# What Can Catastrophic Events Tell Us about Sustainability?

Can Askan Mavi[a*†]

December 16, 2018

[a] *CEE-M, Univ. Montpellier, CNRS, INRA, SupAgro Montpellier, Montpellier, France.*

**Abstract**

This paper aims to analyze the overlooked link between catastrophic events and sustainability through a limit cycle analysis. We use and extend a well known Calvo and Obstfeld (1988, Optimal Time-Consistent Fiscal Policy with Finite Lifetimes, Econometrica) framework in order to distinguish individual's and social planner's discount rates and show that Hopf bifurcation occurs at two critical values for individual discount rate, only if the economy is exposed to catastrophic event risk. This result is important because the role of individual discount rate on aggregate long-term dynamics has been overlooked in the literature. More importantly, the existence of limit cycles implies that consumption and natural resource stock are exposed to cycles in the long run, meaning that the path of utility does not conform to the prominent Sustainable Development criterion. Lastly, we analyze the economic reasons behind limit cycles and show that protecting the environment decreases the likelihood that limit cycles will occur.

*Keywords : Catastrophic damage, Hopf Bifurcation, Limit cycles, Sustainability, Mitigation.*

*JEL Classification : D81, O3, Q54, Q55*

## 1 Introduction

It is widely recognized that uncertain catastrophic events could cause large scale damages (Alley et al. (2003), Field et al. (2012)). Many studies have focused on decision-making regarding the exploitation policy of natural resources under uncertainty (Bretschger and Vinogradova (2017), Tsur and Zemel (1998, 2007, 2016), Clarke and Reed (1994)). In addition, some recent studies concentrate on determining the optimal environmental policy to deal with uncertainty. For this purpose, adaptation and mitigation policies and their implications under uncertainty are a major point of interest (Zemel (2015), Tsur and Zemel (2015), Mavi (2017)).

Apart from studies on uncertainty and resource exploitation, another branch of economics literature concentrates on the relationship between discounting and sustainability, which is a long-standing and important debate. The debate has been intensified in the context of climate change (Stern (2006), Weitzman (2007), Heal (2009)). Some of these studies are related to the role of individual time preferences (see Endress et al.

*Corresponding Author : Can Askan Mavi, CNRS, INRA, SupAgro, 2 Place Pierre Viala, 34060 Montpellier. e-mail : can-askan.mavi@supagro.fr

†This research is supported by the ANR project GREEN-Econ (Grant ANR-16-CE03-0005) and by MENRT/Ecole Doctorale ED465 Panthéon-Sorbonne.

(2014), Schneider et al. (2012), Marini and Scaramozzino (1995, 2008), Burton (1993)). The presence of individual time preferences in an economic model is appealing because the infinitely lived agent model (ILA model hereafter) is criticized for not considering consumer's preferences[1].

The articles cited above which incorporate individual discount rate are based on the seminal framework proposed by Calvo and Obstfeld (1988). The authors introduce individual time preferences (i.e. individual discount rate) in an overlapping generations model (OLG hereafter). Then, they find the aggregate consumption level of all generations at a given time. Once the aggregation is made, the model reduces to the representative agent framework. This framework has been used in environmental economics literature, including the above-cited papers to study some important topics, such as inter-generational equity [2] by the above-cited papers. However, these papers do not analyze the role of individual time preferences on long-term aggregate dynamics. This clearly introduces a dichotomy between the OLG part and ILA model, as the implications of individual discount rate on the long-term dynamics in the ILA model are unknown. One of the aims of this paper is to resolve this dichotomy and analyze in depth the effects of individual discount rate on aggregate dynamics.

How does the present study fit into the literature? On the one hand, studies investigating the long-term impacts of uncertainty on resource exploitation policies do not consider sustainability and intergenerational equity (see Bommier et al. (2015), Zemel (2015)). On the other hand, the strand of literature on sustainability and intergenerational equity does not consider uncertain events (see Marini and Scaramozzino (1995); Endress et al. (2014); Burton (1993)). To the best of our knowledge, the link between sustainability and catastrophic events is overlooked in the environmental economics literature. This paper aims to fill this gap.

In this paper, we have two important motivations: First, we aim to explain the role of individual preferences on the occurrence limit cycles (Hopf bifurcation). Second, we show that limit cycles do not conform to the prominent Sustainable Development criterion. Then, we argue that the Sustainable Development criterion should be revised by policymakers to encompass limit cycles. Otherwise, one should avoid these cycles from a normative point of view.

The contribution of this paper is twofold: First, by extending the Calvo and Obstfeld (1988) framework to account for uncertain events, we show that, for two critical parameter values for the individual discount rate, endogenous cycles (Hopf Bifurcation) arise in the economy in the long run. The mechanism behind limit cycles can be summarized as follows: On the one hand, the economy accumulates physical capital and creates waste. In this sense, the environment is used as a "sink" in the economy. This can be considered the economic goal. On the other hand, because the damage inflicted after a catastrophic event is proportional to the remaining natural resource stock after the event, the economy wishes to protect natural capital. This is the environmental goal. When it becomes difficult to steer between these conflicting goals, it may be optimal to cycle around the optimal steady state[3] (see Wirl (2004)).

What effect does catastrophic event probability have on limit cycles? Note that when there is no chance of a catastrophic event occurring, the above-mentioned trade-off between the economic and environmental goal disappears because the environment has only a proactive role, meaning that the utility from the environment

[1]When there is only a unique social discount rate, we cannot distinguish individual impatience from social planner's impatience level. This is one of the ethical objections to ILA model.

[2]Contributions made in intergenerational equity discussions argue that ILA framework as a utilitarian social welfare function corresponds to a different generation at each point of time (see Schneider et al. (2012)).

[3]Note that cycles around the steady state are optimal.

becomes effective only once the catastrophic event occurs. Therefore, individual preferences cannot cause limit cycles due to the absence of the trade-off between the economic and environmental goal.

To better understand our motivation for using an OLG model, some additional clarifications regarding the link between individual preferences and the occurrence of limit cycles are offered here. In fact, the existence of limit cycles is possible even without an OLG model. One can easily show that limit cycles take place in a representative agent model (see Wirl (2004)). In other words, the main source of the bifurcations is the above-mentioned trade-off, not the structure of the population. However, this is not to say that individual discount rate does not matter for bifurcations. Individual discount rate is important in the sense that it can make it more or less difficult to steer between the economic and environmental goal. For some levels of individual discount rate, it becomes difficult to decide between the environmental and economic goal. It is this difficulty, which depends on individual discount rate, that makes cycles appear. Therefore, we feel it is important to focus on individual discount rate in this study.

Indeed, because our aim is to focus on the importance of individual preferences regarding the sustainability of the economy, we think it is helpful to use the well-known Calvo and Obstfeld (1988) framework, as it allows us to distinguish individual discount rate from the social planner's discount rate.

One may argue that the trade-off between the economic and environmental goal, which is the source of limit cycles, is common in growth models that contain environmental aspects. In such a framework, the occurrence of limit cycles can be understood through the growth condition (i.e., the state grows but belows the social planner's discount rate) as shown in Wirl (2004)[4].

Indeed, limit cycles have been studied extensively in environmental economics. Wirl (1999, 2004) and Bosi and Desmarchelier (2016, 2017) studied the existence of limit cycles in models with representative agent frameworks. None of their studies link limit cycles to equity across generations and sustainability as described by the Sustainable Development criterion.

At this point, the question to be addressed is: What are the implications of limit cycles regarding sustainability? Note that the Sustainable Development criterion requires that the utility of consumption has a non-decreasing path (i.e $\frac{du(C(t))}{dt} \geq 0$). If the economy is exposed to limit cycles due to the trade-off between the environmental and economic goal and/or due to the complementarity of preferences, the Sustainable Development criterion is not respected, as the utility and the dynamics of natural resource stock exhibit cyclical behavior in the long run.

Secondly, contrary to the Calvo and Obstfeld (1988) framework and the articles using this framework, we show that individual discount rate can change the stability properties of the model when the economy is exposed to catastrophic event uncertainty. This result disproves the conventional statement that aggregate dynamics are governed solely by the social planner's discount rate (see Endress et al. (2014), Schneider et al. (2012), Marini and Scaramozzino (1995, 2008), Burton (1993)). Indeed, we show that individual discount rate has an important role regarding long-term aggregate dynamics and, hence, the sustainability of an economy.

Because the first part of the model is an OLG model, there is an intra-generational allocation of consumption which is stable over time. We also show that intra-generational equity can conform to sustainability, as we show that a more even allocation of consumption between generations ensures a stable equilibrium in the long run.

[4]We thank an anonymous referee for this remark.

As stated above, we argue that there are two options: Either a policymaker should revise the Sustainable Development criterion to encompass limit cycles, or one should avoid these cycles in order to respect this criterion. One can argue that the second option is better, as sustainability and intergenerational equity are generally perceived as normative inquiries (Solow (2005, 2006)). As a result, a social planner who pays attention to sustainability and intergenerational equity should seek to avoid limit cycles. We show that the social planner can avoid limit cycles through an environmental policy that aims to protect the environment. This is made possible by the fact that a higher natural resource stock implies a lower marginal utility of consumption. As a result, different levels of individual discount rate are expected to have a relatively small effect on the trade-off between the economic and the environmental goal. Consequently, it would be less likely that an economy would become trapped in limit cycles in the long run.

The remainder of the paper is organized as follows: Section (2) presents the benchmark model and explains all economic mechanisms behind the limit cycles in detail. Section (3) focuses on thresholds and multiple equilibria. Section (4) explains the model with environmental protection that can avoid limit cycles. The last section (5) concludes the paper.

# 2 Model

We use the Calvo and Obstfeld (1988) framework to answer the questions that we posed in the introduction. Our model is a mix of Weil (1989) and Calvo and Obstfeld (1988). Weil (1989) proposes a model with infinitely-lived families in which the population grows. In our paper, the economy consists of many families whose size decreases non-stochastically over time at a constant pace $h$ so that the population remains constant at each time $t$. Indeed, the members within the family die with some probability, but there is no uncertainty about the size of the family that vanishes when time $t$ tends to infinity. The reason to make this assumption of constant population, which differs from what Weil (1989) describes, is as follows: Because Calvo and Obstfeld (1988) discounts the lifetime utilities of representative agents (or families) of each cohort to be born and representative agents currently alive, it is necessary to have a constant population with decreasing size over time. Otherwise, there would be an infinite number of families already alive at date 0. In our framework, at each instant of time $t$, there is a newly-born family of size 1. As of time $t$, the size of the family is $e^{-h(t-s)}$. Then, at each instant of time $t$, there is $\int_0^\infty e^{-h\tau} d\tau$ *unity* of family. In our framework, it is more appropriate to use the term "unit" than "number" for families. At each time $t$, there is a constant $\int_0^\infty e^{-h\tau} d\tau$ unit of family at the aggregate level. However, there is an infinite number of families of unit between 0 and 1[5].

Differently from Calvo and Obstfeld (1988), *the social planner discounts the utility of families but not the utility of a representative individual facing a certain probability of death.* This reasoning is based on the fact that it is difficult to deal with the death probability of individuals when the horizon of the maximization problem of a representative individual is limited to the catastrophic event date (let this date be denoted as $T$). Because the age of death is a random value between $[0; \infty]$[6], an individual can die before or after a catastrophic event. Therefore, one should split the expected utility of a representative individual[7] in two parts, the first of which considers the case where the agent dies before $T$ and the second of which considers the case where the agent dies after $T$. Unfortunately, this yields some tedious calculations and renders the

[5] For example, it is legitimate to say that there is 0.005 unit of family of some age $\tau$.
[6] The age of death can also be bounded by an arbitrary value. See D'Albis and Véron (2011) and Bommier and Lee (2003).
[7] The expectancy term is for the death probability.

attainment of closed-form solutions impossible. However, when the social planner considers the utility of a given family, it is unambiguous that the family does not vanish even though there are people who die within the family (before or after the catastrophic event). Any given family disappears only when time reaches infinity.

Note also that once the social planner allocates the consumption to families of different ages at each time, she does not care about the allocation of the consumption within family. We implicitly assume that each member of the family is identical.

## 2.1 The economy's structure

A family born at time $b$ lives infinitely with a subjective discount rate of $\beta > 0$ which is the individual discount rate of family members[8]. The family maximizes its utility until the unknown catastrophic event date $T$. For the sake of simplicity, we assume a single-occurrence catastrophic event as this is a widely used assumption in the literature (see Clarke and Reed (1994); Polasky et al. (2011); Bommier et al. (2015)). That is to say, once the catastrophic event occurs, there will be no further catastrophes. Here, we assume that the catastrophic event at date $T$ leads to the complete destruction of the economy or a disruption to the economy such that we cannot plan for after date $T$ (see Schumacher (2011)).

Examples of single-occurrence events imply that naturally arising pathogens, organisms, like the destructive biotechnology products, can devastate ecosystem functions. Other illustrative examples include crop failures, human overpopulation, and non-sustainable agriculture.

After the single-occurrence doomsday event has taken place, we assume that consumption reduces to a constant level $c_{min}$ (see Tsur and Zemel (2016)). The subsistence level of consumption is supposed not provide any utility to individuals (i.e $u(c_{min}) = 0$). After the catastrophic event, the economy is exposed to a level of catastrophic damage which is proportional to the resource stock level $S$ at the time of the catastrophic event $T$. The consequences of the catastrophic event are described by the post-event value $\varphi(S)$ , which we discuss throughout the remainder of the paper.

A family born at the birth date $b$ enjoys $c(b,t)$ until the catastrophic event date $T$. The size of the family declines non-stochastically and constantly at rate $h$ . The family exists even after the catastrophic event $T$ but consumption is reduced to $c_{min}$. The family (*and not the representative individual of the family*) maximizes the utility,

$$V_b = \int_b^T u(c(b,t))\, e^{-(\beta+h)(t-b)} dt + \int_T^\infty \underbrace{u(c_{min})}_{=0} e^{-(\beta+h)(t-b)} dt \tag{1}$$

where the second term in (1) is equal to zero. It is important to note that $h$ is the constant death probability for individuals within the family (see Blanchard (1984)) but at the family scale, it represents the rate of decrease of the size of the family. In our framework, the term $e^{-h(t-b)}$ plays a similar role as it does in models where population growth is taken into account in the utility function[9]. Nonetheless, in our case, the population within the family decreases, rather than increases, and the family's discount rate is

[8]All the members of the family have $\beta$ as a discount factor.

[9]In a classical Ramsey growth model, the social planner maximizes $\int_0^\infty u(C(t))\, e^{-\rho t} dt$ where $C(t)$ is the aggregate consumption level. Assume that the population grows at rate $n$, if we want to express the maximization program by variables per capita, we can simply write $\int_0^\infty L(t)\, u(c(t))\, e^{-\rho t} dt = \int_0^\infty u(c(t))\, e^{-(\rho-n)t} dt$ .

then augmented by the rate of decrease. In other words, $c(b,t)$ can be understood as the consumption per capita of the family born at time $b$.

## 2.2 The planner's program

The social planner's objective is composed of two components. The first is an integral for the lifetime utilities of the families to be born, measured from the birth date $b$ until the catastrophic event date $T$. The second is an integral of utilities of the families currently alive. It is assumed that the social planner discounts families at a rate of $\rho > 0$. After the catastrophic event, the state of the economy is described by the post-event value function $\varphi(S)$, which depends on the natural resource stock. After the catastrophic event, the economy is exposed to an amount of damage which is proportional to the level of natural stock. The welfare at time $t = 0$ is

$$W(0) = \int_{-\infty}^{0} \left\{ \int_{0}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} dt \right\} e^{-\rho b} db + \int_{0}^{T} \left\{ \int_{b}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} dt \right\} e^{-\rho b} db + e^{-\rho T} \varphi(S(T)) \tag{2}$$

All generations currently alive and to be born in the future are treated in a symmetric fashion to ensure the time-consistency of the problem. It follows that all families are discounted back to their birth dates rather than to the beginning of the planning horizon. After changing the order of the integration (see Appendix (A.1) for details), we have

$$W(0) = \int_{0}^{T} \left\{ \int_{-\infty}^{t} u(c(b,t)) e^{-(\beta+h)(t-b)} e^{\rho(t-b)} db \right\} e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{3}$$

The flow of utility for the social planner on any date $t$ before the catastrophic event date $T$ is the integral over all families (generations) of instantaneous utilities discounted by the adjusted individual discount factor $(\beta + h)$ times the social planner's discount rate $e^{\rho(t-b)}$. The integral over all these utility flows, discounted by $\rho$ until the catastrophic event date $T$, with the post-value function describing the state of the economy after the catastrophic event gives $W(0)$.

To solve the social planner's program, we slightly reformulate (3). After a change of variables from the birth date $b$ to age $\tau = t - b$ (see Appendix (A.1) for details), $W(0)$ becomes

$$W(0) = \int_{0}^{T} \left\{ \int_{0}^{\infty} u(c(t-\tau,t)) e^{-(\beta+h-\rho)\tau} d\tau \right\} e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{4}$$

The flow of utility on any date before $T$ is the integral over all families of instantaneous utilities discounted by their "family" discount rate and the social planner's discount rate. The integral of these utility flows, until $T$ with the post-event value, yields $W(0)$. The maximization problem of the social planner is solved in two stages. In the first stage, the social planner allocates the consumption between families of different ages for a given $C(t)$. In the second stage, the social planner chooses the aggregate consumption path.

It is important to note what happens after the catastrophic event. Following Tsur and Zemel (2016), we write the post-event value with the corresponding damage function $\psi(S)$

$$\varphi(S) = (u(c_{min}) - \psi(S)) \tag{5}$$

where the damage function $\psi(S)$ is described as

$$\psi(S) = \bar{\psi}(\omega_1 - \omega_2 logS) \tag{6}$$

where $\omega_1$ is the unrecoverable part of the penalty. The parameter $\omega_1$ is supposed to be high enough that some part of the damage is irreversible. The use of the penalty function implies that the more the natural capital stock is protected, the less the economy suffers from the penalty[10].

## 2.3 A static problem: the utility of families

Before finding the social optimum, one should find first the optimal allocation of consumption between families (or generations). Then, after aggregating the consumption of each family, we can take the expectations of the social planner's program in order to take into account the distribution of $T$ since the occurrence date of the catastrophic event is uncertain.

Define aggregate consumption

$$C(t) = \int_0^{\infty} c(t-\tau, t) e^{-h\tau} d\tau \tag{7}$$

Note that the consumption of a family is proportional to its size. Therefore, it is necessary to take decreases in the size of a family into account. Given the level of the aggregate consumption $C(t)$ at time $t$, the social planner should allocate consumption across families. The indirect utility is

$$U(C(t)) = \max_{\{c(t-\tau,t)\}_{\tau=0}^{\infty}} \int_0^{\infty} u(c(t-\tau,t)) e^{-(\beta+h-\rho)\tau} d\tau \tag{8}$$

subject to $\int_0^{\infty} c(t-\tau,t) e^{-h\tau} d\tau \leq C(t)$ (see Appendix (A.2) for the static optimization program). The following graphic illustrates the idea of optimal distribution of the consumption across families:

[10]Tsur and Zemel (1998) use a similar specification for the penalty function. However, authors have a pollution stock instead of the natural capital stock. In this case, the penalty rate increases with respect to the pollution stock after the catastrophic event.

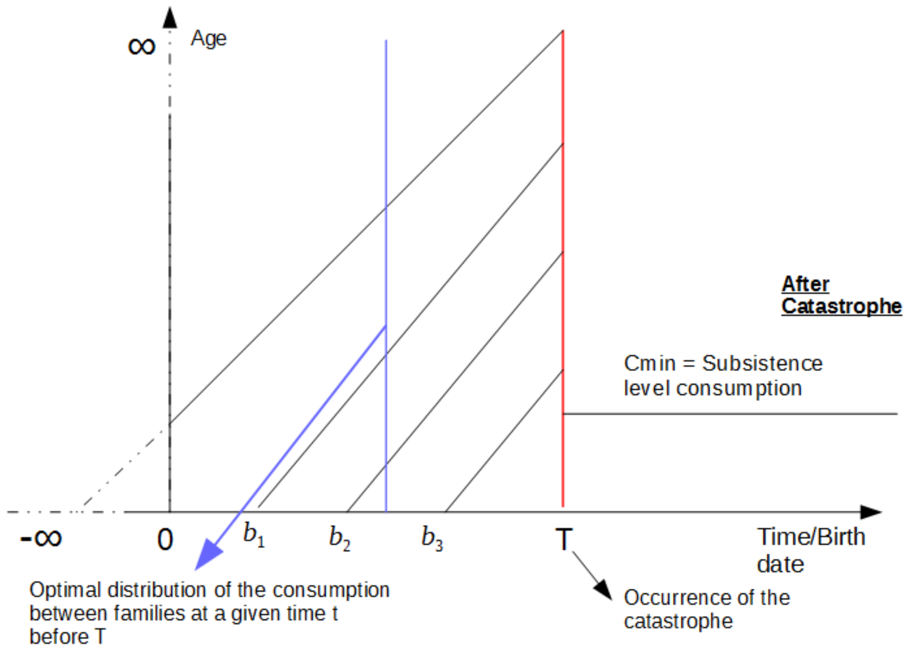


Figure 1: Allocation of the consumption across families

After the aggregation of consumption (see Appendix (A.2)), the social planner's program (4) becomes

$$W(0) = \int_0^T U(C(t))\, e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{9}$$

Once the social planner allocates consumption across families, we need to find the utility of the aggregate level of consumption. With the use of a CRRA-form utility function, we arrive at (see Appendix (A.2) for details.)

$$U(C(t)) = \frac{Z_1^{\sigma} C(t)^{1-\sigma} - Z_2 c_{min}^{1-\sigma}}{1-\sigma} \tag{10}$$

where $Z_1 = \left(\frac{\sigma}{\beta+\sigma h-\rho}\right)$ and $Z_2 = \frac{1}{\beta+h-\rho}$ are the aggregation terms, which include individual discount rate.

### 2.4 A dynamic problem: social optimum

To solve the maximization problem using standard optimal control techniques, one should find the deterministic equivalent of the objective function (9) which is a stochastic expression, as the catastrophic event date $T$ is unknown (see Kamien and Schwartz (1978)). Therefore, we take the expectations of (9) (see Appendix A.3 for details), after which we have

$$W(0) = \int_0^{\infty} \{U(C(t)) + \theta\varphi(S(t))\}\, e^{-(\rho+\theta)t} dt \tag{11}$$

where $\theta$ is the exogenous catastrophic event probability. The economy is subject to two constraints, one for physical capital accumulation $K$ and one for natural resource stock $S$ which regenerates by a logistic growth function. Note that natural capital accumulation is negatively affected by physical capital accumulation, which is similar to Wirl (2004) and Ayong Le Kama (2001). In this sense, nature is considered a sink for waste coming from physical capital. Essentially, this feature creates a trade-off between capital accumulation and its negative effects on nature, which will be analyzed in depth in this study. The social planner maximizes (11) subject to

$$\begin{cases} \dot{K}(t) = f(K(t)) - \delta K(t) - \int_0^\infty c((t-\tau, t)) e^{-h\tau} d\tau \\ \dot{S}(t) = G(S(t)) - \gamma f(K(t)) \end{cases} \tag{12}$$

The social planner maximizes the program (11) subject to physical and natural capital accumulation constraints (12). The current-value Hamiltonian for maximizing $W$ is

$$\mathcal{H} = U(C) + \theta\varphi(S) + \lambda(f(K) - \delta K - c) + \mu(G(S) - \gamma f(K)) \tag{13}$$

The first order conditions and dynamics of the economy are as follows:

$$\begin{cases} U_C = Z_1^\sigma C^{-\sigma} = \lambda \\ \dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} \\ \dot{S} = G(S) - \gamma f(K) \\ \dot{\lambda} = (\rho+\theta)\lambda - (f_K - \delta)\lambda + \mu\gamma f_K \\ \dot{\mu} = (\rho+\theta)\mu - \mu G_S - \theta\varphi_S \end{cases} \tag{14}$$

The regeneration of the environment and the production are given by

$$\begin{cases} G(S) = (1-S)S \\ f(K) = AK \end{cases} \tag{15}$$

By using the functional forms, the steady-state of the economy can be written as a function of the renewable resource stock $S$:

$$\begin{cases} K^*(S) = \frac{(1-S^*)S^*}{\gamma A} \\ \lambda^*(S) = \frac{\gamma A \mu^*(S)}{((A-\delta)-(\rho+\theta))} \\ \mu^*(S) = \frac{\bar{\psi}\omega_2\theta}{S^*((\rho+\theta)-(1-2S^*))} \end{cases} \tag{16}$$

**Proposition 1.** *In an economy with a catastrophic event probability, a first Hopf bifurcations occurs at the critical value of individual discount rate* $\beta = 0.97362$ *and the second Hopf bifurcation takes place at* $\beta = 1.16784$[11].

Proof. See Appendix (A.4)

It is important to determine the economic reasons behind the occurrence of limit cycles. On the one hand, the more that natural capital stock is conserved, the lower the penalty rate is (see Tsur and Zemel (1998)). It follows that the disutility resulting from the penalty decreases with respect to the natural resource stock.

[11]When we choose $\beta$ as a bifurcation parameter, we use following set of parameters; $\sigma = 2$, $\rho = 0.85$, $h = 0.03$, $A = 1$, $\omega_2 = 0.1$, $\delta = 0.001$, $\gamma = 0.0001$, $\bar{\psi} = 0.1$, $\theta = 0.1$. For $\beta = 0.97362$ *and* $\beta = 1.16784$, the Lyapunov numbers are -0.0116453 and -0.0116904 respectively. This means that two Hopf bifurcations are supercritical.

Therefore, the environment plays an important role when a catastrophic event takes place. This creates an incentive to protect the environment, which can be referred to as the *environmental goal*. On the other hand, the resource stock in this model is used as a sink for waste coming from physical capital accumulation[12]. This represents the *economic goal*. When trying to decide which of these two opposing strategies to favor becomes too difficult, limit cycles arise (see Wirl (2004)).

One could argue that this trade-off is usual in growth models involving the environment and does not necessarily cause limit cycles. Indeed, limit cycles and unstable behavior in an economy with waste accumulation are due to the conflicting economic and environmental goals coupled with *the complementarity of preferences over time*[13] (see Dockner and Feichtinger (1991)). To understand complementarity over time, suppose an incremental increase of consumption is implemented at time $t_1$. If this incremental increase shifts the preferences of consumption from $t_3$ to $t_2$ or vice versa [14], there is complementarity of preferences over time. If an incremental change of consumption at time $t_1$ does not shift the preferences of other dates, this means that preferences are intertemporally independent (see Koopmans (1960)).

The model exhibits a complementarity of preferences over time when there is waste coming from physical capital accumulation and catastrophic event probability (see Appendix (A.7)). To understand the implications of complementarity regarding the occurrence of limit cycles, consider an increase of consumption near date $t_1$ which shifts a part of the consumption of date $t_3$ to date $t_2$. At time $t_1$, physical capital accumulation decreases. It follows that the economy accumulates less waste due to there being a lower degree of physical capital accumulation and this leads to a higher natural capital stock. Contrarily, because consumption decreases at date $t_3$ and the economy accumulates more physical capital. Then, more waste accumulates, which harms the environment, and so on. From this mechanism, we can understand the intuition behind optimal limit cycles. Due to complementarity over time, natural resource stock increases and decreases regularly on consecutive dates.

Note that this mechanism does not hold if there is no chance of a catastrophic event occurring or of any waste stemming from physical capital accumulation. We show that, when waste rate $\gamma$ or catastrophic event probability $\theta$ is equal to zero, the complementarity over time vanishes (see Appendix (A.7)). We can also remark that when there is no waste in the economy, the dynamic system (14) reduces to a block recursive system of $(K, \lambda)$ and $(S, \mu)$ which admits a saddle path equilibrium. It is worthwhile to note that limit cycles occur when the synergistic between control and state and/or between states is strong (Wirl (1992)). In the model, it is evident that waste accumulation creates a strong link between two state variables.

Given the separability between control and state variables, another explanation for the occurrence of limit cycles can be offered through the growth condition $\rho + \theta > G_S$ as in Wirl (2004)[15]. We show that the growth condition is necessary for limit cycles to exist (see equation (A.26) in the Appendix (A.4)).

The questions to be addressed are why and how individual preferences cause limit cycles and unstable behavior in an economy. It is evident that individual discount rate $\beta$ is crucial to explain cyclical behavior in an economy, as it changes the difficulty of navigating between the economic and environmental goal. This can be observed through the utility of aggregate consumption, which depends on individual discount.

[12]See Ayong Le Kama (2001) and Wirl (2004) for a similar model.

[13]Dockner and Feichtinger (1991) and Ryder and Heal (1973) show that limit cycles and unstable behavior can stem from interdependent preferences over time.

[14]i.e.. adjacent complementarity and distant complementarity vice versa.

[15]I am grateful to an anonymous referee for pointing this out.

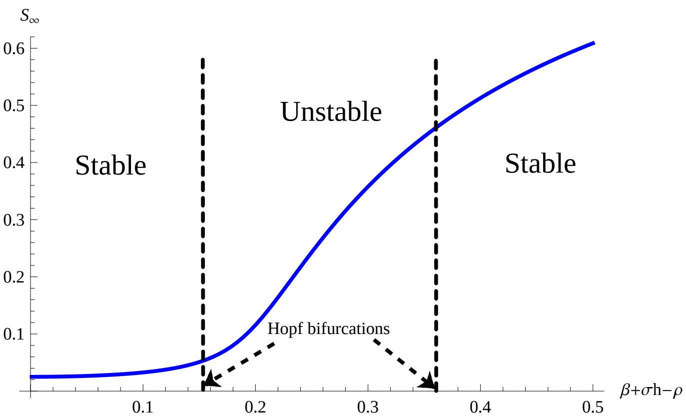


Figure 2: Steady state levels of natural resource stock with respect to the difference between individual discount rate $\beta$ and the discount rate of social planner $\rho$ (dashed lines showing bifurcation points)

Though we try to use simple functional forms, it is always hard to find the critical value analytically for the bifurcation parameter. (see Wirl (1999, 2004))) As in Wirl (2004), we refer to a numerical analysis to study cases in which bifurcation is a possible outcome[16]. Figure (2) shows the steady state values of natural resource stock with respect to individual discount rate $\beta$.

When $\beta = 0.973625$ and $\beta = 1.167842$, we have a pair of purely imaginary eigenvalues where a Hopf bifurcation occurs. Limit cycles are shown to be stable at low and high levels of individual discount rate $\beta$[17]. As mentioned above, it is possible to understand why limit cycles and unstable spirals take place but we are unable to offer an exact economic explanation why Hopf bifurcations take place at two different critical values [18].

According to the Figure (2), the economy becomes more conservative to exploit natural resources when the individual discount rate becomes higher than the social planner's discount rate. This can be explained by the fact that the social planner shifts consumption to younger families that have longer lifespans. A higher individual discount rate causes decreases in the marginal utility of consumption and in the shadow price of the natural resource stock, which is followed by a lower degree of exploitation of natural resources in the long run (see (16)).

The system is stable when the difference between the individual and the social planner's discount rate is close to zero or when this difference is sufficiently high. However, when there is only a modest difference between the individual and the social planner's discount rate, the difficulty regarding the decision between the economic objective and the environmental goal causes limit cycles.

**Definition 1.** *A path of utility respects the Sustainable Development criterion if* $\frac{d}{dt}\left(U\left(C\left(t\right)\right)\right) \geq 0$.

[16]For the numerical exercise in the rest of the analysis, we use the relaxation algorithm proposed by Trimborn et al. (2008). The method consists of determining the solution by an initial guess and improves the initial guess by iteration. Since the iteration improves the solution progressively, it relaxes to the correct solution.

[17]For $\beta = 0.925644$ and $\beta = 0.974355$, the Lyapunov numbers are -0.011645276 and -0.0116452956 respectively. This means that model shows a sub-critical Hopf bifurcation at two different critical individual discount rate $\beta$.

[18]See Wirl (1992, 1994) for a detailed discussion regarding the difficulty of giving intuitive economic explanations for bifurcation points.

The Sustainable Development criterion states that the utility of consumption should follow a non-decreasing or at least a constant path in order to ensure the sustainability of the economy. This means that limit cycles for consumption violate sustainability criterion since consumption decreases at some moments of time $t$. Then, Sustainable Development criterion is violated in the domain $\beta \in [0.973625.., 1.167842..]$.

An important point is that the model reduces to a representative agent model during the second stage. Consequently, we cannot discuss the intergenerational equity as in Schumacher and Zou (2008) for aggregate dynamics because we do not know the length of the life cycle of a generation along time $t$. For example, each generation can see its consumption decrease or increase in the same way. In such a case, the equity between generations can be respected. For this reason, we focus on the link between limit cycles and the sustainability notion in our analysis.

**Remark 1**. *When the social planner treats all generations equally (i.e. $\beta = \rho$), the sustainability criterion is respected.*

As seen in Figure (2), the way in which a social planner allocates consumption across different generations impacts the long-term dynamics (i.e., sustainability) of an economy. Treating generations equally (i.e $\beta = \rho$) ensures sustainability in the long run, as the economy admits a stable equilibrium. This shows a complementarity between intra-generational equity and sustainability. However, when the unequal consumption allocation across families (i.e $\beta \neq \rho$, see two vertical dashed lines where the Hopf bifurcations occur) is moderate, the economy traps in limit cycles which compromise sustainability. In addition, when the social planner distributes the consumption across families in a very uneven manner[19], the economy is stable.

Figure (3) shows limit cycles for given parameters in a phase diagram with a plane $(K, S)$[20]

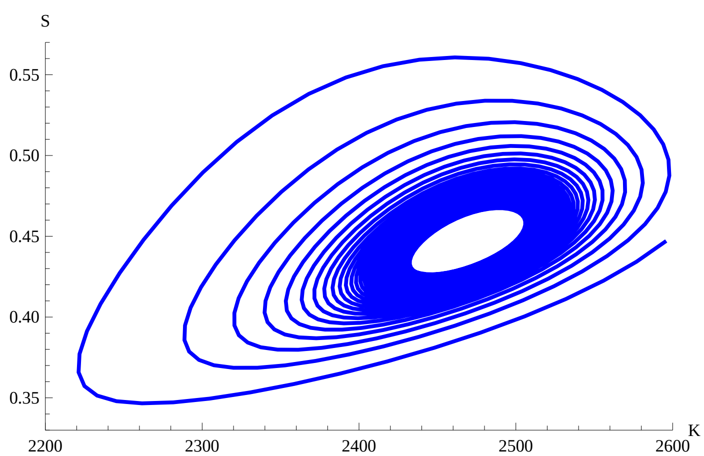


Figure 3: Limit cycles on a phase plane $(K, S)$ with bifurcation parameter $\beta$

A closer intuitive explanation of the mechanism behind limit cycles in this study is proposed by Heal (1982) and Bosi and Desmarchelier (2016) based on the *compensation effect*. Assume that an economy is at a steady state at a given date $t$, and assume that natural capital stock $S$ [21]decreases exogenously. The

[19]This is when the difference between $\beta$ and $\rho$ is big.

[20]The phase diagram and dynamics of aggregate consumption are plotted for the second critical bifurcation point where $\beta = 1.167842$.

[21]Note that Bosi and Desmarchelier (2016) use pollution stock instead of natural capital stock in their model.

degradation of natural capital pushes agents to increase their consumption, as they would like to compensate for the disutility due to the decrease in natural capital stock. It follows that capital accumulation decreases as well, which increases natural capital as there is less waste, and so on; therefore, deterministic cycles arise. Because the objective function in our specification is separable from consumption and natural resource stock, we are unable to justify the limit cycles with a compensation effect. However, we show that even with a non-additive objective function, limit cycles may exist.

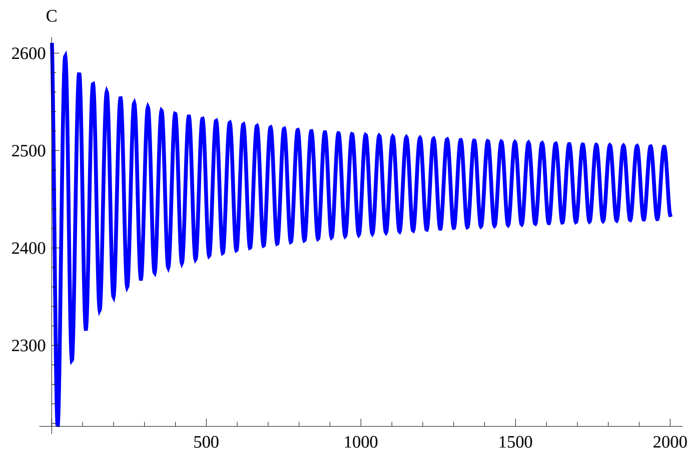


Figure 4: Limit cycles across time for consumption

The model stipulates that the level of natural resource stock and the occurrence of a catastrophic event are not the only factors that can violate the Sustainable Development criterion; even individual preferences can compromise sustainability. Figure (4) shows that at the first bifurcation point, the utility is exposed to cycles and does not converge to a stable point. From a normative point of view, a social planner that pays attention to sustainability should avoid any path that leads to bifurcations. We establish possible policies that could promote the evasion of limit cycles in 4.

#### 2.4.1 The role of the catastrophic event probability

What is the role of catastrophic event probability regarding the limit cycles? To answer this question, we choose $\theta$ as a bifurcation parameter. At $\theta \in [0.0529; 0.0919]$, non-monotonic behavior is exhibited. When the catastrophic event probability is lower than $\theta = 0.0529$ and $\theta = 0.0919$ respectively, the economy admits a saddle-path equilibrium.

Intuitively, one may say that a very high catastrophic event probability would push a society to be more precautionary about the environment as the marginal benefit of the environmental stock increases. Consequently, there would be less physical capital accumulation in such a scenario.

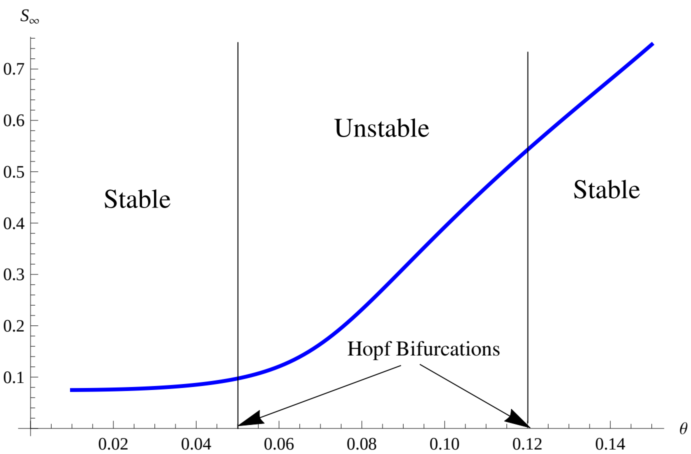


Figure 5: Steady state levels of natural resource stock with respect to catastrophic event probability $\theta$

On the contrary, when there is a relatively low probability of a catastrophic event occurring, a society becomes indifferent to environmental concerns and uses the environment as a sink. In short, when a social planner faces a catastrophic event probability that is somewhere between "too high" and "too low", she cannot make a firm decision between these two strategies. Consequently, it is optimal to cycle around the steady state. As mentioned before, we also show that when the economy does not face any probability of a catastrophic event occurring, the complementarity of preferences over time vanishes (see Appendix (A.7)).

What happens if there is no catastrophic event risk? We analyze an economy without catastrophic event risk to show that the probability of a catastrophic event occurring has important implications for the long-term behavior of the economy. For this purpose, we set the catastrophic event probability to zero. This version of the model is shown to always admit a saddle path stable equilibrium in the long run (see Proposition 1 in the Appendix (A.5)).

**Proposition 3.** *(a) In an economy with no probability of a catastrophic event, individual discount rate $\beta$ has no effect on the stability properties of the economy.*

*(b) The economy always admits a saddle path equilibrium when there is no chance of a catastrophic event occurring.*

*Proof. See Appendix (A.5).*

This result is quite plausible when one thinks about the absence of the trade-off between decreasing the effect of natural stock on penalty rate (environmental goal) and its use as a sink for waste (economic goal). As such, catastrophic event probability plays a crucial role by changing the marginal utility, which, in turn, changes the above-mentioned trade-off between capital accumulation and environment protection. The aggregate dynamics in the economy with no catastrophic event probability are illustrated by the following graphics[22]:

[22]We use following set of parameters ; $\sigma = 2$, $\rho = 0.85$, $h = 0.03$, $A = 1$, $\omega_2 = 0.1$, $\delta = 0.001$, $\gamma = 0.0001$, $\bar{\psi} = 1$,$\beta = 0.83$

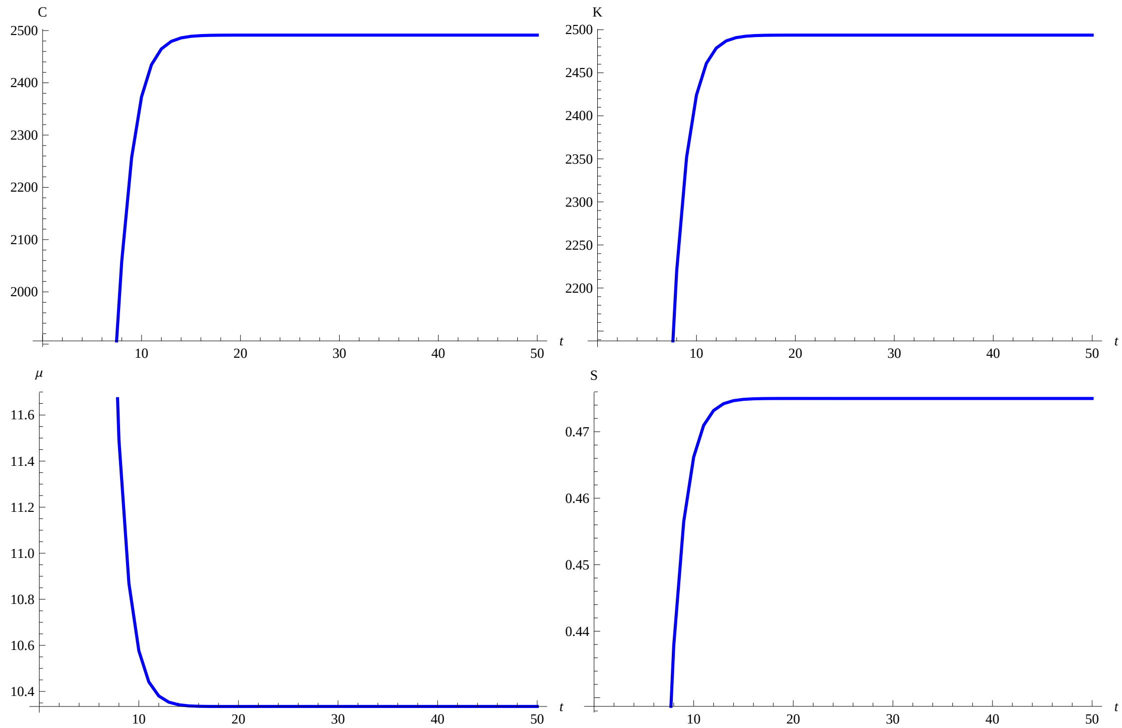


Figure 6: Aggregate dynamics without the catastrophic event probability

As expected, when there is zero probability of a catastrophic event occurring, the economy displays monotonic behavior and converges to a stable steady state in the long run. An important aspect of this version of the model is that a social planner does not face a trade-off between the environmental and economic goal. She considers only the negative effects of capital accumulation on the environment. In turn, the environment simply serves as a sink and does not represent any amenity value. In other words, the economy does not face any cyclical behavior either during the transition to a steady state or in the long run.

# 3 Thresholds and multiple equilibria

As shown in Wirl (2004), in an economy with multiple equilibria, a threshold separates the domains of attraction of two stable steady states in the phase plane of the states. This threshold can be called an indifference curve, along which are two optimal paths about which the decision-maker is indifferent. The existence of a threshold requires an unstable steady state, which is stable for a one-dimensional manifold in the state space in a concave framework.

First, we show an economy with multiple equilibria. Using the equations in (14), we find a polynomial equation that serves to find the different steady states of the economy.

$$G\left(S\right)=\left(1-S\right)S-\frac{\gamma AZ_{1}}{A-\delta}\left(\frac{\gamma A\theta\omega\bar{\psi}}{S\left(\left(\rho+\theta\right)-\left(1-2S\right)\right)\left(\left(A-\delta\right)-\left(\rho+\theta\right)\right)}\right)^{-\frac{1}{\sigma}} \tag{17}$$

It is easy to see that $\lim_{S\to 0}G\left(S\right)=0$ and $\lim_{S\to 1}G\left(S\right)=z<0$, where $z$ is an arbitrary negative value. The numerical illustration of equation (17) is as follows[23]

[23]We use the following parameter values: $\sigma=3$, $\rho=0.5$, $h=0.085$, $A=1.05$, $\omega_2=0.05$, $\delta=0.01$, $\gamma=0.08$, $\bar{\psi}=23500$,

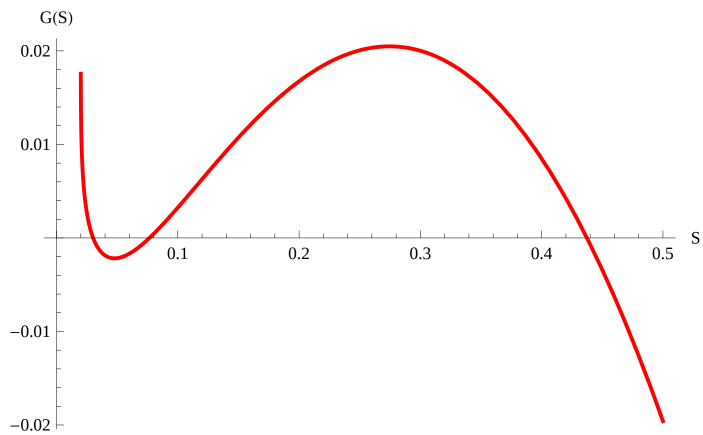


Figure 7: An economy with multiple equilibria

Figure (8) shows the natural capital on the abscissa axis and the physical capital stock on the ordinate axis. Because we have a logistic reproduction function, $K = \frac{G(S)}{\gamma A}$ determines a set of possible equilibrium states. In this example, we have three steady states $S_\infty = 0.0301717, 0.0766351, 0.437338$ (and $K_\infty = 0.34835, 0.842406, 2.92945$ respectively). Of these three steady states, the middle one is unstable but is optimal nonetheless. This means that if the economy starts at this point, it is optimal if it stays there forever.

As mentioned above, there is a one-dimensional stable manifold[24] that separates the domain of the other two stable steady states. This one-dimensional manifold separates the domains of attraction of the steady state of low ($S_\infty = 0.0301717$, $K_\infty = 0.34835$) and of high ($S_\infty = 0.437338$, $K_\infty = 2.92945$) natural capital.

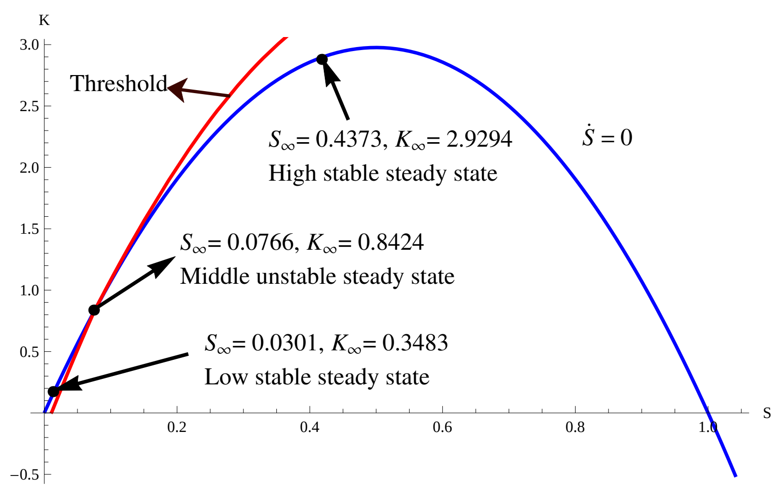


Figure 8: Multiple equilibria and threshold

Figure (8)[25] shows that an economy with any initial conditions $S_0, K_0$ to the right of the threshold converges to the equilibrium with high natural capital stock ($S_\infty = 0.437338$, $K_\infty = 2.92945$) . For all initial

$\beta = 0.5$

[24] Similar to Wirl (2004), we find the threshold by integrating backwards in time.

[25] Since the aim of the diagram is to focus on the threshold (i.e, the indifference curve)) that separates the domains of attraction of two stable steady states, for the sake of space, we do not show the optimal trajectories converging to low and to high steady states.

conditions located at the left of the threshold, the economy converges to the equilibrium with low natural capital stock ($S_\infty = 0.0301717$, $K_\infty = 0.34835$). When the economy is located at the threshold, converging to either of the two stable steady states gives the same payoff. As discussed in Wirl (2004), it is plausible that the threshold is upward-sloping. Suppose that the economy is located at the threshold. In this case, increasing the initial stock of physical capital leads to more waste, thus further degrading the natural capital. Consequently, the economy enters the attraction basin of the lower natural capital for increasing $K_0$ from any point at the threshold.

Limit cycles and thresholds are distinct phenomena. However, they can coexist in a model that contains multiple equilibria. In our model, we check whether limit cycles exist when there is a multiple equilibria economy. For the low equilibrium, the determinant and the sum of principal minors of dimension two are positive and negative respectively[26], meaning that the low equilibrium is saddle point stable with two negative and two positive real roots. For the high equilibrium, both $det\,(J) - \left(\frac{\Omega}{2}\right)^2 - (\rho+\theta)^2\,\frac{\Omega}{2}$ and $det\,(J) - \left(\frac{\Omega}{2}\right)^2$ are positive[27], implying that the high equilibrium is saddle point stable with complex roots and that there are transient oscillations (see Feichtinger et al. (1994)).

# 4 Model with abatement activities

In this section, our focus is on the implications of abatement activities on limit cycles. The social planner's program with abatement activities is

$$W = \int_0^\infty \{U\,(C\,(t)) + \theta\varphi\,(S\,(t))\}\, e^{-(\rho+\theta)t} dt \tag{18}$$

The dynamics of capital accumulation contain the cost of mitigation $M$.

$$\begin{cases} \dot{K}\,(t) = f\,(K\,(t)) - \delta K\,(t) - C\,(t) - M\,(t) \\ \dot{S}\,(t) = G\,(S\,(t)) + \Gamma\,(M\,(t)) - \gamma f\,(K\,(t)) \end{cases} \tag{19}$$

where

$$\Gamma\,(M) = M^\alpha\,,\ \alpha > 0$$

stands for abatement activities, such as reforestation, the desalination of water stock, the enhancement of carbon sinks, etc. The specification of abatement is along the same line as discussed in Chimeli and Braden (2005). Alternatively, the function $\Gamma\,(M)$ can be thought of as an "environmental protection function." The expenditures for environmental protection may be directed not only toward pollution mitigation but also toward the protection of forests and the recovery of degraded areas. Similarly, the abatement activity in this model helps improve environmental quality.

The first order conditions and dynamics of the economy with abatement activity are as follows:

[26]The numerical analysis gives $det\,(J) = 0.000579761$ and $\Omega = -0.0640841$.

[27]The numerical analysis gives $det\,(J) - \left(\frac{\Omega}{2}\right)^2 - (\rho+\theta)^2\,\frac{\Omega}{2} = 0.00799217$ and $det\,(J) - \left(\frac{\Omega}{2}\right)^2 = 0.0178558$.

$$
\begin{cases}
U_C = Z_1^{\sigma} \lambda^{-\frac{1}{\sigma}} = \lambda \\
\dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} - M \\
\dot{S} = G(S) + \Gamma(M) - \gamma f(K) \\
\dot{\lambda} = (\rho + \theta)\lambda - (f_K - \delta)\lambda + \mu \gamma f_K \\
\dot{\mu} = (\rho + \theta)\mu - \mu G_S - \theta \varphi_S
\end{cases}
\tag{20}
$$

The steady-state of the economy as a function of natural resource stock is

$$
\begin{cases}
K^*(S) = \frac{(1-S^*)S^* + \left(\alpha \frac{\mu^*(S^*)}{\lambda^*(S^*)}\right)^{\frac{\alpha}{1-\alpha}}}{\gamma A} \\
M^*(S) = \left(\alpha \frac{\mu(S^*)}{\lambda(S^*)}\right)^{\frac{1}{1-\alpha}} \\
\lambda^*(S) = \frac{\gamma A \mu^*(S^*)}{((A-\delta)-(\rho+\theta))} \\
\mu^*(S) = \frac{\bar{\psi}\omega_2 \theta}{S^*((\rho+\theta)-(1-2S^*))}
\end{cases}
\tag{21}
$$

**Proposition 4.** *(i) Abatement activity decreases the likelihood of limit cycles by increasing the determinant and decreasing the sum of principal minors of dimension two, in which case the economy admits a saddle path stable equilibrium.*

*Proof. See Appendix (A.6).*

We also show numerically that a model with abatement activities always admits a saddle path equilibrium.

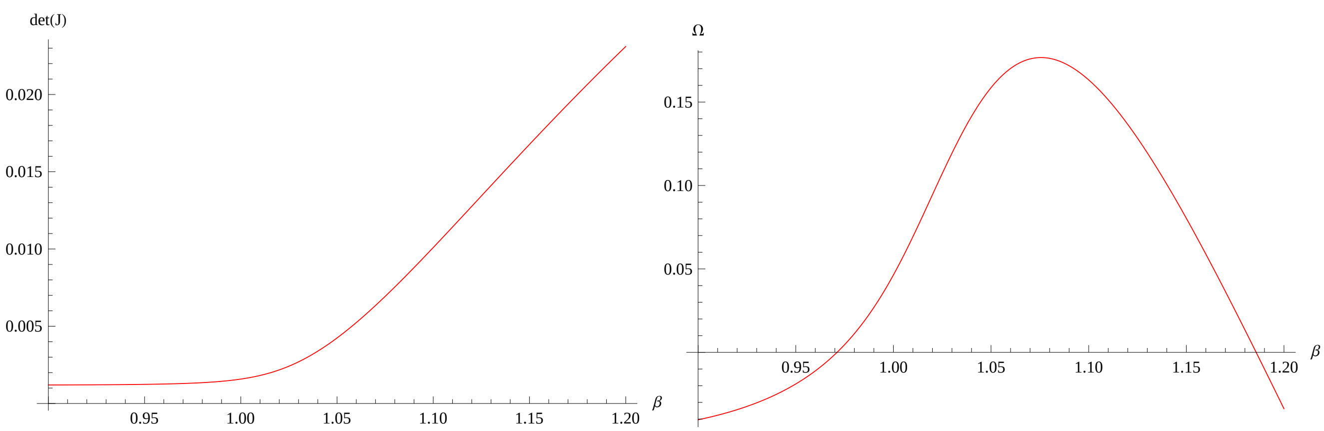


Figure 9: $det(J)$ and $\Omega$ in benchmark model

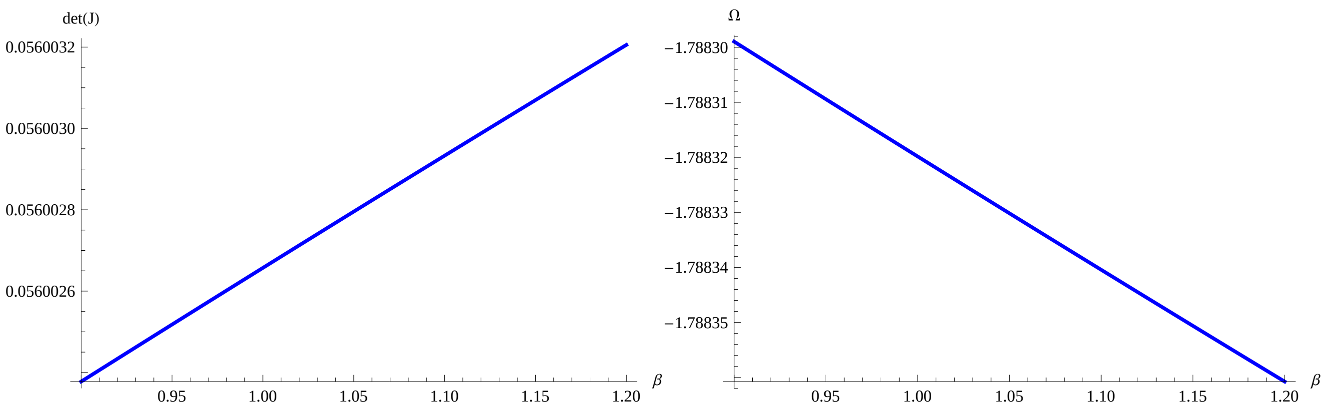


Figure 10: $det\,(J)$ and $\Omega$ in the model with abatement activity

The determinant and the sum of principal minors of dimension two of the model with abatement activity increases and decreases respectively in the model with abatement activity. Unfortunately, the model with abatement activity remains inconclusive when used to determine whether the abatement activity can eliminate limit cycles. For example, a model with very costly abatement activity resembles the case without abatement, and therefore, limit cycles are still possible. However, according to the numerical exercise, there is a better chance that the economy admits a saddle path equilibrium. The economic explanation is as follows: When the steady state level of the natural resource stock increases due to environmental protection activities, the marginal utility of consumption (see (21)) decreases. In this case, any variations in consumption due to the difference between the levels of individual discount rate $\beta$ would have a lower impact on marginal utility. Indeed, when the economy tends toward limit cycles due to a tight trade-off between the economic and environmental goal, the abatement activity relaxes this trade-off by lowering marginal utility. Thus, the dilemma of choosing between the environmental and economic goal is solved, as more weight is given to the environmental goal.

# 5 Conclusion

In this paper, we showed the existence of limit cycles in an economy exposed to the probability of the occurrence of a catastrophic event. The limit cycles are caused by conflicting economic and environmental goals coupled with the complementarity of preferences over time. An interesting finding is that individual time preferences of agents other than the social planner's discount rate are crucial not only for intra-generational equity but also for the sustainability of an economy. When the individual discount rate is close to the social planner's discount rate, intra-generational equity and the sustainability criterion are respected in the long run. This result also disproves a widespread finding in the literature which asserts that aggregate dynamics are governed solely by the social planner's discount rate (see Endress et al. (2014); Schneider et al. (2012)). From a normative point of view, limit cycles are considered undesirable because they compromise the Sustainable Development criterion. Therefore, either a policymaker should revise the Sustainable Development criterion, or limit cycles should be avoided. When augmented by environmental protection activities, the model shows that limit cycles are less likely to occur in the long run.

# A Appendix

## A.1 Change of the order of integration

We use the theorem of Fubini-Tonelli to change the order of the integration and reformulate the social planner's program (2)

$$W(0) = \int_{-\infty}^{0} \left\{ \int_{0}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} \mathbb{1}_{t \geq b} dt \right\} e^{-\rho b} db + \int_{0}^{T} \left\{ \int_{0}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} \mathbb{1}_{t \geq b} dt \right\} e^{-\rho b} db$$

$$+ e^{-\rho T} \varphi(S(T)) \tag{A.1}$$

By using the Chasles relation, it is possible to write down the problem in the following form

$$\int_{-\infty}^{T} \left\{ \int_{0}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} \mathbb{1}_{t \geq b} dt \right\} e^{-\rho b} db + e^{-\rho T} \varphi(S(T)) \tag{A.2}$$

Because the lower and upper bounds of integrals do not depend on the variables of integration, it is easy to change the order of integration

$$\int_{0}^{T} \left\{ \int_{-\infty}^{T} u(c(b,t)) e^{-(\beta+h)(t-b)} e^{\rho(t-b)} \mathbb{1}_{t \geq b} db \right\} e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{A.3}$$

To get rid of the indicator function, and knowing that $t < T$ (when $t > T$, each family has a minimum consumption level) and $b \leq t$, we can write

$$\int_{0}^{T} \left\{ \int_{-\infty}^{t} u(c(b,t)) e^{-(\beta+h)(t-b)} e^{\rho(t-b)} db \right\} e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{A.4}$$

As we know that $\tau = t - b$, we can switch from the generational index $b$ to the age index $\tau$

$$\int_{0}^{T} \left\{ \int_{0}^{\infty} u(c(t-\tau,t)) e^{-(\beta+h-\rho)\tau} d\tau \right\} e^{-\rho t} dt + e^{-\rho T} \varphi(S(T)) \tag{A.5}$$

## A.2 Utility of each generation (family)

In order to find the utility of each family, we write down the Lagrangian as follows:

$$\mathcal{L} = \int_{0}^{\infty} u(c(t-\tau,t)) e^{-(\beta+h-\rho)\tau} d\tau + \lambda(t) \left[ C(t) - \int_{0}^{\infty} c((t-\tau,t)) e^{-h\tau} d\tau \right] \tag{A.6}$$

From the Lagrangian, we can find the utility of each family,

$$u_c(c(t-\tau,t)) e^{-(\beta-\rho)\tau} = u_c(c(t,t)) \tag{A.7}$$

where $u_c(c(t,t))$ is the marginal utility of consumption for a newly-born family. Using a utility function of the CRRA form (10), we have

$$c(t-\tau,t) = c(t,t) e^{-\frac{(\beta-\rho)\tau}{\sigma}} \tag{A.8}$$

Aggregating the consumption of all generations gives

$$C\left(t\right)=\int_{0}^{\infty}c\left(t-\tau,t\right)e^{-h\tau}d\tau=\int_{0}^{\infty}c\left(t,t\right)e^{-\frac{(\beta-\rho)\tau}{\sigma}}e^{-h\tau}d\tau=Z_{1}c\left(t,t\right)\tag{A.9}$$

where $Z_1=\left(\frac{\sigma}{(\beta+\sigma h-\rho)}\right)$. Plugging equation (A.8) into the maximization program (8) yields

$$U\left(C\left(t\right)\right)=\int_{0}^{\infty}\frac{\left[c\left(t,t\right)e^{-\frac{(\beta-\rho)\tau}{\sigma}}\right]^{1-\sigma}-c_{min}^{1-\sigma}}{1-\sigma}e^{-(\beta+h-\rho)\tau}d\tau\tag{A.10}$$

Substituting equation (A.9) in (A.10) gives

$$U\left(C\left(t\right)\right)=\left[\frac{C\left(t\right)}{Z_{1}}\right]^{1-\sigma}\int_{0}^{\infty}\frac{\left[e^{-\frac{(\beta-\rho)\tau}{\sigma}}\right]^{1-\sigma}-c_{min}^{1-\sigma}}{1-\sigma}e^{-(\beta+h-\rho)\tau}d\tau\tag{A.11}$$

After some simple algebra, we have

$$U\left(C\left(t\right)\right)=\frac{Z_{1}^{\sigma}C\left(t\right)^{1-\sigma}-Z_{2}c_{min}^{1-\sigma}}{1-\sigma}\tag{A.12}$$

where $Z_2=\frac{1}{\beta+h-\rho}$.

## A.3 Aggregate economy facing a catastrophic event

Taking the expectations of (9) gives

$$E_{T}\left[\int_{0}^{T}U\left(C\left(t\right)\right)e^{-\rho t}dt+e^{-\rho T}\varphi\left(S\left(T\right)\right)\right]\tag{A.13}$$

Note that the probability distribution and density function are

$$f\left(t\right)=\theta e^{-\theta t}\ \ and\ \ F\left(t\right)=1-e^{-\theta t}\tag{A.14}$$

We write the following expression

$$\int_{0}^{\infty}f\left(T\right)\left[\int_{0}^{T}U\left(C\left(t\right)\right)e^{-\rho t}dt+e^{-\rho T}\varphi\left(S\left(T\right)\right)\right]dT\tag{A.15}$$

$$=\underbrace{\int_{0}^{\infty}f\left(T\right)\left[\int_{0}^{T}U\left(C\left(t\right)\right)e^{-\rho t}dt\right]dT}_{\text{A}}+\underbrace{\int_{0}^{\infty}f\left(T\right)\left[e^{-\rho T}\varphi\left(S\left(T\right)\right)\right]dT}_{\text{B}}\tag{A.16}$$

Integrating by parts $A$

$$dX=f\left(T\right)\implies X=\int_{0}^{T}f\left(s\right)ds$$

$$Y=\int_{0}^{T}U\left(c\left(t\right)\right)e^{-\rho t}dt\implies dY=U\left(c\left(T\right)\right)e^{-\rho T}$$

Using $\int YdX = XY - \int XdY$ yields

$$A = \left[ \left( \int_0^T f(s)\, ds \right) \left( \int_0^T U(c(t))\, e^{-\rho t} dt \right) \right]_{T=0}^{\infty} - \int_0^{\infty} F(T)\, U(c(T))\, e^{-\rho T} dT \tag{A.17}$$

Recall that $\int_0^{\infty} f(s)\, ds = 1$. Part $A$ leads to

$$\int_0^{\infty} U(c(t))\, e^{-\rho t} dt - \int_0^{\infty} F(t)\, U(c(t))\, e^{-\rho t} dt \tag{A.18}$$

Taking the overall sum $A + B$, we have

$$\int_0^{\infty} \left[ (1 - F(t))\, U(c(t)) + f(t)\, \varphi(S(t)) \right] e^{-\rho t} dt \tag{A.19}$$

Inserting the probability distribution and density function gives

$$\int_0^{\infty} \left[ U(c(t)) + \theta \varphi(S(t)) \right] e^{-(\rho+\theta)t} dt \tag{A.20}$$

## A.4 Proof of Proposition 1

The Jacobian matrix of the differential system using the given functional forms is

$$J = \begin{bmatrix} A - \delta & 0 & \frac{Z_1}{\sigma} \lambda^{-\frac{1}{\sigma}-1} & 0 \\ -A\gamma & (1 - 2S) & 0 & 0 \\ 0 & 0 & (\rho+\theta) - (A - \delta) & A\gamma \\ 0 & 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} & 0 & (\rho+\theta) - (1 - 2S) \end{bmatrix} \tag{A.21}$$

Following Dockner and Feichtinger (1991), the characteristic polynomial associated with the Jacobian matrix is

$$\upsilon^4 - trJ\upsilon^3 + b_2\upsilon^2 - b_3\upsilon + det(J) = 0 \tag{A.22}$$

where $b_2$ and $b_3$ are the sums of the second and third order minors of the Jacobian matrix respectively, we have

$$trJ = 2(\rho+\theta) \quad and \quad -b_3 + (\rho+\theta)\, b_2 - (\rho+\theta)^3 = 0 \tag{A.23}$$

The eigenvalues of the Jacobian matrix are calculated from the first order conditions.

$$\upsilon_i = \frac{(\rho+\theta)}{2} \pm \sqrt{\left( \frac{\rho+\theta}{2} \right)^2 - \frac{\Omega}{2} \pm \sqrt{\Omega^2 - det(J)}} \tag{A.24}$$

The sum of the determinants of the second order minors of the Jacobian matrix can be specified as

$$\Omega = \begin{bmatrix} A - \delta & \frac{Z_1}{\sigma} \lambda^{-\frac{1}{\sigma}-1} \\ 0 & (\rho+\theta) - (A - \delta) \end{bmatrix} + \begin{bmatrix} (1 - 2S) & 0 \\ 2\mu + \frac{\theta \bar{\psi} \omega_2}{S^2} & (\rho+\theta) - (1 - 2S) \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & A\gamma \end{bmatrix} \tag{A.25}$$

Then we have

$$\Omega = (A - \delta)\left[ (\rho+\theta) - (A - \delta) \right] + (1 - 2S)\left[ (\rho+\theta) - (1 - 2S) \right] \tag{A.26}$$

and

$$det\left(J\right)=\left[\left(\rho+\theta\right)-\left(1-2S\right)\right]\left[\left(A-\delta\right)\left(1-2S\right)\left[\left(\rho+\theta\right)-\left(A-\delta\right)\right]\right]+\left(A\gamma\right)^{2}\frac{Z_{1}\lambda^{-\frac{1}{\sigma}-1}}{\sigma}\left(2\mu+\frac{\theta\bar{\psi}\omega_{2}}{S^{2}}\right) \quad (A.27)$$

For a Hopf bifurcation to be possible, $\Omega$ should be positive. In this framework, this is possible when the following condition is ensured:

$$\rho+\theta>G_{S}>0 \quad (A.28)$$

In this case, we can observe that an economy is stable when all levels of the natural resource stock exceed the maximum sustainable yield. A Hopf bifurcation occurs in a $4x4$ dimension system when two of the eigenvalues consist of only imaginary parts. This means that the real part of these two eigenvalues crosses zero for some parameters. More precisely, the derivative of the real part of the eigenvalues with respect to the chosen bifurcation parameter is a non-zero value.

### A.4.1 Hopf bifurcation

The conditions to have a Hopf bifurcation

$$det\left(J\right)-\left(\frac{\Omega}{2}\right)^{2}>0 \quad (A.29)$$

$$det\left(J\right)-\left(\frac{\Omega}{2}\right)^{2}-\left(\rho+\theta\right)^{2}\frac{\Omega}{2}=0 \quad (A.30)$$

are necessary and sufficient for all eigenvalues to be complex and two having zero real parts (see Dockner and Feichtinger (1991) and Feichtinger et al. (1994)).

### A.4.2 Saddle point stability

The conditions

$$\left(\frac{\Omega}{2}\right)^{2}\geq det\left(J\right)>0 \quad (A.31)$$

$$\Omega<0 \quad (A.32)$$

are necessary and sufficient to have a saddle point stability, implying two positive and two negative real eigenvalues (see Feichtinger et al. (1994)).

Unfortunately, it is not possible to show analytically where a Hopf bifurcation occurs for the critical parameter $\beta$. Therefore, we conduct a numerical analysis and show that conditions (A.29) and (A.30) hold for two values of the individual discount rate $\beta$. When $\beta=0.973625$ and $\beta=1.167842$, Hopf bifurcation occurs. The two following graphics show numerically the critical parameter values for the individual discount rate $\beta$ where Hopf bifurcations occur.

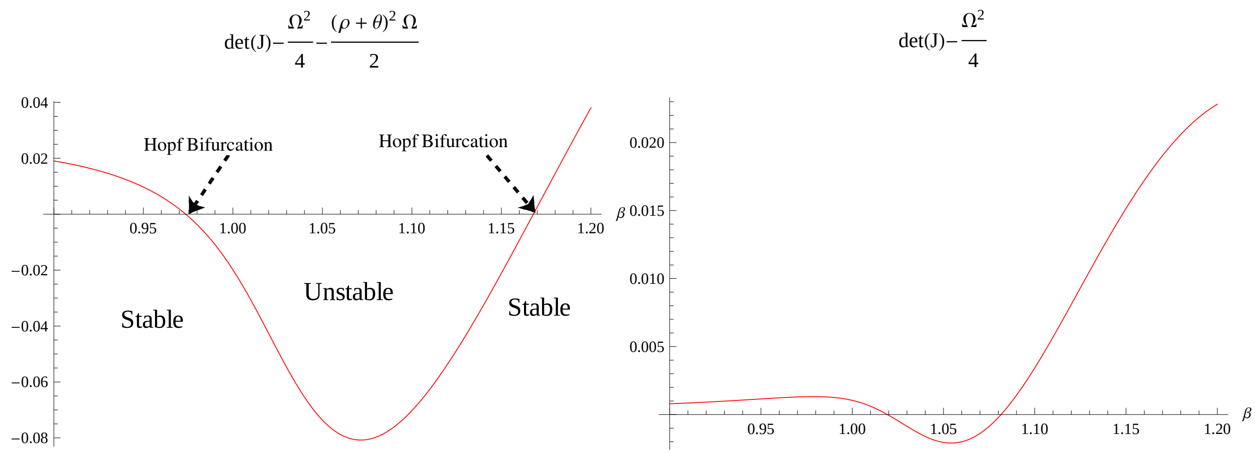


Figure 11: Conditions for Hopf Bifurcation

## A.5 Proof of Proposition 3

The maximization program is indeed the reduced form of the social welfare function

$$W^{NC} = \int_0^\infty U\left(C\left(t\right)\right)e^{-\rho t}dt \tag{A.33}$$

subject to

$$\begin{cases} \dot{K}\left(t\right) = f\left(K\left(t\right)\right) - \delta K\left(t\right) - C\left(t\right) \\ \dot{S} = G\left(S\right) - \gamma f\left(K\left(t\right)\right) \end{cases} \tag{A.34}$$

The differential system describing dynamics of the economy is

$$\begin{cases} \dot{K} = f\left(K\right) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} \\ \dot{S} = G\left(S\right) - \gamma f\left(K\right) \\ \dot{\lambda} = \rho\lambda - \left(f_K - \delta\right)\lambda + \mu\gamma f_K \\ \dot{\mu} = \rho\mu - \mu G_S \end{cases} \tag{A.35}$$

In an economy without any probability of catastrophic events occurring, straightforward calculations allow us to find the steady state equilibrium analytically.

$$\begin{cases} S^* = \frac{1-\rho}{2} \\ K^*\left(S\right) = \frac{(1-S^*)S^*}{\gamma A} \\ \lambda^*\left(S\right) = \left(\frac{(A-\delta)K^*(S)}{Z_1}\right)^{-\sigma} \\ \mu^*\left(S\right) = \frac{\lambda^*(S)[(A-\delta)-\rho]}{\gamma A} \end{cases} \tag{A.36}$$

Then, recall that $det\left(J\right)$ becomes the following when there is no catastrophic event:

$$det\left(J\right) = 2\left(A\gamma\right)^2 \left(\frac{\left(A-\delta\right)}{\left(A-\delta\right)-\rho}\left(\frac{1-\rho}{2}\right)\left(1-\frac{1-\rho}{2}\right)\right) > 0 \tag{A.37}$$

$$\Omega = (A-\delta)\left(\rho - (A-\delta)\right) < 0 \tag{A.38}$$

It is then easy to see that the aggregation term $Z_1$ cancels out when there is no catastrophic event probability. It is also clear that $\Omega$ does not depend on $Z_1$.

## A.6 Proof of Proposition 4

The Jacobian matrix of the differential system with abatement activity at a steady state is

$$J = \begin{bmatrix} A-\delta & 0 & \frac{Z_1}{\sigma}\lambda^{-\frac{1}{\sigma}-1} + \frac{1}{1-\alpha}\left(\alpha\frac{\mu}{\lambda}\right)^{\frac{1}{1-\alpha}}\lambda^{-1} & -\frac{1}{1-\alpha}\left(\alpha\frac{\mu}{\lambda}\right)^{\frac{1}{1-\alpha}}\mu^{-1} \\ -A\gamma & (1-2S) & 0 & 0 \\ 0 & 0 & -\frac{\alpha}{1-\alpha}\left(\alpha\frac{\mu}{\lambda}\right)^{\frac{\alpha}{1-\alpha}}\lambda^{-1} & \frac{\alpha}{1-\alpha}\left(\alpha\frac{\mu}{\lambda}\right)^{\frac{\alpha}{1-\alpha}}\mu^{-1} \\ 0 & 2\mu + \frac{\theta\bar{\psi}\omega_2}{S^2} & 0 & (\rho+\theta)-(1-2S) \end{bmatrix} \tag{A.39}$$

The sum of the determinants of the sub-matrices and determinant of the Jacobian matrix are

$$\Omega = (A-\delta)\left[(\rho+\theta)-(A-\delta)\right] + (1-2S)\left[(\rho+\theta)-(1-2S)\right] - \left(2\mu + \frac{\theta\bar{\psi}\omega_2}{S^2}\right)\frac{\alpha M^{\alpha}}{(1-\alpha)\,\mu} \tag{A.40}$$

and

$$det\,(J) = \left[(\rho+\theta)-(1-2S)\right]\left[(A-\delta)(1-2S)\left[(\rho+\theta)-(A-\delta)\right]\right] + (A\gamma)^2\,\frac{Z_1\lambda^{-\frac{1}{\sigma}-1}}{\sigma}\left(2\mu + \frac{\theta\bar{\psi}\omega_2}{S^2}\right) \tag{A.41}$$

The proof is easy to follow. Let $S^A > S^B$ where $S^A$ is the steady state level of natural stock with abatement activity and $S^B$ is the steady state level of natural stock for the benchmark model without abatement activity. By replacing the steady state value of $\mu$ in (A.40), we can reformulate the sum of principal minors of dimension two:

$$\Omega = (A-\delta)\left[(\rho+\theta)-(A-\delta)\right] + \left(1-2S^A\right)\left[(\rho+\theta)-\left(1-2S^A\right)\right] - \underbrace{\frac{\alpha M^{\alpha}}{(1-\alpha)}\left(\frac{1-(\rho+\theta)}{S^A}\right)}_{>0} \tag{A.42}$$

Notice that $M$ depends only on constant parameters in the long run. The presence of abatement activity decreases the sum of principal minors of dimension two $\Omega$, which decreases the likelihood that unstable spirals and cycles will occur. To see the effect of a higher natural stock level on the determinant of the Jacobian matrix, we look at the first derivative with respect to $S$

$$\frac{\partial\,(det\,(J))}{\partial S} = \underbrace{\left((\rho+\theta)-(A-\delta)\right)}_{<0} 2\,(A-\delta)\left[2\left(1-2S^A\right)-(\rho+\theta)\right] - \underbrace{\frac{(A\gamma)^2}{\sigma} Z_1\left(2\frac{\mu}{\lambda} + \frac{\theta\bar{\psi}\omega_2}{S^2\lambda}\right)\frac{\partial\lambda\left(S^A\right)}{\partial S}}_{<0}$$

$$+ \underbrace{\frac{\left((A-\delta)-(\rho+\theta)\right)\left(1-(\rho+\theta)\right)}{\gamma A S^2}}_{>0} > 0 \tag{A.43}$$

The two last terms are unambiguously negative and positive. The determinant increases unambiguously with respect to $S$ if $2\left(1-2S^{A}\right)-(\rho+\theta)<0$.

### A.7 Proof for complementarity effect between different time periods

This proof aims to support the idea that the complementarity of preferences over time helps explain the existence of limit cycles. An additional objective of the proof is to point out that complementarity vanishes over time when the waste rate coming from physical capital accumulation $\gamma$ is equal to zero.

We write the objective function in the following form:

$$J\left[c\left(.\right)\right]=\int_{0}^{\infty}v\left(C\left(t\right),S\left(t\right)\right)e^{-(\rho+\theta)t}dt \tag{A.44}$$

where $v\left(C\left(t\right),S\left(t\right)\right)=U\left(C\left(t\right)\right)+\theta\varphi\left(S\left(t\right)\right)$

To simplify these calculations, we do not take the quadratic form for regeneration function $G\left(S\right)$ but simply assume a linear regeneration function $G\left(S\right)=mS$.[28] At this point, Wirl (1992, 1994) we show that the occurrence of limit cycles is not linked to the form of the regeneration function. It is shown that a linear regeneration function can also generate limit cycles. Our proof also supports this idea by showing the existence of the complementarity of preferences over time. As such, the use of a linear regeneration function does not conflict with the aim of the proof. We can express physical and natural capital from the constraints

$$S\left(t\right)=e^{mt}\gamma A\int_{t}^{\infty}K\left(s\right)e^{-ms}ds \tag{A.45}$$

$$K\left(t\right)=e^{(A-\delta)t}\int_{t}^{\infty}C\left(s\right)e^{-(A-\delta)s}ds \tag{A.46}$$

To understand the complementarity effects between control and state variables and between state variables, we refer to Volterra derivatives (Ryder and Heal (1973), Dockner and Feichtinger (1991)). This requires an analysis of the marginal rate of substitution between different time periods $t_1$,$t_2$, $t_3$ etc. For example, the marginal utility at time $t_1$ is $J^{'}\left[C\left(.\right),t_1\right]$ which is a Volterra derivative. To summarize, a small incremental increase in consumption in the neighborhood of time $t_1$ can be calculated by using Volterra derivatives. The concept of the Volterra derivative is useful in explaining how a change in consumption at a given date shifts the allocation of consumption between other dates. The marginal rate of substitution between consumption at dates $t_1$ and $t_2$ is

$$R\left[C\left(.\right),t_1,t_2\right]=\frac{J^{'}\left[C\left(.\right),t_1\right]}{J^{'}\left[C\left(.\right),t_2\right]} \tag{A.47}$$

In order to see the effect of an incremental change of consumption near date $t_3$, we take the Volterra derivative of $R\left[C\left(.\right),t_1,t_2\right]$.

$$R^{'}\left[C\left(.\right),t_1,t_2;t_3\right]=\frac{J^{'}\left[C\left(.\right),t_2\right]J^{''}\left[C\left(.\right),t_1,t_3\right]-J^{'}\left[C\left(.\right),t_1\right]J^{''}\left[C\left(.\right),t_2,t_3\right]}{\left(J^{'}\left[C\left(.\right),t_2\right]\right)^{2}} \tag{A.48}$$

If $R^{'}\left[C\left(.\right),t_1,t_2;t_3\right]>0$, an incremental increase of consumption at date $t_3$ shifts consumption from $t_2$ to $t_1$, in which case complementarity exists between $t_1$ and $t_3$. This represents a *distant* complementarity.

[28] Otherwise, in case of the use of logistic growth function for natural regeneration, one should deal with Riccati differential equation which yields tedious calculations.

If $R^{'}[C(.), t_1, t_2; t_3] < 0$, the preferences shift from $t_1$ to $t_2$ where two neighboring dates hold for *adjacent* complementarity. Taking the derivatives of (A.44) , (A.45) and (A.46), we get

$$J^{'}[C(.), t_1] = e^{-(\rho+\theta)t_1} v_C(C(t_1), S(t_1)) + f_C(C(t_1)) e^{-mt_1} \int_{t_1}^{\infty} e^{-(\rho+\theta-m)t} v_s(C(t), S(t))\, dt \tag{A.49}$$

$$J^{''}[C(.), t_1, t_2] = f_C(C(t_1)) f_C(C(t_2)) e^{-m(t_1+t_2)} \int_{t_2}^{\infty} e^{-(\rho+\theta-2m)t} v_{ss}(C(t), S(t))\, dt \tag{A.50}$$

where $f_C$ is the derivative of $S$ with respect to $C$. Regarding the equation (A.45), we know that $S$ is a function of $C$ because $K$ depends on $C$ . For example, when consumption changes marginally, capital accumulation changes and consequently, the trajectory of natural capital accumulation also changes. In this sense, we can simply say that $S = f(C)$ but we can not know the form of this function analytically out of steady state (see Dockner and Feichtinger (1991)). Note also that because we have an additive objective function, $u_{ss}$ does not depend directly on $c$ in our model. In addition, the complementarity over time vanishes when $\theta$ is equal to zero since the second Volterra derivative $J^{''}[C(.), t_1, t_2]$ becomes zero.

For the sake of simplicity, we can restrict our attention to a constant investment path (i.e., steady state) similar to Ryder and Heal (1973) and Dockner and Feichtinger (1991). Using equations (A.45) and (A.46), we can write $S^* = \frac{\gamma A C^*}{m(A-\delta)}$. It is obvious that with this simplification, we can find the form of $f_C = \frac{\gamma A}{m(A-\delta)} > 0$. We write simplified form of equations (A.49) and (A.50),

$$J^{'}[C(.), t_1] = e^{-(\rho+\theta)t_1} \left[ v_C + \frac{f_C v_s}{(\rho+\theta-m)} \right] \tag{A.51}$$

$$J^{''}[C(.), t_1, t_2] = (f_C)^2 e^{-m(t_1+t_2)-(\rho+\theta-2m)t_2} \frac{v_{ss}}{\rho+\theta-2m} \tag{A.52}$$

With all these elements, it is easy to express the effect of the marginal increase of consumption at date $t_3$ on the marginal rate of substitution between $t_1$ and $t_2$,

$$R^{'}[C(.), t_1, t_2; t_3] = \frac{(f_C)^2 \frac{v_{ss}}{\rho+\theta-2m}}{v_C + \frac{f_C v_s}{(\rho+\theta-m)}} e^{(\rho+\theta)(t_2-t_1)} [\alpha(t_3-t_1) - \alpha(t_3-t_2)] \tag{A.53}$$

where $0 < t_1 < t_2$. and $v_{ss} < 0$. We know that the dates $t_2$ and $t_3$ are placed after $t_1$ but we do not know the order of $t_2$ and $t_3$. To understand the effect of a small increase near date $t_3$, we claim that date $t_3$ is situated before $t_2$. Otherwise, a variation of consumption at date $t_3$ would have no effect on the marginal rate of the substitution of consumption between dates $t_1$and $t_2$. This makes sense when we are near date $t_3$, all decisions at $t_1$ and $t_2$ have already been made. Similar to Ryder and Heal (1973), we write

$$\alpha(t) = e^{-(\rho+\theta-m)t}\, for\, t > 0 \tag{A.54}$$

$$\alpha(t) = e^{mt}\, for\, t < 0 \tag{A.55}$$

We observe that $R^{'} < 0$ which means that there exists an adjacent complementarity between dates $t_2$ and $t_3$.

$$t_3 < \frac{(\rho + \theta - m)\, t_1 - m t_2}{(\rho + \theta - 2m)} \tag{A.56}$$

Note that the right-hand side of the inequality increases when Îÿ increases. Limit cycles can appear in the model with both adjacent and distant complementarities, which is in line with the results of Dockner and Feichtinger (1991). However, we do not focus on the type of complementarity here, as doing so is beyond the scope of this study.

It is important to point out the presence of waste in the model. To this end, we write the dynamics of the model when there is no waste coming from the physical capital accumulation.

$$\begin{cases} \dot{K} = f(K) - \delta K - Z_1 \lambda^{-\frac{1}{\sigma}} \\ \dot{S} = G(S) \\ \dot{\lambda} = (\rho + \theta)\, \lambda - (f_K - \delta)\, \lambda \\ \dot{\mu} = (\rho + \theta)\, \mu - \mu G_S - \theta \varphi_S \end{cases} \tag{A.57}$$

We remark that a dynamical system without waste reduces to a block-recursive system. This means that the state and co-state variables $(K, \lambda)$ and $(S, \mu)$ evolve independently. From this feature of the model, one can understand that the dynamics of consumption $c$ and the natural resource stock $S$ are independent of one another. This means that $R^{'}\left[C\left(.\right), t_1, t_2; t_3\right] = 0$ because $f_C = 0$ when there is no waste coming from physical capital accumulation in the economy. Therefore, the complementarity over time vanishes.

## References


Alley, R., J. Marotzke, and W. Nordhaus (2003). Abrupt climate change. *Science 299 (5615) 2005 - 2010.*

Ayong Le Kama, A. (2001). Sustainable growth, renewable resources and pollution. *Journal of Economic Dynamics & Control 25*, 1911–1918.

Blanchard, O. (1984). Debt, deficits and finite horizons. *Journal of Political Economy 93*(2), 223–247.

Bommier, A., B. Lanz, and S. Zuber (2015). Models-as-usual for unusual risks? On the value of catastrophic climate change. *Journal of Environmental Economics and Management 74*, 1–22.

Bommier, A. and R. D. Lee (2003). Overlapping generations models with realistic demography. *Journal of Population Economics 16*(1), 135–160.

Bosi, S. and D. Desmarchelier (2016). Limit Cycles Under a Negative Effect of Pollution on Consumption Demand: The Role of an Environmental Kuznets Curve. *Environmental and Resource Economics*.

Bosi, S. and D. Desmarchelier (2017). Natural cycles and pollution Stefano Bosi - David Desmarchelier Natural cycles and pollution. *FAERE Working Papers*.

Bretschger, L. and A. Vinogradova (2017). Best policy response to environmental shocks : Applying a stochastic framework. *Journal of Environmental Economics and Management*.

Burton, P. S. (1993). Intertemporal Preferences and Intergenerational Equity Considerations in Optimal Resource Harvesting. *Journal of Environmental Economics and Management 24*, 119–132.

Calvo, G. and M. Obstfeld (1988). Optimal Time-Consistent Fiscal Policy with Finite Lifetimes. *Econometrica*.

Chimeli, A. and J. B. Braden (2005). Total factor productivity and the environmental Kuznets curve. *Journal of Environmental Economics and Management 49*(2), 366–380.

Clarke, H. and W. Reed (1994). Consumption/pollution tradeoffs in an environment vulnerable to pollution-related catastrophic collapse. *Journal of Economic Dynamics and Control 18*, 991–1010.

D'Albis, H. and E. Véron (2011). Continuous-Time Overlapping Generations Models.

Dockner, E. and G. Feichtinger (1991). On the optimality of limit cycles in dynamic economic systems. *Journal of Economics 53*, 31–50.

Endress, L., S. Pongkijvorasin, J. Roumasset, and C. Wada (2014). Intergenerational equity with individual impatience in a model of optimal and sustainable growth. *Resource and Energy Economics*.

Feichtinger, G., A. Novak, and F. Wirl (1994). Limit cycles in intertemporal adjustment models. Theory and applications. *Journal of Economic Dynamics and Control 18*(2), 353–380.

Field, C., V. Barros, and T. Stocker (2012). Managing the risks of extreme events and disasters to advance climate change adaptation. *A Special Report of Working Groups I and II of the Intergovernmental Panel on Climate Change. Cambridge University Press, Cambridge, UK..*

Heal, G. (1982). The use of common property resources. *In: Smith VK, Krutilla JV (eds) Explorations in natural resource economics. The Johns Hopkins University Press for Resources for the Future, Baltimore*.

Heal, G. (2009). Climate economics: a meta-review and some suggestions for future research. *Review of Environmental Economics and Policy 3 (1)*, 4 – 21.

Kamien, M. I. and N. L. Schwartz (1978). Optimal exhaustible resource depletion with endogenous technical change. *Review of Economic Studies 45*, 179–196.

Koopmans, T. (1960). Stationary Ordinal Utility and Impatience Author ( s ): Tjalling C . Koopmans. *Econometrica 28*(2), 287–309.

Marini, G. and P. Scaramozzino (1995). Overlapping Generations and Environmental Control. *Journal of Environmental Economics and Management*.

Marini, G. and P. Scaramozzino (2008). Social time preference: A rejoinder. *Journal of Population Economics 21*(3), 623–625.

Mavi, C. A. (2017). Can a hazardous event be another source of poverty traps? *FAERE Working Paper*.

Polasky, S., A. D. Zeeuw, and F. Wagener (2011). Optimal management with potential regime shifts. *Journal of Environmental Economics and Management 62*, 229–240.

Ryder, H. and G. Heal (1973). Optimal Growth with Intertemporally Dependent Preferences. *Review of Economic Studies 40*(1), 1–31.

Schneider, M., C. Traeger, and R. Winkler (2012). Trading off generations: Equity, discounting, and climate change. *European Economic Review*.

Schumacher, I. (2011). When should we stop extracting nonrenewable resources? *Macroeconomic Dynamics 15*, 495–512.

Schumacher, I. and B. Zou (2008). Pollution perception: A challenge for intergenerational equity. *Journal of Environmental Economics and Management 55*(3), 296–309.

Solow, R. (2005). Sustainability, an economist perspective. *In: Stavins, R. (Ed.), Economics of the Environment*, pp. 503 – 513.

Solow, R. (2006). An almost practical step toward sustainability. *In: Oates, W. (Ed.), The RFF Reader in Environmental and Resource Policy. RFF Press, Washington.*, pp. 253 – 264.

Stern, N. (2006). Report on the economics of climate change. avail via the national archives.

Trimborn, T., K. J. Koch, and T. M. Steger (2008). Multidimensional Transitional Dynamics: a Simple Numerical Procedure. *Macroeconomic Dynamics 12*, 301–319.

Tsur, Y. and A. Zemel (1998). Pollution control in an uncertain environment. *Journal of Economic Dynamics and Control*.

Tsur, Y. and A. Zemel (2007). Regulating environmental threats. *Environmental and Resource Economics 39*(3), 297–310.

Tsur, Y. and A. Zemel (2015). Policy tradeoffs under risk of abrupt climate change. *Journal of Economic Behavior and Organization*.

Tsur, Y. and A. Zemel (2016). The management of fragile resources: A long term perspective. *Environmental and Resource Economics 65*(3), 639–655.

Weil, P. (1989). Overlapping Families of Infinitely-Lived Agents. *Journal of Public Economics 38*, 183–198.

Weitzman, M. (2007). A Review of The Stern Review on the Economics of Climate Change. *Journal of Economic Literature 19*(September), 703–724.

Wirl, F. (1992). Cylical Strategies in Two-Dimensional Optimal Control Models : Necessary Conditions and Existence. *Annals of Operations Research 37*, 345–356.

Wirl, F. (1994). A new route to cyclical strategies optimal control models. *Richerche Economiche 48*, 165–173.

Wirl, F. (1999). Complex, dynamic environmental policies. *Resource and Energy Economics 21*(1), 19–41.

Wirl, F. (2004). Sustainable growth, renewable resources and pollution: Thresholds and cycles. *Journal of Economic Dynamics and Control 28*, 1149–1157.

Zemel, A. (2015). Adaptation, mitigation and risk: An analytic approach. *Journal of Economic Dynamics & Control 51*, 133–147.