Dynamic Consistency and Ambiguous Communication*

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Abstract

In most models of ambiguous communication, a Sender can only benefit from ambiguous language if the Receiver behaves dynamically inconsistently. A dynamically inconsistent Receiver might not follow his ex-ante optimal plan after observing an ambiguous message. This paper proposes a novel approach to analyze ambiguous communication by studying dynamically consistent behavior in games with ambiguous strategies. I show that gains from ambiguous communication can be maintained even if players behave dynamically consistently. To achieve this, I define rectangularity, a condition on beliefs that ensures dynamically consistent behavior, for settings where ambiguity arises due to ambiguous strategies. Then, I analyze a Perfect Bayesian Equilibrium in an ambiguous persuasion setting. In this equilibrium, ambiguous communication outperforms standard Bayesian communication even if the Receiver behaves dynamically consistently. Finally, I extend my analysis to settings with ambiguous communication in cheap talk and mechanism design.

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1 Introduction

Ambiguous communication is widely used, e.g., in political and business communication as well as in legalistic or technical writings. One example is fedspeak, a term introduced to describe the wordy and vague language used by chairs of the Federal Reserve Board.

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Furthermore, firms, lobbyists, and politicians spend an extensive amount of time and resources to elaborate on communication strategies – see, for example, McCloskey and Klamer (1995). Thus, even if ambiguous communication increases uncertainty, it seems to play an essential role in the strategic use of communication.

The present paper studies the role of ambiguous communication and its strategic use for dynamically consistent and ambiguity averse players. It has been shown that one or even all players can benefit from ambiguous communication in mechanism design (Bose and Renou (2014)), cheap talk (Kellner and Le Quement (2018)), and persuasion (Beauchêne et al. (2019)). However, in this literature, ambiguous communication is profitable only if players behave dynamically inconsistently. Thus, in these settings, the effect of ambiguity aversion and dynamically inconsistent behavior on the gain of ambiguous communication cannot be separated.

This paper proposes a novel approach to studying dynamically consistent behavior in games with ambiguous communication. In my setting, ex-ante, the Receiver takes into account his knowledge about the information structure and how ambiguous information may manipulate him. Then, I show that the gain of ambiguous communication does not rely on dynamically inconsistent behavior. To study dynamically consistent behavior, I define rectangularity – a condition on the belief set – that implies dynamically consistent behavior. Further, I propose and examine a Perfect Bayesian Equilibrium with rectangular beliefs in settings with strategic ambiguous communication.

In contrast to risk, ambiguity captures uncertainty that cannot be modeled by a single probability measure. One way to model ambiguity averse preferences is the maxmin expected utility model (MEU), proposed by Gilboa and Schmeidler (1989). An agent with MEU preferences is faced with a set of possible beliefs instead of one single belief and maximizes his worst-case expected utility. Further, the literature proposes different updating rules for ambiguity averse agents. I will follow the prior-by-prior Bayesian updating approach (or full Bayesian updating), which assumes that the set of updated beliefs consists of all Bayesian updates of the ex-ante belief set.

Almost all ambiguity averse preferences and updating rules proposed by the literature may lead to dynamically inconsistent behavior. An agent behaves dynamically inconsistently if he does not follow his ex-ante optimal plan after receiving information and updating his beliefs. Roughly speaking, new information can lead to a change in the worst-case belief and, therefore, changes the optimal strategy. Dynamically inconsistent behavior makes it impossible to use standard equilibrium concepts such as Perfect Bayesian Nash Equilibrium or Sequential Equilibrium. Furthermore, it leads to problems, e.g., in the analysis of welfare and value of information, since ex-ante and ex-post decisions are not comparable.

Most of this paper focuses on ambiguous persuasion. However, I show that the same methods and technics can be applied analogously for ambiguous communication in mechanism design and cheap talk. Similar to Beauchêne et al. (2019), I introduce ambiguity in the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by allowing the Sender to choose a set of communication devices. Each communication device can generate a message that reveals information about an unknown (risky) state $\omega \in \Omega$. Sender and Receiver only observe one message without knowing which communication device generated the message. Thus, an ambiguous communication device implies an ambiguous interpretation of the observed message and, therefore, ambiguity about the state ω.

To ensure dynamically consistent behavior, I define rectangularity for settings with ambiguous communication. In these settings, ambiguity arises endogenously due to the ambiguous strategy of the Sender. Intuitively, rectangularity allows players to take into account their knowledge about the information structure and potential future worst-case beliefs. Given the Sender's strategy, a rational Receiver knows which ambiguous message he can observe and how this influences his interim worst-case belief. Formally, rectangularity implies a subjective ex-ante belief set that considers discrepancies between the future and current worst-case beliefs.

To formalize rectangular beliefs, I define beliefs on a state space that depends on the (risky) state ω and the messages. This state space takes the dependence of the ambiguous signal and the ex-ante risky state into account and allows for a non-singleton ex-ante belief set. First, I show that one can restrict the message set to straightforward messages and synonyms. A message set consists of straightforward messages if it only contains recommendations on which action the Receiver should choose. A synonym m' of a message m is a message that induces the same posterior belief or best response of the Receiver as the message m. This result generalizes the well-known Proposition 1 of Kamenica and Gentzkow (2011), which states that one can restrict the message set, without loss of generality, to straightforward messages to the ambiguous setting.

Then, I define rectangular beliefs over the general state space of straightforward messages and states. Given rectangular beliefs, I can extend the usual definition of a Perfect Bayesian Equilibrium to settings with ambiguous communication. I examine a Perfect Bayesian Equilibrium. Further, I show that all results of Beauchêne et al. (2019) can be extended to a Perfect Bayesian Equilibrium with rectangular beliefs. Hence, the gain of ambiguous communication does not rely on dynamically inconsistent behavior. Ambiguous communication is profitable due to ambiguity averse preferences and not due to dynamically inconsistent behavior.

The paper is organized as follows: First, I review the related literature. Section 2 provides an illustrative example to demonstrate the gain of ambiguous communication and the issue of dynamically inconsistent behavior. In Section 3, I formulate the ambiguous persuasion model. Further, Section 4 derives beliefs on the general state space and rectangularity. In Section 5, I define perfect Bayesian equilibria with rectangular beliefs and compare my results to Beauchêne et al. (2019). In Section 6, I discuss rectangular beliefs in an ambiguous cheap talk setting and a mechanism design setting with ambiguous communication. Finally, Section 7 concludes and discusses extensions and related issues.

1.1 Related Literature

This paper contributes to the literature on ambiguous communication. Among others Beauchêne et al. (2019), Kellner and Le Quement (2018) and Bose and Renou (2014) study ambiguous communication in persuasion, cheap talk, and mechanism design. In all three settings, ambiguity arises endogenously due to the ambiguous communication of the Sender.[1] However, in all of these papers, ambiguous communication leads to new equilibria only if players behave dynamically consistently. For example, Beauchêne et al. (2019) claim that there is no gain of ambiguous persuasion compared to Bayesian persuasion if the players behave dynamically consistently.[2]

The present paper proposes a novel approach to studying ambiguous communication. My setting differs from the previous paper by allowing players to consider the discrepancy between their current and future worst-case beliefs.

To implement dynamically consistent behavior, I extend the concept of rectangular beliefs to settings where ambiguity arises due to ambiguous strategies. Epstein and Schneider (2003), Sarin and Wakker (1998) and Riedel et al. (2018) define rectangularity for decision theoretical settings with a fixed information structure. Pahlke (2022) generalizes the concept of rectangularity to multi-stage games with ambiguity about states but non-ambiguous strategies. However, ambiguous beliefs arise in these settings due to exogenous ambiguity about states or types. In the present paper, ambiguous beliefs arise endogenously due to ambiguous strategies. To my knowledge, Muraviev et al. (2017) is the only work analyzing rectangularity for strategic use of ambiguity. However, they only study the relationship between mixed and behavior strategies and do not define an equilibrium concept. Thus, for the first time in this literature, the present paper defines a dynamically consistent equilibrium concept for games with ambiguous strategies.

Concurrent with this work, Cheng (2021) analyzes dynamic consistency for ambiguous persuasion. However, instead of rectangularity, he uses the updating rules of Hanany and Klibanoff (2007). Roughly speaking, these updating rules imply dynamically consistent behavior by assuming that players only update beliefs consistent with the ex-ante worst-case belief. Therefore, the ex-ante optimal choice of a player becomes interim optimal. Cheng (2021) shows that players using the updating rules of Hanany and Klibanoff (2007) can not gain from ambiguous persuasion. I discuss the relation between my work and Cheng (2021) in more detail in Section 7.

Finally, this paper is related to the literature on the value of ambiguous information. Li (2020) and Hill (2020) theoretically study and define the value of ambiguous information. I follow the definition of Hill (2020). In contrast to Li (2020), I show that ambiguous communication can imply a negative or positive value of information. I discuss the relation to these approaches in more detail in Section 7. Further, the value of ambiguous communication has been studied experimentally. Kops and Pasichnichenko (2022) and Ortoleva and Shishkin (2021) find heterogeneous results. While Kops and Pasichnichenko (2022) report a negative value of information for ambiguity averse players, Ortoleva and Shishkin (2021) cannot find a correlation between the negative value of information and ambiguity aversion.

[1] Additionally, to ambiguous communication Bose and Renou (2014) allow for exogenous ambiguity about the state. I discuss a similar generalization for persuasion and cheap talk in Section 7.

[2] See Proposition 5 of Beauchêne et al. (2019).

2 Illustrative Example

The following persuasion example illustrates that ambiguity can lead to a higher expected payoff for the Sender than Bayesian persuasion. Furthermore, I show that the interim equilibrium strategy of the Receiver is not ex-ante ante optimal.

Example 1. *Consider a judge (Receiver) and a prosecutor (Sender). The judge has to decide whether to convict (c) or acquit (a) a defendant. There are two possible states. The defendant can be either guilty (ω_g) or innocent (ω_i). The prosecutor always prefers that the judge chooses to convict. The judge wants to fit the state. He gets a utility of* 1 *for acquitting an innocent defendant and convicting a guilty defendant. However, acquitting a guilty defendant causes reputation costs that imply a utility of* -1. *The payoffs of the judge and the prosecutor are summarized in the following table.*

	ω_g	ω_i
c	$(1, 1)$	$(1, 0)$
a	$(0, -1)$	$(0, 1)$

Figure 1: Payoffs (S, R)

Ex-ante, judge and prosecutor share the belief $p_0 = \mathbb{P}(\omega_g) = 0.2$. *Thus, without any additional information, the judge always acquits the defendant. If the prosecutor fully reveals the state, the defendant would be convicted whenever he is guilty, which occurs with a probability of* 0.2.

First, I derive the prosecutor's optimal signal if she can use standard Bayesian persuasion. Following the approach of Kamenica and Gentzkow (2011), the prosecutor can choose a set of messages and a communication device. Depending on the state, the communication device will generate one message, and the prosecutor has to report the message truthfully. The optimal Bayesian persuasion is given by the communication device π *in Figure 2. It sends either the message* g *or* i. *If the defendant is guilty, the communication device always generates the message* g. *If the defendant is innocent, message* g *occurs with probability* 0.5.

π	ω_g	ω_i
i	0	0.5
g	1	0.5

Figure 2: Optimal Bayesian Persuasion

Then, if the Receiver observes the message i, he believes that the defendant is innocent with probability 1 and chooses a. If the Receiver observes the message g, he believes that the defendant is guilty with probability $\frac{1}{3}$, and he convicts the defendant.[3] *Since the message g is observed with a probability of 0.6, the judge convicts 60 percent of the defendants. Thus, Bayesian persuasion already increases the percentage of convicted defendants from 0 to 60 percent compared to the situation where the Sender does not reveal any information.*

However, if the Sender uses an ambiguous communication device, she can still increase the probability of conviction. Suppose the Sender can create ambiguity by designing two communication devices π and π'. As before, the Receiver only observes one message but, without knowing which of the communication devices generated the message. Consider the two communication devices in Figure 3.

π	ω_g	ω_i
i	0	1
g	1	0

π'	ω_g	ω_i
i	1	0
g	0	1

Figure 3: Optimal Ambiguous Persuasion

Suppose the judge observes the message i. If this message was generated by π his updated belief is $\mathbb{P}^{\pi}(\omega_g|i) = 1$. But if the message was generated by π' his updated belief is $\mathbb{P}^{\pi'}(\omega_g|i) = 0$. Thus, he only knows that the updated belief is either 0 or 1. Since he does not know which communication device generated the message, he faces ambiguity and maximizes his worst-case expected utility. Acquitting generated a worst-case utility of -1, whereas convicting leads to a worst-case utility of 0. Thus, the judge prefers to convict the defendant.

Similarly, if the judge observes the message g, his updated belief is again either 0 or 1, and he convicts the defendant. Thus, both messages imply worst-case beliefs such that the judge always convict the defendant. This signal generates the highest possible utility for the prosecutor. To sum up, without additional information, the prosecutor's expected utility is 0.2, the best Bayesian persuasion implies an expected utility of 0.6, and the best ambiguous persuasion leads to an expected utility of 1. Thus, the Sender gains from ambiguous communication compared to Bayesian persuasion.

However, the strategy 'always convicting' is not ex-ante optimal for the Receiver. His ex-ante worst-case expected utility of convicting is

$$\min\left\{\mathbb{P}^{\pi}(g)\mathbb{P}^{\pi}(\omega_g|g) + \mathbb{P}^{\pi}(i)\mathbb{P}^{\pi}(\omega_g|i), \mathbb{P}^{\pi'}(g)\mathbb{P}^{\pi'}(\omega_g|g) + \mathbb{P}^{\pi'}(i)\mathbb{P}^{\pi'}(\omega_g|i)\right\}$$
$$= \min\left\{\mathbb{P}^{\pi}(g), \mathbb{P}^{\pi'}(i)\right\} = 0.2$$

[3] More precisely, given the belief $\mathbb{P}(guilty) = \frac{1}{3}$, the Receiver is indifferent between acquit and convict. However, I will follow the usual assumption that the Receiver chooses the Sender-preferred action in case of indifference.

and his ex-ante worst-case expected utility of acquitting is

$$\begin{aligned}&\min\left\{\mathbb{P}^{\pi}(g)\mathbb{P}^{\pi}(\omega_g|g)(-1)+\mathbb{P}^{\pi}(i)\mathbb{P}^{\pi}(\omega_i|i),\mathbb{P}^{\pi'}(g)\mathbb{P}^{\pi'}(\omega_g|g)+\mathbb{P}^{\pi'}(i)\mathbb{P}^{\pi'}(\omega_i|i)(-1)\right\}\\ =&\min\left\{\mathbb{P}^{\pi}(g),\mathbb{P}^{\pi'}(i)\right\}=0.6.\end{aligned}$$

Thus, even considering the worst-case analysis, ex-ante, the judge would be better off if he always acquits the defendant. But this requires a strong commitment device. After observing any message, acquitting is no longer optimal without such a commitment device, and he convicts the defendant.

This example illustrates the dynamically inconsistent behavior induced by ambiguity. Ex-ante, any action that is constant with respect to the messages hedges the Receiver against ambiguity. Therefore, choosing his best constant action induces a larger ex-ante worst-case expected utility than his optimal interim strategy. However, after observing the message, he can no longer hedges against ambiguity by ignoring the message. Therefore, he chooses the 'safer' action after observing the ambiguous message. However, a rational Receiver who is aware of ambiguous communication should consider how ambiguous information will change his interim worst-case belief and behavior.

3 Model

The basic setting follows the model of Beauchêne et al. (2019), henceforth BLL, which extends the standard Bayesian persuasion setting of Kamenica and Gentzkow (2011) by an ambiguous communication device. It is a straightforward generalization of the communication device used in Example 1.

As the standard Bayesian persuasion, an ambiguous persuasion game consists of a Sender (she) and a Receiver (he). The utility of both players depends on the risky state $\omega \in \Omega$ and action $a \in A$ chosen by the Receiver, where Ω and A are compact subsets of the Euclidean space. I denote with $u(a, \omega)$ and $\nu(a, \omega)$ the utility of Receiver and Sender, respectively. Further, Sender and Receiver have maxmin preferences á la Gilboa and Schmeidler (1989), i.e., they maximize their worst-case expected utility.

Ex-ante, the state ω is unknown but non-ambiguous. Both players share the same prior state belief $p_0 \in \Delta\Omega$.[4] The Sender tries to persuade the Receiver by choosing a signal that reveals information about the state. A signal consists of a finite set of signal realizations or messages M and a set of communication devices $\Pi = \{\pi_k\}_{k\in K}$.[5] Each communication device is a distribution over the set of messages M for each $\omega \in \Omega$, i.e., $\pi_k(\cdot|\omega) \in \Delta M$ for

[4]Our definition of belief differs from the one of BLL. To avoid confusion, I use the term state belief whenever I refer to beliefs in the sense of BLL.

[5]Please note that I deviate from the model of BLL by defining Π as the set of communication devices. BLL define Π as the convex hull of the set of communication devices. Since Sender and Receiver have maxmin preferences, the minimization problems over $\{\pi_k\}$ or $\text{co}(\{\pi_k\})$ coincide.

all $\omega \in \Omega$. As BLL, I assume that the communication devices π_k have common support for all $k \in K$.

Thus, the only difference to the standard Bayesian persuasion setting is that the Sender chooses a set of communication devices instead of one communication device. Which of the communication devices generates the observed message is ambiguous to both players. After observing a message m, the Receiver updates his prior state belief prior-by-prior using Bayes' rule. Since he does not know which communication device generated the message, he updates p_0 with respect to each communication device π_k. This leads to the following set of posterior state beliefs after observing the message $m \in M$

$$P_m = \left\{ p_m^{\pi_k}(\cdot) \in \Delta\Omega : p_m^{\pi_k}(\cdot) = \frac{p_0(\cdot)\pi_k(m|\cdot)}{\int_\Omega p_0(\omega)\pi_k(m|\omega)\,\mathrm{d}\omega}, \pi_k \in \Pi \right\}. \tag{1}$$

Thus, after observing message m, the Receiver maximizes his interim worst-case expected utility

$$U(a, P_m) = \min_{p_m \in P_m} \mathbb{E}_{p_m}(u(a, \omega)). \tag{2}$$

As usual in the persuasion literature, I assume that the Receiver chooses the Sender preferred action if he has multiple maximizers. I denote with $\hat{a}_m$ the (Sender preferred) best response of the Receiver after observing the message m.

The Sender chooses the signal (M, Π) that maximizes her ex-ante worst-case expected utility

$$\sup_{(M,\Pi)} \min_{\pi \in \Pi} \mathbb{E}_{p_0} \Big[\mathbb{E}_\pi \left[\nu(\hat{a}_m, \omega) | \omega \right] \Big].$$

4 Beliefs and General State Space

In my setting, ambiguity arises due to the ambiguous communication device. Ambiguous interim beliefs only occur due to the combination of a risky state and an ambiguous signal.As shown in Example 1 the interim best response of the Receiver is generally not ex-ante optimal. Intuitively, ex-ante the Receiver can hedge against ambiguity by ignoring the message and playing the same action for all messages. Ignoring the message and playing a constant action requires a strong commitment device for the Receiver. In many applications, such a strong commitment device is unavailable, and a Receiver might fail to commit to the ex-ante optimal action. However, if the Receiver is aware of the information structure and the lack of a commitment device, he might consider his inconsistent beliefs. Knowing that his worst-case belief at the interim stage will contradict his worst-case belief at the ex-ante stage, he might get skeptical about his ex-ante belief and the relation between ambiguous messages and the risky state.

Consider the following two situations at the ex-ante stage:

1) The Receiver does not observe any message. All information about the state $\omega \in \Omega$

is represented by p_0.

2) As in situation 1) the Receiver knows p_0. Additionally, he knows he will receive an ambiguous message before deciding.

In the first situation, the Receiver knows there will be no additional information. Hence, he chooses his optimal action, given the expected utility with respect to p_0. In the second situation, ex-ante, the Receiver has the same information about the state as in situation 1). But, he knows that he will receive additional but ambiguous information before deciding. Furthermore, he knows this ambiguous information influences his interim beliefs and, therefore, his best response. A rational player should consider this knowledge about a game's information structure when deciding at the ex-ante stage. Rectangularity takes the interplay of the prior state belief p_0 and the knowledge about the information structure into account and, therefore, ensures dynamically consistent behavior.

Further, in settings with ambiguous communication, ambiguity arises endogenously due to ambiguous messages. Even if the messages themselves are not payoff-relevant, the ambiguous interpretation of the message induces ambiguity about the state. I define beliefs as joint beliefs on states and messages to consider the relationship between the risky state and ambiguous messages.

4.1 Straightforward Messages

In any persuasion setting, the set of messages M is part of the Sender's strategy. In the Bayesian persuasion setting, Kamenica and Gentzkow (2011) call a signal straightforward if $M \subseteq A$. They show that one can restrict to straightforward signals in a Bayesian persuasion setting without loss of generality. More precisely, for any signal, there exists a straightforward signal that leads to the same expected utility of the Sender in equilibrium. The next proposition generalizes this result to ambiguous persuasion. It shows that the Sender chooses without loss of generality $M = A \cup \tilde{A}$ where $\tilde{A}$ is a duplicated set of A such that there exists a bijection $b(\cdot)$ between A and $\tilde{A}$. Given this result, I can define ex-ante beliefs on the general state space $\Omega \times (A \cup \tilde{A})$.

Proposition 1. *Let* $(M, \Pi) \in \operatorname{argsup} \min_{\pi \in \Pi} \mathbb{E}_{p_0} \left[\mathbb{E}_\pi \left[\nu(\hat{a}_m, \omega) | \omega \right] \right]$. *Let* $\tilde{A}$ *be such that there exist a bijection* $b(\cdot) : A \to \tilde{A}$ *between* A *and* $\tilde{A}$. *Then, there exist* (M', Π') *with* $M' = A \cup \tilde{A}$ *and* $\Pi' = \{\pi_1', \pi_2'\}$ *such that* (M', Π') *generates the same value for the Sender as* (M, Π).

The intuition of the proposition is as follows. BLL show that ambiguous persuasion increases the value for the Sender compared to Bayesian persuasion only if the Sender uses a signal with synonyms. Synonyms are messages that copy the meaning of another message, i.e., they induce the same posterior state belief set or best response of the Receiver. Intuitively, the Sender uses synonyms to hedge himself against ambiguity. Furthermore, they show that for any ambiguous signal, one can find an ambiguous signal which consists only of two communication devices and leads to the same value. Hence, to use straightforward messages as in Kamenica and Gentzkow (2011), I have to duplicate the message space to allow for synonyms, and duplication is enough to generate the same value as any ambiguous signal. Thus, without loss of generality, I can assume that $M = A \cup \tilde{A}$. The detailed proof can be found in Section A.1 in the Appendix.

Due to the assumption that all π_k have common support on M, Sender's strategy (M, Π) is completely characterized by Π. For the rest of the paper, I will use the term strategy of the Sender for such a Π. Furthermore, I denote with $\text{supp}(\Pi) = \text{supp}(\pi_k(\cdot|\omega)) \subset A \cup \tilde{A}$ the support of $\pi_k \in \Pi$ for all $k \in K$.

4.2 Rectangular Beliefs

Given the results from the previous section, I can define beliefs over the general state space $\Omega \times (A \cup \tilde{A})$. Defining beliefs over this general state space allows the Receiver to form a joint belief about the risky state $\omega \in \Omega$ and the ambiguous message $m \in M$, i.e., the Receiver forms beliefs of the events "the state is ω, and I observe message m." The probability of this event depends on the risky state $\omega \in \Omega$ and the ambiguous communication device that generates the message.

Definition 1. *For a strategy Π of the Sender, I define the* ***set of ex-ante beliefs*** *of the Receiver as*

$$\Phi^0_\Pi = \Bigg\{ \rho^k \in \Delta(\Omega \times (A \cup \tilde{A})) : \exists \pi_k \in \Pi \ s.t. \\ \rho^k(\omega, m) = \begin{cases} p_0(\omega)\pi_k(m|\omega) & \text{if } m \in supp(\Pi) \\ 0 & \text{otherwise} \end{cases} \Bigg\}.$$

Note that the strategy of the Sender generates the information structure of the persuasion games. Therefore, it has to influence the joint belief over states and messages, and Φ^0_Π depends on the Sender's strategy Π.

At the interim stage the Receiver observes a message $m \in \text{supp}(\Pi)$. The information structure at the ex-ante stage ($t = 0$) and interim stage ($t = 1$) can be represented by the following partitions

$$\mathcal{F}_0 = \Omega \times (A \cup b(A)),$$
$$\mathcal{F}_1 = \Big\{ \{\Omega \times m\}_{m \in A \cup b(A)} \Big\}.$$

Then, given an observation $\hat{m} \in \text{supp}(\Pi)$ the Receiver updates his ex-ante belief set prior-by-prior using Bayes' formula, i.e., he updates each prior belief in Φ^0_Π with Bayes' formula

$$\rho^k|_{\hat{m}} = \rho^k((\omega, m)|\hat{m}) = \frac{p_0(\omega)\pi_k(m|\omega)}{\int_\Omega p_0(\omega')\pi_k(m|\omega')\,\mathrm{d}\omega'}$$

if $m = \hat{m}$ and zero otherwise. Then, the set of updated beliefs given $\hat{m} \in \text{supp}(\Pi)$ is

$$\text{Bay}(\Phi_\Pi^0|\hat{m}) = \{\rho^k|_{\hat{m}} \text{ with } \rho^k \in \Phi_\Pi^0\}.$$

Remark 1. *Note that $\rho^k((\omega, m)|\hat{m}) = 0$ for $\hat{m} \notin supp(\Pi)$. Furthermore, $\rho^k((\omega, \hat{m})|\hat{m}) = p_{\hat{m}}^{\pi_k}(\omega)$ as defined in Equation (1) for all ω. Therefore, given my definition of beliefs, the Receiver's maximization problem at the interim stage coincides with the maximization problem with state beliefs in Equation (2).*

To define rectangularity, let us first look at the case without ambiguity, i.e., if $\Pi = \{\pi\}$ and $\Phi_\Pi^0 = \{\rho\}$ is a singleton. After observing message m, the updated belief is given by $\rho|_m$. Furthermore, the marginal beliefs of observing $m \in A \cup \tilde{A}$ under ρ is

$$\rho(\Omega, m) = \int_\Omega \rho(\omega, m)\,\mathrm{d}\omega = \int_\Omega p_0(\omega)\pi(m|\omega)\,\mathrm{d}\omega.$$

Then, the structure of Bayes' formula implies that multiplying the updated belief after observing message m with the marginal probability of observing m leads to the prior belief restricted to the events that the message is m. This holds for all messages m and, therefore, for all information sets of the partition defined above. Hence, integrating over all $m \in \text{supp}(\Pi)$ leads to the prior belief ρ

$$\int_{m' \in \text{supp}(\Pi)} \rho(\Omega, m')\rho|_{m'}(\omega, m)\,\mathrm{d}m' = \rho(\Omega, m)\frac{\rho(\omega, m)}{\rho(\Omega, m)} = \rho(\omega, m). \tag{3}$$

The first step follows since $\rho|_{m'}(\omega, m) = 0$ if $m' \neq m$ due to the definition of joint belief.

Now, I generalize these considerations to an ambiguous setting, i.e., Π is not a singleton. Rectangularity requires that any combination of marginal belief and updated belief is a prior belief that the agent considers possible. The Receiver knows which messages he could receive and which updated beliefs potentially exist. Taking this knowledge into account, rectangularity requires that any combination of marginal and updated belief is an element of the ex-ante belief set.

First, remember that $\rho|_{\hat{m}}$ denotes the Bayesian update of the ex-ante belief ρ after observing message $\hat{m}$. I denote with $(\rho|_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in \text{supp}(\Pi)} \text{Bay}(\Phi_\Pi^0|\hat{m})$ a collection of the Bayesian updates of ρ for any message $\hat{m} \in \text{supp}(\Pi)$.

Definition 2. *The pasting of an ex-ante belief $\bar{\rho} \in \Phi_\Pi^0$ and a collection of updated beliefs $(\rho|_{\hat{m}})_{\hat{m}}$ is defined as*

$$\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\omega, m) := \int_{supp(\Pi)} \bar{\rho}(\Omega, \hat{m})\rho|_{\hat{m}}(\omega, m)\,\mathrm{d}\hat{m}$$

The set of ex-ante beliefs is called ***rectangular*** *(or stable under pasting) if it contains all pastings of an ex-ante belief* $\bar{\rho} \in \Phi_{\Pi}^{0}$ *and interim beliefs* $(\rho|_{\hat{m}})_{\hat{m}}$*, i.e.,*

$$\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\cdot) \in \Phi_{\Pi}^{0} \tag{4}$$

for all $\bar{\rho} \in \Phi_{\Pi}^{0}$ *and* $(\rho|_{\hat{m}})_{\hat{m}} \in \bigtimes_{\hat{m} \in supp(\Pi)} Bay(\Phi_{\Pi}^{0}|\hat{m})$*.*

Note that the pasting is always well-defined due to the common support assumption. Compared to Equation (3), the pasting of an ex-ante belief and a collection of updated beliefs does not require marginal and updated beliefs to be derived from the same ex-ante belief, i.e., $\bar{\rho}$ and ρ in Equation (4) can be different. If Φ_{Π}^{0} is singleton, Equation (4) reduces to Equation (3) and rectangularity is always satisfied.

One can always construct the smallest set, which is rectangular and contains Φ_{Π}^{0} by backward induction. This set is called the rectangular hull and is denoted with $\text{rect}(\Phi_{\Pi}^{0})$. Simple calculations show that $\text{Bay}(\Phi_{\Pi}^{0}|\hat{m}) = \text{Bay}(\text{rect}(\Phi_{\Pi}^{0})|\hat{m})$. The same holds for the set of marginal beliefs under Φ_{Π}^{0} and $\text{rect}(\Phi_{\Pi}^{0})$. These two properties are essential for the construction of the rectangular hull. Closing a belief set under the pasting operation does not change the interim and marginal beliefs. For a more detailed explanation of the construction and the properties of the rectangular hull, please see Pahlke (2022) or Epstein and Schneider (2003).

So far, I have focused on the beliefs of the Receiver. The Sender only chooses an action at the ex-ante stage. Therefore, the interim beliefs of the Sender do not influence the equilibria of the game. If the Sender does not know which communication device generated the message, her interim and ex-ante belief sets and the rectangular hull coincide with the Receiver's beliefs. However, even if $\Phi_{\Pi}^{0} \subsetneq \text{rect}(\Phi_{\Pi}^{0})$, the marginal beliefs of observing message m are the same for Φ_{Π}^{0} and $\text{rect}(\Phi_{\Pi}^{0})$. Thus, the ex-ante maximization problem of the Sender given $\text{rect}(\Phi_{\Pi}^{0})$ is the same as given Φ_{Π}^{0}.

Alternatively, I could define an information structure of the Sender that does not influence the ex-ante decision of the Sender but ensures that the ex-ante belief set of the Sender is rectangular for any Π. For example, the Sender could observe which communication device generated the observed message at the interim stage. If the Sender learns which communication device generated the message, Φ_{Π}^{0} is rectangular for all Π. By definition, rectangularity depends on the information structure faced by a player. Therefore, assuming heterogeneous information structures for Sender and Receiver would induce heterogeneous rectangular hulls. However, heterogeneous rectangular beliefs only arise due to heterogeneous information structures. Pahlke (2022) discusses the relation between information structures and common rectangular beliefs in more detail. However, the present paper aims to find a belief formation process that ensures dynamically consistent behavior. Since the Sender can never behave dynamically inconsistent, I do not go into details.

5 Dynamic Consistency and Perfect Bayesian Equilibrium

Finally, I show that rectangularity implies dynamically consistent behavior of the Receiver and, therefore, the existence of a Perfect Bayesian equilibrium.

Definition 3. *A Perfect Bayesian equilibrium with rectangular beliefs consists of a strategy* Π^* *of the Sender, a strategy* $(\hat{a}_m)_{m\in M}$ *of the Receiver, and a belief system* Ψ *for each player. Strategies and belief systems have to satisfy the following conditions:*

- *The belief systems of both players consist of an ex-ante belief set* Ψ_i^0 *and interim belief set* Ψ_i^m *for each message* $m \in A \cup \tilde{A}$ *such that*

$$\Psi_R^0 = \text{rect}(\Phi_{\Pi^*}^0)$$
$$\Psi_S^0 = \Phi_{\Pi^*}^0.$$

  Furthermore, the interim belief sets are derived by Bayes rule whenever possible, i.e., $\Psi_i^m = Bay(\Psi_i^0|m)$ *for all* $m \in \text{supp}(\Pi^*)$.

- *The equilibrium strategy of the Sender* Π^* *with supp*$(\Pi^*) \subseteq A \cup \tilde{A}$ *maximizes her ex-ante worst-case expected utility*

$$\min_{\rho\in\Psi_S^0} \mathbb{E}_\rho\left[\nu(\hat{a}_m,\omega)\right].$$

- *The equilibrium strategy of the Receiver maximizes his interim worst-case expected utility for all* $m \in \text{supp}(\Pi^*)$

$$\min_{\rho|_m\in\Psi_R^m} \mathbb{E}_{\rho|_m}(u(a_m,\omega))$$

  and his ex-ante worst-case expected utility given the ex-ante belief set Ψ_R^0

$$\min_{\rho\in\Psi_R^0} \mathbb{E}_\rho(u(a_m,\omega)).$$

The following proposition shows that I can generalize any consistent planning equilibrium of BLL to a Perfect Bayesian equilibrium using rectangularity.

Proposition 2. *Let* (M,Π) *be the optimal ex-ante choice of the Sender and* $(\hat{a}_m)_{m\in M}$ *the optimal interim choice of the Receiver as in BLL. Then, there exists* (M^*,Π^*), *with* $M^* = A \cup \tilde{A}$ *and* $|\Pi^*| = 2$ *that generate the same value of the Sender as* (M,Π). *Furthermore,* Π^*, $(\hat{a}_m)_{m\in M^*}$ *and* $\Psi_R^0 = \text{rect}(\Phi_{\Pi^*}^0)$, $\Psi_S^0 = \Phi_{\Pi^*}^0$ *and* $(\Psi_i^m)_{m\in M^*} = (Bay(\Psi_i^0|m))_{m\in M^*}$ *are a Perfect Bayesian equilibrium with rectangular beliefs.*

Proof. First, due to Proposition 1, there exists (M^*,Π^*), with $M^* = A\cup\tilde{A}$ and $|\Pi^*| = 2$ that generate the same value of the Sender as (M,Π). The proof of Proposition 1 shows that the Receiver chooses the same action given M or M^* in the sense that any two messages $m, m' \in M$ that are not synonyms of each other but induce the same optimal strategy, i.e., $\hat{a}_m = \hat{a}_{m'}$, are replaced by the same message $\bar{m} \in M^*$. Therefore, even if the message sets M and M^* are different, the Receiver's played actions do not change and $(\hat{a}_m)_{m\in M^*}$ is induced by $(\hat{a}_m)_{m\in M}$.

Furthermore, the Sender never behaves dynamically inconsistently. I only have to show that the Receivers interim best response of BLL is an interim and ex-ante best response given rectangular beliefs. Remember that, by Remark 1, $p^{\pi_k}_{\hat{m}}(\cdot) = \rho^k((\cdot, \hat{m})|\hat{m})$ for all $\hat{m} \in \text{supp}(\Pi)$ and that the set of Bayesian updates given Φ^0_Π or $\text{rect}(\Phi^0_\Pi)$ are the same. Therefore, the interim best response given the state beliefs of BLL is an interim best response given rectangular beliefs. Furthermore, I can rewrite the ex-ante expected utility of the Receiver as

$$\min_{\rho\in\text{rect}(\Phi^0_{\Pi^*})} \int_{\text{supp}(\Pi)} \rho(\Omega, \hat{m})\mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m},$$

where $\rho|_{\hat{m}}$ is the Bayesian update of ρ given message $\hat{m}$. First, I show the following relation of ex-ante and interim worst-case expected utility. Let ρ^* denote the ex-ante worst-case belief given rectangular beliefs. Then,

$$\begin{aligned}
&\int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m})\mathbb{E}_{\rho^*|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m} \\
&\qquad = \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho|_{\hat{m}}\in\text{Bay}(\text{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m}. \qquad (5)
\end{aligned}$$

To prove Equation 5, I first show that the left hand side is greater equal than the right hand side.

$$\begin{aligned}
&\int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \underbrace{\mathbb{E}_{\rho^*|_{\hat{m}}}(u(a_{\hat{m}}, \omega))}_{\geq \min_{\rho|_{\hat{m}}\in\text{Bay}(\text{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}},\omega))} \mathrm{d}\hat{m} \\
&\geq \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho|_{\hat{m}}\in\text{Bay}(\text{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m}.
\end{aligned}$$

To prove the other direction, let $\rho'|_{\hat{m}}$ be the worst-case belief given that the Receiver observed $\hat{m}$. Then, due to rectangularity, there exist $\bar{\rho} \in \text{rect}(\Phi^0_{\Pi^*})$ such that $\rho^* \circ (\rho'|_{\hat{m}})_{\hat{m}} = \bar{\rho}$. Furthermore rectangularity implies, that $\bar{\rho}(\cdot|\hat{m}) = \rho'(\cdot|\hat{m})$ and $\bar{\rho}(\Omega, \hat{m}) = \rho^*(\Omega, \hat{m})$ for all $\hat{m}$. Then,

$$\begin{aligned}
\int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m})\mathbb{E}_{\rho^*|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m} &\leq \int_{\text{supp}(\Pi^*)} \bar{\rho}(\Omega, \hat{m})\mathbb{E}_{\bar{\rho}|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m} \\
&= \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m})\mathbb{E}_{\rho'|_{\hat{m}}}(u(a_{\hat{m}}, \omega))\,\mathrm{d}\hat{m}
\end{aligned}$$

$$= \int_{\text{supp}(\Pi^*)} \rho^*(\Omega, \hat{m}) \min_{\rho|_{\hat{m}} \in \text{Bay}(\text{rect}(\Phi^0_{\Pi^*})|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m}.$$

Combining both directions proves Equation 5. Finally, I show that an interim best response of the Receiver is also an ex-ante best response. I denote with $\hat{a}_{\hat{m}}$ the (Sender preferred) interim best response of the Receiver given message $\hat{m}$, i.e.,

$$\min_{\rho|_{\hat{m}} \in \text{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega)) \geq \min_{\rho|_{\hat{m}} \in \text{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega))$$

for any arbitrary $a_{\hat{m}} \in A$ and all $\hat{m} \in \text{supp}(\Pi^*)$. I have to show that $(\hat{a}_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi^*)}$ is ex-ante optimal. Since $\rho(\Omega, \hat{m}) \geq 0$ for all $\hat{m} \in \text{supp}(\Pi^*)$ and $\rho(\Omega, \hat{m}) = 0$ for all $\hat{m} \notin \text{supp}(\Pi^*)$, Equation 5 implies

$$\begin{aligned}
&\min_{\rho \in \text{rect}(\Phi^0_{\Pi^*})} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho|_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\
&\quad = \min_{\rho \in \text{rect}(\Phi^0_{\Pi^*})} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'|_{\hat{m}} \in \text{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho'|_{\hat{m}}}(u(a_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\
&\quad \leq \min_{\rho \in \text{rect}(\Phi^0_{\Pi^*})} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \min_{\rho'|_{\hat{m}} \in \text{Bay}(\Phi^0_{\Pi^*}|\hat{m})} \mathbb{E}_{\rho'|_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m} \\
&\quad = \min_{\rho \in \text{rect}(\Phi^0_{\Pi^*})} \int_{\text{supp}(\Pi^*)} \rho(\Omega, \hat{m}) \mathbb{E}_{\rho|_{\hat{m}}}(u(\hat{a}_{\hat{m}}, \omega)) \, \mathrm{d}\hat{m}
\end{aligned}$$

for any arbitrary $(a_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi)}$. Here the inequality follows from the interim optimality of $(\hat{a}_{\hat{m}})_{\hat{m} \in \text{supp}(\Pi^*)}$ and the last equality from Equation 5.

Hence, the Receiver's ex-ante best response equals the interim best response, and the interim equilibrium of BLL satisfies ex-ante optimality. ■

To illustrate the previous results, let me return to Example 1.

Example 2 (Example 1 cont.). *Remember that the optimal ambiguous communication device was given by* $\Pi = \{\pi, \pi'\}$ *as depicted in Figure 4. Then, the set of ex-ante beliefs of*

π	ω_g	ω_i
i	0	1
g	1	0

π'	ω_g	ω_i
i	1	0
g	0	1

Figure 4: Optimal Ambiguous Persuasion

the Receiver is $\Phi_{\Pi}^0 = \{\rho, \rho'\}$ *with*

$$\rho(\omega, m) = \begin{cases} p_0 & \text{if } m = g, \omega = \omega_g, \\ 1 - p_0 & \text{if } m = i, \omega = \omega_i, \\ 0 & \text{otherwise}, \end{cases} \qquad \rho'(\omega, m) = \begin{cases} p_0 & \text{if } m = i, \omega = \omega_g, \\ 1 - p_0 & \text{if } m = g, \omega = \omega_i, \\ 0 & \text{otherwise}, \end{cases}$$

where ρ *is constructed from* π *and* ρ' *from* π'. *To construct the rectangular hull, I need to calculate all interim and marginal beliefs:*

$$\rho|_i(\omega, m) = \begin{cases} 1 & \text{if } m = i, \omega = \omega_i, \\ 0 & \text{otherwise}, \end{cases} \qquad \rho|_g(\omega, m) = \begin{cases} 1 & \text{if } m = g, \omega = \omega_g, \\ 0 & \text{otherwise}, \end{cases}$$

$$\rho'|_i(\omega, m) = \begin{cases} 1 & \text{if } m = i, \omega = \omega_g, \\ 0 & \text{otherwise}, \end{cases} \qquad \rho'|_g(\omega, m) = \begin{cases} 1 & \text{if } m = g, \omega = \omega_i, \\ 0 & \text{otherwise}, \end{cases}$$

$$marg(\rho(\cdot, i)) = 1 - p_0, \qquad marg(\rho(\cdot, g)) = p_0,$$

$$marg(\rho'(\cdot, i)) = p_0, \qquad marg(\rho'(\cdot, g)) = 1 - p_0.$$

By combining any marginal and interim belief I obtain the rectangular hull $\text{rect}(\Phi_{\Pi}^0) = \{\rho, \rho', \hat{\rho}, \bar{\rho}\}$ *where* ρ *and* ρ' *are as before and*

$$\bar{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = i, \omega = \omega_i, \\ 1 - p_0 & \text{if } m = g, \omega = \omega_g, \\ 0 & \text{otherwise}, \end{cases} \qquad \hat{\rho}(\omega, m) = \begin{cases} p_0 & \text{if } m = g, \omega = \omega_i, \\ 1 - p_0 & \text{if } m = i, \omega = \omega_g, \\ 0 & \text{otherwise}. \end{cases}$$

Given the rectangular hull, the judge's ex-ante worst-case expected utility of always convicting is

$$\min_{\rho \in \text{rect}(\Phi_{\Pi}^0)} \rho(\omega_g, g) + \rho(\omega_g, i) = p_0 = 0.2$$

and his ex-ante worst-case expected utility of always acquitting is

$$\min_{\rho \in \text{rect}(\Phi_{\Pi}^0)} \rho(\omega_g, g)(-1) + \rho(\omega_g, i)(-1) + \rho(\omega_i, i) + \rho(\omega_i, g) = (-1)(1 - p_0) + p_0 = -0.6.$$

Thus, convicting the defendant is now ex-ante optimal. Intuitively, the rectangular ex-ante worst-case belief combines two worst-case considerations. First, from the ex-ante perspective it maximizes the marginal probability of the message that persuade the judge to convict which is either $marg(\rho(\cdot, i)) = 1 - p_0$, *or* $marg(\rho'(\cdot, g)) = 1 - p_0$. *Second, from the interim perspective, the judge will interpret each message such that he convicts the defendant. Independently of the message, he will always interpret it such that his interim worst-case belief gives probability* 1 *to* ω_g. *Therefore, the ex-ante worst-case belief of the judge, if he plans always to acquit the defendant, is either* $\bar{\rho}$ *or* $\hat{\rho}$. *Both worst-case beliefs* *imply that the joint probability of ω_g and one of the messages is* 0.8.

6 Further Models with Ambiguous Communication

A similar approach to defining rectangularity can be used in various models with ambiguous communication, e.g., in cheap talk or mechanism design. The main task is to define an adequate general state space and generalize beliefs to the general state space. To illustrate the general applicability of my results, I discuss the settings of Bose and Renou (2014) and Kellner and Le Quement (2018) in more detail.

6.1 Ambiguous Mechanism Design

Bose and Renou (2014) analyze a mechanism design setting with ambiguous communication. Their setting consists of a finite set of players N, a finite set of payoff-relevant types Θ_i for each player $i \in N$ and a finite set of alternatives X. Types are privately known. Bose and Renou (2014) allow for exogenous ambiguity about types of opponents, i.e., ex-ante, player i is faced with a set of distribution $P_i \subset \Delta(\Theta_{-i})$ about types θ_{-i}. However, for simplicity, lets assume that P_i is singleton, i.e., $P_i = \{p_i\} \in \Delta(\Theta_{-i})$. I discuss the extension of all three settings to exogenous ambiguity in Section 7. Further, players have maxmin preferences and update beliefs prior-by-prior.

Bose and Renou (2014) study the class of social choice functions $f : \times_{i\in N}\Theta_i \to X$ that is implementable by an ambiguous mechanism. An ambiguous mechanism consists of two steps: The second step called the allocation mechanism is a usual static mechanism specifying a finite set of messages M_i for each player and an allocation rule $g : \times_{i\in N}M_i \to X$. The first step adds an ambiguous communication device before playing the allocation mechanism. An ambiguous communication device consists of a finite set of messages that player i can send to the communication device $\hat{\Omega}_i$, a finite set of messages that player i can receive from the communication device Ω_i, and a set of probability systems Λ. The set of probability systems Λ corresponds to the set of communication devices Π in my setting. Hence, each $\lambda \in \Lambda$ specifies the probability that a profile of messages ω is received by the players given that they send the profile $\hat{\omega}$ to the communication device, i.e., $\lambda : \hat{\Omega} \to \Delta(\Omega)$, where $\hat{\Omega} = \times_{i\in N}\hat{\Omega}_i$ and $\Omega = \times_{i\in N}\Omega_i$.

They define a consistent planning equilibrium, i.e., players may behave dynamically inconsistently. However, similar to ambiguous persuasion, all their results can be extended to dynamically consistent players if players have rectangular beliefs. Here, the general state space is given by $\Theta \times \hat{\Omega} \times \Omega$.[6] Given an ambiguous communication device, the set of ex-ante beliefs of a type θ_i is

$$\Phi^0_\Lambda = \left\{\phi \in \Delta(\Theta \times \hat{\Omega} \times \Omega) : \exists\lambda \in \Lambda \text{ s.t. } \phi(\hat{\theta}, \hat{\omega}, \omega) = \lambda(\hat{\omega})[\omega]p_i[\hat{\theta}_{-i}]\mathbb{1}_{\hat{\theta}_i=\theta_i}\right\}.$$

[6]Note, that $\hat{\Omega}$ and Ω are specified by the mechanism and not part of the player's strategy. Therefore, an analog to Proposition 1 is not needed here.

Now, I can define rectangularity analogously to Definition 2 and extend all results from Bose and Renou (2014) to dynamically consistent players. Thus, if a social choice function is implementable by an ambiguous mechanism of Bose and Renou (2014) for dynamically inconsistent players, then the same social choice function can be implemented by the same ambiguous mechanism for dynamically consistent players with rectangular beliefs.

6.2 Ambiguous Cheap Talk

Kellner and Le Quement (2018) study a cheap talk setting with ambiguous communication. They prove that an ambiguous strategy of the Sender can lead to a pareto improvement compared to the standard non-ambiguous cheap talk. Their setting is based on the standard non-ambiguous cheap talk setting of Crawford and Sobel (1982). The game consists of two players, a Sender and a Receiver. The Sender has private information about a risky payoff-relevant state $\omega \in \Omega = [0, 1]$ and an ambiguous payoff-irrelevant state $\theta \in \Theta$. An Ellsbergian communication strategy is a standard communication strategy $q_\theta(\cdot|\omega) \in \Delta(M)$ for each $\theta \in \Theta$, where M is a finite message space. A strategy of the Receiver is a mapping $M \rightarrow \Delta(\mathbb{R})$. The Receiver's interim belief set is derived by updating the prior state belief p on Ω with respect to each communication strategy $q_\theta(\cdot|\omega)$.

As in the ambiguous persuasion setting, the equilibrium strategy of the Receiver is not ex-ante optimal. However, similarly to the procedure described above, defining beliefs and rectangularity over the general state space $\Omega \times \Theta$ leads to a Perfect Bayesian equilibrium with the same strategies as in the interim equilibrium of Kellner and Le Quement (2018). Thus, ambiguous cheap talk can lead to a pareto improvement compared to the standard non-ambiguous cheap talk, even if players behave dynamically consistently.

7 Conclusion and Discussion

I study dynamically consistent behavior in an ambiguous persuasion setting. First, I show that restricting the message set to straightforward messages and synonyms is without loss of generality. Given this result, I can define beliefs over the more general state space $\Omega \times A \cup \tilde{A}$. This state space allows for the dependence of the risky state and ambiguous signals. Therefore, the Receiver can consider the ambiguous information structure at the ex-ante stage. Then, rectangular beliefs ensure dynamically consistent behavior in ambiguous persuasion and the existence of a Perfect Bayesian equilibrium. Thus, ambiguity induces new equilibria in persuasion settings, even if the players behave dynamically consistently. To conclude, I discuss some related issues and literature.

Ex-ante preferences and commitment device Rectangularity allows players to take their future worst-case beliefs at the ex-ante stage into account and implies that the optimal interim actions become ex-ante optimal. Alternatively, one could assume that the Receiver could commit to his ex-ante optimal action to study dynamically consistent behavior. However, this requires a strong commitment device or a specific updating rule that allows players to ignore all interim beliefs that contradict the ex-ante worst-case belief. Hanany and Klibanoff (2007) propose such updating rules for maxmin preferences.

Concurrent with my work, Cheng (2021) shows that the Sender cannot gain from ambiguous persuasion if the Receiver uses the updating rule of Hanany and Klibanoff (2007). Even if rectangularity and Hanany and Klibanoff (2007) lead to dynamically consistent behavior, they may induce different equilibria. The updating rules of Hanany and Klibanoff (2007) restrict the interim belief set to beliefs that maintain the ex-ante optimal choice interim optimal. In contrast, rectangularity enlarges the ex-ante belief set such that the interim optimal choice becomes ex-ante optimal.

My example can illustrate the difference between the approaches. Using the updating rules of Hanany and Klibanoff (2007), the ex-ante belief set of the judge is Φ^0_Π. Then, ex-ante he would prefer always to acquit the defendant. After updating Φ^0_Π with the updating rules of Hanany and Klibanoff (2007), always acquitting the defendant is still interim optimal. Hence, given the updating rules of Hanany and Klibanoff (2007), the dynamically consistent judge would always acquit the defendant, and the prosecutor cannot benefit from ambiguous persuasion. Given rectangularity, the judge's ex-ante belief set is given by the rectangular hull $\text{rect}(\Phi^0_\Pi)$ and always convicting the defendant is ex-ante and interim optimal. Hence, a dynamically consistent judge with rectangular beliefs will always convict the defendant, and the prosecutor can gain from ambiguous persuasion.

Even if both approaches imply dynamically consistent behavior, the interpretation is different. A Receiver using the updating rules of Hanany and Klibanoff (2007) commits to his ex-optimal choice and ignores any information that would change his worst-case belief. On the other hand, a Receiver with rectangular beliefs considers that he will receive ambiguous information before deciding. Thus, the results of Cheng (2021) do not contradict my result. Further, if Φ^0_Π is rectangular, i.e., $\Phi^0_\Pi = \text{rect}(\Phi^0_\Pi)$, both approaches induce the same equilibria.

Value of Information My work is related to the literature on the negative value of ambiguous information. In my example, the Receiver would prefer to ignore ambiguous information. His ex-ante expected utility given p_0 is higher than his ex-ante expected utility given the rectangular hull. Thus, the Receiver can have a negative value of information.

Ambiguous information induces two effects. On the one hand, an ambiguous communication device generates ambiguous beliefs and, therefore, decreases the worst-case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. The Receiver's value of information is negative if the first effect dominates the second effect. I discuss the (negative) value of information for ambiguous persuasion in greater detail in Section A.2 in the Appendix.

Li (2020) studies the relation between ambiguity aversion and an aversion to (partial) information. She shows that an ambiguity averse decision maker (DM) with maxmin preferences is always (weakly) averse to partial information. However, Li (2020) assumes that the DM's set of acts is the same with and without ambiguous information. In contrast, my setting implies that given p_0, the DM can only choose from constant acts. Given an ambiguous communication device, the DM can choose any act which is measurable with respect to the information partition induced by the communication device. These are precisely the two effects I describe above. On the one hand, an ambiguous information device induces ambiguity, which decreases the utility of an ambiguity averse Receiver. On the other hand, anticipating this information at the ex-ante stage allows the Receiver to choose an action for each message that could occur with positive probability. Li (2020) focuses only on the first effect. Therefore, her result about partial information aversion of maxmin preferences does not imply that the value of information of the Receiver is always negative.[7]

Further ambiguity preferences Maxmin preferences á la Gilboa and Schmeidler (1989) assume that players maximize their worst-case expected utility. Thus, only the worst-case belief influences the expected utility of the Sender and Receiver. The literature proposes more general models as, e.g., the α-maxmin preferences (Ghirardato et al. (2004)) or the smooth ambiguity preferences (Klibanoff et al. (2005)).

Generalizing dynamically consistent behavior in ambiguous communication to settings with α-maxmin preferences or smooth ambiguity preferences consists of two parts. First, generalizing ambiguous communication, and second, generalizing rectangularity or further conditions to imply dynamically consistent behavior.

The first part is discussed by Beauchêne et al. (2019) for ambiguous persuasion. They show that the Sender's preferences can be extended to a broad class, including ambiguity neutral and ambiguity loving. Intuitively, since the Sender can design the signal such that she perfectly hedges against ambiguity, her preferences are not influential. This result directly extends to my setting with dynamically consistent behavior. The Sender only chooses an action at the ex-ante stage and never behaves dynamically inconsistently. Thus, the issue of dynamically inconsistent behavior never occurs for the Sender.

The Receiver may behave dynamically inconsistently, and extending his preferences is more complex. Beauchêne et al. (2019) show, with an example that the Sender can still gain from ambiguous persuasion if the Receiver has α-maxmin preferences. A Receiver with α-maxmin preferences maximizes a weighted sum of worst- and best-case expected utility. However, extending rectangularity to α-maxmin preferences is not straightforward. Recently, Chandrasekher et al. (2022) propose an extension of rectangularity to a more general class of multiple-prior preferences that includes α-maxmin. Extending this definition to ambiguous communication with α-maxmin preferences could be interesting for future research.

To study dynamically consistent behavior in ambiguous communication with smooth ambiguity preferences, one could follow the approach of Hanany and Klibanoff (2009). They propose the smooth rule to update the beliefs of dynamically consistent players with smooth ambiguity preferences. However, in contrast to rectangularity and similar to the updating rules used by Cheng (2021) in an ambiguous communication model, the

[7] The same consideration applies to the cheap talk setting of Kellner and Le Quement (2018). Therefore, the result of Li (2020) does not contradict the pareto improvement result of Kellner and Le Quement (2018).

smooth rule implies that the ex-ante optimal action becomes interim optimal. Therefore, the equilibrium predictions in a setting with smooth ambiguity averse preferences and the smooth rule might differ from maxmin preferences with rectangular beliefs.

Additional exogenous ambiguity Extending the setting to additional exogenous ambiguity is straightforward. Suppose that additionally to ambiguous communication, players are faced with ambiguity about the payoff-relevant state ω. Then, their ex-ante state belief p_0 is replaced by a set of state beliefs $P_0 \subseteq \Delta\Omega$. We can extend Definition 1 of joint beliefs on states and messages as follows:

$$\Phi^0_\Pi = \Bigg\{ \rho^k \in \Delta(\Omega \times (A \cup \tilde{A})) : \exists \pi_k \in \Pi \text{ and } p_0 \in P_0 \text{ s.t. } \rho^k(\omega, m) = \begin{cases} p_0(\omega)\pi_k(m|\omega) & \text{if } m \in \text{supp}(\Pi) \\ 0 & \text{otherwise} \end{cases} \Bigg\}.$$

Then, the extension of all results follows straightforward. Thus, my results do not rely on the assumption that ambiguity arises only due to ambiguous communication.

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A Appendix

A.1 Proofs

Proof of Proposition 1. Corollary 1 of BLL shows that there exists π_1 and π_2 such that $(M, \{\pi_1, \pi_2\})$ generates the same value as (M, Π). Hence, I have to show that (M', Π') generates the same value as $(M, \{\pi_1, \pi_2\})$. I first look at the case where the Sender does not use synonyms.

i) Sender does not use synonyms.

   Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there does not exist $m, m' \in M$ with $m \neq m'$ such that $\hat{a}_m = \hat{a}_{m'}$. Remember that p_m^{π} denotes the posterior state belief of the Receiver given the message m and the communication device π. Furthermore, $\hat{a}_m$ denotes Receivers best response given message $m \in M$ and the communication devices $\{\pi_1, \pi_2\}$. Since $(M, \{\pi_1, \pi_2\})$ does not use synonyms, there exists at most one $m \in M$ for each $a \in A$ such that $a = \hat{a}_m$. I define $\bar{\pi}_i(\cdot|\omega) \in \Delta M'$ with $M' = A$ such that

$$\bar{\pi}_i(a|\omega) = \begin{cases} \pi_i(m|\omega) & \text{if } \exists m \in M \text{ with } a = \hat{a}_m, \\ 0 & \text{otherwise.} \end{cases}$$

   Then, the posterior state belief $p_m^{\pi_i}$ equals the posterior state belief $p_a^{\bar{\pi}_i}$ if $a = \hat{a}_m$. Therefore, $(M, \{\pi_1, \pi_2\})$ and $(M', \{\bar{\pi}_1, \bar{\pi}_2\})$ generate the same set of posterior state beliefs and the same best response of the Receiver. Since the best response does not change, the value of the Sender is the same for both signals.

ii) Sender uses synonyms.

   If $(M, \{\pi_1, \pi_2\})$ uses synonyms I can split M in M_1 and M_2 such that there exist a bijection between M_1 and M_2 and $M_1 \cup M_2 = M$. Then $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$ with

$$\hat{\pi}_i(m|\omega) = \frac{\pi_i(m|\omega)}{\sum_{m \in M_1} \pi_i(m|\omega)}$$

   defines a signal that does not use synonyms. Thus, as in Case i), there exists $(M_1', \{\bar{\pi}_1, \bar{\pi}_2\})$ with $M_1' = A$ that generates the same value as $(M_1, \{\hat{\pi}_1, \hat{\pi}_2\})$. Similar one can define the restriction of π_i to M_2 and find $(M_2', \{\tilde{\pi}_1, \tilde{\pi}_1\})$ with $M_2' = \tilde{A}$, that generates the same value as M_2 and the restriction of π_i to M_2. Then, $(M', \{\pi_1', \pi_2'\})$ with $M' = M_1' \cup M_2'$ and

$$\pi_i'(a|\omega) = \begin{cases} \bar{\pi}_i(a|\omega) \sum_{m \in M_1} \pi_i(m|\omega) & \text{if } a \in A, \\ \tilde{\pi}_i(a|\omega) \sum_{m \in M_2} \pi_i(m|\omega) & \text{if } a \in \tilde{A}, \end{cases}$$

   generates the same value as $(M, \{\pi_1, \pi_2\})$.

■

A.2 Value of Information

Example 1 shows that the Receiver is better off by making his decision based on p_0. Therefore, he would prefer getting no additional information than getting ambiguous information. This result is consistent with the recent literature on the (negative) value of information under ambiguity, e.g., Li (2020) or Hill (2020). However, BLL show in their subsections 6.3 and 6.4 that the Receiver may benefit from listening to an ambiguous device.

I denote with $U^0(a)$ the ex-ante expected utility of action a of the Receiver without any additional information, i.e.,

$$U^0(a) = \int_\Omega u(a,\omega)p_0(\omega)\,\mathrm{d}\omega.$$

Definition 4. *A communication device* Π *has a positive value of information for the Receiver if*

$$\max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\operatorname{rect}(\Phi^0_\Pi)} \mathbb{E}_\rho(u(a_m,\omega)) \geq \max_{a\in A} U^0(a).$$

Ambiguous information induces two effects. On the one hand, an ambiguous communication device generates ambiguous beliefs and, therefore, decreases the worst-case expected utility of the Receiver. On the other hand, the communication device still reveals information about the state. This information allows the Receiver to choose an action that better fits the state and increases his expected utility. Then, the value of information is positive if the second effect exceeds the negative effect of ambiguity and ambiguity aversion.

BLL say that a communication device satisfies a participation constrain if

$$\max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\pi\in\Pi} \int_\Omega\int_M \pi(m|\omega)u(a_m,\omega)\,\mathrm{d}m\,p_0(\omega)\,\mathrm{d}\omega \geq \max_{a\in A} U^0(a).$$

They call this condition a participation constraint since it ensures that the Receiver is willing to pay attention to the communication device. If the participation constraint is not satisfied, the Receiver would be better off ignoring the communication device, ex-ante. Since $\Phi^0_\Pi \subseteq \operatorname{rect}(\Phi^0_\Pi)$, it follows

$$\begin{aligned}
&\max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\pi\in\Pi} \int_\Omega\int_M \pi(m|\omega)u(a_m,\omega)\,\mathrm{d}m\,p_0(\omega)\,\mathrm{d}\omega \\
&= \max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\Phi^0_\Pi} \mathbb{E}_\rho(u(a_m,\omega)) \\
&\geq \max_{(a_m)_{m\in\operatorname{supp}\Pi}\in A^{|\operatorname{supp}\Pi|}} \min_{\rho\in\operatorname{rect}(\Phi^0_\Pi)} \mathbb{E}_\rho(u(a_m,\omega)).
\end{aligned}$$

Hence, any communication device with a positive value of information satisfies the participation constraint of BLL.

BLL characterize a condition that guarantees that the Receiver benefits from listening to a communication device (see BLL Proposition 8). I now translate this condition to my setting. I denote with a_0 the default actions, i.e., the action that maximizes $U^0(a)$.

Definition 5. *Let $\hat{a}_m$ denote the interim optimal action of the Receiver given the belief set* $Bay(\text{rect}(\Phi^0_\Pi)|m)$*. A message m is value-increasing (to the Receiver) if* $\mathbb{E}_{\rho|_m}(u(\hat{a},\omega)) \geq U^0(a_0)$ *for all* $\rho|_m \in Bay(\text{rect}(\Phi^0_\Pi)|m)$*.*

BLL show that a communication device Π satisfies the participation constraint if Π uses only value-increasing messages. The following proposition proves a stronger result and very intuitive result: A communication device that increases the worst-case expected utility of the Receiver for any message has a positive value of information.

Proposition 3. *If Π only uses value-increasing messages, then Π has a positive value of information for the Receiver.*

Proof. Since $\mathbb{E}_{\rho|_m}(u(\hat{a},\omega)) \geq U^0(a_0)$ for all $\rho|_m \in \text{Bay}(\text{rect}(\Phi^0_\Pi)|m)$ it follows that

$$\min_{\rho|_m \in \text{Bay}(\text{rect}(\Phi^0_\Pi)|m)} \mathbb{E}_{\rho|_m}(u(\hat{a},\omega)) \geq U^0(a_0). \tag{A.1}$$

Then, rectangularity and Equation A.1 imply

$$\begin{aligned}
&\max_{(a_m)_{m\in \text{supp}\,\Pi} \in A^{|\text{supp}\,\Pi|}} \min_{\rho \in \text{rect}(\Phi^0_\Pi)} \mathbb{E}_\rho(u(a_m,\omega)) \\
&= \min_{\rho \in \text{rect}(\Phi^0_\Pi)} \int_{\text{supp}(\Pi)} \rho(\Omega,m) \min_{\rho'|_m \in \text{Bay}(\Phi^0_\Pi|m)} \mathbb{E}_{\rho'|_m}(u(\hat{a}_m,\omega))\,\mathrm{d}m \\
&\geq \min_{\rho \in \text{rect}(\Phi^0_\Pi)} \int_{\text{supp}(\Pi)} \rho(\Omega,m) U^0(a_0)\,\mathrm{d}m \\
&= U^0(a_0).
\end{aligned}$$

■

A.3 Ambiguous Cheap Talk

A.3.1 Setting

Kellner and Le Quement (2018) study a standard cheap talk game à la Crawford and Sobel (1982) with two players, a Sender S and a Receiver R. The Sender has private information about a payoff-relevant state ω and a payoff-irrelevant state θ. The state ω follows a uniform distribution on $[0,1]$ whereas θ can take two values, either θ_1 or θ_2. The distribution of θ is fully ambiguous, i.e., $\mathbb{P}(\theta=\theta_1)=p\in[0,1]$ and $\mathbb{P}(\theta=\theta_2)=1-\mathbb{P}(\theta=\theta_1)$. The state θ is payoff-irrelevant, but it allows the Sender to send an ambiguous signal by conditioning her strategy on θ.

The game consists of two stages. First, the Sender observes ω and θ and sends a message m. Then, the Receiver observes m, updates his beliefs, and chooses an action $a\in\mathbb{R}$. The utility of both players depends on the payoff-relevant state ω and the action a chosen by the Receiver:

$$U^S(a, \omega, b) = -(\omega + b - a)^2 \quad \text{where } b > 0$$
$$U^R(a, \omega) = -(\omega - a)^2.$$

The Receiver is ambiguity averse and has maxmin preferences as defined by Gilboa and Schmeidler (1989).

Kellner and Le Quement (2018) generalize the partitional communication strategy of Crawford and Sobel (1982) to the ambiguous setting in the following way: An Ellsbergian partitional communication strategy $\left(\{t_i\}_{i=0}^N, \{c_i\}_{i=0}^{N-1}\right)$ consists of two profiles of thresholds $\{t_i\}_{i=0}^N$ and $\{c_i\}_{i=0}^{N-1}$ with $t_0 = 0 < t_1 < \cdots < t_{N-1} < t_N = 1$ and $c_i \in (t_i, t_{i+1}]$ for all $i = 0, \ldots, N-1$. Given these threshold profiles the Sender sends a signals m_i^j with $j = A, B$ if $\omega \in (t_i, t_{i+1}]$ for all $i = 0, \ldots, N$. Further, she sends m_i^A if (ω, θ) is in $(t_i, c_i) \times \theta_1$ or $[c_i, t_{i+1}] \times \theta_2$ and m_i^B otherwise. Figure A.1 illustrates the signal structure. By observing a signal m_i^j the Receiver learns that ω is in the interval $(t_i, t_{i+1}]$. But due

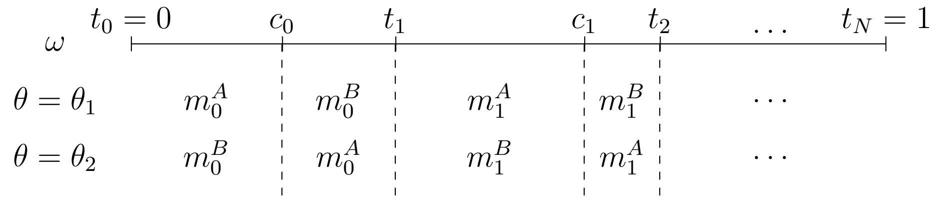

Figure A.1: Signal structure of an Ellsbergian partitional communication strategy

to the ambiguous payoff irrelevant state θ, the Receiver has only ambiguous information about the subintervals (t_i, c_i) and $[c_i, t_{i+1}]$.

Kellner and Le Quement (2018) show that given an Ellsbergian partitional communication strategy, the best interim response of the Receiver after observing m_i^j is the same for m_i^A and m_i^B. More explicitly, after observing m_i^j the best response of the Receiver is $a^*(t_i, t_{i+1}, c_i) = \frac{t_i + t_{i+1} + c_i}{3}$. However, the strategy $a^*(t_i, t_{i+1}, c_i)$ is not ex-ante optimal. Ex-ante the Receiver would prefer the strategy $a^{ea}(t_i, t_{i+1}, c_i) = \frac{t_i + t_{i+1}}{2}$, which equals the equilibrium strategy in the standard cheap talk game of Crawford and Sobel.

Kellner and Le Quement (2018) define an Ellsbergian equilibrium as follows: The Sender chooses her ex-ante best response depending on (ω, θ) and the Receiver maximizes his interim expected utility given the signal m_i^j. They show that if an informative equilibrium exists in the sense of Crawford and Sobel (1982), then an Ellsbergian equilibrium exists that pareto dominates it. However, as in the Bayesian persuasion setting of Beauchêne et al. (2019), the Receiver might behave dynamically inconsistently. Ex-ante, he would prefer to ignore the ambiguous message and choose a strategy that is constant with respect to θ_i. Hence, as in the ambiguous persuasion setting, the effect of dynamically inconsistent behavior and ambiguity averse preferences cannot be separated in the model of Kellner and Le Quement (2018).

A.3.2 Rectangular beliefs

Like in the ambiguous persuasion setting, I define beliefs on a general state space. Here this state space is given by $\Omega \times \Theta$.[8] Let $\rho(\omega, \theta)$ be the joint density of ω and θ and remember that $\mathbb{P}(\theta = \theta_1) = p \in [0, 1]$. Furthermore, let Φ^0 denote the set of ex-ante beliefs

$$\Phi^0 := \{\rho^p(\omega, \theta) = p\mathbb{1}_{\theta=\theta_1} + (1-p)\mathbb{1}_{\theta=\theta_2} \text{ s. t. } p \in [0,1]\}.$$

I can define rectangularity similar to the ambiguous persuasion setting. Instead of repeating the definition, I directly construct the rectangular hull.

Fix an Ellsbergian partitional communication strategy for the Sender. The updated beliefs $\rho|_m(\cdot,\cdot)$ given message m and the marginal probability $\mathbb{P}^\rho(m)$ of observing message m are given by

$$\begin{aligned}
\mathbb{P}^\rho(m_i^A) &= p(c_i - t_i) + (1-p)(t_{i+1} - c_i), \\
\mathbb{P}^\rho(m_i^B) &= (1-p)(c_i - t_i) + p(t_{i+1} - c_i), \\
\rho^p|_{m_i^A}(\omega, \theta) &= \frac{p\mathbb{1}_{\theta=\theta_1,\omega\in(t_i,c_i)} + (1-p)\mathbb{1}_{\theta=\theta_2,\omega\in[c_i,t_{i+1}]}}{p(c_i - t_i) + (1-p)(t_{i+1} - c_i)}, \\
\rho^p|_{m_i^B}(\omega, \theta) &= \frac{(1-p)\mathbb{1}_{\theta=\theta_1,\omega\in[c_i,t_{i+1}]} + p\mathbb{1}_{\theta=\theta_2,\omega\in(t_i,c_i)}}{(1-p)(c_i - t_i) + p(t_{i+1} - c_i)}.
\end{aligned}$$

Then, the pasting of an ex-ante belief $\bar{\rho}$ and a collection of updated beliefs $(\rho|_{\hat{m}})_{\hat{m}}$ is

$$\begin{aligned}
\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\omega, \theta) &:= \mathbb{P}^{\bar{\rho}}(m_i^A)\rho^p|_{m_i^A}(\omega, \theta) + \mathbb{P}^\rho(m_i^B)\rho^p|_{m_i^B}(\omega, \theta) \\
&= \begin{cases}
p\frac{\bar{p}(c_i-t_i)+(1-\bar{p})(t_{i+1}-c_i)}{p(c_i-t_i)+(1-p)(t_{i+1}-c_i)} & \text{if } (\omega,\theta) \in (t_i, c_i) \times \theta_1, \\
(1-p)\frac{\bar{p}(c_i-t_i)+(1-\bar{p})(t_{i+1}-c_i)}{p(c_i-t_i)+(1-p)(t_{i+1}-c_i)} & \text{if } (\omega,\theta) \in [c_i, t_{i+1}] \times \theta_2, \\
p\frac{\bar{p}(t_{i+1}-c_i)+(1-\bar{p})(c_i-t_i)}{p(t_{i+1}-c_i)+(1-p)(c_i-t_i)} & \text{if } (\omega,\theta) \in [c_i, t_{i+1}] \times \theta_1, \\
(1-p)\frac{\bar{p}(t_{i+1}-c_i)+(1-\bar{p})(c_i-t_i)}{p(t_{i+1}-c_i)+(1-p)(c_i-t_i)} & \text{if } (\omega,\theta) \in (t_i, c_i) \times \theta_2,
\end{cases}
\end{aligned}$$

and the rectangular hull is given by

$$\text{rect}(\Phi^0) = \big\{\bar{\rho} \circ (\rho|_{\hat{m}})_{\hat{m}}(\cdot) \text{ s.t. } p, \bar{p} \in [0,1]\big\}.$$

[8]In this setting, the ambiguous payoff irrelevant part of the state space is exogenously given, i.e., the Sender cannot choose the amount of communication devices. Therefore, I can define joint beliefs directly on the state space $\Omega \times \Theta$, and a similar result to Proposition 1 is not needed.

A.3.3 Dynamically Consistent Behavior

Remember that in the setting of Kellner and Le Quement (2018), the interim best response of the Receiver to the Ellsbergian communication strategy $\left(\{t_i\}_{i=0}^{N}, \{c_i\}_{i=0}^{N-1}\right)$ is given by $a^*(t_i, t_{i+1}, c_i) = \frac{t_i+t_{i+1}+c_i}{3}$. However, the ex-ante optimal strategy of the Receiver is $a^{ea}(t_i, t_{i+1}, c_i) = \frac{t_i+t_{i+1}}{2}$. The next proposition shows that rectangular beliefs imply dynamically consistent behavior.

Proposition 4. *Suppose the ex-ante belief set of the Receiver is given by* $\text{rect}(\Phi^0)$*. Then, the strategy* $a^*(t_i, t_{i+1}, c_i) = \frac{t_i+t_{i+1}+c_i}{3}$ *is an ex-ante and interim best response to the Ellsbergian communication strategy* $\left(\{t_i\}_{i=0}^{N}, \{c_i\}_{i=0}^{N-1}\right)$.

Proof. As in the previous setting, it follows straightforwardly that the prior-by-prior Bayesian update of the rectangular hull $\text{rect}(\Phi^0)$ equals the prior-by-prior Bayesian update of Φ^0. Thus, Lemma 1 of Kellner and Le Quement (2018) proves that $a^*(t_i, t_{i+1}, c_i) = \frac{t_i+t_{i+1}+c_i}{3}$ is interim optimal. I only have to prove ex-ante optimality.

Similar to above, one can show that rectangularity establishes the following relation between ex-ante worst-case expected utility and interim worst-case expected utilities:

$$\underbrace{\min_{\rho\in\text{rect}(\Phi^0)} \mathbb{E}^{\rho}\left[U^R(a(m),\omega)\right]}_{\text{ex-ante worst-case exp. utility}}$$
$$= \min_{\bar{p}\in[0,1]} \sum_{i=1}^{N-1} \mathbb{P}^{\bar{p}}(m_i^A) \underbrace{\min_{p\in[0,1]} \mathbb{E}^{\rho}\left[U^R(a(m_i^A),\omega)|m_i^A\right]}_{\text{interim worst-case exp. utility at } m_i^A} + \mathbb{P}^{\bar{p}}(m_i^B) \underbrace{\min_{p\in[0,1]} \mathbb{E}^{\rho}\left[U^R(a(m_i^B),\omega)|m_i^B\right]}_{\text{interim worst-case exp. utility at } m_i^B}.$$

Then, since $a^*(t_i, t_{i+1}, c_i)$ is a best-response at any interim information set, it is as well ex-ante optimal. ■

Applying Proposition 4, I can extend all results from Kellner and Le Quement (2018) to a Perfect Bayesian Equilibrium with rectangular beliefs. Therefore, given rectangular beliefs, all results of Kellner and Le Quement (2018) can be generalized to dynamically consistent behavior. In particular, if an informative equilibrium exists in the sense of Crawford and Sobel (1982), then a Perfect Bayesian Equilibrium exists with rectangular beliefs that pareto dominates it. Thus, the pareto improvement implied by ambiguous cheap talk does not rely on dynamically inconsistent behavior.