\beta))})(1 - 1/\beta v)] \leq 1$$

Take the inequality ≤ 1. It follows that :

$$(1/\beta v)(1-\beta+\beta)/(1-\beta) \leq 1 + (\frac{\beta}{(1-\beta))})$$

or

$$(1/\beta v) \leq 1$$

Take the inequality $-1 \leq []$. Then,

$$(1/\beta v)(1-\beta+\beta)/(1-\beta) \geq -1 + (\frac{\beta}{(1-\beta))}$$

or

$$(1/\beta v) \geq (-1+2\beta)$$

or

$$v \leq (1/\beta)[\frac{1}{(2\beta-1)}]$$

■

Indeed one conjecture that this necessary condition is sufficient, as soon as one specify the initial set of beliefs as avoiding "sweeping" beliefs (i.e alternating expectations of high and low inflation). In the sense of our general discussion of section 1, this is like choosing an appropriate topology for the neighbourhood of the steady state (sweeping beliefs being considered as "non close" to the initial one[13]). The proof would consist in showing that the initial beliefs induces a smaller deviation from the targeted inflation, not only at the first period but at any period, and then to iterate the argument using the CK assumption.

The result is striking : the range of $v = \varphi'$, insuring "eductive stability" is rather small : with β close to 1, the condition looks roughly as :

$$(1/\beta) \leq v \leq (1/\beta)[(1+2(1-\beta))$$

and for the sake of illustration with a high $\beta = 0,95$, this is roughly

$$(1,05) \leq v \leq (1,05)(1,1) = (1,15)$$

More generally, for small r*, the "window" for the reaction coefficient is, to the first order approximation $[1+r^*, 1+2r^*]$

[13]This is reminiscent of the distinction between C_0 and C_1 topology in the discussion of Section 3.

Hence the analysis suggests that standard Taylor rules are too reactive.

It is also particularly striking, but not surprising, that a plausible intuition within the determinacy viewpoint, i.e the equilibrium is more determinate, and in sense more expectationnaly stable whenever v increases, is plainly wrong here; there is a small window, above $1/\beta$, (and shrinking with β and vanishing when β tends to 1), for expectational stability.

7 Conclusion.

The conclusion is necessarily provisionnal, since an outsider's random walk in monetary models, (although starting from a hopefully well established base camp), has to be confronted with criticism. It has also to be enriched in order to develop an intuition somewhat missing in the present state of the author's understanding of the specialized issues that have been touched. This outsider's walk has however attempted to raise interesting questions for insiders and then will hopefully open new fronts of thinking.

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