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Margin for error semantics and signal perception

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Abstract A joint modelling of objective worlds and subjective perceptions within two-dimensional semantics eliminates the margin for error principle and solves the epistemic sorites paradox. Two objective knowledge modalities can be defined in two-dimensional frames accounting for subjective perceptions: "necessary knowledge" (NK) and "possible knowledge" (PK), the latter being better suited to the interpretation of knowledge utterances. Two-dimensional semantics can in some cases be reduced to one-dimensional ones, by defining accessibility relations between objective worlds that reflect subjective perceptions: NK and PK are respectively equivalent to $\Box\Box$ and $\Diamond\Box$ in some one-dimensional frame, and to $\Box$ and another modality in some other.

Keywords Margin for error · Sorites paradox · Intransitive frames · Positive introspection · Possible worlds semantics

1 A reminder: alternative solutions to the epistemic sorites paradox

1.1 The epistemic sorites paradox

The epistemic sorites paradox can be stated as follows (see, e.g., Williamson 1994). Consider an imperfectly perceptive individual who knows that the height of a tree is below h meters. Given the individual's limited perceptual abilities, such knowledge is

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possible only if the true height is below $h - \eta$ for some $\eta > 0$. Writing $K\Phi$ for "the individual knows Φ", this margin for error principle can be expressed as follows:

$$\text{There exists } \eta > 0 \text{ such that for all } k,\ \ K(h \leq k) \longrightarrow h \leq k - \eta \tag{1}$$

Knowledge is usually considered to satisfy positive introspection, in the sense that if someone knows something, he knows that he knows it:

$$K\Phi \longrightarrow KK\Phi. \tag{2}$$

Combining (1) and (2), by virtue of the closure of K under implication, leads to the following chain of implications:

$$K(h \leq k) \underset{(2)}{\longrightarrow} KK(h \leq k) \underset{(1)}{\longrightarrow} K(h \leq k - \eta) \rightarrow \cdots \rightarrow K(h \leq k - n\eta) \tag{3}$$

for any positive integer n. According to this reasoning, an individual knowing that the height of a tree is less than 20 meters must also know that it is below x meters, however small x may be.

1.2 Williamson's solution: removing positive introspection

Williamson's solution relies on a possible worlds semantics in which the accessibility relation is intransitive, reflecting the intuitive intransitivity of the indistinguishability relation. In its simplest version, the fixed-margin semantics, the set of worlds is endowed with a metric d, and there exists some $\alpha > 0$ such that worlds w and w' are accessible to each other (denoted $w\Re w'$) if and only if $d(w, w') \leq \alpha$. The standard definition of knowledge in possible worlds semantics (namely, $w \models_{(W,\Re)} K\Phi$ if and only if $w' \models_{(W,\Re)} \Phi$ for all w' such that $w\Re w'$) then need not imply that knowledge satisfies positive introspection. Consider for instance the case where $W=\mathbb{R}$, $d(w, w') = |w - w'|$ and $\alpha = 1$. Assume that at world w, the value of some magnitude h (say, the height of a tree) is equal to w. The aforementioned assumptions imply that $w \models_{(W,\Re)} K(h \leq w+1)$, because $h \leq w+1$ is true at all worlds accessible from w (i.e., all worlds located between $w - 1$ and $w + 1$). But knowledge does not satisfy positive introspection: $w \models_{(W,\Re)} \neg KK(h \leq w + 1)$, because $K(h \leq w + 1)$ is true only at worlds w' such that $w' \leq w$, and is therefore not true at all worlds accessible to w (for instance, it is not true at any world w' such that $w' > w$).

The lack of transitivity of margin for error frames is essential to the solution of the paradox, because knowledge necessarily satisfies positive introspection in transitive frames.

1.3 Signal-based solutions

Another approach has been proposed by authors such as Bennett (1998), Mott (1998), Halpern (2004) and Dutant (2007). It is based on considering not only the objective worlds that propositions refer to, but also the subjective signals perceived by individuals (called "precisifications" by Bennett, owing to his more semantic outlook).[1] The main idea is that when interpreting an individual's statement about his knowledge regarding the objective world, one should make inferences about the objective world but also about the individual's subjective perception.

The core of this approach is thus the modeling of individual perceptions. It can be seen as a way of checking Williamson's informal claim that the perceptual margin for error principle is a consequence of the limitations on individuals' perceptual abilities (Williamson 1994, p. 226). As is explained hereafter, this approach solves the sorites paradox because it leads to the conclusion that the margin for error principle as stated in (1) is unwarranted even if perceptual abilities are limited.

The argument goes as follows, when formulated in the terms of Williamson (2000b) canonical example of Mr Magoo trying to estimate the height of a tree. A world corresponds to a given height (let "X" denote the world in which the true height is X meters) but Mr Magoo's perception is imprecise: when the true height is H, the height he perceives (i.e., the signal) can be anywhere between $H - 1$ and $H + 1$. Having perceived the signal S, Mr Magoo thus knows that the true height is between $S - 1$ and $S + 1$. We refer to this world and signal structure as the "imperfect sight case with a margin of error of 1" hereafter.

Under these assumptions, Mr Magoo's uttering of "I know that the height of the tree is less than or equal to 20 m" does not imply that the tree is less than 19 m high. Indeed, perceiving 19 would cause Mr Magoo to know that the true height is between 18 and 20, and thus less than or equal to 20. But since Mr Magoo could perceive 19 when the height is 20, he could know that the height is less than or equal to 20 when the true height is 20. Therefore, the margin for error principle does not hold.[2]

One might find this conclusion puzzling in the presence of a margin for error in perceptions, since, as has been shown by Williamson, the knowledge operator in a margin-for-error Kripkean semantics satisfies (1) (see Sect. 1.2). As we show below, the reason behind this apparent paradox is that a sentence like "I know, given my perception, that the height of the tree is less than 20" cannot be interpreted with reference to a standard knowledge modality, as expressed by the modality $\Box$ in a Kripke model the possible worlds of which coincide with the objective worlds that are the object of propositions.

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[1] For the sake of completeness, Bonnay and Egré's approach to the sorites paradox should also be mentioned. While they do not challenge positive introspection (unlike Williamson) nor the margin for error principle (like Williamson), they constrain the interplay of these two assumptions. In their centered semantics (Bonnay and Egré 2009), (1) and (2) hold but they cannot be combined in a way leading to (3) because an individual's knowledge of his own knowledge does not lead him to know the margin of error associated to his knowledge.

[2] This approach has been criticized in Williamson's (2000a) reply to Mott (1998). The crux of Williamson's argument is that one should not only assume that perceptual abilities are limited (which is the point of signal-based models) but also that individuals' knowledge of their perceptual limitations is subject to the margin for error principle, i.e., arguments relying on the assumption that individuals know the distribution of signals (conditional on the state of the world) should be rejected. We do not view this objection as convincing because it begs the question: it amounts to saying that perceptual knowledge is subject to the margin for error principle because one's knowledge of one's perceptual limitations is.

2 The goal of this paper

2.1 One more look at the signal-based solution to the epistemic sorites paradox

In order to pin down the source of the discrepancy between signal-based reasoning and Williamson's margin-for-error semantics, one might want to describe Mr Magoo's utterances regarding his knowledge without explicit reference to signals (perceived heights), within a margin for error semantics in which a world coincides with a given (true) height. Following Williamson's notations, let MS_α denote the frame defined by the set of worlds $W = \mathbb{R}$ and the accessibility relation $\Re_\alpha$: $x\Re_\alpha y$ iff $|x - y| \leq \alpha$. At first glance, both MS_1 and MS_2 seem to be suitable candidates for the representation of the signal structure outlined above, i.e., the case of imperfect sight with a margin for error of 1. MS_1's appeal comes from the observation that the difference between perceived and actual height is at most 1, and MS_2 's from the observation that at world w, considering all the possible signals (in the interval $[w - 1, w + 1]$), the only worlds that can be safely ruled out are those not belonging to $[w - 2, w + 2]$.

Since the signal-based reasoning sketched above (Sect. 1.3) leads us to discard the margin for error principle, whereas this principle is satisfied by the knowledge modality $\Box$ in all MS_α ($\alpha > 0$), it must be the case that the informal use of the notion of knowledge in the above signal-based argument is at odds with the knowledge modality in margin for error semantics. The reason is the following. In MS_2, a proposition is known at world w if it is true at all worlds belonging to $[w - 2, w + 2]$. Since the set of signals compatible with world w is $[w - 1, w + 1]$, the $[w - 2, w + 2]$ interval coincides with the set of worlds that are compatible with some signal compatible with w. In order words, $\Box\Phi$ in MS_2, or equivalently $\Box\Box\Phi$ in MS_1, can be interpreted as: "for *any* signal compatible with w, Φ is true at all worlds compatible with this signal", or equivalently "at world w, one is bound to know Φ, whatever the observed signal."

However, in our view, this modality is at odds with the way in which the utterance "I know Φ" is understood in natural language. Mr Magoo's utterance need not imply that for *any* signal compatible with w, Φ is true at all worlds compatible with this signal, but rather that Φ is true at all worlds compatible with *the signal actually observed* by Mr Magoo before speaking. In other words, Mr Magoo's utterance only implies that at world w, Φ *can* be known (for some signal compatible with w), not that Φ is *necessarily* known.

2.2 The quest for an equivalence between two-dimensional subjective frames and one-dimensional objective frames

The above observation seems to imply that two-dimensional frames, comprising an "objective" dimension corresponding to the objective worlds that propositions deal with, and a "subjective" one corresponding to perceived signals, are a better tool than one-dimensional frames that leave perceptions aside. According to this view, margin for error semantics are bound to lead to errors if they are taken to be a reduced form for a richer underlying theory encompassing both objective reality and subjective perception. Another possible approach is to try to bridge the gap between the two approaches.

For instance, Egré and Bonnay (2010) prove an equivalence result between their one-dimensional, "objective" centered semantics and Halpern's two-dimensional semantics, and they address the similarity between Halpern's DR modality and Williamson's clarity modality.

The goal of this paper is to further investigate the nature and scope of the equivalence between the two approaches. One can state the problem as follows. A two-dimensional subjective frame can be thought of as a standard Kripke frame, in which a world (labeled an "extended world" in the next few sentences, to avoid ambiguity) is a pair comprising an objective world and a subjective signal, and two extended worlds are mutually accessible if they have the same subjective component. In order for the equivalence question to make any sense, we are interested in propositions the truth of which only depends on the objective world ("objective propositions"). Properly defined knowledge modalities applied to objective propositions can be studied within such a two-dimensional frame.[3] Our question is whether it is possible to study such "objective modalities" within a one-dimensional objective frame, i.e., one in which the set of worlds coincides with that of objective worlds. In the above example, "necessary knowledge" in the subjective frame is equivalent to the standard $\Box$ modality in one objective frame (MS_2), and to the $\Box\Box$ modality in another (MS_1). The aim of this paper is to generalize this type of equivalence result, both to different world and signal structures, and to modalities other than necessary knowledge.

In particular, since we believe that, in a subjective frame, the proper way of interpreting an individual's statement about his knowledge is through a modality meaning "at world w, Φ *can* be known (for some signal compatible with w)", rather than "is necessarily known", we would like to assess whether a modality having this meaning exists a one-dimensional objective frame.

Technically, this question amounts to an attempt to convert a given signal structure, i.e., a given relation between objective worlds and subjective signals, into an accessibility relation on a set of objective worlds.

In the simple case where only one signal is compatible with a given objective world, the signal structure can obviously be expressed through a transitive accessibility relation defined over the set of objective worlds. It suffices to define the accessibility relation as follows: two objective worlds are mutually accessible if and only if they are associated to the same signal. Conversely, any transitive frame can be interpreted as reflecting a signal structure, where the signal observed at a given world coincides with the equivalence class to which this world belongs, with respect to the accessibility relation (which is an equivalence relation if it is transitive in addition to being reflexive and symmetric).

Our question can thus be rephrased as follows: does a similar equivalence between signal structures and accessibility relations over objective worlds obtain when several signals are compatible with a given objective world (which seems a reasonable assumption if the goal is to account for imprecise perceptions)? Or, in other words: do non-transitive frames (such as margin for error frames) have a signal-detection

[3] This is Halpern's (2004) and Dutant's (2007) approach, as well as Egré and Bonnay's (2010) to some extent.

foundation, and if they do, how does this foundation affect the epistemic interpretation of modalities?

In our view, the quest for such equivalence results is justified by reasons going beyond the goal of narrowing the gap between two well-established approaches. On the one hand, one-dimensional objective semantics are handier than two-dimensional ones, if only because in two-dimensional semantics, the restriction of one's interests to objective propositions is a factor of complexity. On the other hand, two-dimensional semantics are relevant if one believes that the source of vagueness is the presence of noise in individual perceptions. Using one-dimensional objective semantics without explicitly modeling signals can lead to mistakes, as the above discussion shows.[4] Our goal is thus to combine the advantages of both approaches, namely, the simplicity of one-dimensional objective semantics and the greater accuracy of two-dimensional subjective semantics, and to cast light on the epistemic interpretation of modalities in non-transitive frames.

3 Two-dimensional subjective frames

3.1 Definition of the two-dimensional frame associated to a signal structure

The following definitions amount to a restatement of Halpern's model. Consider a set of objective worlds W, a set of signals S, a function c from W to $2^S \backslash \emptyset$ (to be interpreted as "$c(w)$ is the set of signals that can be observed at world w") and a function γ from S to 2^W (to be interpreted as "$\gamma(s)$ is the set of worlds at which s can be observed") such that $w \in \gamma(s)$ iff $s \in c(w)$. We say that w and s are compatible iff $w \in \gamma(s)$ (or, equivalently, $s \in c(w)$). We can then define a frame such that

- the set of possible worlds, called "extended worlds", is the set of all pairs (w, s) such that $w \in W$ and $s \in c(w)$; and
- the accessibility relation is defined as follows: (w, s) and (w', s') are mutually accessible if and only if $s = s'$.

We refer to this frame as (W, S, γ), and we call such frames "two-dimensional subjective frames" hereafter.[5]

3.1.1 The standard knowledge modality in a two-dimensional subjective frame

The accessibility relation defined above is transitive. The knowledge modality it induces (defined by $(w, s) \models_{(W,S,\gamma)} \Box\Phi$ iff Φ is true at all worlds compatible with s) thus has all the usual properties (including positive introspection). It is not an objective modality, in the sense that even if a proposition Φ is objective (i.e., its truth value only

[4] As Dutant (2007) points out, the root of the problem is that if knowledge statements result from perceptions, interpreting them in terms of modal propositions over objective worlds only may result in the loss of relevant information about perceptions.

[5] This expression is not literally exact, since these frames comprise both an objective dimension and a subjective one. We choose to call them "two-dimensional subjective frames" in order to stress their difference with one-dimensional frames comprising only an objective dimension.

depends on the objective world), $\Box\Phi$ is not an objective proposition. This can be seen by going back to the example of Mr Magoo trying to estimate the height of a tree: Mr Magoo knows that the height is below 20 at the extended world (19,18), but not at the extended world (19,20), even though the objective world is the same (19).

3.1.2 Objective knowledge modalities: possible knowledge and necessary knowledge

Since we are looking for an equivalence between two-dimensional subjective frames and one-dimensional objective frames (i.e., frames such that the set of worlds coincides with W, the set of objective worlds), we must start by defining objective knowledge modalities in two-dimensional subjective frames—the expression "objective modality" referring to modalities M having the following characteristics: (i) M applies to objective propositions; and (ii) for any objective proposition Φ, $M\Phi$ is an objective proposition.

In order to define objective knowledge modalities, it is natural to try to derive them from the standard, but non-objective, knowledge modality. Two modalities stand out as natural candidates, which we denote possible knowledge ("PK") and necessary knowledge ("NK").

Definition Consider an objective proposition Φ. Φ is *possibly known* at w (denoted $w \models_{(W,S,\gamma)} PK\Phi$) iff there exists $s \in c(w)$ such that for all $w' \in \gamma(s)$, Φ is true at w'. In other words, Φ is possibly known at w iff there exists a signal s compatible with w such that $(w, s) \models_{(W,S,\gamma)} \Box\Phi$.

Definition Consider an objective proposition Φ. Φ is *necessarily known* at w (denoted $w \models_{(W,S,\gamma)} NK\Phi$) iff for any $s \in c(w)$ and any $w' \in \gamma(s)$, Φ is true at w'. In other words, Φ is *necessarily known* at w iff for any signal s compatible with w, $(w, s) \models_{(W,S,\gamma)} \Box\Phi$.

Φ is possibly known at w iff there exists a signal compatible with w such that the observation of this signal, combined with an individual's knowledge of the compatibility relation and of the truth values of Φ at all worlds, is enough to allow the individual to conclude that Φ is true. Φ is necessarily known at w if this is the case, not for some signal compatible with w, but for all of them.

NK and PK respectively coincide with the DR and $\neg D\neg R$ modalities in Halpern's two-dimensional semantics, with a slight abuse of notation.[6] These two modalities coincide with the standard knowledge modality $\Box$ if, for each w, $c(w)$ is a singleton.

In the light of the above discussion, we view possible knowledge as a better modality than necessary knowledge in order to describe what can be inferred from Mr Magoo's utterance "I know Φ". Without knowing the state of the world or the signal observed by Mr Magoo, the only conclusion one can draw from such an utterance is that for some signal s compatible with the true world w, $(w, s) \models_{(W,S,\gamma)} \Box\Phi$, which is equivalent to $w \models_{(W,S,\gamma)} PK\Phi$.

[6] Formally, the evaluation of a proposition in Halpern's model is made at an extended world but the validity of propositions starting with DR or $\neg D\neg R$ (unlike that of those starting with the R modality) only depends on the objective world.

The relationship between possible knowledge, necessary knowledge, and the non-objective knowledge modality (□) can be clarified by noticing two implications that follow immediately from the definition of PK and NK:

Remark For any extended world (w, s), $w \models_{(W,S,\gamma)} NK\Phi \implies (w, s) \models_{(W,S,\gamma)} \Box\Phi \implies w \models_{(W,S,\gamma)} PK\Phi$.

3.2 A few properties of the PK and NK modalities

Proposition 1 *NK satisfies factivity and closure under implication and conjunction. NK need not satisfy positive nor negative introspection.*

Proposition 2 *PK satisfies factivity and positive introspection. PK need not satisfy negative introspection.*

Proposition 3 *If the following two conditions hold: there exists $(s, s') \in S \times S$ such that $\gamma(s) \cap \gamma(s') \neq \varnothing$ (A1); and there exists no $s'' \in S$ such that $\gamma(s'') \subset \gamma(s) \cap \gamma(s')$ (A2), then PK does not satisfy closure under implication nor under conjunction. This implies that there exists no Kripke frame such that PK coincides with the standard knowledge modality □ on that frame.*

3.2.1 A comment on the failure of PK to satisfy closure

The failure of PK to satisfy closure under conjunction is compatible with the closure of individual knowledge. This is because the PK modality bears only an indirect relationship to an individual's actual knowledge: for any objective proposition Φ, $PK\Phi$ is a objective proposition about Φ, meaning that Φ can be known for some signal compatible with the true world. It is not a statement about whether Φ is actually known. That Φ is known given the signal that was observed can be expressed by means of the □ modality (in the two-dimensional frame). However, while $(w, s) \models_{(W,S,\gamma)} \Box\Phi$ ("Φ is known given the observed signal") implies $w \models_{(W,S,\gamma)} PK\Phi$ ("Φ can be known for some signal"), these two propositions do not coincide. The objective modality PK is thus related to but different from the (non-objective) knowledge modality □. In particular, the failure of PK to satisfy closure should not be construed as meaning that knowledge does not satisfy closure. On the contrary, □, which is the standard knowledge modality in the two-dimensional frame, satisfies closure.

PK is not closed under conjunction because it is possible that, at a given objective world w, Φ be possibly known after some signal s compatible with w is observed, and Ψ be possibly known after some signal s' compatible with w is observed, while [Φ and Ψ] cannot be known because no single signal allows one to know both Φ and Ψ. As an illustration, consider again Mr Magoo's tree. At $w = 20$, the proposition "$H \leq 20$" is possibly known (because $H \leq 20$ is known if the perceived height is 19), and, likewise, "$H \geq 20$" is possibly known (because $H \geq 20$ is known if the perceived height is 21). But the conjunction of these propositions, namely "$H = 20$" is not possibly known, because no signal provides so precise information.

This example highlights why closure cannot be satisfied by the possible knowledge modality: closure under conjunction would in essence amount to assuming that knowledge can result from the joint observation of all the signals compatible with the true world, which, under many standard signal structures, would be enough to know which world one is in. Closure under conjunction would thus eliminate vagueness.

These remarks show that the lack of closure under conjunction of PK is in accordance with the meaning of the proposition "Φ can be known" in natural language.[7]

3.2.2 *A comment on the failure of NK to satisfy positive introspection*

The failure of NK to satisfy positive introspection is in accordance with the meaning of necessary knowledge in natural language. While it sounds strange to claim that an individual can know something without knowing that he knows it (at least when discussing conscious knowledge, as is relevant since the starting point of the epistemic sorites paradox is a knowledge utterance), it makes perfect sense to claim that an individual can know Φ necessarily (in the sense that *any* possible signal would have caused him to know Φ) without knowing (and, a fortiori, without necessarily knowing) that his knowledge of Φ was necessary. If the height of the tree is 20, Mr Magoo necessarily knows that the height is less than 22 (because perceived height can be at most 21). However, if he perceives 21, Mr Magoo, while knowing the true height to be below 22 (and knowing that he knows it, knowing that he knows that he knows it, etc.), cannot know that this knowledge is necessary. Since the signal 21 is compatible with a true height of 22, and the knowledge that the height is below 22 is possible, but not necessary at world 22, Mr Magoo cannot know that his knowledge is necessary.

This observation supports the case for an interpretation of margin for error semantics as a signal-free reduced form for signal-based semantics, in which the $\Box$ modality (in a signal-free, objective frame) does not mean "knowing" but rather is equivalent to "necessarily knowing" in the corresponding non-reduced signal-based frame. Without such an interpretation, the failure of $\Box$ to satisfy positive introspection in margin for error semantics is problematic because it is at odds with common-sense intuitions about what it means to (consciously) know something.[8] With our interpretation of $\Box$ (or $\Box\Box$, depending on the frame) as a modality expressing necessary knowledge (as is the case in the simple case of fixed-margin semantics, see above and see the generalization in Sect. 4), the absence of positive introspection does not run counter to intuition any more.

3.2.3 *The failure of PK and NK to satisfy negative introspection*

The failure of PK to satisfy negative introspection can be illustrated as follows, using the same framework as above. Consider the proposition "the height of the tree is not 20", denoted Φ, and consider a signal s compatible with world 20, assuming without loss of generality that $s \leq 20$. $\neg PK\Phi$ is true at world 20 because PK is factive. s is compatible with world $s-1$, at which $PK\Phi$ is true because world $s-1$ is compatible

7 Similar comments can be made about the failure of PK to satisfy closure under implication.

8 Unease with the lack of positive introspection in Williamson's margin semantics is one of the reasons motivating Bonnay and Egré's search for an alternative solution to the epistemic sorites paradox.

with the signal $s-2$, the observation of which would allow Mr Magoo to rule out all heights strictly greater than $s-1$, including 20. Therefore, whatever the observed signal compatible with world 20, Mr Magoo cannot rule out that at the true world, $PK\Phi$ be true. Therefore, at world 20, $\neg PK\Phi$ is true but $PK\neg PK\Phi$ is not.

With the same notation, $\neg NK\Phi$ is true at world 20 because NK is factive. However, the proposition $NK\neg NK\Phi$ does not hold at world 20, because the signal 19 is compatible with world 20, and it does not allow Mr Magoo to infer $\neg NK\Phi$. $s=19$ is indeed compatible with world 18, at which $NK\Phi$ is true.

3.3 PK, NK, and the margin for error principle

In order to state general results about the margin for error principle, we need to impose some more structure on frames. Consider a set of worlds W assumed to be an open interval of $\mathbb{R}$ and a set of signals $S=W$. We interpret a world as a true value for some parameter (e.g., the height of a tree) and a signal as a perceived value. Assume that there exist two continuous functions v and V from W to W such that

$$v \text{ and } V \text{ are strictly increasing;} \tag{4}$$

$$\text{for any } w\in W,\ v(w)<w<V(w); \tag{5}$$

$$v(W)=V(W)=W; \tag{6}$$

and

$$\text{for any } w\in W,\ c(w)=[v(w),V(w)]. \tag{7}$$

(4) and (6) imply that v and V are bijections from W to W and, combined with (7), that $\gamma(s)=\left[V^{-1}(s),v^{-1}(s)\right]$.[9] They correspond to cases where perceptions are about some real-valued parameter and the observed signal is within an interval containing the true parameter, with the lower and upper bounds of the interval moving in the same direction as the true parameter.

We show below that under these assumptions, PK does not satisfy the margin for error principle, and we state an additional condition on the signal structure that is sufficient for NK to satisfy it.[10]

Definition An operator $Op.$ satisfies the margin for error principle iff there exists $\eta>0$ such that for any $k\in\mathbb{R}$: (i) $Op\cdot(w\le k)\longrightarrow w\le k-\eta$ and (ii) $Op\cdot(w\ge k)\longrightarrow w\ge k+\eta$.

Proposition 4 *If* (4)–(7) *hold, then* PK *does not satisfy the margin for error principle.*

Proposition 5 *If* (4)–(7) *hold, and there exists* $\alpha>0$ *such that for any* $w\in W$, $V(w)-w>\alpha$ *and* $w-v(w)>\alpha$, *then* NK *satisfies the margin for error principle.*

[9] Notice that Assumption (5) cannot hold unless W is open, given (4) and (6).

[10] The example of Sect. 1.3 is a particular case of this model, with $v(w)=w-1$ and $V(w)=w+1$.

4 Representing two-dimensional subjective frames through one-dimensional objective frames

4.1 A general representation result for the NK modality

The following equivalence result is very general. It does not hinge on any of the assumptions of Sect. 3.3.

Proposition 6 *Consider a two-dimensional subjective frame (W, S, γ), and the binary relation $\Re_\gamma$ over W defined as follows: $w \Re_\gamma w'$ iff there exists a signal compatible with both w and w' (i.e., there exists s such that $w \in \gamma(s)$ and $w' \in \gamma(s)$). Then (i) $\Re_\gamma$ is reflexive and symmetric; and (ii) NK in the two-dimensional subjective frame (W, S, γ) is equivalent to the $\Box$ modality in the one-dimensional objective frame $(W, \Re_\gamma)$: $w \models_{(W,S,\gamma)} NK\Phi$ iff $w \models_{(W,\Re_\gamma)} \Box\Phi$.*

In the case of imperfect sight with a margin of error of 1, $(W, \Re_\gamma)$ coincides with MS_2 and Proposition 6 boils down to the observation (see Sect. 2.1 above) that $\Box$ in MS_2 is equivalent to necessary knowledge—an equivalence already noted by several authors in slightly different terms. This example shows that $\Re_\gamma$ need not satisfy transitivity.

4.2 The lack of a general representation result for the PK modality

We argue above (Sect. 3) that the PK modality is the most relevant one in order to interpret knowledge utterances. This view implies that signal-free margin for error semantics cannot be seen as a reduced form for richer signal-based semantics unless PK can be expressed as a modality in a signal-free margin for error semantics, just like NK.

However, Proposition 6 has no counterpart for PK. This is because the binary relation binary relation $\Re_\gamma$ defined over the set of objective worlds W does not contain information that is sufficient for the recovery of the PK modality, as the following example shows. Consider the following two-dimensional frames, $F1$ and $F2$. In both frames, $W = \mathbb{R}$. However, the signal structure is different. In F_1, the signal structure is characterized by $S_1 = \mathbb{R}$ and for any $w \in W$, $c_1(w) = [w-1, w+1]$. In F_2,

$$S_2 = \mathbb{R} \times \{\text{“}imprecise\text{”}\} \cup \mathbb{R} \times \{\text{“}precise\text{”}\} \text{ and}$$
$$\text{for any } w \in W,\ c_2(w) = [w-1, w+1] \times \{\text{“}imprecise\text{”}\} \cup \{w\} \times \{\text{“}precise\text{”}\}.$$

The signal structure in F_2 can be described as follows: it is known whether the signal is precise or imprecise. If it is precise, it is known to coincide with the true world. If it is imprecise, it is known that it can be anywhere within a distance of 1 of the true world.

Both subjective frames give rise to the same binary relation over W. Defining $\Re_{i\gamma}$ as the binary relation over W derived from F_i ($i = 1$ or $i = 2$) according to the definition in Proposition 6:

$$w\Re_{1\gamma}w' \iff w\Re_{2\gamma}w' \iff w' \in [w-2, w+2]\,.$$

However, PK is not identical in both frames:

$$\left[w \models_{F_1} PK\Phi\right] \iff \left[\exists x \text{ s.t. (i) } |x-w| \leq 1 \text{ and (ii) } \forall w' \in [x-1, x+1]\,, w' \models \Phi\right],$$

while

$$\left[w \models_{F_2} PK\Phi\right] \iff [w \models \Phi]\,.$$

This possibility that two two-dimensional subjective frames, with two corresponding PK modalities that are different, can lead to the same binary relation in the one-dimensional objective frame (defined as per Proposition 6), implies that this binary relation does not contain sufficient information to allow one to retrieve the PK modality.

4.3 A partial representation result for the PK modality

There exists a representation result for PK under the assumptions of Sect. 3.3 (i.e., Assumptions (4)–(7)). Stating it requires one to define a specific modality in one-dimensional frames.

Definition Given a Kripke frame $(W, \Re)$, we define the $\triangledown$ modality as follows: $w \models_{(W,\Re)} \triangledown\Phi$ iff there exists $T \subset W$ s.t. (a) $w \in T$; (b) $T = \cap_{w' \in T} \left\{w'' \text{ s.t. } w'\Re w''\right\}$; (c) For all $w' \in T$, $w' \models \Phi$.

Remark (i) $\triangledown$ is a weaker modality than $\Box$. (ii) If $\Re$ is transitive, $\triangledown$ coincides with $\Box$. (iii) $\triangledown$ satisfies positive introspection.

Proposition 7 *Given a two-dimensional subjective frame (W, S, γ), if (4)–(7) hold, then PK in the two-dimensional subjective frame (W, S, γ) is equivalent to the $\triangledown$ modality in the one-dimensional objective frame $(W, \Re_\gamma)$: $w \models_{(W,S,\gamma)} PK\Phi$ iff $w \models_{(W,\Re_\gamma)} \triangledown\Phi$.*

4.4 Additional representation results

We show hereafter that under some additional assumptions, another representation of PK and NK as modalities in a one-dimensional objective frame exists. The one-dimensional frame involved in this representation is different from the one mentioned above, $(W, \Re_\gamma)$. We believe that these representations are interesting because they involve the usual interpretations of the $\Diamond$ and $\Box$ modalities.

We make hereafter the same assumptions as in Sect. 3.3 (i.e., (4)–(7)). We also make the following, additional assumption:

$$V = v^{-1}. \tag{8}$$

(8) means that whenever the world x and the signal y are compatible, so are the world y and the signal x. In other words, if x can look like y, then y can look like x.

Given the monotonicity of v and V (Assumption (4)), (8) is equivalent to the assumption that if $x \in c(y)$, then $y \in c(x)$.[11]

We define the binary relation $\Re'_\gamma$ as follows: for any pair of objective worlds $(w, w') \in W \times W$, $w\Re'_\gamma w'$ iff $w' \in c(w)(= [v(w), V(w)])$. (By comparison, (8) implies that $\Re_\gamma$, as defined in Subsection 4.1, is such that $w\Re_\gamma w'$ iff $w' \in [v(v(w)), V(V(w))]$.)

Notice that if $v(w) = w - \alpha$ and $V(w) = w + \alpha$, then

- (8) holds;
- $(W, \Re'_\gamma)$ coincides with MS_α; and
- $(W, \Re_\gamma)$ coincides with $MS_{2\alpha}$.

(5) implies that $\Re'_\gamma$ is reflexive and (8) is equivalent to $\Re'_\gamma$ being symmetric (see the proof in the Appendix).

The following representation result holds.

Proposition 8 *Given a two-dimensional subjective frame (W, S, γ), if (4)–(8) hold, then (i) PK in the two-dimensional subjective frame (W, S, γ) is equivalent to the $\Diamond\Box$ modality in the one-dimensional objective frame $(W, \Re'_\gamma)$ ($w \models_{(W,S,\gamma)} PK\Phi$ iff $w \models_{(W,\Re'_\gamma)} \Diamond\Box\Phi$); and (ii) NK in (W, S, γ) is equivalent to the $\Box\Box$ modality in $(W, \Re'_\gamma)$ ($w \models_{(W,S,\gamma)} NK\Phi$ iff $w \models_{(W,\Re'_\gamma)} \Box\Box\Phi$).*

This result is in line with the standard meaning of the modalities $\Diamond$ and $\Box$: if $\Diamond$ means "possibly" and $\Box$ can mean both "necessarily" and "is it known that", then $\Diamond\Box$, which is proved to be equivalent to possible knowledge, can be translated literally as "it is possible to know", and $\Box\Box$, which is proved to be equivalent to necessary knowledge, as "it is necessarily known".

5 Concluding remarks

Table 1 wraps up the main results and compares the possible knowledge and necessary knowledge operators.

Our results can be summarized as follows. Mechanically extending the standard definition of knowledge in terms of accessible worlds to the case of nontransitive frames (such as margin for error frames) leads to mistakes and paradoxes because, unlike in the case of transitive frames, there is no obvious bijection between an accessibility relation and an underlying, unmodeled signal structure. Once signals and their perception are explicitly modeled, there is no single compelling definition of knowledge any more, but rather two: necessary knowledge and possible knowledge. These modalities, which are defined with reference to two-dimensional frames and "extended worlds" comprising both an objective and a subjective dimension, are equivalent to some modalities in standard, one-dimensional objective frames. Possible knowledge,

[11] See the proof in the Appendix.

Table 1 Summary of the main results

	Possible knowledge	Necessary knowledge
Equivalent in Halpern's semantics	¬D¬R	DR
Matches intuition about Mr. Magoo's utterance	Yes	No
Factivity	Yes	Yes
Closure under implication	No	Yes
Positive introspection	Yes	No
Negative introspection	No	No
Satisfies margin for error principle	No	Yes
	(under Assumptions (4)–(7))	
Equivalent modality in some objective frame	$\Diamond\Box$	$\Box\Box$
	(under Assumptions (4)–(8))	
Equivalent modality in some other objective frame	$\triangledown$ (under Assumptions (4)–(7))	$\Box$

which satisfies positive introspection, but not closure nor the margin for error principle (which solves the sorites paradox) is in our view well-suited to the interpretation of statements of knowledge.

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Appendix: Proofs

Proof of Proposition 3

We prove first the lack of closure under conjunction. Consider two propositions P and Q such that P is true only at all worlds belonging to $\gamma(s)$ and Q is true only at all worlds belonging to $\gamma(s')$. By construction, both P and Q are possibly known at all worlds belonging to $\gamma(s) \cap \gamma(s')$, which is not empty (by $A1$). But, since $P \wedge Q$ is true only at all worlds belonging to $\gamma(s) \cap \gamma(s')$, and $(A2)$ implies that no signal s'' is such that all worlds compatible with s'' belong to $\gamma(s) \cap \gamma(s')$, $P \wedge Q$ is not possibly known at any world. To prove the lack of closure under implication, it suffices to prove that the closure of PK under implication would imply its closure under conjunction. Assume indeed that closure under implication holds. Since $(P \to (Q \to (P \text{ and } Q)))$ is valid, it is possibly known. Then if both $PK(P)$ and $PK(Q)$ hold, it follows from closure under implication that $PK(P \wedge Q)$ holds. □

Proof of Proposition 4 The set of worlds compatible with $v(w_0)$ is

$$\gamma(v(w_0)) = \left[V^{-1}(v(w_0)), v^{-1}(v(w_0))\right] = \left[V^{-1}(v(w_0)), w_0\right].$$

Therefore, at all worlds compatible with the signal $v(w_0)$, which is compatible with the world w_0, "$w \leq w_0$" is true. Therefore, PK("$w \leq w_0$") is true at w_0, implying that PK does not satisfy part (i) of the definition of the margin for error principle. The proof regarding part (ii) is identical. □

Proof of Proposition 5 Assume that for any $w \in W$, $V(w) - w > \alpha$ and $w - v(w) > \alpha$ for some $\alpha > 0$. Let Φ_k denote a proposition that is true at world w iff $w \leq k$.

$$\begin{aligned}
w \ \models \ & _{(W,S,\gamma)} NK\Phi_k \\
& \Longleftrightarrow \forall s \in [v(w), V(w)] \forall w' \in \left[V^{-1}(s), v^{-1}(s)\right], \ w' \models_{(W,S,\gamma)} \Phi_k \\
& \Longleftrightarrow \forall w' \in \left[V^{-1}(v(w)), v^{-1}(V(w))\right], \ w' \models_{(W,S,\gamma)} \Phi_k \\
& \Longrightarrow \forall w' \in [w - 2\alpha, w + 2\alpha], \ w' \models_{(W,S,\gamma)} \Phi_k \\
& \Longrightarrow (w + 2\alpha) \models_{(W,S,\gamma)} \Phi_k \\
& \Longleftrightarrow w \leq k - 2\alpha.
\end{aligned}$$

A similar proof holds for propositions of the form Ψ_k, where Ψ_k is true at world w iff $w \geq k$. □

Proof of Proposition 6 $w \models_{(W,S,\gamma)} NK\Phi \Longleftrightarrow \forall s \in c(w), \forall w' \in \gamma(s), w' \models \Phi \Longleftrightarrow \forall w'$ s.t. $(\exists s$ s.t. $(w, w') \in (c(w))^2)$, $w' \models \Phi \Longleftrightarrow w \models_{(W,\Re)} \Box\Phi$. □

Proof of Proposition 7 Notice first that if Assumptions (4)–(7) hold, then the binary relation $\Re_\gamma$ over W, as defined in Proposition 7, can be described as follows: $w\Re_\gamma w'$ iff $w' \in \left[V^{-1}(v(w)), v^{-1}(V(w))\right]$.

Proof of the implication $w \models_{(W,S,\gamma)} PK\Phi \Longrightarrow w \models_{(W,\Re_\gamma)} \nabla\Phi$

$w \models_{(W,S,\gamma)} PK\Phi \Longrightarrow \exists s$ s.t. $s \in c(w)$ and $\forall w' \in \gamma(s)$, $w' \models \Phi$. Define $T = \gamma(s) = \left[V^{-1}(s), v^{-1}(s)\right]$. $s \in c(w)$ implies $w \in T$. Also,

$$\begin{aligned}
\bigcap_{w' \in T} \left\{w'' \text{ s.t. } w'\Re_\gamma w''\right\} &= \bigcap_{w' \in T} \left[V^{-1}(v(w')), v^{-1}(V(w'))\right] \\
&= \left[V^{-1}(v(Max(T))), v^{-1}(V(Min(T)))\right] \\
&= \left[V^{-1}(v(v^{-1}(s))), v^{-1}(V(V^{-1}(S)))\right] \\
&= \left[V^{-1}(s), v^{-1}(s)\right] = T,
\end{aligned}$$

so that (b) holds. Therefore, there exists $T \subset W$ satisfying (a), (b) and (c).

Proof of the implication $w \models_{(W,\Re_\gamma)} \nabla\Phi \Longrightarrow w \models_{(W,S,\gamma)} PK\Phi$

Assume that there exists $T \subset W$ satisfying (a), (b) and (c). (b) implies that

$$T = \bigcap_{w' \in T} \left[V^{-1}(v(w')), v^{-1}(V(w'))\right]. \tag{9}$$

As an intersection of closed bounded intervals, T is therefore a closed bounded interval. (9) implies that

$$T = \left[V^{-1}(v(Max(T))), v^{-1}(V(Min(T)))\right], \text{ so that}$$
$$Min(T) = V^{-1}(v(Max(T))) \text{ and } Max(T) = v^{-1}(V(Min(T))).$$

These equalities imply that $V(Min(T)) = v(Max(T))$. Let s denote the common value of $V(Min(T))$ and $v(Max(T))$. The above equalities imply that $T = [Min(T), Max(T)] = \left[V^{-1}(s), v^{-1}(s)\right] = \gamma(s)$. Therefore, (b) and (c) imply that $w' \models \Phi$ for all $w' \in \gamma(s)$. Also, (a) and the identity $T = \gamma(s)$ imply that $w \in \gamma(s)$. Therefore, (a), (b) and (c) imply the existence of a signal s compatible with w such that for all w' compatible with s, $w' \models \Phi$. This is equivalent to $w \models_{(W,S,\gamma)} PK\Phi$.
Proof of the claim that Assumption (8) is equivalent to $\Re'_\gamma$ being symmetric The symmetry of $\Re'_\gamma$ is equivalent to the following equivalence: for any (w, w'), $w \in c(w') \iff w' \in c(w)$. If (8) holds and $w \in c(w') = \left[v(w'), V(w')\right]$, then $w' \in \left[V^{-1}(w'), v^{-1}(w')\right] \underset{(8)}{=} \left[v(w'), V(w')\right] = c(w')$.

Assume conversely that $\Re'_\gamma$ is symmetric. The following two chains of implications hold:

$$V(w) \in c(w) \underset{symmetry}{\Longrightarrow} w \in c(V(w)) = [v(V(w)), V(V(w))] \Longrightarrow v(V(w)) \leq w;$$

and

$$v(V(w)) \in c(V(w)) \underset{symmetry}{\Longrightarrow} V(w) \in c(v(V(w))) = [v(v(V(w))), V(v(V(w)))]$$
$$\Longrightarrow V(v(V(w))) \geq V(w) \underset{(4)}{\Longrightarrow} v(V(w)) \geq w;$$

which together imply $v(V(w)) = w$, i.e., (8). □
Proof or Proposition 8 (i) $w \models_{(W,S,\gamma)} PK\Phi \iff \exists s$ s.t. $(s \in c(w)$ and $\forall w' \in \gamma(s),\ w' \models \Phi) \iff \exists s$ s.t. $(s \in [v(w), V(w)]$ and $\forall w' \in [w(s), V(s)],\ w' \models \Phi) \iff \exists s$ s.t. $(w\Re'_\gamma s$ and $\forall w'$ s.t. $s\Re'_1 w',\ w' \models \Phi) \iff \exists s$ s.t. $(w\Re'_\gamma s$ and $s \models_{(W,\Re'_1)} \Box\Phi) \iff w \models_{(W,\Re'_\gamma)} \Diamond\Box\Phi$. (ii) $w \models_{(W,S,\gamma)} NK\Phi \iff \forall s$ s.t. $s \in c(w)$ and $\forall w' \in \gamma(s),\ w' \models \Phi \iff \forall s$ s.t. $s \in [v(w), V(w)]$ and $\forall w' \in [v(s), V(s)],\ w' \models \Phi \iff \forall s$ s.t. $w\Re'_\gamma s$ and $\forall w'$ s.t. $s\Re'_\gamma w',\ w' \models \Phi \iff \forall s$ s.t. $w\Re'_\gamma s$, $s \models_{(W,\Re'_\gamma)} \Box\Phi \iff w \models_{(W,\Re'_\gamma)} \Box\Box\Phi$. □

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